Reflected rough differential equations*

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Abstract

In this paper, we study reflected differential equations driven by continuous paths with finite p-variation ($1 \le p < 2$) and p-rough paths ($2 \le p < 3$) on domains in Euclidean spaces whose boundaries may not be smooth. We define reflected rough differential equations and prove the existence of a solution. Also we discuss the relation between the solution to reflected stochastic differential equation and reflected rough differential equation when the driving process is a Brownian motion.

Keywords: reflected stochastic differential equation, rough path, Skorohod equation

1 Introduction

In [2], we proved the strong convergence of the Wong-Zakai approximations of the solutions to reflected stochastic differential equations defined on domains in Euclidean spaces whose boundaries may not be smooth. The driving stochastic process in the equation is a Brownian motion. Recently, many researchers have been studying differential equations driven by more general stochastic processes and irregular paths. This is due to the development of rough path theory which gives new meaning of stochastic integrals. The aim of this paper is to study reflected differential equations driven by rough paths and prove the existence of solutions. We use the Euler approximation of the differential equations by modifying the idea of Davie [5]. When the equation has reflection term, the Euler approximation becomes an implicit Skorohod equation and it is not trivial to see the existence of the solutions. Hence, we need stronger assumptions than those given in [2] on the boundary of the domain to prove the existence of solutions. At the moment, we neither have uniqueness of solutions nor continuity theorem with respect to driving paths.

The paper is organized as follows. In Section 2, we introduce several conditions on the boundary under which reflected rough differential equations are studied and prepare necessary lemmas. In Section 3, we study the reflected differential equations driven by continuous path of finite p-variation with $1 \le p < 2$. The meaning of the integral in this equation is justified by the Young integrals. We prove the existence of solutions by using Davie's approach [5]. This problem was already studied when D is a half space in [9]. Our existence theorem is valid for more general domains. In Section 4, we study the case where the driving path is p-rough path with $2 \le p < 3$. In this case, we consider stronger assumptions than that in previous sections.

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First, we define reflected rough differential equations and prove the existence of a solution and give some estimates of the solution. Also we explain the reason of the difficulty to prove the uniqueness of solutions and continuity theorems with respect to driving rough paths. At the end of this section, we prove the existence of a measurable solution mapping for geometric rough paths. In Section 5, we go back to reflected SDEs driven by Brownian motion. We explain the relation between the solution to reflected rough differential equation and the solution to reflected stochastic differential equation.

2 Preliminary

First, we prepare necessary definitions and results for our purposes. The following conditions on the connected domain $D \subset \mathbb{R}^d$ are standard assumptions for reflected SDE and can be found in [13, 19, 22] and we will study our equations on domains which satisfy these conditions. We will introduce other conditions later. For other references of reflected SDEs related with this paper, we refer the readers to [2, 25, 6, 7, 8, 16, 17, 18, 20, 21]. In [1], we study Wong-Zakai approximations ([23]) in the two cases, (i) the domain is convex, (ii) the conditions (A) and (B) are satisfied which are not contained in the result in [2].

Recall that the set $\mathcal{N}_x$ of inward unit normal vectors at the boundary point $x \in \partial D$ is defined by

$$\mathcal{N}_x = \cup_{r>0} \mathcal{N}_{x,r},$$
$$\mathcal{N}_{x,r} = \left\{ \boldsymbol{n} \in \mathbb{R}^d \mid |\boldsymbol{n}| = 1, B(x - r\boldsymbol{n}, r) \cap D = \emptyset \right\},$$

where $B(z, r) = \{y \in \mathbb{R}^d \mid |y - z| < r\}$, $z \in \mathbb{R}^d$, $r > 0$.

Definition 2.1. (A) *There exists a constant $r_0 > 0$ such that*

$$\mathcal{N}_x = \mathcal{N}_{x,r_0} \neq \emptyset \quad \textit{for any } x \in \partial D.$$

(B) *There exist constants $\delta > 0$ and $\beta \geq 1$ satisfying:*
for any $x \in \partial D$ there exists a unit vector l_x such that

$$(l_x, \boldsymbol{n}) \geq \frac{1}{\beta} \qquad \textit{for any } \boldsymbol{n} \in \cup_{y \in B(x,\delta)\cap\partial D} \mathcal{N}_y.$$

(C) *There exists a C_b^2 function f on $\mathbb{R}^d$ and a positive constant γ such that for any $x \in \partial D$, $y \in \bar{D}$, $\boldsymbol{n} \in \mathcal{N}_x$ it holds that*

$$(y - x, \boldsymbol{n}) + \frac{1}{\gamma} ((Df)(x), \boldsymbol{n}) |y - x|^2 \geq 0.$$

We use the following quantities of paths w_t as in [2].

$$\|w\|_{\infty,[s,t]} = \max_{s \leq u \leq v \leq t} |w_u - w_v|, \tag{2.1}$$

$$\|w\|_{[s,t]} = \sup_{\Delta} \sum_{k=1}^{N} |w_{t_k} - w_{t_{k-1}}|, \tag{2.2}$$

where $\Delta = \{s = t_0 < \cdots < t_N = t\}$ is a partition of the interval $[s, t]$. When the domain D satisfies the conditions (A) and (B), the Skorohod problem associated with a continuous path $w \in C([0, T] \to \mathbb{R}^d)$:

$$\xi_t = w_t + \phi(t), \quad \xi_t \in \bar{D} \quad 0 \le t \le T, \tag{2.3}$$

$$\phi(t) = \int_0^t 1_{\partial D}(\xi_s) \boldsymbol{n}(s) d\|\phi\|_{[0,s]}, \qquad \boldsymbol{n}(s) \in \mathcal{N}_{\xi_s} \text{ if } \xi_s \in \partial D \tag{2.4}$$

can be uniquely solved. See [19]. When the mapping $w \mapsto \xi$ is unique, we write $\Gamma(w)_t = \xi_t$ and $L(w)(t) = \phi(t)$. The following lemma can be proved by a similar proof to that of Lemma 2.3 in [2].

Lemma 2.2. *Assume conditions* (A) *and* (B) *hold. Let w_t be a continuous path of finite p-variation such that*

$$|w_t - w_s| \le \omega(s, t)^{1/p} \qquad 0 \le s \le t \le T,$$

where $p \ge 1$ and $\omega(s, t)$ is the control function of w_t. Then the local time term ϕ of the solution to the Skorohod problem associated with w has the following estimate.

$$\|\phi\|_{[s,t]} \le \beta \left(\left\{ \delta^{-1} G(\|w\|_{\infty,[s,t]}) + 1 \right\}^p \omega(s, t) + 1 \right) \left(G(\|w\|_{\infty,[s,t]}) + 2 \right) \|w\|_{\infty,[s,t]}, \tag{2.5}$$

where

$$G(a) = 4 \left\{ 1 + \beta \exp \left\{ \beta \left(2\delta + a \right) / (2r_0) \right\} \right\} \exp \left\{ \beta \left(2\delta + a \right) / (2r_0) \right\}, \quad a \in \mathbb{R}.$$

3 Reflected differential equations driven by continuous paths of finite p-variation with $1 \le p < 2$

Let x_t $(0 \le t \le T)$ be a continuous path of finite p-variation on $\mathbb{R}^n$ with the control function $\omega(s, t)$, where $1 \le p < 2$. We prove the existence of a solution y_t which is also a continuous path with finite p-variation to the reflected differential equation driven by x:

$$y_t = y_0 + \int_0^t \sigma(y_s) dx_s + \Phi(t), \qquad y_0 \in \bar{D}, \tag{3.1}$$

where $\sigma \in C_b^1(\mathbb{R}^d, \mathbb{R}^n \otimes \mathbb{R}^d)$. The integral in this equation is a Young integral [24]. The following is a main result in this section. See Remark 4.6.

Theorem 3.1. *Assume that* (A) *and* (B) *hold. Then there exists a solution* (y, Φ) *to* (3.1) *and satisfies*

$$|y_t - y_s| \le C\omega(s, t)^{1/p} \tag{3.2}$$

$$\|\Phi\|_{[s,t]} \le C\omega(s, t)^{1/p}. \tag{3.3}$$

Here C is a constant which depends on $\omega(0, T)$ and σ and r_0, β, δ in Definition 2.1.

We solve this equation by using the Euler approximation. Let $\Delta : 0 = t_0 < t_1 < \cdots < t_N = T$ be a partition of $[0, T]$. We define y^Δ by the solution to the Skorohod equation:

$$y_t^\Delta = y_{t_{k-1}}^\Delta + \sigma(y_{t_{k-1}}^\Delta)(x_t - x_{t_{k-1}}) + \Phi^\Delta(t) - \Phi^\Delta(t_{k-1}) \quad t_{k-1} \le t \le t_k.$$

Let

$$I_s^\Delta(t) = y_t^\Delta - y_s^\Delta - \sigma(y_s^\Delta)(x_t - x_s) - \left(\Phi^\Delta(t) - \Phi^\Delta(s)\right) \quad s \le t. \tag{3.4}$$

By the definition, we have $I_{t_k}^\Delta(t) = 0$ for all $t_k \le t \le t_{k+1}$ and For any $s \le t \le u$,

$$I_s^\Delta(u) - I_s^\Delta(t) - I_t^\Delta(u) = \left(\sigma(y_t^\Delta) - \sigma(y_s^\Delta)\right)(x_u - x_t).$$

Also we write $\pi^\Delta(t) = \max\{t_k \mid t_k \le t, 0 \le k \le N\}$ for $0 \le t \le T$.

In the following lemma, we use a constant in the estimate (2.5). Let C_0 be a positive number such that $C_0 > 1$ and

$$\|\phi\|_{[s,t]} \le C_0 \left(\omega(s,t) + 1\right) \left(e^{C_0 \omega(s,t)^{1/p}} + 1\right) \omega(s,t)^{1/p} \tag{3.5}$$

holds. Hence for any positive δ, if $\omega(s,t)$ is sufficiently small, $\|\phi\|_{[s,t]} \le (2+\delta) C_0 \omega(s,t)^{1/p}$ holds.

Lemma 3.2. *Let $1 \le p < \gamma \le 2$. Let $C_1 = 3C_0\|\sigma\|_\infty$, $C_2 = 1 + 4C_0\|\sigma\|_\infty$ and $M = \frac{2C_2\|D\sigma\|_\infty}{1-2^{1-(\gamma/p)}}$. For sufficiently small $\varepsilon (\le 1)$ which depends only on $\|\sigma\|_\infty, \|D\sigma\|_\infty$ and C_0 such that for any $t_k \le s \le t$ with $\omega(t_k, t) \le \varepsilon$,*

$$|I_{t_k}^\Delta(t)| \le M \omega(t_k, t)^{\gamma/p} \tag{3.6}$$

$$\|\Phi^\Delta\|_{[s,t]} \le C_1 \omega(s,t)^{1/p}. \tag{3.7}$$

Proof. Note that if (3.6) and (3.7) hold, then by taking ε to be sufficiently small, we have for t with $\omega(t_k, t) \le \varepsilon$,

$$|y_t^\Delta - y_{t_k}^\Delta| \le \left(M\varepsilon^{(\gamma-1)/p} + 3C_0\|\sigma\|_\infty + \|\sigma\|_\infty\right) \omega(t_k,t)^{1/p} \le C_2 \omega(t_k,t)^{1/p}. \tag{3.8}$$

Let K be a positive integer. Consider a claim which depends on K: The estimates (3.6) and (3.7) hold for all t_k and t, where $t_k \le t \le t_{k+K}$ and $0 \le k \le N-1$. We prove this claim by an induction on K. Let $K = 1$. Then $I_{t_k}^\Delta(t) = 0$ for all $t_k \le t \le t_{k+1}$. Also by taking ε to be sufficiently small,

$$\|\Phi^\Delta\|_{[s,t]} \le 3C_0\|\sigma\|_\infty \omega(s,t)^{1/p} \quad \text{for } t_k \le s \le t,\ \omega(t_k,t) \le \varepsilon.$$

Suppose the claim holds for all K which is smaller than or equal to $K'-1$. We prove the case $K = K'$. Let t_l be the largest partition point such that $t_k \le t_l < t \le t_{k+K'}$ and $\omega(t_k, t_l) \le \frac{1}{2}\omega(t_k, t)$. There are two cases, (a) $t_l < \pi^\Delta(t)$ and (b) $t_l = \pi^\Delta(t)$. We consider the case (a). In this case, $t_l < t_{l+1} \le \pi^\Delta(t)$. By the definition, we have $\omega(t_k, t_{l+1}) \ge \frac{1}{2}\omega(t_k, t)$. By the superadditivity of ω, we have

$$\omega(t_{l+1}, t) \le \frac{1}{2}\omega(t_k, t).$$

We have

$$|I_{t_k}^{\Delta}(t)| \le |I_{t_k}^{\Delta}(t_l)| + |I_{t_l}^{\Delta}(t_{l+1})| + |I_{t_{l+1}}^{\Delta}(t)| + |\sigma(y_{t_{l+1}}^{\Delta}) - \sigma(y_{t_l}^{\Delta})||x_t - x_{t_{l+1}}| \\ + |\sigma(y_{t_l}^{\Delta}) - \sigma(y_{t_k}^{\Delta})||x_t - x_{t_l}|$$

By the assumption of the induction, we have

$$|I_{t_k}^{\Delta}(t_l)| \le M\omega(t_k, t_l)^{\gamma/p}, \quad |I_{t_{l+1}}^{\Delta}(t)| \le M\omega(t_{l+1}, t)^{\gamma/p}$$
$$|\sigma(y_{t_{l+1}}^{\Delta}) - \sigma(y_{t_l}^{\Delta})||x_t - x_{t_{l+1}}| \le C_2\|D\sigma\|_\infty \omega(t_l, t_{l+1})^{1/p}\omega(t_{l+1}, t)^{1/p}$$
$$|\sigma(y_{t_l}^{\Delta}) - \sigma(y_{t_k}^{\Delta})||x_t - x_{t_l}| \le C_2\|D\sigma\|_\infty \omega(t_k, t_l)^{1/p}\omega(t_l, t)^{1/p}$$

Therefore

$$|I_{t_k}^{\Delta}(t)| \le M\left(2^{1-(\gamma/p)} + (1 - 2^{1-(\gamma/p)})\varepsilon^{(2-\gamma)/p}\right)\omega(t_k, t)^{\gamma/p} \le M\omega(t_k, t)^{\gamma/p}.$$

In the case of (b), by using the assumption of the induction, we obtain

$$\begin{aligned} |I_{t_k}^{\Delta}(t)| &\le |I_{t_k}^{\Delta}(t_l)| + |I_{t_l}^{\Delta}(t)| + |\sigma(y_{t_l}) - \sigma(y_{t_k})||x_t - x_{t_l}| \\ &\le M\omega(t_k, t_l)^{\gamma/p} + C_2\|D\sigma\|_\infty \omega(t_k, t_l)^{1/p}\omega(t_l, t)^{1/p} \\ &\le M\left(2^{-\gamma/p} + 2^{-1}\left(1 - 2^{1-(\gamma/p)}\right)\varepsilon^{(2-\gamma)/p}\right)\omega(t_k, t)^{\gamma/p} \\ &\le 2^{-1}M\omega(t_k, t)^{\gamma/p}. \end{aligned}$$

Next we show $\|\Phi^{\Delta}\|_{[t_k,t]} \le C_1\omega(t_k, t)^{1/p}$ for t_k, s, t with $t_k \le s \le t \le t_{k+K'}$ and $\omega(t_k, t) \le \varepsilon$. To this end, we note that $\Phi^{\Delta}(t) - \Phi^{\Delta}(t_k) = L(z^{\Delta})(t)$, where $z_t^{\Delta} = I_{t_k}^{\Delta}(t) + y_{t_k}^{\Delta} + \sigma(y_{t_k}^{\Delta})(x_t - x_{t_k})$. By (3.5), it suffices to estimate z_t^{Δ}. Take s, t such that $t_k \le s < t \le t_{k+K'}$ and $\omega(t_k, s) \le \varepsilon, \omega(t_k, t) \le \varepsilon$. We estimate $I_{t_k}^{\Delta}(t) - I_{t_k}^{\Delta}(s)$ by using

$$I_{t_k}^{\Delta}(t) - I_{t_k}^{\Delta}(s) = I_s^{\Delta}(t) + \left(\sigma(y_s^{\Delta}) - \sigma(y_{t_k}^{\Delta})\right)(x_t - x_s). \tag{3.9}$$

Let t_m be the largest number such that $t_m \le s$. That is $t_m = \pi^{\Delta}(s)$. Then we have two cases, (a) $t_k \le t_m \le s < t_{m+1} < t$ and (b) $t_k \le t_m \le s < t \le t_{m+1}$. First we consider the case (a). We have

$$I_s^{\Delta}(t) = I_s^{\Delta}(t_{m+1}) + I_{t_{m+1}}^{\Delta}(t) + (\sigma(y_{t_{m+1}}^{\Delta}) - \sigma(y_s^{\Delta}))(x_t - x_{t_{m+1}}).$$

Since $I_s^{\Delta}(t_{m+1}) = -\left(\sigma(y_s^{\Delta}) - \sigma(y_{t_m}^{\Delta})\right)(x_{t_{m+1}} - x_s)$, we have

$$|I_s^{\Delta}(t_{m+1})| \le C_2\|D\sigma\|_\infty \omega(t_m, s)^{1/p}\omega(s, t_{m+1})^{1/p} \le C_2\|D\sigma\|_\infty \varepsilon^{1/p}\omega(s, t)^{1/p}.$$

By the hypothesis of the induction, $|I_{t_{m+1}}^{\Delta}(t)| \le M\omega(t_{m+1}, t)^{\gamma/p} \le M\varepsilon^{(\gamma-1)/p}\omega(s, t)^{1/p}$. Also,

$$\begin{aligned} |(\sigma(y_{t_{m+1}}^{\Delta}) - \sigma(y_s^{\Delta}))(x_t - x_{t_{m+1}})| &\le 2C_2\|D\sigma\|_\infty \omega(t_m, t_{m+1})^{1/p}\omega(t_{m+1}, t)^{1/p} \\ &\le 2\varepsilon^{1/p}C_2\|D\sigma\|_\infty \omega(s, t)^{1/p}. \end{aligned}$$

By the assumption of the induction and (3.8), we have $|y^\Delta_s - y^\Delta_{t_k}| \le C_2\omega(t_k,s)^{1/p}$. Hence

$$
\begin{aligned}
|I^\Delta_s(t)| &\le \left(3C_2\|D\sigma\|_\infty \varepsilon^{1/p} + M\varepsilon^{(\gamma-1)/p}\right)\omega(s,t)^{1/p}\\
|I^\Delta_{t_k}(t) - I^\Delta_{t_k}(s)| &\le \left(4C_2\|D\sigma\|_\infty \varepsilon^{1/p} + M\varepsilon^{(\gamma-1)/p}\right)\omega(s,t)^{1/p}
\end{aligned}
$$

and

$$
|z^\Delta_t - z^\Delta_s| \le \left(4C_2\|D\sigma\|_\infty \varepsilon^{1/p} + M\varepsilon^{(\gamma-1)/p} + \|\sigma\|_\infty\right)\omega(s,t)^{1/p} \tag{3.10}
$$

We consider the case (b). In this case, $I^\Delta_s(t) = -\left(\sigma(y^\Delta_s) - \sigma(y^\Delta_{t_m})\right)(x_t - x_s)$. Noting $m \le k + K' - 1$ and using the assumption of the induction, we have

$$
\begin{aligned}
|y^\Delta_s - y^\Delta_{t_k}| &\le |y^\Delta_s - y^\Delta_{t_m}| + |y^\Delta_{t_m} - y^\Delta_{t_k}|\\
&\le C_2\omega(t_m,s)^{1/p} + C_2\omega(t_m,t_k)^{1/p}\\
&\le 2C_2\varepsilon^{1/p}.
\end{aligned}
$$

So, we have

$$
|z^\Delta_t - z^\Delta_s| \le \|\sigma\|_\infty\omega(s,t)^{1/p} + 3C_2\|D\sigma\|_\infty\varepsilon^{1/p}\omega(s,t)^{1/p}. \tag{3.11}
$$

(3.5), (3.10) and (3.11) implies that for sufficiently small ε

$$
\|\Phi^\Delta\|_{[s,t]} \le 3C_0\|\sigma\|_\infty\omega(s,t)^{1/p} \quad \text{for } t_k \le s \le t \text{ with } \omega(t_k,s) \le \varepsilon,\ \omega(t_k,t) \le \varepsilon.
$$

□

Actually, the proof of Lemma 3.2 shows

Lemma 3.3. *Let $\Delta = \{t_k\}$ be a partition of $[0,T]$. Let C_1, C_2 be the same numbers as in Lemma 3.2. Then for sufficiently small $0 < \varepsilon \le 1$ which depends only on $\|\sigma\|_\infty, \|D\sigma\|_\infty$ and C_0 such that for any s,t with $\omega(t_k,s) \le \varepsilon, \omega(t_k,t) \le \varepsilon$ for some t_k, we have*

$$
|y^\Delta_t - y^\Delta_s| \le C_2\omega(s,t)^{1/p}. \tag{3.12}
$$

By Lemma 3.3, we can prove the following.

Lemma 3.4. *Let ε be a positive number in Lemma 3.3. Let $\Delta = \{t_k\}_{k=0}^N$ be a partition of $[0,T]$ such that*

$$
\sup_{0\le k\le l\le N-1} |\omega(t_k,t_{l+1}) - \omega(t_k,t_l)| \le \varepsilon/2. \tag{3.13}
$$

Then there exists $C > 0$ such that for any $0 \le s \le t \le T$ the following estimates hold. The constant C depends only on σ, p and D.

(1) $|y^\Delta_t - y^\Delta_s| \le C\left(1 + \omega(0,T)\right)\omega(s,t)^{1/p}$

(2) $\|\Phi^\Delta\|_{[s,t]} \le C\left(1 + \omega(0,T)\right)\omega(s,t)^{1/p}$.

Proof of Lemma 3.4. We note that the statement is true if $t_k \le s \le t \le t_{k+1}$ for some k by Lemma 2.2. Let us consider general cases. We define a subsequence $\{s_k\}_{k=0}^{N'}$ of $\{t_k\}$ in the following way. Let $s_0 = t_0 = 0$. When s_k is defined, we define s_{k+1} is the smallest t_i such that $\omega(s_k, t_i) > \varepsilon/2$ and $t_i > s_k$. If there does not exist such a t_i, we set $s_{k+1} = t_N$. By the assumption (3.13), $\omega(s_k, s_{k+1}) \le \max(\omega(s_k, t_{i-1}) + \varepsilon/2, \varepsilon/2) \le \varepsilon$. Hence by Lemma 3.3,

$$|y^{\Delta}_{s_{k+1}} - y^{\Delta}_{s_k}| \le C_2 \omega(s_k, s_{k+1})^{1/p}.$$

By the superadditivity of ω, we have

$$\omega(0,T) \ge \sum_{k=0}^{N'-1} \omega(s_k, s_{k+1}) \ge (N'-1)\varepsilon/2$$

which implies $N' \le 1 + 2\omega(0,T)/\varepsilon$. For $0 \le s < t \le T$, let us choose the numbers l, m so that $s_l \le s < s_{l+1} \le s_m \le t < s_{m+1}$. Then

$$\begin{aligned}
|y^{\Delta}_t - y^{\Delta}_s| &\le |y^{\Delta}_t - y^{\Delta}_{s_m}| + \sum_{k=l+1}^{m-1} |y^{\Delta}_{s_{k+1}} - y^{\Delta}_{s_k}| + |y^{\Delta}(s_{l+1}) - y^{\Delta}(s)| \\
&\le C_2 \omega(s_m, t)^{1/p} + \sum_{k=l+1}^{m-1} C_2 \omega(s_k, s_{k+1})^{1/p} + C_2 \omega(s, s_{l+1})^{1/p} \\
&\le C_2 \left(2\omega(0,T)/\varepsilon + 3\right) \omega(s,t)^{1/p}.
\end{aligned}$$

For Φ^{Δ}, similarly, we have

$$\begin{aligned}
\|\Phi^{\Delta}\|_{[s,t]} &= \|\Phi^{\Delta}\|_{[s_m,t]} + \sum_{k=l+1}^{m-1} \|\Phi^{\Delta}\|_{[s_k,s_{k+1}]} + \|\Phi^{\Delta}\|_{[s,s_{l+1}]} \\
&\le C_1 \omega(s_m, t)^{1/p} + \sum_{k=l+1}^{m-1} C_1 \omega(s_k, s_{k+1})^{1/p} + C_1 \omega(s, s_{l+1})^{1/p} \\
&\le C_1 \left(2\omega(0,T)/\varepsilon + 3\right) \omega(s,t)^{1/p}.
\end{aligned}$$

These estimates complete the proof. □

Proof of Theorem 3.1. Let us consider a sequence of partitions $\Delta(n) = \{t(n)_k\}$ of $[0,T]$ such that

(a) the estimate (3.13) holds for all $\Delta(n)$,

(b) $\lim_{n\to\infty} \max_{k\ge 0} |t(n)_{k+1} - t(n)_k| = 0$.

These partitions exist because the mapping $(s,t) \mapsto \omega(s,t)$ is continuous. By Lemma 3.4, there exists a subsequence $y^{\Delta(n_k)}$ and $\Phi^{\Delta(n_k)}$ converge uniformly to continuous paths y^{∞} and Φ^{∞} respectively which also satisfy (3.2) and (3.3). Then these subsequences converge in p'-variation norm for any $p' > p$. The solution $y^{\Delta(n_k)}$ satisfies

$$y^{\Delta(n_k)}_t = y_0 + \int_0^t \sigma\left(y^{\Delta(n_k)}(\pi^{\Delta}(u))\right) dx_u + \Phi^{\Delta(n_k)}(t).$$

By taking the limit $n_k \to \infty$ and by the continuity theorem of Young integral and the continuity of the Skorohod map, we see that $(y^{\infty}, \Phi^{\infty})$ is a solution to the equation. □

Before closing this section, we make a simple remark on the continuity of the solution map $x \mapsto y$ when x is a bounded variation path.

Remark 3.5. Let x_t and x'_t be continuous bounded variation paths on $\mathbb{R}^n$ starting at 0. Let D be the domain which satisfies (A), (B), (C). Let us consider two reflected ODEs and their solutions y_t, y'_t:

$$\begin{aligned} y_t &= y_0 + \int_0^t \sigma(y_s)dx_s + \Phi(t) \\ y'_t &= y_0 + \int_0^t \sigma(y'_s)dx'_s + \Phi'(t) \end{aligned}$$

Let $m_t = |y_t - y'_t|^2 e^{-\frac{2}{\gamma}(f(y_t)+f(y'_t))}$. Then by calculating dm_t as in [13, 19], and by the Gronwall inequality, we obtain

$$\sup_{0\le s\le t} |y_s - y'_s| \le Ce^{C'\left(\|x\|_{[0,t]}+\|x'\|_{[0,t]}\right)}\|x - x'\|_{[0,t]}.$$

This implies the solution map $x \mapsto y$ is a Lipschitz continuous map between the set of bounded variation paths and the set of continuous paths.

4 Reflected differential equations driven by p-rough path with $2 \le p < 3$

In this section, we prove the existence of a solution to reflected differential equations driven by rough path. We mainly follow the formulation of rough path in [14, 15, 5]. See also [4, 10, 11, 12]. First, we define reflected differential equation driven by rough path.

Definition 4.1. *Let D be a connected domain in $\mathbb{R}^d$ for which the condition* (A) *holds. Let $2 \le p < 3$. Let $X_{s,t} = (1, X^1_{s,t}, X^2_{s,t}) \in \Omega_p(\mathbb{R}^n)$ $(0 \le s \le t \le T)$ be a p-rough path. Let $Y_{s,t} = (1, Y^1_{s,t}, Y^2_{s,t}) \in \Omega_p(\mathbb{R}^d)$ be a p-rough path and $\Phi(t)$ $(0 \le t \le T)$ be a continuous bounded variation path on $\mathbb{R}^d$. Let $\sigma \in C^2_b(\mathbb{R}^d, \mathbb{R}^n \otimes \mathbb{R}^d)$. The pair (Y, Φ) is called a solution to a rough differential equation on D driven by X with normal reflection with the starting point $y_0 \in \bar{D}$:*

$$dY_t = \sigma(Y_t)dX_t + d\Phi(t) \quad 0 \le t \le T, \quad Y_0 = y_0, \tag{4.1}$$

if the following hold.

(1) *Let $Y_t = y_0 + Y^1_{0,t}$. Then $Y_t \in \bar{D}$ $(0 \le t \le T)$ and it holds that there exists a Borel measurable map $s(\in [0,T]) \mapsto \boldsymbol{n}(s) \in \mathbb{R}^d$ such that $\boldsymbol{n}(s) \in \mathcal{N}_{Y_s}$ if $Y_s \in \partial D$ and*

$$\Phi(t) = \int_0^t 1_{\partial D}(Y_s)\boldsymbol{n}(s)d\|\Phi\|_{[0,s]} \quad 0 \le t \le T. \tag{4.2}$$

(2) *$Y_{s,t}$ is a solution to the following rough differential equation.*

$$dY_t = \hat{\sigma}(Y_t)d\hat{X}_t \quad 0 \le t \le T, \quad Y_0 = y_0, \tag{4.3}$$

where $\hat{\sigma}(x)$ is a linear mapping from $\mathbb{R}^n \oplus \mathbb{R}^d$ to $\mathbb{R}^d$ defined by $\hat{\sigma}(x)(\xi, \eta) = \sigma(x)\xi + \eta$ and the driving rough path $\hat{X} \in \Omega_p(\mathbb{R}^n \oplus \mathbb{R}^d)$ is given by

$$\hat{X}^1_{s,t} = (X^1_{s,t}, \Phi(t) - \Phi(s))$$
$$\hat{X}^2_{s,t} = \Big(X^2_{s,t}, \int_s^t X^1_{s,u} \otimes d\Phi(u), \int_s^t (\Phi(u) - \Phi(s)) \otimes dX^1_{s,u}, \int_s^t (\Phi(u) - \Phi(s)) \otimes d\Phi(u)\Big).$$

Note that if $X_{s,t}$ is a rough path defined by a continuous path X_t of finite q-variation with $1 \le q < 2$, then the solution Y_t coincides with the solution in the sense of Section 2. Below, we assume $\sigma \in C_b^2$. To solve this equation, we consider the Euler approximation modifying the Davies' approximation for rough differential equations without reflection terms. Let $\Delta : 0 = t_0 < t_1 < \cdots < t_N = T$ be a partition of $[0, T]$. Let us consider a Skorohod problem :

$$\begin{aligned} y^\Delta_t &= y^\Delta_{t_{k-1}} + \sigma(y^\Delta_{t_{k-1}})(x_t - x_{t_{k-1}}) + (D\sigma)(y^\Delta_{t_{k-1}})(\sigma(y^\Delta_{t_{k-1}})X^2_{t_{k-1},t}) \\ &\quad + (D\sigma)(y^\Delta_{t_{k-1}})\left(\int_{t_{k-1}}^t (\Phi^\Delta(r) - \Phi^\Delta(t_{k-1})) \otimes dx_r\right) + \Phi^\Delta(t) - \Phi^\Delta(t_{k-1}) \\ &\qquad y^\Delta_t \in \bar{D}, \quad y^\Delta_0 = y_0, \quad t_{k-1} \le t \le t_k, \quad 1 \le k \le N, \end{aligned} \tag{4.4}$$

where $x_t = X^1_{0,t}$. That is, the pair $(y^\Delta_t, \Phi^\Delta_t - \Phi^\Delta_{t_{k-1}})$ is the solution to the Skorohod problem associated with the continuous path

$$\begin{aligned} &y^\Delta_{t_{k-1}} + \sigma(y^\Delta_{t_{k-1}})(x_t - x_{t_{k-1}}) + (D\sigma)(y^\Delta_{t_{k-1}})(\sigma(y^\Delta_{t_{k-1}})X^2_{t_{k-1},t}) \\ &\quad + (D\sigma)(y^\Delta_{t_{k-1}})\left(\int_{t_{k-1}}^t (\Phi^\Delta(r) - \Phi^\Delta(t_{k-1})) \otimes dx_r\right) \qquad t_{k-1} \le t \le t_k. \end{aligned}$$

Since this is an implicit Skorohod problem, the existence of the solution is not trivial. In view of this, we consider the following condition (D) and assumptions (H1) and (H2) on D.

Assumption 4.2. (D) *Condition* (A) *is satisfied and there exist constants $K_1 \ge 0$ and $0 < K_2 < r_0$ such that*

$$|\bar{x} - \bar{y}| \le (1 + K_1\varepsilon)|x - y|$$

holds for any $x, y \in \mathbb{R}^d$ with $|x - \bar{x}| \le K_2$, $|y - \bar{y}| \le K_2$, where $\varepsilon = \max\{|x - \bar{x}|, |y - \bar{y}|\}$. Here $\bar{x}$ denotes the nearest point of x in $\bar{D}$.

(H1) *The condition* (A) *holds and the Skorohod problem* (2.3), (2.4) *is uniquely solved for any w. Moreover, there exists a positive constant C_D such that for all continuous paths w on $\mathbb{R}^d$*

$$\|L(w)\|_{[s,t]} \le C_D \|w\|_{\infty,[s,t]} \quad 0 \le s \le t \le T.$$

(H2) *The condition* (A) *holds and the Skorohod problem* (2.3), (2.4) *is uniquely solved for any w. Moreover, there exists a positive constant C'_D such that for all continuous paths w, w' on $\mathbb{R}^d$*

$$\|L(w) - L(w')\|_{\infty,[0,t]} \le C'_D \left\{\|w - w'\|_{[0,t]} + |w(0) - w'(0)|\right\}.$$

Remark 4.3. It is proved in [22] that the condition (H1) holds if D is convex and there exists a unit vector $l \in \mathbb{R}^d$ such that

$$\inf\{(l, \boldsymbol{n}(x)) \mid \boldsymbol{n}(x) \in \mathcal{N}_x, x \in \partial D\} > 0.$$

The condition (H2) holds if the conditions (B) and (D) are satisfied. This is due to [19].

About the existence and uniqueness of solutions to (4.4), we have the following.

Lemma 4.4. *Let η_t be a continuous path on $\mathbb{R}^d$ with $\eta_0 = 0$ and x_t be a continuous path of finite p-variation on $\mathbb{R}^n$ with $x_0 = 0$ for some $p \geq 1$. Let F be a linear mapping from $\mathbb{R}^d \otimes \mathbb{R}^n$ to $\mathbb{R}^d$. We consider the following implicit Skorohod equation:*

$$y_t = y_0 + \eta_t + F\left(\int_0^t \Phi(r) \otimes dx_r\right) + \Phi(t) \qquad y_0 \in \bar{D} \quad 0 \leq t \leq T, \tag{4.5}$$

where $y_t \in \bar{D}$ $(0 \leq t \leq T)$ and $\Phi(t)$ is a continuous bounded variation path which satisfies

$$L\left(y_0 + \eta_\cdot + F\left(\int_0^\cdot \Phi(r) \otimes dx_r\right)\right)(t) = \Phi_t \quad 0 \leq t \leq T, \qquad \Phi_0 = 0.$$

(1) *Assume* (H2) *are satisfied and x_t is bounded variation. Then there exists a unique solution $(y_t, \Phi(t))$ to* (4.5)*.*

(2) *Assume* (H1) *holds. There exists a solution $(y_t, \Phi(t))$ to* (4.5)*.*

Proof. (1) By (H2), we see the unique existence of Φ, by a standard iteration procedure on continuous path spaces with the norm $\| \ \|_{\infty,[0,T]}$ considering the equation in the small interval, if necessary. This arguments produce the solution for the whole interval $[0, T]$.
(2) First we prove the existence of a solution on a small interval $[0, T']$, where $T' < T$. We specify T' later. Let $\Delta = \{t_k\}_{k=0}^N$ be a partition of $[0, T']$. We consider the Euler approximation of y.

$$y_t^\Delta = y_{t_k}^\Delta + \eta_t - \eta_{t_k} + F\left(\Phi^\Delta(t_k) \otimes (x_t - x_{t_k})\right) + \Phi^\Delta(t) - \Phi^\Delta(t_k) \quad t_k \leq t \leq t_{k+1}.$$

That is, y^Δ, Φ^Δ satisfies

$$y_t^\Delta = y_0 + \eta_t + F\left(\int_0^t \Phi^\Delta(\pi^\Delta(r)) \otimes dx_r\right) + \Phi^\Delta(t) \qquad 0 \leq t \leq T'.$$

Let $0 \leq s < t \leq T'$. If $t_{k-1} \leq s < t \leq t_k$ for some k, then

$$\int_s^t \Phi^\Delta(\pi^\Delta(r)) \otimes dx_r = \Phi^\Delta(t_{k-1}) \otimes (x_t - x_s). \tag{4.6}$$

We consider the case where $0 \leq t_{k-1} \leq s < t_k < \cdots < t_l \leq t < t_{l+1} \leq T'$. Then

$$\begin{aligned}\int_s^t \Phi^\Delta(\pi^\Delta(r)) \otimes dx_r = {} & \Phi^\Delta(t_{k-1}) \otimes (x_{t_k} - x_s) + \Phi^\Delta(t_l) \otimes (x_t - x_{t_l}) \\ & + \Phi^\Delta(t_{l-1}) \otimes x_{t_l} - \Phi^\Delta(t_k) \otimes x_{t_k} \\ & + \sum_{m=k}^{l-2} \left(\Phi^\Delta(t_m) - \Phi^\Delta(t_{m+1})\right) \otimes x_{t_{m+1}}. \end{aligned} \tag{4.7}$$

Therefore we have for all $0 \le s < t \le T'$,

$$\left| \int_s^t \Phi^\Delta(\pi^\Delta(r)) \otimes dx_r \right| \le 3\|\Phi^\Delta\|_{[0,T']}\|x\|_{\infty,[s,t]} + 2\|\Phi^\Delta\|_{[s,t]}\|x\|_{\infty,[0,T']} \le 5\|\Phi^\Delta\|_{[0,T']}\|x\|_{\infty,[0,T']}. \tag{4.8}$$

Hence by (H1),

$$\|\Phi^\Delta\|_{[0,T']} \le C_D \left(\|\eta\|_{\infty,[0,T']} + 5\|F\|\|x\|_{\infty,[0,T']}\|\Phi^\Delta\|_{[0,T']} \right).$$

Therefore if $\|x\|_{\infty,[0,T']} \le 1/\left(10 C_D \|F\|\right)$,

$$\|\Phi^\Delta\|_{[0,T']} \le 2C_D\|\eta\|_{\infty,[0,T']}. \tag{4.9}$$

Substituting this into (4.8), we obtain for any $0 \le s \le t \le T'$,

$$\left| \int_s^t \Phi^\Delta(\pi^\Delta(r)) \otimes dx_r \right| \le 6C_D\|\eta\|_{\infty,[0,T']}\|x\|_{\infty,[s,t]} + 2\|\Phi^\Delta\|_{[s,t]}\|x\|_{\infty,[0,T']}. \tag{4.10}$$

Hence, again by applying (H1), we obtain

$$\|\Phi^\Delta\|_{[s,t]} \le C_D\|\eta\|_{\infty,[s,t]} + 6C_D^2\|F\|\,\|\eta\|_{\infty,[0,T']}\|x\|_{\infty,[s,t]} + 2C_D\|F\|\,\|\Phi^\Delta\|_{[s,t]}\|x\|_{\infty,[0,T']}. \tag{4.11}$$

Consequently, if

$$\|x\|_{\infty,[0,T']} \le \left(10C_D\|F\|\right)^{-1} \tag{4.12}$$

then

$$\|\Phi^\Delta\|_{[s,t]} \le \frac{5}{4}C_D\|\eta\|_{\infty,[s,t]} + 10C_D^2\|F\|\,\|\eta\|_{\infty,[0,T']}\|x\|_{\infty,[s,t]} \qquad 0 \le s \le t \le T'.$$

Now we choose T' so that (4.12) holds. Then $\{\Phi^\Delta\}_\Delta$ is a family of equicontinuous and bounded functions on $[0,T']$ and so there exists a sequence $|\Delta_n| \to 0$ such that Φ^{Δ_n} converges to a certain Φ uniformly on $[0,T']$. By the estimate (4.9), this convergence takes place for all p-variation norm ($p > 1$) on $[0,T']$. Therefore $F\left(\int_0^t \Phi^{\Delta_n}(\pi^{\Delta_n}(r)) \otimes dx_r\right)$ converges uniformly to $F\left(\int_0^t \Phi(r) \otimes dx_r\right)$. Here we use the property of Young integrals. Also $y_t^{\Delta_n}$ converges uniformly. We denote the limit by y. Then $(y_t, \Phi(t))$ $(0 \le t \le T')$ is a solution to (4.5). Next, we need to construct a solution after time T'. For $t \ge T'$, (4.5) reads

$$y_t = y_{T'} + (\eta_t - \eta_{T'}) + F\left(\Phi_{T'} \otimes (x_t - x_{T'})\right) + F\left(\int_{T'}^t (\Phi(r) - \Phi(T')) \otimes dx_r\right) + \Phi_t - \Phi_{T'}. \tag{4.13}$$

Since T' depends only on C_D and $\|F\|$, by iterating the above procedure, we can get a solution defined on $[0,T]$. □

By the above lemma, we see that there exist a solution (y^Δ, Φ^Δ) to the implicit Skorohod equation (4.4). Using this approximation solution, we can prove the existence of a solution to reflected rough differential equations. Now we state our main theorem in this section.

Theorem 4.5. *Assume* (H1) *and* $\sigma \in C_b^3$. *Let* ω *be the control function of* $X_{s,t}$, *i.e., it holds that*

$$|X^i_{s,t}| \le \omega(s,t)^{i/p} \qquad 0 \le s \le t \le T, \quad i = 1, 2.$$

Then there exists a solution (Y, Φ) *to the reflected rough differential equation* (4.1) *such that for all* $0 \le s \le t \le T$,

$$|Y^i_{s,t}| \le C(1 + \omega(0,T))^3 \omega(s,t)^{i/p}, \qquad i = 1, 2, \tag{4.14}$$

$$\|\Phi\|_{[s,t]} \le C(1 + \omega(0,T))^3 \omega(s,t)^{1/p}, \tag{4.15}$$

where the positive constant C *depends only on* σ, C_D, p.

In the proof of this theorem, we use Lyons' continuity theorem. That is why we assume $\sigma \in C_b^3$. However, it may not be necessary. Actually $\sigma \in C_b^2$ is sufficient for the proof of Lemma 4.7 and Lemma 4.8. Here we make remarks on this theorem together with Theorem 3.1.

Remark 4.6. (1) At the moment, I do not prove the uniqueness yet and it is not clear to see whether the functional $X \mapsto \Phi$, $X \mapsto Y$ is continuous or not. Actually, at the moment, I do not know the existence of Borel measurable selection of the mapping. We consider this measurable selection problem for geometric rough path at the end of this section. If there are no boundary terms, the functional $X \mapsto Y$ is continuous and this is known as Lyons' continuity theorem and universal limit theorem. If the continuity theorem would hold, then by applying it to the case of Brownian rough path, it would imply the strong convergence of Wong-Zakai approximation which was proved in [2] under general conditions on the boundary. We discuss the relation between the solution to reflected rough differential equation driven by Brownian rough path and the solution to reflected SDE driven by Brownian motion later.
(2) We consider the case where D is a half space. In this simplest case too, we have difficulties to prove the uniqueness of solutions and continuity theorems with respect to driving paths (rough paths) in the equations (3.1) and (4.1). We explain the reason. When D is a half space, the Skorohod mapping Γ is given explicitly and it is globally Lipschitz continuous in the set of continuous path spaces with the sup-norm. This nice result is used in the studies [3, 6]. However, it is not Lipschitz continuous in the λ-Hölder continuous path spaces C^λ. This is pointed out by Ferrante and Rovira [9] who studied reflected differential equations driven by Hölder continuous paths on half spaces. This implies the difficulty of the study of the uniqueness of solutions to reflected differential equations as pointed out in their paper. We may need to restrict the set of solutions to reflected rough differential equations to obtain the uniqueness. In the usual rough differential equations, we have locally Lipschitz continuities of the solutions with respect to the driving rough paths. On the other hand, it is not difficult to show that Γ is Hölder continuous mapping in C^λ. Hence it may be possible to prove such a weaker continuity of the solution mapping for reflected rough differential equation.

To prove this theorem, we argue similarly to the case $1 \le p < 2$. When $\Phi^\Delta(t)$ is defined, let

$$\begin{aligned} J^\Delta_s(t) &= I^\Delta_s(t) - D\sigma(y^\Delta_s)(\sigma(y^\Delta_s))(X^2_{s,t}) \\ &\quad - (D\sigma)(y^\Delta_s)\left(\int_s^t \left(\Phi^\Delta(r) - \Phi^\Delta(s)\right) \otimes dx_r\right) \quad s \le t. \end{aligned}$$

The definition of $I_s^\Delta(t)$ is similar to (3.4) just replacing Φ^Δ by a solution to (4.4). By the definition of y^Δ, we have $J_{t_k}^\Delta(t) = 0$ for $t_k \le t \le t_{k+1}$. We define $J^\Delta(s,t,u) = J_s^\Delta(u) - J_s^\Delta(t) - J_t^\Delta(u)$. By an easy calculation, we have for $s \le t \le u$,

$$
\begin{aligned}
J^\Delta(s,t,u) = &\Big(\sigma(y_t^\Delta) - \sigma(y_s^\Delta) - (D\sigma)(y_s^\Delta)(y_t^\Delta - y_s^\Delta) + (D\sigma)(y_s^\Delta)(I_s^\Delta(t))\Big)(x_t - x_u) \\
&+ \big((D\sigma)(y_t^\Delta)(\sigma(y_t^\Delta)) - (D\sigma)(y_s^\Delta)(\sigma(y_s^\Delta))\big)\,(X_{t,u}^2) \\
&+ \big((D\sigma)(y_t^\Delta) - (D\sigma)(y_s^\Delta)\big)\left(\int_t^u \big(\Phi^\Delta(r) - \Phi^\Delta(t)\big) \otimes dx_r\right).
\end{aligned}
$$

This relation plays important role as in [5] and the proof in Lemma 3.2 in the calculation below.

Lemma 4.7. *Suppose* (H1) *hold. Let* $2 \le p < \gamma \le 3$. *There exist positive constants* M *and* ε *which depend only on* σ *and* C_D *such that if* $\omega(t_k,t) \le \varepsilon(\le 1)$ *and* $t_k \le s \le t$, *then*

$$
|J_{t_k}^\Delta(t)| \le M\omega(t_k,t)^{\gamma/p} \tag{4.16}
$$

$$
\|\Phi^\Delta\|_{[s,t]} \le C_3\omega(s,t)^{1/p}, \tag{4.17}
$$

where $C_3 = 2C_D\|\sigma\|_\infty$. *The constant* M *is specified in* (4.20).

Proof. If (4.16) and (4.17) hold, then

$$
\begin{aligned}
|y_t^\Delta - y_{t_k}^\Delta| &\le \Big(M\varepsilon^{(\gamma-1)/p} + \|\sigma\|_\infty + C_3 + \|D\sigma\|_\infty\|\sigma\|_\infty\varepsilon^{1/p} + 2C_3\|D\sigma\|_\infty\varepsilon^{1/p}\Big)\,\omega(t_k,t)^{1/p} \\
&\le C_4\omega(t_k,t)^{1/p},
\end{aligned}
$$

where $C_4 = 1 + C_3 + \|\sigma\|_\infty$ and we have used the relation

$$
\int_{t_k}^t \big(\Phi^\Delta(r) - \Phi^\Delta(t_k)\big) \otimes dx_r = \big(\Phi^\Delta(t) - \Phi^\Delta(t_k)\big) \otimes (x_t - x_{t_k}) - \int_{t_k}^t d\Phi^\Delta(r) \otimes (x_r - x_{t_k}).
$$

Also

$$
\begin{aligned}
|I_{t_k}^\Delta(t)| &\le M\omega(t_k,t)^{\gamma/p} + \|D\sigma\|_\infty\|\sigma\|_\infty\omega(t_k,t)^{2/p} + 2C_3\|D\sigma\|_\infty\omega(t_k,t)^{2/p} \\
&\le C_5\omega(t_k,t)^{2/p},
\end{aligned}
$$

where $C_5 = 1 + 2C_3\|D\sigma\|_\infty + \|D\sigma\|_\infty\|\sigma\|_\infty$. Let

$$
\begin{aligned}
z_t^\Delta = &\, y_{t_k}^\Delta + \sigma(y_{t_k}^\Delta)(x_t - x_{t_k}) + (D\sigma)(y_{t_k}^\Delta)(\sigma(y_{t_k}^\Delta)X_{t_k,t}^2) \\
&+ (D\sigma)(y_{t_k}^\Delta)\left(\int_{t_k}^t (\Phi^\Delta(r) - \Phi^\Delta(t_k)) \otimes dx_r\right) + J_{t_k}^\Delta(t) \qquad t \ge t_k.
\end{aligned}
$$

Then $\Phi^\Delta(t) - \Phi^\Delta(t_k) = L(z^\Delta)(t)$ for $t \ge t_k$. We use this relation to estimate Φ^Δ. Let K be a positive integer. Consider a claim which depends on K: The estimates (4.16) and (4.17) hold for all t_k and t, where $t_k \le t \le t_{k+K}$ and $0 \le k \le N-1$. We prove this claim by an induction on K. Let $K = 1$. By the definition, $J_{t_k}^\Delta(t) = 0$ for any $t_k \le t \le t_{k+1}$. We estimate the bounded variation norm of Φ^Δ. Let $t_k \le s \le t \le t_{k+1}$. Noting Chen's identity

$$
X_{t_k,t}^2 - X_{t_k,s}^2 = X_{s,t}^2 + (x_s - x_{t_k}) \otimes (x_t - x_s). \tag{4.18}
$$

and by (H1),

$$\|\Phi^\Delta\|_{[s,t]} \le C_D \left(\|\sigma\|_\infty + 2\|D\sigma\|_\infty \|\sigma\|_\infty \varepsilon^{1/p}\right) \omega(s,t)^{1/p} + C_D \varepsilon^{1/p} \|D\sigma\|_\infty \|\Phi^\Delta\|_{[s,t]}$$
$$+ C_D \|D\sigma\|_\infty \|\Phi^\Delta\|_{[t_k,t]} \omega(s,t)^{1/p}$$

which implies for sufficiently small ε,

$$\|\Phi^\Delta\|_{[s,t]} \le 2C_D \|\sigma\|_\infty \omega(s,t)^{1/p}.$$

Suppose the claim holds for all K which is smaller than or equal to $K'-1$. We prove the case $K = K'$. Let t_l be the largest partition point such that $t_k \le t_l < t \le t_{k+K'}$ and $\omega(t_k, t_l) \le \frac{1}{2}\omega(t_k, t)$. There are two cases, (a) $t_l < \pi^\Delta(t)$ and (b) $t_l = \pi^\Delta(t)$. We consider the case (a). In this case, $t_l < t_{l+1} \le \pi^\Delta(t)$. By the definition, we have $\omega(t_k, t_{l+1}) \ge \frac{1}{2}\omega(t_k, t)$. By the superadditivity of ω, we have

$$\omega(t_{l+1}, t) \le \frac{1}{2}\omega(t_k, t). \tag{4.19}$$

We have

$$|J^\Delta_{t_k}(t)| \le |J^\Delta_{t_k}(t_l)| + |J^\Delta_{t_l}(t_{l+1})| + |J^\Delta_{t_{l+1}}(t)|$$
$$+ |J^\Delta(t_k, t_l, t)| + |J^\Delta(t_l, t_{l+1}, t)|$$

By the assumption of the induction and the choice of t_l,

$$|J^\Delta_{t_k}(t_l)| \le 2^{-\gamma/p} M \omega(t_k, t)^{\gamma/p}, \quad |J^\Delta_{t_{l+1}}(t)| \le 2^{-\gamma/p} M \omega(t_k, t)^{\gamma/p}.$$

By the assumption of the induction, we have

$$|J^\Delta(t_k, t_l, t)| \le (C_4/2)\|D^2\sigma\|_\infty \omega(t_k, t_l)^{2/p} \omega(t_l, t)^{1/p} + C_5 \|D\sigma\|_\infty \omega(t_k, t_l)^{2/p} \omega(t_l, t)^{1/p}$$
$$+ C_4 \left(\|D^2\sigma\|_\infty \|\sigma\|_\infty + \|D\sigma\|_\infty^2\right) \omega(t_k, t_l)^{1/p} \omega(t_l, t)^{2/p}$$
$$+ 2C_4 C_3 \|D^2\sigma\|_\infty \omega(t_k, t_l)^{1/p} \omega(t_l, t)^{2/p}.$$

Here we have used that if $t_k < t_l$ we can use the assumption of the induction and so,

$$\left|\int_{t_l}^t \left(\Phi^\Delta(r) - \Phi^\Delta(t_l)\right) \otimes dx_r\right| = \left|\left(\Phi^\Delta(t) - \Phi^\Delta(t_l)\right) \otimes (x_t - x_{t_l}) - \int_{t_l}^t d\Phi^\Delta(r) \otimes (x_r - x_{t_l})\right|$$
$$= 2\|\Phi^\Delta\|_{[t_l,t]} \omega(t_l, t)^{1/p}$$
$$\le 2C_3 \omega(t_l, t)^{2/p}.$$

Similarly,

$$|J^\Delta(t_l, t_{l+1}, t)| \le (C_4^2/2)\|D^2\sigma\|_\infty \omega(t_l, t_{l+1})^{2/p} \omega(t_{l+1}, t)^{1/p} + C_5 \|D\sigma\|_\infty \omega(t_l, t_{l+1})^{2/p} \omega(t_{l+1}, t)^{1/p}$$
$$+ C_4 \left(\|D^2\sigma\|_\infty \|\sigma\|_\infty + \|D\sigma\|_\infty^2\right) \omega(t_l, t_{l+1})^{1/p} \omega(t_{l+1}, t)^{2/p}$$
$$+ 2C_3 C_4 \|D^2\sigma\|_\infty \omega(t_l, t_{l+1})^{1/p} \omega(t_{l+1}, t)^{2/p}.$$

Consequently,

$$|J^{\Delta}_{t_k}(t)| \le 2^{1-(\gamma/p)} M\omega(t_k,t)^{\gamma/p} + \varepsilon^{(3-\gamma)/p} C_6 \omega(t_k,t)^{\gamma/p},$$

where

$$C_6 = C_4^2 \|D^2\sigma\|_\infty + 2C_5\|D\sigma\|_\infty + 2C_4\left(\|D^2\sigma\|_\infty\|\sigma\|_\infty + \|D\sigma\|_\infty^2\right) + 4C_3C_4\|D^2\sigma\|_\infty.$$

Therefore, if M satisfies

$$M \ge \frac{C_6}{1-2^{1-(\gamma/p)}}, \tag{4.20}$$

then the desired estimate for $J^{\Delta}_{t_k}(t)$ holds. In the case of (b), by using the assumption of the induction and noting $J^{\Delta}_{t_l}(t)=0$, we obtain

$$\begin{aligned}
|J^{\Delta}_{t_k}(t)| &\le |J^{\Delta}_{t_k}(t_l)| + |J^{\Delta}_{t_l}(t)| + |J^{\Delta}(t_k,t_l,t)| \\
&\le M\omega(t_k,t_l)^{\gamma/p} + |J^{\Delta}(t_k,t_l,t)| \\
&\le 2^{-\gamma/p} M\omega(t_k,t)^{\gamma/p} + \left(\varepsilon^{(3-\gamma)/p}/2\right) C_6\omega(t_k,t)^{\gamma/p}.
\end{aligned}$$

Hence, under the condition (4.20), the desired estimate for $J^{\Delta}_{t_k}(t)$ holds. We show $\|\Phi^{\Delta}\|_{[s,t]} \le C_3\omega(s,t)^{1/p}$ for $t_k \le s < t \le t_{k+K'}$ with $\omega(t_k,t)\le\varepsilon$. We have

$$J^{\Delta}_{t_k}(t) - J^{\Delta}_{t_k}(s) = J^{\Delta}_{s}(t) + J^{\Delta}(t_k,s,t).$$

Let t_m be the largest number such that $t_m \le s$. Then we have two cases, (a) $t_k \le t_m \le s < t_{m+1} < t$ and (b) $t_k \le t_m \le s < t \le t_{m+1}$. We consider the case (a). We can apply the assumption of the induction to t_k, s and we obtain,

$$\begin{aligned}
|J^{\Delta}(t_k,s,t)| \le\ & 2^{-1}C_4^2\|D^2\sigma\|_\infty C_4^2\omega(t_k,s)^{2/p}\omega(s,t)^{1/p} \\
&+ C_5\|D\sigma\|_\infty\omega(t_k,s)^{2/p}\omega(s,t)^{1/p} \\
&+ C_4\left(\|D^2\sigma\|_\infty\|\sigma\|_\infty + \|D\sigma\|_\infty^2\right)\omega(t_k,s)^{1/p}\omega(s,t)^{2/p} \\
&+ 2C_4\|D^2\sigma\|_\infty\omega(t_k,s)^{1/p}\|\Phi^{\Delta}\|_{[s,t]}\omega(s,t)^{1/p}.
\end{aligned}$$

We have

$$J^{\Delta}_s(t) = J^{\Delta}_s(t_{m+1}) + J^{\Delta}_{t_{m+1}}(t) + J^{\Delta}(s,t_{m+1},t).$$

Since $J^{\Delta}_s(t_{m+1}) = -J^{\Delta}(t_m,s,t_{m+1})$,

$$\begin{aligned}
|J^{\Delta}_s(t_{m+1})| \le\ & 2^{-1}\|D^2\sigma\|_\infty\omega(t_m,s)^{2/p}\omega(s,t_{m+1})^{1/p} + C_5\|D\sigma\|_\infty\omega(t_m,s)^{2/p}\omega(s,t_{m+1})^{1/p} \\
&+ C_4\left(\|D^2\sigma\|_\infty\|\sigma\|_\infty + \|D\sigma\|_\infty^2\right)\omega(t_m,s)^{1/p}\omega(s,t_{m+1})^{2/p} \\
&+ 2C_4\|D^2\sigma\|_\infty\omega(t_m,s)^{1/p}\|\Phi^{\Delta}\|_{[s,t_{m+1}]}\omega(s,t_{m+1})^{1/p}.
\end{aligned}$$

Note that

$$\begin{aligned}
|I^{\Delta}_s(t_{m+1})| &\le |I^{\Delta}_{t_m}(t_{m+1})| + |I^{\Delta}_{t_m}(s)| + |\sigma(y^{\Delta}_s) - \sigma(y^{\Delta}_{t_m})|\cdot|x_{t_{m+1}} - x_s| \\
&\le C_5\omega(t_m,t_{m+1})^{2/p} + C_5\omega(t_m,s)^{2/p} + \|D\sigma\|_\infty C_4\omega(t_m,s)^{1/p}\omega(s,t_{m+1})^{1/p} \\
&\le (2C_5 + C_4\|D\sigma\|_\infty)\varepsilon^{1/p}\omega(t_k,t)^{1/p} \\
|y^{\Delta}_s - y^{\Delta}_{t_{m+1}}| &\le 2C_4\omega(t_m,t_{m+1})^{1/p}.
\end{aligned}$$

Hence,

$$\begin{aligned}|J^{\Delta}(s,t_{m+1},t)| &\le 2C_4^2\|D^2\sigma\|_\infty\omega(t_m,t_{m+1})^{2/p}\omega(t_{m+1},t)^{1/p}\\&\quad+\|D\sigma\|_\infty(2C_5+C_4\|D\sigma\|_\infty)\varepsilon^{1/p}\omega(t_k,t)^{1/p}\omega(t_{m+1},t)^{1/p}\\&\quad+2C_4\left(\|D^2\sigma\|_\infty\|\sigma\|_\infty+\|D\sigma\|_\infty^2\right)\omega(t_m,t_{m+1})^{1/p}\omega(t_{m+1},t)^{2/p}\\&\quad+4C_4\|D^2\sigma\|_\infty\omega(t_m,t_{m+1})^{1/p}\|\Phi^{\Delta}\|_{[t_{m+1},t]}\omega(t_{m+1},t)^{1/p}.\end{aligned}$$

By the assumption of induction,

$$|J^{\Delta}_{t_{m+1}}(t)|\le M\omega(t_{m+1},t)^{\gamma/p}.$$

Because

$$\int_s^t\left(\Phi^{\Delta}(r)-\Phi^{\Delta}(t_k)\right)\otimes dx_r=\left(\Phi^{\Delta}(t)-\Phi^{\Delta}(t_k)\right)\otimes(x_t-x_s)-\int_s^t d\Phi^{\Delta}(r)\otimes(x_r-x_s),$$

we have

$$\begin{aligned}\left|\int_s^t\left(\Phi^{\Delta}(r)-\Phi^{\Delta}(t_k)\right)\otimes dx_r\right| &\le \|\Phi^{\Delta}\|_{[t_k,t]}\omega(s,t)^{1/p}+\|\Phi^{\Delta}\|_{[s,t]}\omega(s,t)^{1/p}\\&\le C_3\varepsilon^{1/p}\omega(s,t)^{1/p}+\varepsilon^{1/p}\|\Phi^{\Delta}\|_{[s,t]}.\end{aligned}$$

By Chen's identity and putting the estimates above together, by (H1), for sufficiently small ε, we have

$$\begin{aligned}\|\Phi^{\Delta}\|_{[s,t]} &\le C_D\|z^{\Delta}\|_{\infty,[s,t]}\\&\le C_D\|\sigma\|_\infty\omega(s,t)^{1/p}+\varepsilon^{1/p}C_D\left(2\|D\sigma\|_\infty\|\sigma\|_\infty+C_3\|D\sigma\|_\infty\right)\omega(s,t)^{1/p}\\&\quad+C_D\|D\sigma\|_\infty(C_6\varepsilon^{2/p}+2\varepsilon^{1/p})\|\Phi^{\Delta}\|_{[s,t]}\\&\quad+C_DC_7\varepsilon^{1/p}\omega(s,t)^{1/p},\end{aligned}\tag{4.21}$$

where C_7 depends only on p,σ,D. Therefore, for sufficiently small ε, we obtain $\|\Phi^{\Delta}\|_{[s,t]}\le C_3\omega(s,t)^{1/p}$. We consider the case (b). Since $I^{\Delta}_{t_k}(s)=I^{\Delta}_{t_k}(t_m)+I^{\Delta}_{t_m}(s)+\left(\sigma(y_{t_m})-\sigma(y_{t_k})\right)(x_s-x_{t_m})$, by using the assumption of the induction, we have

$$\begin{aligned}|I^{\Delta}_{t_k}(s)| &\le C_5\omega(t_k,t_m)^{2/p}+C_5\omega(t_m,s)^{2/p}+C_4\|D\sigma\|_\infty\omega(t_k,t_m)^{1/p}\omega(t_m,s)^{1/p}\\&\le(2C_5+C_4\|D\sigma\|_\infty)\varepsilon^{2/p}.\end{aligned}$$

Since $J^{\Delta}_s(t)=-J^{\Delta}(t_m,s,t)$, we have

$$\begin{aligned}|J^{\Delta}_s(t)| &\le 2^{-1}C_4^2\|D^2\sigma\|_\infty\omega(t_m,s)^{2/p}\omega(s,t)^{1/p}+C_5\|D\sigma\|_\infty\omega(t_m,s)^{2/p}\omega(s,t)^{1/p}\\&\quad+C_4\left(\|D^2\sigma\|_\infty\|\sigma\|_\infty+\|D\sigma\|_\infty^2\right)\omega(t_m,s)^{1/p}\omega(s,t)^{2/p}\\&\quad+2C_4\|D^2\sigma\|_\infty\omega(t_m,s)^{1/p}\omega(s,t)^{1/p}\|\Phi^{\Delta}\|_{[s,t]}.\end{aligned}$$

Therefore, by the same argument as the case (a), we complete the proof of the case (b) and the proof of the lemma is finished. □

Actually, the above proof shows stronger estimates similarly to the case $1 \le p < 2$. For $t_k \le s \le t$ with $\omega(t_k, t) \le \varepsilon$

$$|J^{\Delta}_{t_k}(t) - J^{\Delta}_{t_k}(s)| \le C_8 \varepsilon^{1/p} \omega(s,t)^{1/p}.$$

Thus taking smaller ε if necessary, we have

$$|y^{\Delta}_t - y^{\Delta}_s| \le C_4 \omega(s,t)^{1/p} \quad \text{for } t_k \le s \le t \text{ with } \omega(t_k,t) \le \varepsilon.$$

For general s, t, we have the following estimates.

Lemma 4.8. *Let ε be a positive number specified in the above argument. Let $\Delta = \{t_k\}_{k=1}^N$ be a partition of $[0,T]$ which satisfies (3.13). Then there exists $C > 0$ such that for any $0 \le s \le t \le T$ the following estimates hold. The constant C depends only on σ, p and D.*

(1) $|y^{\Delta}_t - y^{\Delta}_s| \le C\,(1 + \omega(0,T))\,\omega(s,t)^{1/p}$

(2) $\|\Phi^{\Delta}\|_{[s,t]} \le C\,(1 + \omega(0,T))\,\omega(s,t)^{1/p}$.

Proof. The proof of this lemma is similar to that of Lemma 3.4. □

Now we prove our main theorem.

Proof of Theorem 4.5. Let $\widehat{X^{\Delta}}$ be the naturally defined p-rough path associated with the p-rough path X and Φ^{Δ} as in Definition 4.1 (2). Thanks to the above lemma, this family of p-rough path has a common control function $C\omega$ for some positive constant C which is independent of Δ. Let $p' > p$. Since the two-parameter function $(s,t) \mapsto \widehat{X^{\Delta}}_{s,t}$ and y^{Δ}_t are equicontinuos (we need Chen's identity to prove the equicontinuity of the former), there exist subseqeunces $\widehat{X^{\Delta_n}}, y^{\Delta_n}$, a p-rough path $\hat{X} \in \Omega_p(\mathbb{R}^n \oplus \mathbb{R}^d)$, a continuous path y and a positive decreasing sequence $\delta_n \downarrow 0$ such that

$$\left|\widehat{X^{\Delta_n}}_{s,t} - \hat{X}_{s,t}\right| \le \delta_n \omega(s,t)^{1/p'} \quad 0 \le s \le t \le T,$$
$$\lim_{n\to\infty} \max_{0 \le t \le T} |y^{\Delta_n}_t - y_t| = 0,$$

where Δ_{n+1} is a subdivision of Δ_n and $|\Delta_n| \to 0$. We denote the limit of $\Phi^{\Delta_n}(t)$ by $\Phi(t)$. Clearly, the estimate (4.15) holds for this Φ. The limit $\hat{X}$ is naturally defined rough path by X and Φ as in Definition 4.1 (2). Also we have for all $0 \le s \le t \le T$,

$$\Bigg| y_t - y_s - \sigma(y_s)(x_t - x_s) - (\Phi(t) - \Phi(s)) \\ - (D\sigma)(y_s)(\sigma(y_s) X^2_{s,t}) - (D\sigma)(y_s)\left(\int_s^t (\Phi(r) - \Phi(s)) \otimes dx_r\right)\Bigg| \le C\omega(s,t)^{\gamma/p}. \tag{4.22}$$

(4.16) shows (4.22) for $s = t_k \in \cup_n \Delta_n$ with $\omega(t_k, t) \le \varepsilon$. By the denseness of $\cup_n \Delta_n$ and the continuity of the functions on the both sides in (4.22), we see that this estimate holds for s, t with $\omega(s,t) \le \varepsilon$. When $\omega(s,t) \ge \varepsilon$, the estimate clearly holds. This shows y_t is a solution to

$$dy_t = \hat{\sigma}(y_t) d\hat{X}_t$$

in the sense of Davie [5]. Also we can find a p-rough path $Y_{s,t} \in \Omega_p(\mathbb{R}^d)$ so that $y_t = y_0 + Y^1_{0,t}$ and the equation (4.1) is satisfied. We refer the reader for the construction of $Y_{s,t}$ to [5]. We write $Y_t = y_0 + Y^1_{0,t}$. Since $\hat{X}_{s,t}$ has the control function $C(1+\omega(0,T))\omega(s,t)$, the estimate on the rough differential equations implies the estimate (4.14). We have to show Y_t and $\Phi(t)$ is the solution to the Skorohod problem associated with the first level path $y_0 + \int_0^t \sigma(Y_s)dX^1_s$. To this end, we consider the solution $Y^{\Delta_n}_{s,t}$ associated with $\widehat{X^{\Delta_n}}_{s,t}$. Let $Y^{\Delta_n}_t = y_0 + (Y^{\Delta_n})^1_{0,t}$. Since $\widehat{X^{\Delta_n}}$ converges to $\hat{X}$ in $\Omega_{p'}(\mathbb{R}^n \oplus \mathbb{R}^d)$, by Lyons' continuity theorem of solutions to rough differential equations, we have $\lim_{n\to\infty} \|Y^{\Delta_n} - y\|_{\infty,[0,T]} = 0$. Hence

$$\lim_{n\to\infty} \|Y^{\Delta_n} - y^{\Delta_n}\|_{\infty,[0,T]} = 0.$$

Also by Lyons' continuity theorem for the integrals of p'-rough path,

$$\lim_{n\to\infty} \left\| \int_0^{\cdot} \sigma(Y^{\Delta_n}_s)dX^1_s - \int_0^{\cdot} \sigma(Y_s)dX^1_s \right\|_{\infty,[0,T]} = 0.$$

Let $z^{\Delta_n}_t = y^{\Delta_n}_t - \Phi^{\Delta_n}_t$. Then $(y^{\Delta_n}_t, \Phi^{\Delta_n}_t)$ is the solution to the Skorohod problem associated with $z^{\Delta_n}_t$. Since $z^{\Delta_n}_t = y^{\Delta_n}_t - Y^{\Delta_n}_t + y_0 + \int_0^t \sigma(Y^{\Delta_n}_s)dX^1_s$, $z^{\Delta_n}_t$ converges to $y_0 + \int_0^t \sigma(Y_s)dX^1_s$ uniformly. By the continuity of the Skorohod map (see [19]), this shows the desired result. □

Let $G\Omega_p(\mathbb{R}^n)$ be the set of geometric rough paths with $2 \le p < 3$. That is, this set is the closure of the set of smooth rough paths in p-variation norm. For solutions to reflected rough differential equations driven by geometric rough path, we can prove the existence of measurable selection of the solution mapping.

Theorem 4.9. *Assume D satisfies* (H1) *and $\sigma \in C^3_b$. There exists a universally measurable map $I : G\Omega_p(\mathbb{R}^n) \to G\Omega_p(\mathbb{R}^d) \times V_p(\mathbb{R}^d)$ such that the following hold. Here $V_p(\mathbb{R}^d)$ denotes the set of continuous paths of finite p-variation in $\mathbb{R}^d$ defined on* $[0,T]$.
(1) *For any $X \in G\Omega_p(\mathbb{R}^n)$, $I(X)$ is a solution to* (4.1) *and satisfies the estimates* (4.14) *and* (4.15).
(2) *There exists a sequence of smooth rough paths $\{X_N\} \subset G\Omega_p(\mathbb{R}^n)$ such that $\lim_{N\to\infty} X_N = X$ and $\lim_{N\to\infty} I(X_N) = I(X)$, where the convergences take place in the topology $G\Omega_p(\mathbb{R}^n)$ and the product topology of $G\Omega_{p'}(\mathbb{R}^d) \times V_p(\mathbb{R}^d)$ for all $p < p' < 3$ respectively.*

Proof. For any $X \in G\Omega_p(\mathbb{R}^n)$, there exists a sequence of smooth rough paths $\{X_N\}$ such that $\lim_{N\to\infty} \|X_N - X\|_p = 0$. Let (Y_N, Φ_N) be the solution to reflecting rough differential equation driven by X_N. Let $\widetilde{X_N} \in G\Omega_p(\mathbb{R}^n \oplus \mathbb{R}^d \oplus \mathbb{R}^d)$ be the smooth rough path associated with (X_N, Φ_N, Y_N) similarly to Definition 4.1 (2). By the estimate (4.14) and (4.15), there exists a subsequence $\widetilde{X_{N_k}}$ which converges to an element $\widetilde{X}$ in $G\Omega_p(\mathbb{R}^n \oplus \mathbb{R}^d \oplus \mathbb{R}^d)$ in the topology of $G\Omega_{p'}(\mathbb{R}^n \oplus \mathbb{R}^d \oplus \mathbb{R}^d)$ for any $p < p' < 3$. A pair of the rough path and the bounded variation path $\pi(\tilde{X}) = (Y, \Phi) \in G\Omega_p(\mathbb{R}^d) \otimes V_p(\mathbb{R}^d)$ which is obtained by a projection of $\widetilde{X}$ is a solution to (4.1) driven by X. This follows from the Lyon's continuity theorem and the continuity of the Skorohod map. Let $\Theta \subset G\Omega_p(\mathbb{R}^n \oplus \mathbb{R}^d \oplus \mathbb{R}^d) \times G\Omega_p(\mathbb{R}^n)$ be the set consisting of all limit points $(\widetilde{X}, X)$. Then clearly Θ is a closed subset. Hence by the measurable selection theorem, there exists a universally measurable map $\mathcal{I} : G\Omega_p(\mathbb{R}^n) \to G\Omega_p(\mathbb{R}^n \oplus \mathbb{R}^d \oplus \mathbb{R}^d)$ such that $\{(\mathcal{I}(X), X) \mid X \in G\Omega_p(\mathbb{R}^n)\} \subset \Theta$. The mapping $I(X) = \pi(\mathcal{I}(X))$ is the desired map. □

5 Reflected stochastic differential equation

In this section, we consider stronger topology than p-variation topology of geometric rough path. The set of geometric rough paths $G\Omega_p(\mathbb{R}^n)$ is the closure of the set of smooth rough paths defined by continuous bounded variation paths with respect to the distance d_p below and consists $X_{s,t} = (1, X^1_{s,t}, X^2_{s,t})$ where $X^1_{s,t}, X^2_{s,t}$ are $\mathbb{R}^n$ and $\mathbb{R}^n \otimes \mathbb{R}^n$-valued continuous maps satisfying Chen's identity and

$$\sup_{0\le s<t\le T} \frac{|X^i_{s,t}|}{|t-s|^{i/p}} < \infty. \tag{5.1}$$

The distance is given by

$$d_p(X, X') = \sum_{i=1}^{2} \sup_{0\le s<t\le T} \frac{|X^i_{s,t} - (X')^i_{s,t}|}{|t-s|^{i/p}}, \qquad X, X' \in G\Omega_p(\mathbb{R}^n).$$

$(G\Omega_p(\mathbb{R}^n), d_p)$ is a complete separable metric space. Let $W^n = C([0,T] \to \mathbb{R}^n \mid B(0) = 0)$ be the classical Wiener space. That is, W^n is a probability space with the Wiener measure μ. The coordinate process $t \mapsto B(t)$ is a realization of Brownian motion. Let

$$B^N(t) = B(t^N_{k-1}) + \frac{B(t^N_k) - B(t^N_{k-1})}{\Delta_N}(t - t^N_{k-1}) \quad t^N_{k-1} \le t \le t^N_k,$$

where $t^N_k = kT/(2^N)$ $(1 \le k \le 2^N)$, $\Delta_N = 2^{-N}T$ and $\Delta_k B^N = B(t^N_k) - B(t^N_{k-1})$. We may omit superscript N in the notation t^N_k. Consider a smooth rough path $B^N_{s,t}$ over B^N. Then we can see that there exists a subset $\Omega \subset W^n$ such that $\mu(\Omega) = 1$ and any $B \in \Omega$ satisfies that $B^N_{s,t}$ converges in $G\Omega_p(\mathbb{R}^n)$. The limit which is denoted by $B_{s,t}$ is called a Brownian rough path. We can take control function $\omega(s,t)$ such that $\omega(s,t) = C(X)(t-s)$, where $C(X) = \left(d_p(0,X)^p + d_p(0,X)^{p/2}\right)$ and $X_{s,t} = B_{s,t}, B^N_{s,t}$. It is not difficult to see that

$$E[C(B)^q] + \sup_N E[C(B^N)^q] < \infty \quad \text{for any } q \ge 1. \tag{5.2}$$

Now, again, we assume $\sigma \in C^2_b(\mathbb{R}^d, \mathbb{R}^n \otimes \mathbb{R}^d)$ throughout this section. Let Y^N be the solution to reflected ODE on a domain $D \subset \mathbb{R}^d$:

$$dY^N(t) = \sigma(Y^N(t))dB^N(t) + d\Phi^N(t), \quad Y^N(0) = y_0. \tag{5.3}$$

Under the assumption (H1), by Theorem 4.5 in the Section 4, we have

$$|Y^N(B)^i_{s,t}| \le g(d_p(0, B^N))(t-s)^{i/p} \quad i = 1, 2, \tag{5.4}$$

$$\|\Phi^N(B)\|_{[s,t]} \le g(d_p(0, B^N))(t-s)^{1/p}, \tag{5.5}$$

where g is a polynomial function. Therefore, by the same reasoning as in the proof of Theorem 4.5, for any $B \in \Omega$, there exists a subsequence $N_k(B) \uparrow +\infty$ such that $Y^{N_k(B)}(B)_{s,t}$ and $\Phi^{N_k(B)}(B)(t)$ converge in the topology of p'-rough path and p'-variation path respectively. The limit is a solution to reflected rough differential equation driven by $B_{s,t}$. However, we cannot conclude that the limit and the solution is unique by this argument. On the other hand, the solution Y^N is the Wong-Zakai approximation of $Y^S(t)$ which is the solution to the reflected SDE driven by Brownian motion:

$$dY^S(t) = \sigma(Y^S(t)) \circ dB(t) + d\Phi^S(t), \quad Y^S(0) = y_0,$$

where $\circ dB(t)$ denotes the Stratonovich integral and Φ^S is the local time term. We use the notation Y^S to distinguish the solution in the sense of Itô calculus from the solution in the sense of rough path. Note that in [2], we used the notation $X^N(t)$ for the Wong-Zakai approximation. Let us consider the case $D = \mathbb{R}^d$. Then Lyon's continuity theorem and the coincidence of the solution in the sense of Itô's SDE and rough differential equations, imply that the Wong-Zakai approximation of the solution converges to the solution in the sense of Itô calculus uniformly. However, we cannot do such a thing if $\partial D \neq \emptyset$ because we do not prove the continuity theorem yet. In [2], we proved that $Y^N(t)$ converges to $Y^S(t)$ uniformly on $[0,T]$ for almost all B. By the results in [2], we can prove the following lemma.

Lemma 5.1. *Assume conditions* (A), (B), (C) *are satisfied for D. Then for any $\varepsilon > 0$, there exists a positive constant $C_\varepsilon(T)$ independent of N such that*

$$E\left[\max_{0\le t\le T}\left|\int_0^t \sigma(Y^N(s))dB^N(s) - \int_0^t \sigma(Y^S(s)) \circ dB(s)\right|^2\right] \le C_\varepsilon(T)\cdot 2^{-(1-\varepsilon)N/6}.$$

Thanks to the lemma above, applying the Borel-Cantelli lemma, we see that there exists a full measure subset $\Omega' \subset W^n$ such that

$$\max_{0\le t\le T}\left|\int_0^t \sigma(Y^N(s))dB^N(s) - \int_0^t \sigma(Y^S(s)) \circ dB(s)\right| \to 0 \quad \text{for all } B \in \Omega'.$$

Hence by the continuity property of the Skorohod mapping, $\Phi^N(t)$ also converges to $\Phi^S(t)$ uniformly for all $B \in \Omega'$. Therefore, $Y^N(B)_{s,t}$ converges to a certain p-rough path $Y(B)_{s,t}$ for all $B \in \Omega' \cap \Omega$, without taking subsequences. The pair $(Y(B)_{s,t}, \Phi^S(t))$ is a solution to rough differential equation driven by $B \in \Omega \cap \Omega'$ and $Y^S(t) = y_0 + Y(B)^1_{0,t}$. Also (5.4) and (5.5) imply the following.

Theorem 5.2. *Assume* (A), (B), (C), (H1). *Then we have for $0 \le s \le t \le T$,*

$$|Y^S(t) - Y^S(s)| \le C\,(1 + d_p(0,B))^3\,|t-s|^{1/p},$$
$$\|\Phi^S\|_{[s,t]} \le C\,(1 + d_p(0,B))^3\,|t-s|^{1/p},$$

where C is a positive constant which depends only on σ, D, p.

Proof of Lemma 5.1. In this proof, we use the estimate obtained in [2]. Note that some notation there are different from those in this paper. Take points such that $t_l < t \le t_{l+1}$. We have

$$\left|\int_0^t \sigma(Y^N(s))dB^N(s) - \int_0^{t_l} \sigma(Y^N(s))dB^N(s)\right| \le C|\Delta_l B^N|.$$

Hence

$$
\begin{aligned}
&E\left[\max_{0\le s\le t}\left|\int_0^s \sigma(Y^N(u))dB^N(u) - \int_0^s \sigma(Y^S(u))\circ dB(u)\right|^2\right] \\
&\le 3E\left[\max_{0\le k\le l}\left|\int_0^{t_k} \sigma(Y^N(s))dB^N(s) - \int_0^{t_k} \sigma(Y^S(s))\circ dB(s)\right|^2\right] \\
&\quad + 3CE\left[\max_k |\Delta_k B^N|^2\right] + 3E\left[\max_{|u-v|\le 2^{-N}T, 0\le u\le v\le T}\left|\int_u^v \sigma(Y^S(s))\circ dB(s)\right|^2\right] \\
&\le 3E\left[\max_{0\le k\le l}\left|\int_0^{t_k} \sigma(Y^N(s))dB^N(s) - \int_0^{t_k} \sigma(Y^S(s))\circ dB(s)\right|^2\right] + C_\varepsilon \left(2^{-N}T\right)^{1-\varepsilon},
\end{aligned}
$$

where ε is any positive number. Let $\pi^N(t) = \max\{t_k^N \mid t_k^N \le t\}$.

$$
\begin{aligned}
&\int_0^{t_l} \sigma(Y^N(s))dB^N(s) - \int_0^{t_l} \sigma(Y^S(s))\circ dB(s) \\
&= \int_0^{t_l} \left(\sigma(Y^N(\pi^N(s))) - \sigma(Y^S(s))\right) dB(s) \\
&\quad + \left\{\sum_{k=1}^{l} \int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s} (D\sigma)\left(Y^N(u)\right)\left(\sigma(Y^N(u))\frac{\Delta_k B^N}{\Delta_N}du\right)\left(\frac{\Delta_k B^N}{\Delta_N}\right) ds \right.\\
&\quad \left. - \int_0^{t_l} \frac{1}{2}\mathrm{tr}\,(D\sigma)\left(Y^S(s)\right)(\sigma(Y^S(s)))ds\right\} \\
&\quad + \sum_{k=1}^{l} \int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s} (D\sigma)\left(Y^N(u)\right)\left(\sigma(Y^N(u))d\Phi^N(u)\right)\left(\frac{\Delta_k B^N}{\Delta_N}\right) ds \\
&=: I_1^N(t_l) + I_2^N(t_l) + I_3^N(t_l).
\end{aligned}
$$

Noting

$$
\begin{aligned}
I_1^N(t) = &\int_0^t \Big(\sigma(Y^N(\pi^N(s))) - \sigma(Y^S(\pi^N(s)))\Big)dB(s) \\
&+ \int_0^t \Big(\sigma(Y^S(\pi^N(s))) - \sigma(Y^S(s))\Big)dB(s),
\end{aligned}
$$

and by using Burkholder-Davis-Gundy's inequality and estimates in Theorem 2.9 and Lemma 4.5 in [2], we obtain

$$
E\left[\max_{0\le s\le t} |I_1^N(s)|^2\right] \le Ct\left(2^{-N}T\right)^{(1-\varepsilon)/6} + C\cdot 2^{-N}tT.
$$

$$
\begin{aligned}
I_2^N(t_l) &= \frac{1}{2}\int_0^{t_l} \left(\mathrm{tr}\,(D\sigma)\,(Y^N(\pi^N(s)))(\sigma(Y^N(\pi^N(s)))) - \mathrm{tr}\,(D\sigma)\,(Y^S(s))(\sigma(Y^S(s)))\right) ds \\
&\quad + \sum_{k=1}^{l} \int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^{s} \Bigg\{ (D\sigma)\,(Y^N(u)) \left(\sigma(Y^N(u)) \frac{\Delta_k B^N}{\Delta_N} \right) \\
&\quad - (D\sigma)\,(Y^N(\pi^N(u))) \left(\sigma(Y^N(\pi^N(u))) \frac{\Delta_k B^N}{\Delta_N} \right) \Bigg\} du \left(\frac{\Delta_k B^N}{\Delta_N} \right) ds \\
&\quad + \frac{1}{2} \sum_{k=1}^{l} \Bigg\{ (D\sigma)\,(Y^N(t_{k-1}))\left(\sigma(Y^N(t_{k-1}))\Delta_k B^N\right)\left(\Delta_k B^N\right) \\
&\quad - \sum_{i=1}^{n} (D\sigma)\,(Y^N(t_{k-1}))\left(\sigma(Y^N(t_{k-1}))e_i\right)(e_i)\, 2^{-N}T \Bigg\}, \\
&=: I_{2,1}^N(t_l) + I_{2,2}^N(t_l) + I_{2,3}^N(t_l)
\end{aligned}
$$

where e_i is a unit vector in $\mathbb{R}^d$ whose i-th element is equal to 1. We have

$$
\begin{aligned}
|I_{2,1}^N(t_l)| &\leq C \int_0^{t_l} |Y^N(s) - Y^S(s)| ds \\
&\quad + C t_l \max_{0 \leq u \leq v \leq T, |v-u| \leq 2^{-N}T} \left(|Y^N(v) - Y^N(u)| + |Y^S(v) - Y^S(u)| \right),
\end{aligned}
$$

$$
|I_{2,2}^N(t_l)| \leq C \sum_{k=1}^{l} \max_{0 \leq u \leq v \leq T, |v-u| \leq 2^{-N}T} |Y^N(v) - Y^N(u)| \cdot |\Delta_k B^N|^2.
$$

By Burkholder-Davis-Gundy's inequality, we have

$$
E\left[\max_{1 \leq k \leq l} |I_{2,3}^N(t_k)|^2 \right] \leq CE\left[\sum_{k=1}^{l} \eta_k \right],
$$

where

$$
\eta_k = \sum_{i=1}^{n} \left((\xi_k^i)^2 - 2^{-N}T \right)^2 + \sum_{1 \leq i < j \leq n} (\xi_k^i)^2 (\xi_k^j)^2.
$$

Here $\xi_k^i = B^i(t_k) - B^i(t_{k-1})$ $(1 \leq i \leq n)$ which is the increment of the i-th element of the Brownian motion. By the estimates in Lemma 2.8, Theorem 2.9 and Lemma 4.5 in [2] and arguing similarly to pages 3813 and 3814 in [2], we have

$$
\begin{aligned}
E\left[\max_{1 \leq k \leq l} |I_{2,1}^N(t_k)|^2 \right] &\leq C t_l^2 \left(2^{-N}T \right)^{(1-\varepsilon)/6} + C t_l^2 \left(2^{-N}T \right)^{1-\varepsilon} \\
E\left[\max_{1 \leq k \leq l} |I_{2,2}^N(t_k)|^2 \right] &\leq C \left(2^{-N}T \right)^{1-\varepsilon} \\
E\left[\max_{1 \leq k \leq l} |I_{2,3}^N(t_k)|^2 \right] &\leq C \cdot 2^{-N}T.
\end{aligned}
$$

Finally, since $\max_{0\le t\le T} |I_3^N(t)| \le C\|\Phi^N\|_{[0,T]} \max_k |\Delta_k B^N|$ we have

$$E\left[\max_{0\le t\le T} |I_3^N(t)|^2\right] \le (2^{-N}T)^{1-\varepsilon}$$

which completes the proof. □

Finally, we discuss the relation between the solution to reflected rough differential equation which is obtained as a limit of the Euler approximation defined in (4.4) and Y^S. For each $B_{s,t}$, we see the existence of the solution $y^{\Delta}(B,t)$. However, it is not trivial to see that a certain version of $y^{\Delta}(B,t)$ is a semimartingale. Therefore we need the following proposition.

Proposition 5.3. *Assume D is convex and satisfies* (H1). *Let $\{B_t(w)\}$ be an $\mathcal{F}_t$-Brownian motion on a probability space $(S,\mathcal{F},P)$ and $\eta_t(w)$ be a continuous $\mathcal{F}_t$-semimartingale with $E[\|\eta\|^q_{\infty,[s,t]}] \le C_q(t-s)^{q/2}$ for all $q \ge 1$ and $0 \le s \le t \le T$. We consider the following equation.*

$$Y_t(w) = y_0 + \eta_t(w) + F\left(\int_0^t \Phi(r,w) \otimes dB_r(w)\right) + \Phi(t,w) \qquad 0 \le t \le T, \tag{5.6}$$

where $Y_t(w)$ is an $\mathcal{F}_t$-adapted continuous process and $\Phi(t,w)$ is an $\mathcal{F}_t$-adapted continuous bounded variation process, and $(Y_t(w), \Phi(t,w))$ is the solution to the Skorohod problem associated with $y_0 + \eta_t(w) + F\left(\int_0^t \Phi(r,w) \otimes dB_r(w)\right)$. For this problem, there exists a unique solution.

Proof. We consider again an Euler approximation. Let $\Delta = \{t_k\}$ be a partition of $[0,T]$. We write $|\Delta| = \max_k(t_k - t_{k-1})$ and $\pi^{\Delta}(t) = \max\{t_k \mid t_k \le t\}$. Let Y_t^{Δ} be the solution to the Skorohod equation:

$$Y_t^{\Delta} = Y_{t_{k-1}}^{\Delta} + \eta_t - \eta_{t_{k-1}} + F\left(\Phi^{\Delta}(t_{k-1}) \otimes (B_t - B_{t_{k-1}})\right) + \Phi^{\Delta}(t) - \Phi^{\Delta}(t_{k-1}) \qquad t_{k-1} \le t \le t_k.$$

Then $Y^{\Delta}, \Phi^{\Delta}$ satisfy

$$Y_t^{\Delta} = y_0 + \eta_t + \int_0^t F\left(\Phi^{\Delta}(\pi^{\Delta}(t)) \otimes dB_t\right) + \Phi^{\Delta}(t).$$

Let $q \ge 2$. By the assumption (H1), we have

$$E\left[\|\Phi^{\Delta}\|^q_{\infty,[s,t]}\right] \le C_q(t-s)^{q/2} + C_q(t-s)^{(q-2)/2} \int_s^t E\left[|\Phi^{\Delta}(\pi^{\Delta}(u))|^q\right] du.$$

Hence by considering the case where $s = 0$, we have

$$E\left[\|\Phi^{\Delta}\|^q_{\infty,[0,t]}\right] \le C_q t^{q/2} + C_q t^{(q-2)/2} \int_0^t E\left[\|\Phi^{\Delta}\|^q_{\infty,[0,u]}\right] du$$

and by the Gronwall inequality, we get $E\left[\|\Phi^{\Delta}\|^q_{\infty,[0,t]}\right] \le C_q T^{q/2} \exp\left(T^{q/2}\right)$. Thus, we obtain

$$E\left[\|\Phi^{\Delta}\|^q_{\infty,[s,t]}\right] \le C_q\left(1 + T^{q/2}\exp\left(T^{q/2}\right)\right)(t-s)^{q/2} \quad 0 \le s \le t \le T. \tag{5.7}$$

Let Δ' be another partition of $[0,T]$. Define

$$Z(t) = Y^{\Delta}(t) - Y^{\Delta'}(t),$$
$$k(t) = |Z(t)|^2.$$

By the Itô formula, we have

$$\begin{aligned} dk(t) = 2 \Big(Z(t), F\Big(\Big(\Phi^{\Delta}(\pi^{\Delta}(t)) - \Phi^{\Delta'}(\pi^{\Delta'}(t))\Big) \otimes dB_t\Big)\Big) \\ + \sum_{i=1}^{d} \Big| F\left(\Phi^{\Delta}(\pi^{\Delta}(t)) - \Phi^{\Delta'}(\pi^{\Delta'}(t)), e_i\right)\Big|^2 dt \Big\} \\ + 2\left(Z(t), d\Phi^{\Delta}(t) - d\Phi^{\Delta'}(t)\right). \end{aligned} \tag{5.8}$$

By the convexity of D, we obtain

$$E\left[|Y_t^{\Delta} - Y_t^{\Delta'}|^2\right] \le C_F \int_0^t E\left[|\Phi^{\Delta}(u) - \Phi^{\Delta'}(u)|^2\right] du + C(|\Delta| + |\Delta'|)t,$$

where we have used the estimate (5.7) and the positive constant C_F depends on the (Hilbert-Schmidt) norm of F. Combining the above inequality and the identity

$$\Phi^{\Delta}(t) - \Phi^{\Delta'}(t) = Y^{\Delta}(t) - Y^{\Delta'}(t) - \int_0^t F\Big(\Big(\Phi^{\Delta}(u) - \Phi^{\Delta'}(u)\Big) \otimes dB_u\Big), \tag{5.9}$$

we obtain

$$\begin{aligned} E\left[|Y_t^{\Delta} - Y_t^{\Delta'}|^2\right] \le C(|\Delta| + |\Delta'|)t + 2C_F \int_0^t E[|Y_u^{\Delta} - Y_u^{\Delta'}|^2] du \\ + 2C_F^2 \int_0^t \int_0^u E[|\Phi^{\Delta}(r) - \Phi^{\Delta'}(r)|^2] dr du. \end{aligned}$$

Iterating this procedure, we have

$$E[|Y_t^{\Delta} - Y_t^{\Delta'}|^2] \le C(|\Delta| + |\Delta'|)t + C \int_0^t E[|Y_s^{\Delta} - Y_s^{\Delta'}|^2] ds.$$

By the Gronwall inequality, we obtain

$$E[|Y_t^{\Delta} - Y_t^{\Delta'}|^2] \le C(|\Delta| + |\Delta'|)e^{Ct}t.$$

Therefore, by (5.9),

$$E\left[|\Phi^{\Delta}(t) - \Phi^{\Delta'}(t)|^2\right] \le 2C(|\Delta| + |\Delta'|)e^{Ct}t + C_F \int_0^t E\left[|\Phi^{\Delta}(s) - \Phi^{\Delta'}(s)|^2\right] ds$$

and

$$E\left[|\Phi^{\Delta}(t) - \Phi^{\Delta'}(t)|^2\right] \le 2C(|\Delta| + |\Delta'|)e^{(C+C_F)t}t.$$

Therefore L^2-limit $Y_t := \lim_{|\Delta|\to 0} Y_t^{\Delta}$ and $\Phi(t) := \lim_{|\Delta|\to 0} \Phi^{\Delta}(t)$ exist. Moreover there exists a subsequence Δ such that $\int_0^t F\left(\Phi^{\Delta}(\pi^{\Delta}(s)) \otimes dB(s)\right)$ converges to $\int_0^t F\left(\Phi(s) \otimes dB(s)\right)$ $0 \le t \le T$ uniformly P-a.s. ω. Thus, by the continuity of the Skorohod mapping, we see that the pair (Y, Φ) is a solution. We prove the uniqueness. Let (Y, Φ) and (Y', Φ') be solutions to (5.6). Then by a similar calculation to (5.8), we have

$$E\left[|Y(t) - Y'(t)|^2\right] \le C_F \int_0^t E[|\Phi(s) - \Phi'(s)|^2] ds.$$

By arguing similarly to the above, we complete the proof. □

We consider solutions to (4.4) when $X_{s,t} = B_{s,t}$. By applying the above proposition, we see that the solution $(y^{\Delta}(B), \Phi^{\Delta}(B))$ is unique in the set of semimartingales. We obtain the following convergence speed of $y^{\Delta}(B)$. Below we denote $y^{\Delta}(B)$ by y^{Δ} simply.

Theorem 5.4. *Assume that D is convex and* (H1) *is satisfied. Let $\Delta_N = \{2^{-N}kT\}_{k=0}^{2^N}$. There exists a full measure set $\Omega' \subset \Omega$ such that for any $B \in \Omega'$, y^{Δ_N} and Φ^{Δ_N} converge to Y^S and Φ^S uniformly respectively and*

$$E\left[\sup_{0\le t\le T} |y_t^{\Delta_N} - Y^S(t)|^2\right] \le C_T \Delta_N^{4/p}. \tag{5.10}$$

Proof. Semimartingales $y_t^{\Delta_N}$ and $\Phi^{\Delta_N}(t)$ satisfied the following reflected SDE

$$dy_t^{\Delta_N} = \sigma\left(y_t^{\Delta_N}\right) \circ dB(t) + R_N(t) + \Phi^{\Delta_N}(t), \qquad y^{\Delta_N}(B)_0 = y_0$$

where

$$R_N(t) = \sum_{k=1}^{l-1} M_{t_{k-1},t_k} + M_{t_{l-1},t} \quad t_{l-1} \le t \le t_l$$

and

$$\begin{aligned} M_{t_{k-1},t} = & \int_{t_{k-1}}^t \int_{t_{k-1}}^s \left\{\left(\left(D\sigma(y_u^{\Delta_N})\right)(\sigma(y_u^{\Delta_N})) - \left(D\sigma(y_{t_{k-1}}^{\Delta_N})\right)(\sigma(y_{t_{k-1}}^{\Delta_N}))\right) \circ dB(u)\right\} \circ dB(s) \\ & + \int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^s \left\{\left(\left(D\sigma(y_u^{\Delta_N})\right) - \left(D\sigma(y_{t_{k-1}}^{\Delta_N})\right)\right) d\Phi^{\Delta_N}(u)\right\} \circ dB(s) \qquad t_{k-1} \le t \le t_k. \end{aligned}$$

$R_N(t)$ is an $\mathbb{R}^d$-valued semimartingale and its quadratic variation satisfies that for any unit vectors ξ and $q \ge 1$,

$$\langle (R_N, \xi)\rangle_T \le C_N \Delta_N^{4/p},$$
$$\sup_N E\left[C_N^q\right] < \infty.$$

These estimates follow from Lemma 4.8 and the estimate on the control function ω of $B_{s,t}$. Set

$$Z^N(t) = Y^S(t) - y_t^{\Delta_N}, \quad k_N(t) = |Z^N(t)|^2.$$

Then by a similar calculation to the proof of Theorem 3.1 in [2], we obtain for $0 \le T' \le T$,

$$E\left[\sup_{0\le t\le T'} k_N(t)\right] \le C_T \Delta_N^{4/p} + C_T \int_0^{T'} E\left[\sup_{0\le s\le t} k_N(s)\right] dt$$

which shows (5.10) and $E\left[\sup_{0\le t\le T} k_N(t)\right] \le C'_T \Delta_N^{4/p}$ and $\sup_{0\le t\le T} |Y^S(t) - y_t^{\Delta_N}| \to 0$ for almost all B. These estimates imply that

$$\lim_{N\to\infty} \sup_{0\le t\le T} \left| \int_0^t \sigma\left(y^{\Delta_N}(s)\right) \circ dB(s) - \int_0^t \sigma(Y^S(s)) \circ dB(s) \right| = 0 \quad a.s.$$

and so we have $\lim_{N\to\infty} \sup_{0\le t\le T} |\Phi^{\Delta_N}(t) - \Phi^S(t)| = 0 \quad a.s.$ □

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