SEMI-CLASSICAL LIMIT OF THE LOWEST EIGENVALUE OF A SCHRÖDINGER OPERATOR ON A WIENER SPACE : I. UNBOUNDED ONE PARTICLE HAMILTONIANS

by

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Abstract. — We study a semi-classical limit of the lowest eigenvalue of a Schrödinger operator on a Wiener space. The Schrödinger operator is a perturbation of the second quantization operator of an unbounded self-adjoint operator by a C^3-potential function. This result is an extension of [**1**].

1. Introduction

In [**1**], we studied the semi-classical limit of the lowest eigenvalue of Schrödinger operators which are perturbations of the number operator. In that case, one particle Hamiltonian (the coefficient operator of the second order differential operator) is identity operator. However, we need to study the case where the coefficient operator is unbounded to study $P(\phi)$-type Hamiltonians. For example, the typical coefficient operator is $\sqrt{m^2 - \Delta}$, where $m > 0$ and Δ is the Laplace-Bertlami operator on $\mathbb{R}$. In this paper, we study the asymptotics of the lowest eigenvalue of a Schrödinger operator in the case where the coefficient operator is unbounded linear operator and the potential function is C^3. In $P(\phi)$-type model cases, the potential functions are defined by using a renormalization and they are not continuous. In [**2**], we studied Schrödinger operators on path spaces over Riemannian manifolds. In that case, the differential operators are variable coefficient ones and the coefficient operators are not bounded linear because they contain stochastic integrals. Moreover, the dependence on the path of the coefficients are discontinuous in the natural topology. The discontinuity comes from the discontinuity of solutions of stochastic differential equations as a functional of Brownian motion. Thus, we need to consider two kind of discontinuity for potential functions and coefficient operators in that case. But, the difficulties are

2000 ***Mathematics Subject Classification.*** — 81Q20, 35J10, 35P15, 60H07.
Key words and phrases. — Semi-classical limit, Quantum field theory, Schrödinger operator, $P(\phi)$-type Hamiltonian.

different from that of the $P(\phi)$-type potentials. We will study semi-classical limit of the lowest eigenvalue of a $P(\phi)_2$-Hamiltonian on a finite interval in [**3**].

2. Preliminaries

Let (W, H, μ) be an abstract Wiener space. That is,

(i) H is a separable Hilbert space and W is a separable Banach space. Moreover H is continuously and densely embedded into W,

(ii) μ is the unique Gaussian measure on W such that for any $\varphi \in W^*$,

$$\int_W e^{\sqrt{-1}\varphi(w)} d\mu(w) = e^{-\frac{1}{2}\|\varphi\|_H^2}.$$

Here we use the natural inclusion and the identification by the Riesz theorem $W^* \subset H^* \simeq H$.

In this paper, we assume that W is a Hilbert space. This is equivalent to that there exists a positive self-adjoint trace class operator S such that W is a completion of H with respect to the Hilbert norm $\|\sqrt{S}h\|_H$. That is, $\|h\|_W = \|\sqrt{S}h\|_H$ for all $h \in H$. We denote the sets of bounded linear operators, Hilbert-Schmidt operators, trace class operators on H by $L(H), L_1(H), L_2(H)$. Also we denote their operator norms, trace norms, Hilbert-Schmidt norms by $\| \ \|, \| \ \|_1, \| \ \|_2$, respectively. For $\lambda > 0$, we define the new measure μ_λ on W by $\mu_\lambda(E) = \mu\left(\sqrt{\lambda}E\right)$ $(E \subset W)$. Now we define our Schrödinger operators.

Definition 2.1. — *Let A be a strictly positive self-adjoint operator on H. That is, we assume that $\inf \sigma(A) > 0$, where $\sigma(A)$ denotes the spectral set of A. We denote $c_A = \inf \sigma(A^2)$. We denote by $\mathfrak{F}C_A^\infty(W)$ the space of all smooth cylindrical functions $f(w) = F(\varphi_1(w), \ldots, \varphi_n(w))$ $(F \in C_b^\infty(\mathbb{R}^n), \varphi_i \in W^* \cap_{n\in\mathbb{N}} \mathrm{D}(A^n))$. For such a f, we define $Df(w) = \sum_{i=1}^n \partial_i F(w)\varphi_i \in H$. Here we use the identification $\varphi_i \in W^* \subset H^* \simeq H$ and $\partial_i F(w)$ denotes the partial derivative with respect to the i-th variable. Moreover we define $D_A f(w) = \sum_{i=1}^n \partial_i F(w) A\varphi_i$. We define a Dirichlet form on $L^2(W, d\mu_\lambda)$ by $\mathcal{E}_{\lambda,A}(f,f) = \int_W \|D_A f(w)\|_H^2 d\mu_\lambda(w)$. $-L_{\lambda,A}$ denotes the generator. Let V be a real-valued measurable function on W such that $V \in \cap_{\lambda>0} L^1(W, \mu_\lambda)$. Under the assumption that for all $\lambda > 0$, $\mathcal{E}_{\lambda,A,V}(f,f) = \mathcal{E}_{\lambda,A}(f,f) + \int_W \lambda^2 V(w) f(w)^2 d\mu_\lambda(w)$ $(f \in \mathfrak{F}C_A^\infty(W))$ is a lower bounded symmetric form, we denote the generator of the smallest closed extension by $-L_{\lambda,A,V}$. Also let $E_0(\lambda, A, V) = \inf \sigma(-L_{\lambda,A,V})$.*

Remark 2.2. — (1) $-L_{\lambda,A}$ can be viewed as the second quantization of A^2 on H. Let $H = H^{1/2}(\mathbb{R})$ be the Hilbert space with the norm $\|h\|_H^2 = \int_{\mathbb{R}} |(m^2 - \Delta)^{1/4} h(x)|^2 dx$, where $m > 0$. Consider $A = (m^2 - \Delta)^{1/4}$ on H. In this case, $-L_{1,A}$ is the time 0 field free Hamiltonian in $P(\phi)_2$-model. However note that $-L_{1,A}$ is

usually identified with the second quantization of $\sqrt{m^2-\Delta}$ on $H^*=H^{-1/2}(\mathbb{R})$. See also Example 3.3.
(2) In [**5, 1**], the Schrödinger operator with semi-classical parameter λ is defined in a different way. Let $V_\lambda(w)=\lambda V\left(\frac{w}{\sqrt{\lambda}}\right)$. The semi-classical limit of $-L_{1,A}+V_\lambda$ on $L^2(W,d\mu)$ is studied in the above papers. However note that this operator is unitarily equivalent to $-L_{\lambda,A,V}/\lambda$ on $L^2(W,\mu_\lambda)$. We adopt the similar definition to $-L_{\lambda,A,V}$ in the case of Schrödinger operators on path spaces over Riemannian manifolds because the scaling $w/\sqrt{\lambda}$ can not defined on the curved spaces but the measure corresponding to μ_λ can be defined on curves spaces too. See Remark 5.3 in [**1**] and [**2**].

Let us introduce the following assumptions on potential functions of Schrödinger operators.

Assumption 2.3. — The following assumptions (A1),(A2) are standard in semi-classical analysis. (A4) assures that the symmetric form $\mathcal{E}_{\lambda,A,V}$ is bounded from below by Corollary 2.8 (2). Note that (A5) implies that A is an unbounded operator.

(A1) V is a C^2-function on H. Let $U(h)=\frac{1}{4}\|Ah\|_H^2+V(h)$ $(h\in \mathrm{D}(A))$. Then $\min_{h\in\mathrm{D}(A)}U(h)=0$ and the zero point set is a finite set $N=\{h_1,\dots,h_n\}$.
(A2) $\frac{1}{2}D^2U(h_i)=\frac{1}{4}A^2+K_i$ is a strictly positive self-adjoint operator on H, where $K_i=\frac{1}{2}D^2V(h_i)\in L(H,H)$.
(A3) V can be extended to a C^3-function on W such that for any $R>0$ and $0\le k\le 3$

$$\sup\left\{\|D^kV(w)\|_{L(W\times\cdots\times W,\mathbb{R})}\mid \|w\|_W\le R\right\}\le C(R)<\infty.$$

(A4) V can be extended to a continuous function on W and there exists $p>1$ such that

$$\limsup_{\lambda\to\infty}\lambda^{-1}\log\int_W e^{-\frac{2p\lambda}{c_A}V(w)}d\mu_\lambda(w)<\infty,$$

(A5) There exists $\gamma_0>1$ such that $A^{-\gamma_0}\in L_2(H)$.

For $r>0$ and $z\in W, k\in H$, we denote $B_r(z)=\{w\in W\mid \|w-z\|_W\le r\}$ and $B_{r,H}(k)=\{h\in H\mid \|h-k\|_H\le r\}$.

Lemma 2.4. — (1) *Suppose that* (A4) *holds or* $\inf\{V(h)\mid h\in H\}>-\infty$. *Then we have* $\displaystyle\lim_{\|h\|_H\to\infty}\left(\frac{c_A}{4}\|h\|_H^2+V(h)\right)=+\infty$.
(2) *Assume* (A1), *the same assumptions in* (1) *and for any* $L>0$, $\sup\{|V(h)|\mid \|h\|_H\le L\}<\infty$. *Then for any* $\varepsilon>0$,

$$\kappa(\varepsilon):=\inf\left\{U(h)\mid h\in\{\cup_{i=1}^n B_\varepsilon(h_i)\}^c\right\}>0.$$

Proof. — (1) If $\inf\{V(h)\mid h\in H\}>-\infty$, the statement is trivial. We assume (A4). Let C be a positive number such that $\limsup_{\lambda\to\infty}\lambda^{-1}\log\int_W e^{-\frac{2p\lambda}{c_A}V}d\mu_\lambda<C$. Take

$R > 0$. Then for sufficiently large λ, we have

$$\begin{aligned}
&\frac{1}{\lambda}\log\int_W \exp\left(-\frac{2p\lambda}{c_A}\left(R\wedge V(w)\vee(-R)\right)\right)d\mu_\lambda(w)\\
&\quad\le\frac{1}{\lambda}\log\left(\int_W\left(e^{-\frac{2p\lambda}{c_A}R}+\exp\left(-\frac{2p\lambda}{c_A}(V(w)\vee(-R))\right)d\mu_\lambda(w)\right)\right)\\
&\quad\le\frac{1}{\lambda}\log\left(e^{\lambda C}+e^{-\frac{2p\lambda}{c_A}R}\right)\le C+\frac{\log 2}{\lambda}.
\end{aligned}$$

By the Large deviation estimate, we have

$$\sup_h\left(-\frac{1}{2}\|h\|_H^2-\frac{2p}{c_A}\left((-R)\vee V(h)\wedge R\right)\right)\le C.$$

Since R is an arbitrary number, we get

$$-\frac{c_A}{4}\|h\|_H^2-pV(h)\le\frac{C\cdot c_A}{2}\quad\text{for all } h\in H.$$

Suppose that there exists $\{h_n\}$ such that $\|h_n\|_H\to\infty$ and $\sup_n\left(\frac{c_A}{4}\|h_n\|_H^2+V(h_n)\right)=: l<+\infty$. Then $\lim_{n\to\infty}V(h_n)=-\infty$. Hence

$$\frac{c_A}{4}\|h_n\|_H^2+pV(h_n)=\frac{c_A}{4}\|h_n\|_H^2+V(h_n)+(p-1)V(h_n)\le l+(p-1)V(h_n)\to-\infty.$$

This is a contradiction. So we are done.

(2) By the result in (1), we need to prove that for sufficiently large positive number L,

$$\inf\{U(h)\mid h\in B_{L,H}(0)\cap(\cup_{i=1}^n B_\varepsilon(h_i))^c\}>0.$$

Suppose that there exists $\{\varphi_l\}\subset B_{L,H}(0)\cap(\cup_{i=1}^n B_\varepsilon(h_i))^c$ such that $\lim_{l\to\infty}U(\varphi_l)=0$. By the assumption, there exists a subsequence $\{\varphi_{l(i)}\}$ which converges to a certain element $\varphi_\infty\in H$ weakly. Since $\frac{1}{4}\|A\varphi_{l(i)}\|_H^2=U(\varphi_{l(i)})-V(\varphi_{l(i)})$, $\sup_i\|A\varphi_{l(i)}\|_H<\infty$ holds. Hence again by choosing a subsequence $\{\varphi_{p(i)}\}$, $A\varphi_{p(i)}$ also converges to some ϕ_∞ weakly. By the Banach-Saks theorem, we see that $\varphi_\infty\in \mathrm{D}(A)$ and $A\varphi_\infty=\phi_\infty$. On the other hand, since the embedding $H\subset W$ is compact, $\lim_{i\to\infty}\|\varphi_{p(i)}-\varphi_\infty\|_W=0$ which implies $\lim_{i\to\infty}V(\varphi_{p(i)})=V(\varphi_\infty)$. Since $\|A\varphi_\infty\|_H^2\le\liminf_{i\to\infty}\|A\varphi_{p(i)}\|_H^2$, we obtain $U(\varphi_\infty)\le\liminf_{i\to\infty}U(\varphi_{p(i)})=0$. This implies $\varphi_\infty\in N$ and $\varphi_{p(i)}\in B_\varepsilon(h_j)$ for some large i and $1\le j\le n$. This is a contradiction. □

Lemma 2.5. — *Let A be a strictly positive self-adjoint operator and K be a trace class self-adjoint operator on H. Assume that A^2+K is also a strictly positive operator. Then $\sqrt{A^2+K}-A\in L_1(H)$ and*

$$\left\|\sqrt{A^2+K}-A\right\|_1\le\frac{\|K\|_1}{\min\left\{\inf\sigma(\sqrt{A^2+K}),\inf\sigma(A)\right\}}.$$

Proof. — We prove this in three steps: (i) $A=I+T$ and T is a trace class operator, (ii) A is a bounded linear operator, (iii) General cases.

(i) We denote $S_1 = \sqrt{A^2+K}$ and $S_0 = A$. Note that $S_1 - S_0 = \sqrt{A^2+K} - A$ is a trace class operator. We denote the all eigenvalues and corresponding complete orthonormal system of $S_1 - S_0$ by $\{\alpha_n\}$ and $\{e_n\}$. Then

$$\begin{aligned}
|(Ke_n, e_n)| &= |\left((S_1^2 - S_0^2)e_n, e_n\right)| \\
&= \left|\left(\left(S_1(S_1-S_0) + (S_1-S_0)S_1 - (S_1-S_0)^2\right)e_n, e_n\right)\right| \\
&= |\alpha_n\left((S_1+S_0)e_n, e_n\right)| \\
&\geq |\alpha_n| \inf\sigma(S_1+S_0).
\end{aligned}$$

This implies that

$$\|\sqrt{A^2+K} - A\|_1 = \sum_{n=1}^{\infty} |\alpha_n| \leq \frac{\|K\|_1}{\inf\sigma(\sqrt{A^2+K}+A)}.$$

(ii) Let $\{u_m\}$ be all eigenvectors of K which is a c.o.n.s. of H. Set $P_m h = \sum_{i=1}^m (h, u_i)u_i$ and $A_m = \sqrt{P_m A^2 P_m + P_m^\perp}$. Then $A_m^2 \to A^2$, $A_m \to A$ converge strongly. On the other hand, $A_m^2 + K = P_m(A^2+K)P_m + P_m^\perp(I_H + P_m^\perp K P_m^\perp)P_m^\perp$. Hence for sufficiently large m, we have

$$\min\left\{\inf\sigma(\sqrt{A_m^2+K}), \inf\sigma(A_m)\right) \geq \min\left(\inf\sigma(\sqrt{A^2+K}), 1/2, \inf\sigma(A)\right).$$

Since $A_m - I_H$ is a trace class operator, by (i),

$$\|\sqrt{A_m^2+K} - A_m\|_1 \leq \frac{\|K\|_1}{\min\left(\inf\sigma(A^2+K), \inf\sigma(A), 1/2\right)}.$$

By taking the limit $m \to \infty$, we see that $\sqrt{A^2+K} - A \in L_1(H)$. Therefore again by the same argument as in (i), we can prove (ii).

(iii) Let $\chi_n(x)$ be a function such that $\chi_n(x) = 1$ for $x \leq n$ and $\chi_n(x) = 0$ for $x > n$. Then $\chi_n(A)$ is a projection operator which commutes with A. Let $A_n = A\chi_n(A) + (1 - \chi_n(A))$ and $K_n = \chi_n(A)K\chi_n(A)$. Then

$$\begin{aligned}
\sqrt{A^2+K_n} - A &= \sqrt{A^2\chi_n(A) + \chi_n(A)K\chi_n(A)} - A\chi_n(A) \\
&= \sqrt{A_n^2+K_n} - A_n \in L(\mathrm{Im}(\chi_n(A)))
\end{aligned}$$

By (ii), we have

$$\begin{aligned}
\|\sqrt{A^2+K_n} - A\|_1 &\leq \frac{\|K_n\|_1}{\inf\sigma\left(\sqrt{A^2\chi_n(A) + \chi_n(A)K\chi_n(A)} + A\chi_n(A)\right)} \\
&\leq \frac{\|K_n\|_1}{\min\left(\inf\sigma(\sqrt{A^2+K}), \inf\sigma(A)\right)}.
\end{aligned} \tag{2.1}$$

For $l > n > m$,

$$\begin{aligned}
\left(\sqrt{A_n^2+K_n} - A_n\right) - \left(\sqrt{A_m^2+K_m} - A_m\right) &= \sqrt{A^2+K_n} - \sqrt{A^2+K_m} \\
&= \sqrt{A_l^2+K_n} - \sqrt{A_l^2+K_m}.
\end{aligned}$$

This and (ii) implies that $\sqrt{A_n^2+K_n}-A_n$ converges in the trace norm. It is not difficult to check that the strong limit is equal to $\sqrt{A^2+K}-A$. Therefore, (2.1) implies the conclusion. □

Proposition 2.6. — *Let A be a strictly positive self-adjoint operator. For a trace class self-adjoint operator K on H and $h\in \mathrm{D}(A^2)$, we set*

$$V_{K,h}(w)=\frac{1}{4}\|Ah\|_H^2-\frac{1}{2}(A^2h,w)+(K(w-h),w-h)\,.$$

We assume that A^2+4K is a strictly positive self-adjoint operator and AKA can be extended to a trace class operator. Then $\mathcal{E}_{\lambda,A,V_{K,h}}$ is a symmetric form bounded from below and $E_0(\lambda,A,V_{K,h})=\lambda e(A,K)$ holds, where

$$e(A,K)=\frac{1}{2}\mathrm{tr}\left(\sqrt{A^4+4AKA}-A^2\right). \tag{2.2}$$

Moreover it is the lowest eigenvalue of $-L_{\lambda,A,V_{K,h}}$ and the corresponding normalized positive eigenfunction is

$$\begin{aligned}\Omega_{\lambda,A,V_{K,h}}(w) &= \det\left(I_H+T_K\right)^{1/4}\\ &\quad\times\exp\left\{-\frac{\lambda}{4}\left(\left(A^{-1}\{A^4+4AKA\}^{1/2}A^{-1}-I_H\right)(w-h),(w-h)\right)\right\}\\ &\quad\times\exp\left(\frac{\lambda}{2}(h,w)-\frac{\lambda}{4}\|h\|_H^2\right),\end{aligned}$$

where $T_K=A^{-1}(\sqrt{A^4+4AKA}-A^2)A^{-1}$.

Proof. — If A is bounded linear operator, the proof is a straightforward calculation. Suppose that A is unbounded. Let A_n and K_n be the operators which are defined in the proof of (iii) in Lemma 2.5. Then $AK_nA=A_nK_nA_n$. Thus $\left(A^{-1}\{A^4+4AK_nA\}^{1/2}A^{-1}-I_H\right)\in L_1(H)\cap_k \mathrm{D}(A^k)$. Therefore for sufficiently large n, $\Omega_{\lambda,A,V_{K_n,h}}\in L^2(\mu_\lambda)$ and the simple calculation shows that

$$-L_{\lambda,A,V_{K_n,h}}\Omega_{\lambda,A,V_{K_n,h}}=\lambda e(A,K_n)\Omega_{\lambda,A,V_{K_n,h}}.$$

Letting $n\to\infty$, we have

$$-L_{\lambda,A,V_{K,h}}\Omega_{\lambda,A,V_{K,h}}=\lambda e(A,K)\Omega_{\lambda,A,V_{K,h}}.$$

To prove that $\lambda e(A,K)=\inf\sigma\left(-L_{\lambda,A,V_{K,h}}\right)$, we note that for any $f\in\mathfrak{F}C_A^\infty(W)$, it holds that

$$\begin{aligned}\mathcal{E}_{\lambda,A,V_{K,h}}(f,f)=\int_W \|D_A(f\Omega^{-1}_{\lambda,A,V_{K,h}})\|_H^2\Omega_{\lambda,A,V_{K,h}}(w)^2d\mu_\lambda(w)&\\ +\lambda e(A,K)\|f\|^2_{L^2(\mu_\lambda)}.&\end{aligned}$$

□

We use the following estimate to prove a lower bound in Lemma 3.4. We refer the reader to [**7, 14, 12**] for this estimate.

***Theorem 2.7* (NGS estimate).** — *Let $\mathcal{E}(f,f)$ be a closed form on $L^2(X,m)$, where $(X,\mathcal{F},m)$ is a probability space. Assume that there exists $\alpha>0$ such that for any $f\in \mathrm{D}(\mathcal{E})$,*

$$\int_X f(x)^2\log\left(f(x)^2/\|f\|^2_{L^2(X,m)}\right)dm(x)\le \alpha\mathcal{E}(f,f).$$

Then for any bounded measurable function V, it holds that

$$\mathcal{E}(f,f)+\int_X V(x)f(x)^2dm(x)\ge -\frac{1}{\alpha}\log\left(\int_X e^{-\alpha V(x)}dm(x)\right)\|f\|^2_{L^2(X,m)}. \tag{2.3}$$

The following follows from the above estimate and Gross's logarithmic Sobolev inequality [**7**]: For any $f\in\mathfrak{F}C_I^\infty(W)$,

$$\int_W f(w)^2\log\left(f(w)^2/\|f\|^2_{L^2(\mu_\lambda)}\right)d\mu_\lambda(w)\quad\le\quad\frac{2}{\lambda}\int_W\|Df(w)\|_H^2d\mu_\lambda(w).$$

Originally NGS(=Nelson, Glimm, Segal) estimate (2.3) was proved by the hypercontractivity of the corresponding semigroup. See [**14**]. Corollary 2.8 (2) is proved by Lemma 4.5 in [**2**] which follows from Gross's log-Sobolev inequalities and finite dimensional approximations.

Corollary 2.8. — (1) *It holds that*

$$E_0(\lambda,A,V)\ge -\frac{\lambda c_A}{2}\log\left(\int_W\exp\left(-\frac{2\lambda}{c_A}V\right)d\mu_\lambda(w)\right).$$

(2) *Suppose that there exists a Hilbert-Schmidt operator T such that $A=I+T$. Then*

$$\begin{aligned}(2.4)\ E_0(\lambda,A,V)&\\ \ge -\frac{\lambda}{2}&\log\left\{\int_W\exp\left(-2\lambda V(w)-\lambda:(Tw,w):_{\mu_\lambda}-\frac{\lambda}{2}\|Tw\|_H^2\right)d\mu_\lambda(w)\right\}\\ &+\frac{\lambda}{2}\log\det{}_{(2)}(I_H+T)-\frac{\lambda}{2}\mathrm{tr}\left(T^2\right).\end{aligned}$$

In (2.4), $:(Tw,w):_{\mu_\lambda}$ is defined by the limit $\lim_{n\to\infty}\left\{(P_nTP_nw,w)-\frac{1}{\lambda}\mathrm{tr}P_nTP_n\right\}$, where P_n is a projection on to a finite dimensional subspace of H such that $P_n\uparrow I_H$. $\det{}_{(2)}$ denotes the Carleman-Fredholm determinant.

3. Results

***Theorem 3.1* (Bounded case).** — *We assume that A is a bounded linear operator and satisfies the assumptions* (A1),(A2),(A3),(A4). *Then we have*

$$\lim_{\lambda\to\infty}\frac{E_0(\lambda,A,V)}{\lambda}=\min_{1\le i\le n}e(A,K_i). \tag{3.1}$$

In the unbounded case, we can prove the following. The assumption is too strong to cover the $P(\phi)$-type Hamiltonian. We will relax the assumptions and discuss such a case in a separate paper.

***Theorem 3.2* (Unbounded case).** — *Assume* (A5). *Let* $\gamma \geq 1+\gamma_0$ *and* $S = A^{-2\gamma}$. *Then* AK_iA *is a trace class operator and* (2.2) *is well-defined. Furthermore, we assume that* (A1),(A2), (A3),(A4) *hold. Then the asymptotics* (3.1) *holds.*

Example 3.3. — Let $I = [-\frac{l}{2}, \frac{l}{2}]$ $(l > 0)$ be an interval of $\mathbb{R}$. Let $-\Delta$ be the Laplacian with periodic boundary condition on $X = L^2(I \to \mathbb{R}, dx)$. Let $m > 0$. For $\alpha \in \mathbb{R}$, let $H^\alpha = D((m^2-\Delta)^{\alpha/2})$ and $\|h\|_{H^\alpha} = \|(m^2-\Delta)^{\alpha/2}h\|_X$.
(1) Let $H = H^{1/2}$. Then for any $\varepsilon > 0$, we can take $W = H^{-\varepsilon}$. Let $0 < \varepsilon < 1/2$. Then using the inclusion and the identification $H^{1/2} \subset H^{\varepsilon} = (H^{-\varepsilon})^*$, we can see that μ satisfies that $\int_{W\ H^{-\varepsilon}} (w,h)^2_{H^\varepsilon}\,d\mu(w) = \|(m^2-\Delta)^{-1/4}h\|_X^2$ for $h \in H$. Let $U : X \to H^{1/2}$ be the natural isometry operator and define $A = U(m^2-\Delta)^{1/4}U^{-1}$. This is a standard example in $P(\phi)_2$-model on finite interval. Let $P(u) = \sum_{k=0}^{2M} a_k u^k$ be a polynomial with real coefficients with $a_{2M} > 0$. For $h \in H$, $\tilde{V}(h) = \int_I P(h(x))dx$ is well-defined by the Sobolev embedding theorem. However $H^{-\varepsilon}$ is the space of distribution and $P(w(x))$ is not defined for $w \in H^{-\varepsilon}$. Actually, it should be defined by $\int_I : P(w(x)) :_{\mu_\lambda} dx$ where $: P(w(x)) :$ denotes the Wick product. However this is not a smooth function on $W = H^{-\varepsilon}$ and cannot be covered by Theorem 3.2. This will be studied in [**3**].
(2) Let $H = H^2$. Then μ can be defined on $W = H^1$. For $0 < \delta < 1/2$, let $A = U(m^2-\Delta)^{\frac{1}{2}(\frac{1}{2}-\delta)}U^{-1}$, where U is the natural isometry from X to H. Let $Q(u) = \frac{1}{4}m^{1-2\delta}u^2 + P(u)$, where $P(u)$ is the polynomial defined in (1). Let $\{c_1, \ldots, c_n\}$ be the minimum points of Q and asssume that $Q''(c_i) > 0$ $(1 \leq i \leq n)$. Again let $\tilde{V}(h) = \int_I P(h(x))dx$ for $h \in H$. Then we see that $\tilde{V}(h) - l\min Q$ can be extended to a smooth function $V(w)$ on W. Then the zero point set of $U(h) = \frac{1}{4}\|Ah\|_H^2 + V(h)$ is the set of the constant functions $\{c_1, \ldots, c_n\}$. For this V and A, all assumptions in Theorem 3.2 hold with $\gamma_0 = 1 + \frac{4\delta}{1-2\delta}$ and $\gamma = 1 + \gamma_0$.

We prove these theorems after preparations. Here we just prove $AK_iA \in L_1(H)$ under (A5). Since $V \in C^2(W)$, there exists a bounded linear operator $\hat{K}_i$ on W such that $D^2V(h_i)(u,v) = \left(\hat{K}_i u, v\right)_W$ for any $u, v \in W$. By the definition of the norm of W, there exists $\tilde{K}_i \in L(H)$ such that $\hat{K}_i = A^\gamma \tilde{K}_i A^{-\gamma}$. Thus for any $u, v \in H \subset W$,

$$D^2V(h_i)(u,v) = \left(\hat{K}_i u, v\right)_W = \left(A^{-\gamma}A^\gamma \tilde{K}_i A^{-\gamma}u, A^{-\gamma}v\right)_H = \left(A^{-\gamma}\tilde{K}_i A^{-\gamma}u, v\right)_H.$$

This shows $K_i = A^{-\gamma}\tilde{K}_i A^{-\gamma}$ and $AK_iA = A^{1-\gamma}\tilde{K}_i A^{1-\gamma}$. Because $\gamma - 1 \geq \gamma_0$, $A^{1-\gamma}$ is a Hilbert-Schmidt operator and this implies AK_iA is a trace class operator on H.

In our main theorems, we may assume that $c_A = 1$. Because, if Theorems hold in the case where $c_A = 1$, then it implies that $E_0\left(\lambda, \frac{A}{\sqrt{c_A}}, \frac{V}{c_A}\right) = e\left(\frac{A}{\sqrt{c_A}}, \frac{V}{c_A}\right)$. This shows the general cases.

The proof of upper bound is standard. Let χ be a smooth function on $\mathbb{R}$ satisfying $0 \le \chi(x) \le 1$, $\chi(x) = 1$ for $x \in [-1, 1]$ and $\chi(x) = 0$ for $|x| \ge 2$. For $2/3 < \delta < 1$, set

$$\tilde{\Omega}_{\lambda,A,V_{K_i,h_i}}(w) = Z_\lambda \Omega_{\lambda,A,V_{K_i,h_i}}(w)\chi\left(\lambda^\delta \|w - h_i\|_W^2\right).$$

Here Z_λ is a normalization constant which makes the L^2-norm to be equal to 1. It holds that $\lim_{\lambda\to\infty} Z_\lambda = 1$. Since h_i is a minimizer of U, for any $k \in \mathrm{D}(A)$, $\frac{1}{2}\left(Ah_i, Ak\right)_H + DV(h_i)(k) = 0$. The fact $DV(h_i) \in H^*$ implies that $h_i \in \mathrm{D}(A^2)$ and $DV(h_i) = -\frac{1}{2}A^2 h_i$. Using this and by the Taylor expansion, we have

$$
\begin{aligned}
V(w) &= V(h_i) + DV(h_i)(w-h_i) + (K_i(w-h_i), w-h_i) \\
&\quad + \frac{1}{3!}DV^3(w + \theta(w-h_i))((w-h_i)^{\otimes 3}) \\
&= \frac{1}{4}\|Ah_i\|_H^2 - \frac{1}{2}\left(A^2 h_i, w\right) + (K_i(w-h_i), w-h_i) + R_{h_i}(w) \\
&= V_{K_i,h_i}(w) + R_{h_i}(w).
\end{aligned}
\tag{3.2}
$$

Here we denote the remainder term by $R_{h_i}(w)$. If $\chi(\lambda^\delta\|w - h_i\|_W^2) \ne 0$, then $|R_{h_i}(w)| \le C\lambda^{-3\delta/2}$. This and the tail estimate of the Gaussian measure shows that

$$\mathcal{E}_{\lambda,A,V}\left(\tilde{\Omega}_{\lambda,A,V_{K_i,h_i}}, \tilde{\Omega}_{\lambda,A,V_{K_i,h_i}}\right) = E_0(\lambda, A, K_i) + O(\lambda^{2-\frac{3}{2}\delta}).$$

This proves the upper bound.

To prove the lower bound estimates, it suffices to prove the following Lemma 3.4. Let R be a sufficiently large positive number. Set $\chi_{i,R}(w) = \chi\left(R\|w - h_i\|_W^2\right)$ $(1 \le i \le n)$ and $\chi_{0,R}(w) = \sqrt{1 - \sum_{i=1}^n \chi_{i,R}(w)^2}$.

Lemma 3.4. — *Let us assume that the conditions of either Theorem* 3.1 *or Theorem* 3.2 *hold.*

(1) *There exists a constant $C > 0$ such that for all i, $\chi_{i,R} \in \mathrm{D}(D_A)$ and $\|D_A\chi_{i,R}(w)\|_H^2 \le CR$ μ_λ-a.e. w. Moreover it holds that*

$$
\begin{aligned}
&\mathcal{E}_{\lambda,A,V}(f,f) \\
&\quad = \sum_{i=0}^n \mathcal{E}_{\lambda,A,V}(f\chi_{i,R}, f\chi_{i,R}) - \sum_{i=0}^n \int_W \|D_A\chi_{i,R}(w)\|_H^2 f(w)^2 d\mu_\lambda(w).
\end{aligned}
\tag{3.3}
$$

(2) *For $1 \le i \le n$,*

$$\mathcal{E}_{\lambda,A,V}(f\chi_{i,R}, f\chi_{i,R}) \ge \lambda(1 + g(\lambda))e(A, K_i)\|f\chi_{i,R}\|_{L^2(\mu_\lambda)}^2,$$

where $\lim_{\lambda\to\infty} g(\lambda) = 0$.

(3) *There exists a constant $C > 0$ such that*

$$\mathcal{E}_{\lambda,A,V}(f\chi_{0,R}, f\chi_{0,R}) \ge C\lambda^2\|f\chi_{0,R}\|_{L^2(\mu_\lambda)}^2.$$

The essential part of this lemma is in (3). In the case where $A = I +$ Hilbert-Schmidt operator, we can apply the same method as in [**1**] without any modification by using Corollary 2.8 (2) to prove (3). In general cases, we need to approximate A by such kind of operators.

Lemma 3.5. — *Assume that A is a bounded linear operator and* (A1),(A3),(A4) *hold. Also we assume that $c_A = 1$. Let R be a sufficiently large positive number such that*

$$\inf\left\{\frac{1}{4}\|h\|_H^2 + V(h) \mid \|h\|_W \geq R\right\} \geq 1$$

and ε be a small positive number. Set $D_{\varepsilon,R} = B_R(0) \cap \left(\cup_{i=1}^n B_{3\varepsilon}(h_i)\right)^c$. Then there exists a self-adjoint operator $T_\varepsilon \in L_1(H)$ and a positive number $\delta(\varepsilon)$ such that
(1) *it holds that for any $h \in \mathrm{D}(A)$, $\|Ah\|_H^2 \geq \|(I_H + T_\varepsilon)h\|_H^2$,*
(2)

$$\inf\left\{\frac{1}{4}\,\|(I_H + T_\varepsilon)h\|_H^2 + V(h) \;\middle|\; h \in D_{\varepsilon,R} \cap H\right\} \quad \geq \quad \delta(\varepsilon).$$

Proof. — It holds that for a large positive number L,

$$\inf\left\{\frac{1}{4}\,\|h\|_H^2 + V(h) \;\middle|\; h \in D_{\varepsilon,R} \cap B_{L,H}(0)^c\right\} \geq 1.$$

Hence we prove the lemma on $D_{\varepsilon,R} \cap B_{L,H}(0)$. For a natural number k, we define $A_k = \sum_{i=2^k}^{\infty} \frac{i}{2^k} 1_{I_{k,i}}(A)$, where $I_{k,i} = \left\{x \in \mathbb{R} \mid \frac{i}{2^k} \leq x < \frac{i+1}{2^k}\right\}$. Then

$$0 \leq \|Ah\|_H^2 - \|A_k h\|_H^2 \leq \frac{3}{2^k}\|Ah\|_H^2 \leq \frac{3}{2^k}\|A\|^2\|h\|_H^2.$$

By Lemma 2.4 (2), for sufficiently large k_0,

$$\inf\left\{\frac{1}{4}\,\|A_{k_0}h\|_H^2 + V(h) \;\middle|\; h \in D_{\varepsilon,R} \cap B_{L,H}(0)\right\} \geq \frac{1}{2}\kappa(\varepsilon),\ \frac{3}{2^{k_0}}\|A\|^2 L^2 \leq \frac{1}{4}\varepsilon^2\|\sqrt{S}\|^{-2}.$$

Note that there exists a family of finite dimensional projection operators on H such that $P_n \uparrow I_H$ and $A_{k_0}P_n = P_n A_{k_0}$ for all $n \geq 1$. Hence, it holds that for any $h \in H$ and n

$$\|A_{k_0}h\|_H^2 = \|A_{k_0}P_n h\|_H^2 + \|A_{k_0}P_n^\perp h\|_H^2 \geq \|A_{k_0}P_n h\|_H^2 + \|P_n^\perp h\|_H^2.$$

Let $h \in B_{L,H}(0)$. Then $|V(h) - V(P_n h)| \leq \|DV(P_n h + \theta P_n^\perp h)\|_{W^*}\|P_n^\perp h\|_W$ $(0 < \theta < 1)$. Noting

$$\begin{aligned}
\|P_n h + \theta P_n^\perp h\|_W &\leq L\|\sqrt{S}\|, \\
\|P_n^\perp h\|_W &= \|\sqrt{S}P_n^\perp h\|_H \leq \|\sqrt{S}P_n^\perp\|_2\|h\|_H, \\
\lim_{n\to\infty}\|\sqrt{S}P_n^\perp\|_2 &= 0,
\end{aligned}$$

by (A2),

$$\lim_{n\to\infty}\sup\left\{|V(h) - V(P_n h)| \mid h \in B_{L,H}(0)\right\} = 0.$$

Now we take a natural number n_0 such that

$$\sup\{|V(h) - V(P_{n_0}h)| \mid h \in B_{L,H}(0)\} \leq \frac{1}{4}\min\left(\kappa(\varepsilon), 1, \varepsilon^2\|\sqrt{S}\|^{-2}\right).$$

Let $h \in D_{\varepsilon,R} \cap B_{L,H}(0)$. Then three cases are possible for $P_{n_0}h$ such that (i) $P_{n_0}h \in D_{\varepsilon/3,R} \cap B_{L,H}(0)$, (ii) $P_{n_0}h \in B_R(0)^c$, (iii) $P_{n_0}h \in \cup_{i=1}^n B_\varepsilon(h_i)$.

In the case of (i),

$$\frac{1}{4}\|A_{k_0}P_{n_0}h\|_H^2 + V(h) = \frac{1}{4}\|A_{k_0}P_{n_0}h\|_H^2 + V(P_{n_0}h) + (V(h) - V(P_{n_0}h)) \geq \frac{1}{4}\kappa(\varepsilon).$$

If (ii) happens, then

$$\frac{1}{4}\|A_{k_0}P_{n_0}h\|_H^2 + V(h) = \frac{1}{4}\|A_{k_0}P_{n_0}h\|_H^2 + V(P_{n_0}h) + (V(h) - V(P_{n_0}h)) \geq 3/4.$$

In the case where $P_{n_0}h \in B_\varepsilon(h_i)$ for some i,

$$\|P_{n_0}^\perp h\|_W = \|h - P_{n_0}h\|_W = \|h - h_i\|_W - \|h_i - P_{n_0}h\|_W \geq 2\varepsilon.$$

Thus $\|P_{n_0}^\perp h\|_H \geq \|\sqrt{S}\|^{-1}\|P_{n_0}^\perp h\|_W \geq 2\varepsilon\|\sqrt{S}\|^{-1}$. Therefore, we have for $h \in D_{\varepsilon,R} \cap B_{L,H}(0)$ satisfying (iii),

$$\begin{aligned}
&\frac{1}{4}\|A_{k_0}P_{n_0}h\|_H^2 + \frac{1}{4}\|P_{n_0}^\perp h\|_H^2 + V(h) \\
&\quad= \frac{1}{4}\|P_{n_0}^\perp h\|_H^2 + \frac{1}{4}\|AP_{n_0}h\|_H^2 + V(P_{n_0}h) - \frac{1}{4}\left(\|AP_{n_0}h\|_H^2 - \|A_{k_0}P_{n_0}h\|_H^2\right) \\
&\qquad+ (V(h) - V(P_{n_0}h)) \\
&\quad\geq \varepsilon^2\|\sqrt{S}\|^{-2} - \frac{3}{2^{k_0}}\|A\|^2L^2 - \frac{1}{4}\varepsilon^2\|\sqrt{S}\|^{-2} \geq \frac{1}{2}\varepsilon^2\|\sqrt{S}\|^{-2}.
\end{aligned}$$

Consequently,

$$\inf\left\{\frac{1}{4}\|A_{k_0}P_{n_0}h\|_H^2 + \frac{1}{4}\|P_{n_0}^\perp h\|_H^2 + V(h) \;\middle|\; h \in D_{\varepsilon,R}\right\} \geq \delta(\varepsilon).$$

This implies that the operator $T_\varepsilon = (A_{k_0} - I_H)P_{n_0}$ satisfies the desired properties. □

In Theorem 3.2, we assume $\gamma \geq 1 + \gamma_0$. But $\gamma \geq \gamma_0$ is sufficient for $\chi_{i,R} \in \mathrm{D}(D_A)$.

Lemma 3.6. — (1) *Assume that* A *is bounded. Then* $\|w\|_W \in \mathrm{D}(D_A)$ *and* $\|D_A\|w\|_W\|_H \leq \|A\sqrt{S}\|$.
(2) *Assume* (A5) *and let* $S = A^{-2\gamma}$, *where* $\gamma \geq \gamma_0$. *Then* $\|w\|_W \in \mathrm{D}(D_A)$ *and* $\|D_A\|w\|_W\|_H \leq \|A^{1-\gamma}\|$.

Proof. — (1) We have $D\|w\|_W = \frac{Sw}{\|w\|_W}$. So $D_A\|w\|_W = \frac{ASw}{\|w\|_W}$ and $\|D_A\|w\|_W\|_H \leq \|A\sqrt{S}\|$.
(2) This is proved in the same way as in (1). □

Lemma 3.7. — *Assume* (A1),(A3),(A4),(A5) *and* $c_A = 1$. *Let* $\gamma \geq \gamma_0$ *and* $S = A^{-2\gamma}$. *Then the same results as in Lemma* 3.5 *hold .*

Proof. — For $a>0$ let $\psi_a(x)$ be the positive function such that $\psi_a(x)=1$ for $x\le a$ and $\psi_a(x)=a/x$ for $x\ge a$. Then for $h\in H$

$$\begin{aligned}
\|\psi_a(A)h\|_W^2 &= \|\psi_a(A)A^{-\gamma}h\|_H^2 \le \|A^{-\gamma}h\|_H^2=\|h\|_W^2,\\
\|\psi_a(A)h-h\|_W^2 &\le \|(\psi_a(A)-1)A^{-\gamma}h\|_H^2 \le \frac{1}{a^{2\gamma}}\|h\|_H^2.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|\psi_a(A)h-h_i\|_W &= \|\psi_a(A)h-h+h-h_i\|_W\\
&\ge \|h-h_i\|_W-\frac{1}{a^\gamma}\|h\|_H
\end{aligned}$$

Thus, if $\|h-h_i\|_W\ge 3\varepsilon$ and $\|h\|_H\le \frac{3a^\gamma}{2}\varepsilon$, hold, then $\|\psi_a(A)h-h_i\|_W\ge\frac{3\varepsilon}{2}$. Let $A^{(a)}=A\psi_a(A)$. Let L be a positive number such that for h with $\|h\|_H\ge L$, $\frac14\|h\|_H^2+V(h)\ge\frac12\kappa(\varepsilon)$. Now let a be a positive number satisfying that

$$\frac{C(L)}{a^\gamma}L\le \min\left(\frac12\kappa(\varepsilon),\frac14\varepsilon^2\|\sqrt{S}\|^{-2}\right),\qquad \frac{3a^\gamma}{2}\varepsilon\ge L.$$

Here $C(L)$ is the number which appeared in (A3). Then for such an a, for h with $\|h\|_H\le L$, by the above estimates, we have

$$\begin{aligned}
|V(h)-V(\psi_a(A)h)| &\le \frac{C(L)}{a^\gamma}L\le\frac12\kappa(\varepsilon),\\
\frac14\|A\psi_a(A)h\|_H^2+V(\psi_a(A)h) &\ge \kappa(\varepsilon).
\end{aligned}$$

Consequently, we have, for sufficiently large a,

$$\inf\left\{\frac14\|A^{(a)}h\|_H^2+V(h)\mid h\in D_{\varepsilon,R}\cap H\right\}\ge\frac12\kappa(\varepsilon).$$

Therefore, it suffices for us to do the same calculation as in the bounded case replacing A by $A^{(a)}$. But of course, the norm of W is still defined by $S=A^{-2\gamma}$. Note that $(A^{(a)})_{k_0}$ is defined first and next P_{n_0} is defined by $(A^{(a)})_{k_0}$. Case (iii) requires some additional care. That is, we use the following estimate:

$$\begin{aligned}
&\frac14\|(A^{(a)})_{k_0}P_{n_0}h\|_H^2+\frac14\|P_{n_0}^\perp h\|_H^2+V(h)\\
&\quad=\frac14\|P_{n_0}^\perp h\|_H^2+\frac14\|A^{(a)}P_{n_0}h\|_H^2+V(P_{n_0}h)-\frac14\left(\|A^{(a)}P_{n_0}h\|_H^2-\|(A^{(a)})_{k_0}P_{n_0}h\|_H^2\right)\\
&\qquad\quad+(V(h)-V(P_{n_0}h))\\
&\quad\ge\varepsilon^2\|\sqrt{S}\|^{-2}+\frac14\|A\psi_a(A)P_{n_0}h\|_H^2+V(\psi_a(A)P_{n_0}h)+(V(P_{n_0}h)-V(\psi_a(A)P_{n_0}h))\\
&\qquad\quad-\frac{3}{2^{k_0}}\|A^{(a)}\|^2L^2-\frac14\varepsilon^2\|\sqrt{S}\|^{-2}\\
&\quad\ge\frac14\varepsilon^2\|\sqrt{S}\|^{-2}.
\end{aligned}$$

Therefore, it suffices to put $T_\varepsilon=\left(\left(A^{(a)}\right)_{k_0}-I_H\right)P_{n_0}$. □

Proof of Lemma 3.4. — (1) The first assertion is proved in Lemma 3.6. (3.3) can be proved by a simple calculation

(2) In the Taylor expansion (3.2) when $\chi_{i,R}(w) \neq 0$, we have $|R_{h_i}(w)| \leq C\|w - h_i\|_B^3 \leq CR^{-1/2}\|w - h_i\|_W^2$. This implies

$$\mathcal{E}_{\lambda,A,V}(f\chi_{i,R}, f\chi_{i,R}) \geq \lambda e(A, K_i - CR^{-1/2}S)\|f\chi_{i,R}\|^2_{L^2(\mu_\lambda)}.$$

Here S is the trace class operator which defines the norm of W. Using the fact that

$$\lim_{R\to\infty} e(A, K_i - CR^{-1/2}S) = e(A, K_i)$$

which follows from Lemma 2.5, we complete the proof of (2).

(3) Let ρ be a continuous function on W such that (i) $0 \leq \rho(w) \leq 1$, (ii) ρ is 0 near the neighborhood $U(N)$ of the zero point set N, (iii) ρ is 1 in $V(N)^c$, where $V(N)$ is a neighborhood of N such that $U(N) \subset V(N)$. Moreover assume that $\{w \mid \chi_{0,R}(w) \neq 0\} \subset \{w \mid \rho(w) = 1\}$. Let r be a small positive number. Then

$$\begin{aligned}\mathcal{E}_{\lambda,A,V}(f\chi_{0,R}, f\chi_{0,R}) &= \mathcal{E}_{\lambda,A,V-r\rho}(f\chi_{0,R}, f\chi_{0,R}) + \int_W r\lambda^2\rho f^2\chi_{0,R}^2 d\mu_\lambda \\ &= \mathcal{E}_{\lambda,A,(V-r\rho)\rho}(f\chi_{0,R}, f\chi_{0,R}) + \int_W r\lambda^2 f^2\chi_{0,R}^2 d\mu_\lambda.\end{aligned}$$

L^2-norm of the second term on the right-hand side is $r\lambda^2\|f\chi_{0,R}\|^2$. To estimate the first term, we use again IMS localization formula. We write $g_0 = f\chi_{0,R}$. Let $\varphi_0(w) = \chi\left(\frac{\|w\|_W^2}{R^2}\right)$ and $\varphi_1(w) = \sqrt{1 - \varphi_0(w)^2}$. Then

$$\mathcal{E}_{\lambda,A,(V-r\rho)\rho}(g_0, g_0) = \sum_{i=0,1} \mathcal{E}_{\lambda,A,(V-r\rho)\rho}(g_0\varphi_i, g_0\varphi_i) - \sum_{i=0,1} \int_W \|D_A\varphi_i\|_H^2 g_0^2 d\mu_\lambda.$$

We use Corollary 2.8 (2) to estimate the term containing $g_0\varphi_0$. Let $\tilde{\varphi}_0(w) = \chi\left(\frac{\|w\|_W^2}{3R^2}\right)$. We can find a positive number ε' and R' such that $\{w \in W \mid \rho(w)\tilde{\varphi}_0(w) \neq 0\} \subset D_{\varepsilon',R'}$. Let $T_{\varepsilon'}$ be a trace class operator which satisfies the property in Lemma 3.5 for $D_{\varepsilon',R'}$. Then

$$\begin{aligned}\mathcal{E}_{\lambda,A,(V-r\rho)\rho}(g_0\varphi_0, g_0\varphi_0) &\geq \mathcal{E}_{\lambda,I_H+T_{\varepsilon'},(V-r\rho)\rho\tilde{\varphi}_0}(g_0\varphi_0, g_0\varphi_0) \\ &\geq -\frac{\lambda}{2}\log I(\lambda)\|g_0\varphi_0\|^2_{L^2(\mu_\lambda)} \\ &\quad + \left(\frac{\lambda}{2}\log\det{}_{(2)}(I_H + T_{\varepsilon'}) - \frac{\lambda}{2}\mathrm{tr}\left(T_{\varepsilon'}^2\right)\right)\|g_0\varphi_0\|^2_{L^2(\mu_\lambda)},\end{aligned}$$

where

$$\begin{aligned}I(\lambda) = \int_W \exp\Big(&-2\lambda\left((V(w) - r\rho(w))\,\rho(w)\tilde{\varphi}_0(w)\right. \\ &\left. - \lambda : (T_{\varepsilon'}w, w) :_\lambda - \frac{\lambda}{2}\|T_{\varepsilon'}w\|_H^2\Big) d\mu_\lambda(w).\end{aligned}$$

Let $U_{\varepsilon'}(h)=\frac{1}{4}\|(I_H+T_{\varepsilon'})h\|_H^2+(V(h)-r\rho(h))\rho(h)\tilde{\varphi}_0(h)$. Then

$$\begin{aligned} U_{\varepsilon'}(h) &= \frac{1}{4}\|(I_H+T_{\varepsilon'})h\|_H^2(1-\rho(h)\tilde{\varphi}_0(h)) \\ &\quad +\left\{\frac{1}{4}\|(I_H+T_{\varepsilon'})h\|_H^2+(V(h)-r\rho(h))\right\}\rho(h)\tilde{\varphi}_0(h). \end{aligned}$$

By the property of $T_{\varepsilon'}$, by taking r to be sufficiently small, we have

$$\left\{\frac{1}{4}\|(I_H+T_{\varepsilon'})h\|_H^2+(V(h)-r\rho(h))\right\}\rho(h)\tilde{\varphi}_0(h)\geq 0 \qquad \text{for all } h\in H.$$

Therefore by the Large deviation estimate, for such an r, $\lim_\lambda \frac{1}{\lambda}\log I(\lambda)\leq 0$. This shows that for any $c>0$ it holds that for large λ

$$\mathcal{E}_{\lambda,A,(V-r\rho)\rho}(g_0\varphi_0,g_0\varphi_0)\geq -c\lambda^2\|g_0\varphi_0\|_{L^2(\mu_\lambda)}^2.$$

Next, we give a lower bound estimate for the another term. Let $\tilde{\varphi}_1(w)=\sqrt{1-\chi\left(\frac{3\|w\|_W^2}{R^2}\right)^2}$. Then $\{w\mid g_0(w)\varphi_1(w)\neq 0\}\subset\{w\mid \tilde{\varphi}_1(w)=1\}$. By using Corollary 2.8 (1),

$$\mathcal{E}_{\lambda,A,(V-r\rho)\rho}(g_0\varphi_1,g_0\varphi_1) \quad\geq\quad -\frac{\lambda}{2}\log\left(\int_W \exp\left(-2\lambda(V-r\rho)\rho\tilde{\varphi}_1\right)d\mu_\lambda\right)\|g_0\varphi_1\|_{L^2(\mu_\lambda)}^2.$$

If R is sufficiently large and r is small, then

$$\inf\left\{\frac{1}{4}\|h\|_H^2+(V(h)-r\rho(h))\,\rho(h)\tilde{\varphi}_1(h)\;\middle|\;\tilde{\varphi}_1(h)\neq 0, h\in H\right\}>0.$$

Thus, by the Large deviation results, for any $c>0$ it holds that for large λ

$$\mathcal{E}_{\lambda,A,(V-r\rho)\rho}(g_0\varphi_1,g_0\varphi_1)\geq -c\lambda^2\|g_0\varphi_1\|_{L^2(\mu_\lambda)}^2.$$

These prove (3). $\square$

Remark 3.8. — Let $\tilde{V}$ be a bounded measurable function on W. Assume that A^4+4AKA is strictly positive and AKA is a trace class operator. Let

$$c_{A,K}=\inf\sigma\left(\sqrt{A^4+4AKA}\right).$$

Then it holds that for any $f\in\mathfrak{F}C_A^\infty(W)$,

$$\begin{aligned} &\mathcal{E}_{\lambda,A,V_{k,h}+\tilde{V}}(f,f) \\ &\quad\geq E_0(\lambda,A,V_K)\|f\|_{L^2(\mu_\lambda)}^2 \\ &\quad -\frac{\lambda c_{A,K}}{2}\log\left(\int_W \exp\left(-\frac{2\lambda}{c_{A,K}}\tilde{V}(w)\right)\Omega_{\lambda,A,V_{K,h}}(w)^2 d\mu_\lambda(w)\right)\|f\|_{L^2(\mu_\lambda)}^2. \end{aligned}$$

By this estimate, we can prove local estimates near N in Lemma 3.4 (2) using the Laplace method. This proof could be extended to the case of Schrödinger operators with more general potential functions.

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