UNIVERSAL CONNECTION, STOCHASTIC DIFFERENTIAL EQUATION, CALCULUS ON LOOP SPACE

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1. Introduction

Let (M, g) be a compact Riemannian manifold of dimension n. The loop space $L_x(M) := C([0,1] \to M; \gamma(0) = \gamma(1) = x)$ is equipped with the pinned Brownian motion measure ν_x which is defined by the Levi Civita Laplacian. The gradient operator D on $L_x(M)$ is defined through a Riemannian connection ∇ on (M, g) as follows: for $F(\gamma) = f(\gamma(t_1), \ldots, \gamma(t_m)) \in \mathcal{F}C_b^\infty(L_x(M))$,

$$DF(\gamma)_t = \sum_{i=1}^m \tau(\gamma)_{t_i}^{-1}(\nabla_{(i)} f)(\gamma(t_1), \ldots, \gamma(t_i), \ldots, \gamma(t_m))(t \wedge t_i - tt_i), \tag{1.1}$$

where $\mathcal{F}C_b^\infty(L_x(M))$ denotes the set of the smooth cylindrical functions and $\nabla_{(i)}$ denotes the covariant derivative with respect to the $i-$th variable. Also $\tau(\gamma)_t \ : \ T_x M \to T_{\gamma(t)} M$ is the stochastic parallel translation along γ. Among Riemannian connections, we restrict ourselves to the torsion skew symmetric (=TSS in short) connection, Namely the torsion T must satisfy that

$$g(T(X, Y), Y) = 0.$$

for any vector fields X, Y. Then for the associated derivative D on $L_x(M)$ is a closable operator on $L^2(L_x(M), \nu_x)$. This is an immediate consequence of the integration by parts formula [D]. We denote the closure by the same notation D and the domain of the smallest closed extension by $\mathbb{D}$. Next we consider the topology on $L_x(M)$. Let us assume that M is isometrically embedded into the Euclidean space $\mathbb{R}^d$. As is well-known, $\nu_x(W^{\alpha,2m} \cap L_x(M)) = 1$. Here

$$W^{\alpha,2m} = \left\{ \gamma \in L^{2m}([0,1] \to \mathbb{R}^d) \,\middle|\, \|\gamma\|_{\alpha,2m}^{2m} = \|\gamma\|_{L^{2m}}^{2m} + \int_0^1 \int_0^1 \frac{|\gamma(t) - \gamma(s)|^{2m}}{|t-s|^{1+2m\alpha}} dtds < \infty \right\},$$

where we assume that $0 < \alpha < 1/2$ $m \in \mathbb{N}$ and $2\alpha m > 1$. We consider $L_x(M)$ as the topological subspace of $W^{\alpha,2m}$.

The following is our main theorem.

Theorem 1.1. *Let $F \in \mathbb{D}$ and O be an open connected set in $L_x(M)$. If*

$$DF(\gamma) = 0 \qquad \nu_x - \text{a.s. } \gamma \in O,$$

then

$$F(\gamma) = \text{constant} \qquad \nu_x - \text{a.s.} \gamma \in O.$$

Remark 1.2. *The theorem holds for the topology of the uniform convergence topology.*

In the previous paper [A2], we proved the theorem for the derivative coming from Levi-Civita connection. In the case of Levi-Civita, the proof is based on a property of a certain SDE which is called a gradient Brownian system. In general cases, we use special SDEs which are corresponding to gradient Brownian system and found by [ELL]. In the next section, we recall the Le Jan-Watanabe connection and we explain how to get SDEs using the idea of the existence of the Narasimhan and Ramanan's universal connection which are due to [ELL].

2. Le Jan-Watanabe connection, Universal Connection and ELL's SDE

At first, we recall the Le Jan-Watanabe connection according to [ELL]. Let (M, g) be a compact Riemannian manifold of dimension d. We consider a bundle homomorphism from the trivial product bundle $\pi : M \times \mathbb{R}^N \to M$ to the tangent bundle $\pi : TM \to M$,

$$\sigma : M \times \mathbb{R}^N \to TM \tag{2.1}$$

We assume that

(i) the map $\sigma(x) : \mathbb{R}^N \to T_x M$ is surjective.
(ii) the map $\sigma(x) : \ker \sigma(x)^{\perp} \to T_x M$ is isometry.

Cleary it holds that

$$\sigma(x)\sigma^*(x) = id_{T_x M}$$

For given σ, we shall consider an SDE such that

$$dX(t, x, w) = \sigma(X(t, x, w)) \circ dw(t) \tag{2.2}$$

$$X(0, x, w) = x \in M \tag{2.3}$$

Then by [ELL],

Theorem 2.1. *Let us set for $Z \in \mathfrak{X}(M)$*

$$\nabla Z(x) = \sigma(x) d\ (\sigma(x)^* Z(x)) \tag{2.4}$$

Then the followings hold.

(1) ∇ *defines a Riemannian connection satisfying that*

$$\nabla_{\xi} \sigma(x)(v) = 0 \tag{2.5}$$

for any $\xi \in \ker^{\perp} \sigma(x)$ and $v \in \mathbb{R}^m$.

(2) *If the connection ∇ is TSS, then the diffusion measure which is defined by $X(t)$ is the same as the Brownian motion measure coming from the Levi-Civita Laplacian.*

The connection associated with the SDE (2.2) in the above theorem is called a Le Jan-Watanabe connection. Next we recall the universal connection due to Narasimhan and Ramanan [N-R]. Let $V_{N,n}$ be the Stiefel manifold which consists of orthonormal $n-$frames in the Euclidean space $\mathbb{R}^N$. Note that using unit vector in $\mathbb{R}^N$,

$$e_i = i \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \tag{2.6}$$

for $[v] = (v_1, \dots, v_n) \in V_{N,n}$ there exits an unique $A \in M(N, n; \mathbb{R})$ (=the sets of all real (N, n)-matrices) such that

$$(v_1, \dots, v_n) = (e_1, \dots, e_N)A \tag{2.7}$$

So we have the inclusion map $\rho : V_{N,n} \to M(N, n; \mathbb{R})$. Also we denote by $G_{N,n}$ the Grassmann manifold, the space of $n-$dimensional subspaces of $\mathbb{R}^N$. Let us consider the principal bundle with structure group $O(n)$, $\pi : V_{N,n} \to G_{N,n}$. Then we have a connection $1-$form $\sqsupseteq = \rho^* d\rho$ on $V_{N,n}$.

The following is due to Narasimhan and Ramanan:

Theorem 2.2. *Let us put $N = (n+1)(2n+1)n^3$. Let α be a connection form on the total space $O(M)$ of the orthonormal frame bundle $\pi : O(M) \to M$. Then there exists a bundle homomorphism $\Phi : O(M) \to V_{N,n}$ such that*

$$\Phi^* \sqsupseteq = \alpha \tag{2.8}$$

For any covariant derivative ∇ whose connection is a Riemannian connection on (M, g), we can find an SDE whose Le Jan-Watanabe connection is ∇. This is also due to Elworthy, Le Jan and X-M. Li. Let us explain how to construct σ using Φ. Let us put $N = (n+1)(2n+1)n^3$ as in the above theorem. Let U be an open set in M and take a smooth section $s : U \to O(M)$. We denote $s(x) = (e_1(x), \dots, e_n(x))$. We consider the following $(N, n)-$matrix for $x \in U$:

$$A(x) = \rho \circ \Phi \circ s(x) \tag{2.9}$$

Using $A(x)$, we define $\sigma(x) : \mathbb{R}^N \to T_x M$,

$$\sigma(x) \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_N \end{pmatrix} = (e_1(x), \dots, e_n(x)) \; {}^t A(x) \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_N \end{pmatrix}. \tag{2.10}$$

Since Φ is a bundle homomorphism, it is easy to check that this definition is well-defined and this is a desired one.

Remark 2.3. *Let us consider the Levi-Civita case. We have the isometric embedding* $\iota : M \hookrightarrow \mathbb{R}^d$. *This* ι *naturally induces a bundle homomorphism* $\Phi : O(M) \to V_{d,n}$. σ *which is defined through the relation (2.10) is just the projection operator* $P(x) : \mathbb{R}^d \to T_xM$. *This is nothing but the gradient Brownian system.*

Below we denote the adjoint connection of ∇ by $\hat{\nabla}$ whose definition is given by

$$\hat{\nabla}_X Y = \nabla_X Y - T(X, Y).$$

Also we denote the parallel translation of $\hat{\nabla}$ by $\hat{\tau}(\gamma)_t : T_xM \to T_{\gamma(t)}M$. The following lemma is almost trivial.

Lemma 2.4.
(1) ∇ *is TSS* $\iff \hat{\nabla}$ *is a metric connection.*
(2) *If* ∇ *is Levi-Civita, then* $\nabla = \hat{\nabla}$.

To state a theorem due to [ELL], we introduce more notations. Also we fix a metric connection $\tilde{\nabla}$ on the bundle $\amalg_{x\in M} \ker \sigma(x) \to M$ and denote the parallel translation by $\tilde{\tau}(\gamma)_t : \ker \sigma(x) \to \ker \sigma(\gamma(t))$. So we have two vector bundles $\pi : TM \to M$, $\pi : \amalg_{x\in M} \ker \sigma(x) \to M$ and three metric connections over them, $\nabla, \hat{\nabla}, \tilde{\nabla}$. Let us denote the curvature tensors by $\mathrm{R}, \hat{\mathrm{R}}, \tilde{\mathrm{R}}$ respectively. We denote the Ricci curvatures in the similar way. For a section s of our vector bundles, we denote the trivialization by

$$\begin{aligned}
\overline{s(\gamma)}_t &= \tau(\gamma)_t^{-1} s(\gamma(t)) \in T_xM \\
\overline{s(\gamma)}^{\wedge}_t &= \hat{\tau}(\gamma)_t^{-1} s(\gamma(t)) \in T_xM \\
\overline{s(\gamma)}^{\sim}_t &= \tilde{\tau}(\gamma)_t^{-1} s(\gamma(t)) \in \ker \sigma(x)
\end{aligned}$$

Theorem 2.5 (Elworthy, Le Jan, X-M. Li). *Let us assume that* ∇ *is TSS.*
(1) *Then* $\hat{v}(t) = \hat{\tau}(X)_t^{-1} \partial_x X(t, x, w) : T_xM \to T_xM$ *satisfies the following SDE.*

$$\begin{aligned}
d\hat{v}(t) &= \overline{\nabla\sigma(X)}^{\wedge}_t(\hat{v}(t), d\beta(t)) - \frac{1}{2}\overline{\mathrm{Ric}(X)}^{\wedge}_t \hat{v}(t)dt \\
\hat{v}(0) &= id
\end{aligned}$$

where

$$\beta(t) = \int_0^t \tilde{\tau}(X)_s^{-1} \left\{ id_{\mathbb{R}^N} - \sigma^*(X(s))\sigma(X(s)) \right\} dw(s)$$

(2) $\beta(t)$ *is a standard Brownian motion on* $\ker \sigma(x)$ *and* $X(t)$, $\beta(t)$ *are independent.*

The proof of Theorem 1.1 is carried out in almost similar way as in [A2] using Theorem 2.4. In the case of Levi-Civita, since the adjoint connection is the same connection, the calculation is more easy. We give the sketch of the proof of Theorem 1.1 in the next section.

3. Proof of main theorem

The solution $X(t,x,w)$ has the regularity property as in Theorem 2.12 in [A2]. Let us denote

$$S_x = \left\{ w \in W_0^N \mid X(1,x,w) = x \right\}.$$

We denote the $H-$derivative along S_x by D_S and the normalized probability measure of $\delta_x(X(1,x,w))d\mu(w)$ by $d\mu_x$. Let us denote by $\mathbb{D}(S_x)$ the domain of the Dirichlet form which is the closure of the pre Dirichlet form

$$\mathcal{E}_S(F,F) = \int_{S_x} |D_S F(w)|^2 d\mu_x$$

with the pre domain $\mathcal{F}C_b^\infty(W_0^N)$. (Precisely the quotient space by the identification that $F \sim G \Leftrightarrow F(w) = G(w)\ \mu_x - a.s.w.$) Actually it is proved in [A1] that the selfadjoint extension is unique. Using Theorem 2.4, we get as in [AE]

Lemma 3.1. *For any $F \in \mathbb{D}$, $F \circ X(\cdot,x,w) \in \mathbb{D}(S_x)$ holds and there exists a constant C which depends only on σ such that*

$$\int_{X^{-1}(O)} |D_S\left\{F \circ X(\cdot,x,w)\right\}|^2 d\mu_x \le C \int_O |DF(\gamma)|^2 d\nu_x,$$

where

$$X^{-1}(O) = \{w \in S_x \mid X(\cdot,x,w) \in O\}.$$

$X^{-1}(O)$ is well-defined because of the quasi continuity of X. We can prove the above lemma the same as [AE] after that $\hat{\nabla}$ is also a metric connection.

Theorem 1.1 follows from Lemma 3.1, the following theorem and the fact that the image measure of μ_x by $X(\cdot,x,\cdot)$ is ν_x immediately.

Theorem 3.2. *Let O be an open connected set in $L_x(M)$. If $F \in \mathbb{D}(S_x)$ satisfies $D_S F(w) = 0\ \mu_x - a.s.w \in X^{-1}(O)$, then*

$$F(w) = constant. \quad \mu_x - a.s.w \in X^{-1}(O)$$

We apply Kusuoka's theorem to prove Theorem 3.2. Below (B,H,μ) denotes an abstract Wiener space and we denote the $H-$ derivative by the same notation D on $L_x(M)$. For further notation, we refer to [K]. Here we just recall the $H-$connectivity of $U \subset B$. $U \subset B$ is called an $H-$connected set if $H(w) = \{h \in H \mid w + h \in U\}$ is a connected set in H for any $w \in U$.

Theorem 3.3. *Let U be an $H-$connected set in B. Let us consider the symmetric form $\mathcal{E}_U$ with the domain $\mathbb{D}(\mathcal{E}_U)$ such that*

$$\mathbb{D}(\mathcal{E}_U) = \left\{ u \in W^1(U,\mu) \middle| \int_U \left(|Du(w)|_H^2 + |u(w)|^2\right) d\mu < \infty \right\},$$

$$\mathcal{E}_U(u,u) = \int_U |Du(w)|_H^2 d\mu.$$

Then $\mathcal{E}_U$ is a Dirichlet form and the following property (P) *holds:*

(P) *If $\{u_n\}_{n=1}^{\infty} \subset \mathbb{D}(\mathcal{E}_U)$ satisfies that*

$$\begin{cases} \sup_n \|u_n\|_{L^2(m)} & < \infty \\ \int_U u_n d\mu & = 0 \\ \mathcal{E}(u_n, u_n) & \to 0 \end{cases}$$

then

$$u_n \to 0 \quad \textit{in probability.}$$

The property (P) is stronger than the irreducibility of the Dirichlet form. For example, the unpredictability property holds under (P). See [M1,M2]. Actually we can prove (P) for loop case. We will study this problem in the forthcoming paper.

It suffices to prove the following lemma to reduce the proof of Theorem 3.2 to Theorem 3.3.

Lemma 3.4. (1) *For any open set $O \in L_x(M)$, there exists a measurable subset $\mathcal{U}_O \subset W_0^N$ and the measurable map $\Phi : \mathcal{U}_O \to X^{-1}(O)$, and positive constants C_1, C_2 such that if $F \in \mathbb{D}(S_x)$, then $\hat{F} = F|_{X^{-1}(O)} \circ \Phi \in \mathbb{D}(\mathcal{E}_{\mathcal{U}_O})$ holds and*

$$\int_{\mathcal{U}_O} \hat{F}(w) d\mu = \int_{X^{-1}(O)} F(w) g(w) d\mu_x \tag{3.1}$$

$$\int_{\mathcal{U}_O} |D\hat{F}|^2 d\mu \leq C_1 \int_{X^{-1}(O)} |D_S F(w)|^2 d\mu_x \tag{3.2}$$

where g satisfies that $0 < g(w) \leq C_2 \ \mu_x a.s.w.$
(2) *If O is connected, then $\mathcal{U}_O$ is $H-$connected.*

In the proof of the above lemma, we need to calculate the derivative of the trivialization of $\overline{s(\gamma)}_t$. Let us explain it briefly. In general, assume we are given a vector bundle $\pi : E \to M$ with a metric and a Riemannian connection ∇^E on it. Also we fix a Riemannian connection ∇ on $TM \to M$. Then for a smooth section $s : M \to E$ and a semimartingale $\gamma(t)$ $(0 \leq t \leq 1)$ on M starting at $x \in M$, we have the trivialization $\overline{s(\gamma)}_u \in E_x$ using the stochastic parallel translation along γ. This defines a map from the space of smooth based loop with the base point x to E_x. The derivative which is defined the same as (1.1) using the parallel translation τ which is coming from ∇ of $\overline{s(\gamma)}_u$ can be calculated using the curvature tensor R^E as follows:

Lemma 3.5.

$$(D\overline{s(\gamma)}_t, h)_{H^1} = (\overline{(\nabla^E s)(\gamma)}_t, h(t)) + \left\{ \int_0^t \overline{\mathrm{R}^E(\gamma)}_u (h(u), \circ db(u)) \right\} \overline{s(\gamma)}_t, \tag{3.3}$$

where $b(u)$ denotes the stochastic development of $\gamma(u)$ to $T_x M$.

Note that in the above formula, $\overline{\mathrm{R}^E(\gamma)}_u$ is the trivialization of the curvature tensor using ∇^E and ∇ and they are in $E \otimes E^* \otimes T_x M^* \otimes T_x M^*$.

Proof. At first we assume that $\gamma(t)$ is a $C^\infty-$curve and we will prove (3.3) replacing $\circ db(u)$ by $\dot{b}(u)\ du$. Let $\{\gamma(s,t)\}_{0\le s,t\le 1}$ be a smooth variation of $\gamma(t)$, where $\gamma(0,t)=\gamma(t)$ and $\gamma(s,0)=\gamma(0,0)=x$ for any s. Let us denote

$$v(t)=\frac{\partial}{\partial s}\gamma(s,t)\Big|_{s=0}=\tau(\gamma)_t h(t)\in T_{\gamma(t)}M.$$

Let us denote the parallel translation along $\gamma(s,\cdot)$ and $\gamma(\cdot,t)$ respectively by

$$\begin{aligned}&\tau\{(s,t)\to(s,t')\}:E_{\gamma(s,t)}\to E_{\gamma(s,t')}\\ &\tau\{(s,t)\to(s',t)\}:E_{\gamma(s,t)}\to E_{\gamma(s',t)}.\end{aligned}$$

Then we see that

$$\begin{aligned}T(s,t)_\varepsilon:=&\ \tau\{(s,t+\varepsilon)\to(s,t)\}\circ\tau\{(s+\varepsilon,t+\varepsilon)\to(s,t+\varepsilon)\}\\ &\circ\tau\{(s+\varepsilon,t)\to(s+\varepsilon,t+\varepsilon)\}\circ\tau\{(s,t)\to(s+\varepsilon,t)\}\\ =&\ id|_{E_{\gamma(s,t)}}-\varepsilon^2\mathrm{R}^E(\gamma(s,t))(v(t),\frac{\partial}{\partial t}\gamma(0,t))+M(s,t,\varepsilon^3):E_{\gamma(s,t)}\to E_{\gamma(s,t)}\end{aligned}\tag{3.4}$$

where there exists a constant C which is independent of (s,t) such that $\|M(s,t,\varepsilon^3)\|\le C\varepsilon^3$. Next note that

$$\begin{aligned}\overline{s(\gamma(\varepsilon,\cdot))}_t-\overline{s(\gamma(0,\cdot))}_t=&\ \tau\{(\varepsilon,t)\to(\varepsilon,0)\}\Big(s(\gamma(\varepsilon,t))-\tau\{(0,t)\to(\varepsilon,t)\}s(\gamma(0,t))\Big)\\ &+\Big\{\tau\{(\varepsilon,t)\to(\varepsilon,0)\}\circ\tau\{(0,t)\to(\varepsilon,t)\}-\tau\{(0,t)\to(0,0)\}\Big\}s(\gamma(0,t))\\ =&\ I_1(\varepsilon)+I_2(\varepsilon).\end{aligned}$$

The first term of (3.3) comes from

$$\lim_{\varepsilon\to 0}\frac{I_1(\varepsilon)}{\varepsilon}=\tau(\gamma)_t^{-1}\nabla^E_{v(t)}s(\gamma(t)).$$

We can rewrite $I_2\left(\frac{t}{n}\right)$ as follows

$$I_2\left(\frac{t}{n}\right)=\sum_{k=1}^n\tau(\gamma)^{-1}_{\frac{k}{n}t}\left(id|_{E_{\gamma(0,\frac{k}{n}t)}}-T\Big(0,\frac{k}{n}t\Big)_{\frac{1}{n}t}\right)\tilde{T}_{\frac{1}{n}t}\Big(\frac{k}{n}t\Big)s(\gamma(0,t))\tag{3.5}$$

where

$$\tilde{T}_{\frac{1}{n}t}\Big(\frac{k}{n}t\Big)=\tau\Big\{\Big(\frac{1}{n}t,\frac{k}{n}t\Big)\to\Big(0,\frac{k}{n}t\Big)\Big\}\circ\tau\Big\{\Big(\frac{1}{n}t,\frac{k}{n}t\Big)\to\Big(\frac{1}{n}t,\frac{k}{n}t\Big)\Big\}\circ\tau\Big\{(0,t)\to\Big(\frac{1}{n}t,t\Big)\Big\}.$$

The equation (3.5) can be proved by the induction with respect to the number of the points of the partition of $[0,1]$. Making use of (3.4) and (3.5), we have

$$\begin{aligned}\frac{n}{t}I_2\left(\frac{1}{n}t\right)=&\sum_{k=1}^n\tau(\gamma)^{-1}_{\frac{k}{n}t}\frac{t}{n}\mathrm{R}^E\left\{\gamma\Big(\frac{k}{n}t\Big)\right\}\left(v\Big(\frac{k}{n}t\Big),\dot{\gamma}\Big(\frac{k}{n}t\Big)\right)\tau\left\{(0,t)\to\Big(0,\frac{k}{n}t\Big)\right\}s(\gamma(0,t))\\ &+\tilde{M}\left(\frac{1}{n}\right),\end{aligned}$$

where $\|\tilde{M}\left(\frac{1}{n}\right)\| \le \frac{C}{n}$. Letting $n \to \infty$, $\frac{n}{t} I_2\left(\frac{t}{n}\right)$ tends to

$$\int_0^t \tau(\gamma)_s^{-1} \mathrm{R}^E(\gamma(s))(v(s), \dot{\gamma}(s)) \tau\{(0,t) \to (0,s)\} s(\gamma(t))\, ds.$$

This completes the proof in smooth case. We can get the formula (3.3) using the approximation of the Brownian curve by the smooth curve. Then (3.3) is valid replacing $\dot{b}(v)dv$ by the Stratonovich integral $\circ db(v)$ which is the stochastic development of $\gamma(t)$. □

Proof of Lemma 3.4. We will use the notation in [A2]. We defined vector fields on W_0^N,

$$A_\xi(w) = \tilde{V}_\xi(w) \varphi\left(\rho(\xi)\theta_m(w)^2/\varepsilon\right)$$

in §3 in [A2] using the gradient Brownian system. Here we take ε to be sufficiently small. We define $A_\xi(w)$ in general cases in the same way. Explicitly

$$\tilde{V}_\xi(w)_t = \int_0^t \overline{\widehat{\sigma(X)}}_s^{\,*} (\hat{v}_s^{-1})^* \hat{v}_1^* \left\{ \int_0^1 \hat{v}_1 \hat{v}_u^{-1} (\hat{v}_u^{-1})^* \hat{v}_1^* du \right\}^{-1} ds\, \overline{V_\xi(X)}_1.$$

We denote the flow generated by $A_\xi(w)$ by $U_\xi(t,w)$ and set $U(1,w) = U_{X(1,x,w)}(1,w)$. $U(1,w)$ has the property that if $w \in S_\xi = \{w \in W_0^N \mid X(1,x,w) = \xi\}$ and for all $t \in [0,1]$, $\phi\left(\rho(\xi)\theta_m(U_\xi(t,w))^2/\varepsilon\right) = 0$ holds, then $U_\xi(1,w) \in S_x$. The existence of the flow can be proved by the Cruzeiro's theorem using the formula (3.6) below.

Using Lemma 3.5, we can calculate the $H-$derivative of $\hat{v}(t)$ as the functional of $w(t)$ as follows:

$$\begin{aligned}
(D\hat{v}(t), h) &= \hat{v}_t \int_0^t \hat{v}_s^{-1} \overline{\widehat{\nabla\hat{\nabla}\sigma(X)}}_s(\hat{h}(s))(\hat{v}_s, d\beta(s)) \\
&+ \hat{v}_t \int_0^t \hat{v}_s^{-1} \int_0^s \overline{\widehat{\hat{\mathrm{R}}(X)}}_u(\hat{h}(u), \circ d\hat{B}(u))(\overline{\widehat{\nabla\sigma(X)}}_s(\hat{v}_s, d\beta_s)) \\
&+ \hat{v}_t \int_0^t \hat{v}_s^{-1} \overline{\widehat{\nabla\sigma(X)}}_s(\hat{v}_s, \overline{\widetilde{N(X)}}_s \dot{h}(s)) ds \\
&+ \hat{v}_t \int_0^t \hat{v}_s^{-1} \overline{\nabla\sigma(X)}_s \left(\hat{v}_s, \left\{\overline{\widetilde{\tilde{\nabla} N(X)}}_s(\bar{h}(s))\right\} d\beta(s)\right) \\
&+ \hat{v}_t \int_0^t \hat{v}_s^{-1} \overline{\widehat{\nabla\sigma(X)}}_s \left(\hat{v}_s, \int_0^s \overline{\widetilde{\tilde{\mathrm{R}}(X)}}_u(\bar{h}(u), \circ dB(u)) d\beta(s)\right) \\
&- \frac{1}{2}\hat{v}_t \int_0^t \hat{v}_s^{-1} \overline{\widehat{\hat{\nabla}\mathrm{Ric}(X)}}_s(\hat{h}(s)) \hat{v}_s ds \\
&- \frac{1}{2}\hat{v}_t \int_0^t \hat{v}_s^{-1} \int_0^s \overline{\widehat{\mathrm{R}(X)}}_u(\hat{h}(u), \circ d\hat{B}(u)) \overline{\widehat{\mathrm{Ric}(X)}}_s \hat{v}_s ds,
\end{aligned} \tag{3.6}$$

where

$$\begin{aligned}
N(x) &= id_{\mathbb{R}^N} - \sigma^*(x)\sigma(x) : \mathbb{R}^N \to \ker \sigma(x) \\
\bar{h}(t) &= v_t \int_0^t v_s^{-1}\overline{\sigma(X)}_s \dot{h}(s) ds \\
v_t &= \tau(X)_t^{-1} \partial_x X(t,x,w) \\
\hat{h}(t) &= \hat{v}_t \int_0^t \hat{v}_s^{-1}\overline{\sigma(X)}_s^{\wedge} \dot{h}(s) ds \\
B(t) &= \int_0^t \tau(X)_s^{-1}\sigma(X(s,x,w)) \circ dw(s) \\
\hat{B}(t) &= \int_0^t \hat{\tau}(X)_s^{-1}\sigma(X(s,x,w)) \circ dw(s).
\end{aligned}$$

One crucial point is that since $\hat{\nabla}$ is again a Riemannian connection, $\hat{B}(t)$ is again a standard Brownian motion on T_xM like $B(t)$. This observation is important to check the assumption of Cruzeiro's theorem.

As in [A1], we can see that there exists a $\mathbb{D}^\infty-$function $C(w)$ and $0 < \alpha < 1$,

$$|D^* A_\xi(w)| \le \rho(\xi)^\alpha C(w). \tag{3.7}$$

Also note that (see Lemma 3.5 in [A2]) for $w \in W_0^N$ with $\rho(X(1,x,w))^{1/2} < \frac{1}{3}$, we have

$$|P(U(1,w))DU(1,w)|_H \le D(w)\rho(X(1,x,w))^\beta + E. \tag{3.8}$$

where D is a positive $\mathbb{D}^\infty-$function and E is a constant. Define

$$\tilde{C}(\xi,w) = 1 + |(C+D)\,(U_\xi(1,w))|^2 + \int_0^1 \left|\frac{d}{dt}(C+D)(U_\xi(t,w))\right|^2 dt.$$

Using $\tilde{C}(\xi,w)$, let us define for sufficiently large $m > 0$,

$$\mathcal{U}_O = \left\{ w \in W_0^N \middle| \left\{\Theta\left(w, X(1,x,w)\right) + \tilde{C}\left(X(1,x,w),w\right)\right\}^m \rho\left(X(1,x,w)\right) < \frac{\varepsilon}{200} \right.$$
$$\left. X(\cdot,x,U(1,w)) \in O \right\}.$$

Note that ε appears in the definition of A_ξ. This is a variant of $\mathcal{U}_O$ in §3 in [A2]. We refer the definition of Θ to it. Let $\Phi(w) = U(1,w)$. We will prove (3.1).

$$\int_{\mathcal{U}_O} F \circ \Phi(w) d\mu = \int_{X^{-1}(O)} F(w) f(w) d\mu_x(w),$$

where

$$f(w) \leq \int_M \exp\left(\int_0^1 D^* A_\xi(U_\xi(-s,w))ds\right) \chi_{N(w)}(\xi) d\xi$$

and

$$N(w) = \left\{\xi \in M \middle| \left(\Theta(U_\xi(-1,w),\xi) + \tilde{C}(\xi, U_\xi(-1,w))\right)^m \rho(\xi) < \frac{\varepsilon}{200}, \right\}.$$

This equation can be checked by the calculation as in the page 94 in [A1]. By the property of $N(w)$, we have $0 < \text{ess.sup } f(w) \leq \exp\left((\frac{\varepsilon}{200})^\alpha\right) vol(M)$. This proves (3.1). (3.2) follows from (3.1) and the estimate (3.8). $H-$connectivity of $\mathcal{U}_O$ is proved by the way similar to Corollary 4.7 in [A2]. In the course of the proof of (2), Lemma 2.13 in [A2] is important. In the present case, taking $\sigma^*(x) : T_xM \to \mathbb{R}^N$ instead of $\iota_* : T_xM \to \mathbb{R}^d$, we see that the conclusion of Lemma 2.13 holds. We refer to §4 in [A2] for the detail of the proof of (2). □

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