Topics in Mathematical Science (S.Aida)

1. Review of lecture on 12/3

Consider the following heat equation:

$$
\begin{aligned}
\frac{\partial}{\partial t}u(t,x) &= \frac{1}{2}a(x)\frac{\partial^2}{\partial x^2}u(t,x) + b(x)\frac{\partial}{\partial x}u(t,x) \qquad (*) \\
u(0,x) &= f(x)
\end{aligned}
$$

Theorem 1 Assume
(A1) $a(x), b(x) \in C^\infty(\mathbb{R})$
(A2) There exist $0 < C_1 < C_2 < \infty$ such that $C_1 \le a(x) \le C_2$ and $|b(x)| \le C_2$ for all $x \in \mathbb{R}$.

Then
(1) For any bounded function f, there exists the unique solution $u(t,x)$ to $(*)$.
(2) There exists a positive function $p(t,x,y)$ $(t > 0, x, y \in \mathbb{R})$ such that $p(t,x,y)$ is a C^∞ function of (t,x,y) and

$$u(t,x) = \int_{\mathbb{R}} p(t,x,y) f(y) dy.$$

Remark $p(t,x,y)$ is called the fundamental solution of $(*)$.

Theorem 2 Let $B_t(w)$ $(t \ge 0, w \in \Omega)$ be one dimensional Brownian motion. Let a and b be the function in Theorem 1. Let $\sigma(x) = \sqrt{a(x)}$ and consider the following SDE:

$$
\begin{aligned}
\dot{X}(t) &= \sigma(X(t))\dot{B}_t(w) + b(X(t)) \qquad (*') \\
X(0) &= x \in \mathbb{R}.
\end{aligned}
$$

We denote the solution to $(*')$ by $X(t,x,w)$. Then

$$u(t,x) = E[f(X(t,x,w))]$$

This theorem and Theorem 1 imply that the law of $X(t,x,w)$ has the density function and it is equal to $p(t,x,y)$. But Theorem 2 itself does not say anything about the existence of the density function of $X(t,x,w)$ and the existence of $p(t,x,y)$. So we need to consider the following problem:

Problem Let $(\Omega, \mathfrak{B}, P)$ be a probability space. Let $X : \Omega \to \mathbb{R}$ be a random variable. Find a sufficient condition under which the law of X has the probability density function. In that case, can you represent the density function by using X?

Theorem 3 (Paul Malliavin, Shinzo Watanabe....) Let us consider the situation in Theorem 1 and Theorem 2. Moreover we assume that $a'(x)$ and $b'(x)$ are bounded functions, that is there exists a positive number C_3 such that $|a'(x)| \le C_3$ and $|b'(x)| \le C_3$ for all x. Then the law of $X(t,x,w)$ has the density function. Moreover we can define the expectation $E[\delta_y(X(t,x,w))]$ rigorously by using **integration by parts formula** and it holds that

$$p(t,x,y) = E[\delta_y(X(t,x,w))]$$

Now we consider a finite dimensional Gaussian space.

Let $\Omega = \mathbb{R}^N$ (N-dimensional euclidean space), $\mathfrak{B} = \mathfrak{B}(\mathbb{R}^N)$ (=Borel σ-field). Note that $\mathfrak{B}$ is a set of events of Ω. The probability measure on Ω which we consider is the following Gaussian measure.

$$P(dw) = \frac{1}{\sqrt{2\pi}^N} \exp\left(-\frac{1}{2}\sum_{i=1}^N w_i^2\right) dw_1 \cdots dw_N.$$

Theorem 4 Let $X(w)$ be a random variable on Ω which is a function from Ω to $\mathbb{R}$. Assume

(i)

$$\int_\Omega \|\nabla X(w)\|^{-8} P(dw) < \infty,$$

(ii)

$$\int_\Omega \left(|X(w)|^4 + \|\nabla X(w)\|^4 + \|\nabla^2 X(w)\|^4\right) P(dw) < \infty.$$

Then X has a density ρ such that

$$\begin{aligned}\rho(y) &= \int_{\{w\in\Omega\ :\ X(w)\ge y\}} \left(\frac{-\Delta X(w) + (w, \nabla X(w))}{\|\nabla X(w)\|^2}\right. \\ &\qquad \left. +2\frac{\sum_{1\le i,j\le N} \frac{\partial^2}{\partial w_i \partial w_j} X(w) \frac{\partial}{\partial w_i} X(w) \frac{\partial}{\partial w_i} X(w)}{\|\nabla X(w)\|^4}\right) P(dw) < \infty. \qquad (1)\end{aligned}$$

Here

$$\nabla X(w) = \left(\frac{\partial X}{\partial w_1}, \cdots, \frac{\partial X}{\partial w_N}\right) \qquad (w, \nabla X(w)) = \sum_{i=1}^N w_i \frac{\partial X}{\partial w_i}$$

$$\nabla^2 X(w) = \left(\frac{\partial X}{\partial w_i \partial w_j}\right)_{1\le i,j\le N} \qquad \Delta X(w) = \sum_{i=1}^N \frac{\partial^2 X}{\partial w_i^2}$$

$$\|\nabla X(w)\|^2 = \sum_{i=1}^N \left(\frac{\partial X}{\partial w_i}\right)^2 \qquad \|\nabla^2 X(w)\|^2 = \sum_{1\le i,j\le N} \left(\frac{\partial^2 X}{\partial w_i \partial w_j}\right)^2$$

2. Menu of Today's lecture

1. To show that the right-hand side of (1) is equal to $E[\delta_y(X(w))]$ by using integration by parts formula.

2. To show that Theorem 3 is obtained by Theorem 4 in the case where $N = \infty$.

To this end, we introduce

(i) Fourier series expansion of Brownian path

(ii) Definition of derivatives in ∞-dimensional spaces (Stochastic calculus of variation!)

3. If we have time, I will comment multi-dimensional cases.

NOTE:

The law of a random variable has the probability density function $\rho(x)$

$\overset{\text{def}}{\Longleftrightarrow}$ For any $A \subset \mathbb{R}$,

$$P(\{w \in \Omega \mid X(w) \in A\}) = \int_A \rho(x)dx$$

This is equivalent to that for any bounded continuous function φ,

$$E[\varphi(X)] = \int_{\mathbb{R}} \varphi(x)\rho(x)dx$$

hold.