COH formula and Dirichlet Laplacians on small domains of pinned path spaces

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May 20, 2010

Abstract

We consider a small domain of a pinned path space over a compact Riemannian manifold. We establish a Clark-Ocone-Haussman formula for functions which belong to H^1-Sobolev space on the domain with the Dirichlet boundary condition and apply it to obtain spectral gap estimate for the Dirichlet Laplacians.

1 Introduction

Let (M, g) be a compact Riemannian manifold. Let $P_x(M) = C([0,1] \to M \mid \gamma(0) = x)$ be the set of continuous paths starting at $x \in M$. The generator of the Dirichlet form defined by the H-derivative and the Brownian motion measure on $P_x(M)$ is a generalization of the Ornstein-Uhlenbeck(=OU) operator on a Wiener space. It was proved in [14, 18, 6, 1] that the Poincaré inequality(=PI) and the logarithmic Sobolev inequality(=LSI) hold for the Dirichlet form. The Clark-Ocone-Haussman(=COH) formula is one of main ingredient of [14, 6]. Let $y \in M$ and consider a pinned path space $P_{x,y}(M) = \{\gamma \in P_x(M) \mid \gamma(1) = y)$. The same as $P_x(M)$, the OU operator is defined on $P_{x,y}(M)$ with the pinned Brownian motion measure. It was proved in [3] that if M is a manifold with a pole and the Riemannian metric is asymptotically flat then LSI holds in pinned case too. However one cannot expect that the PI hold on pinned path spaces over a simply connected compact Riemannian manifold generally. In fact, Eberle [11] gave such kind of examples. On the other hand, the COH formula on a certain class of pinned path spaces, including the case where M is a hyperbolic space $\mathbb{H}^n$ $(n \geq 2)$, were studied in [2] and it was proved that LSI with variable coefficient Dirichlet forms on them were valid. Also another form of COH formula can be found in [15]. Recently, Chen, Li and Wu [8] proved that LSI with a variable coefficient imply weak log-Sobolev inequality(=WLSI) if the coefficient function satisfies some conditions. An WLSI contains a non-increasing function β on $(0, \infty)$ and Cattiaux, Gentil and Guillin [7] proved that if $\beta(s) = O(|\log s|)$ as $s \to 0$, PI hold. Thus [8] proved that PI holds on the pinned path space over the hyperbolic space $\mathbb{H}^n$. However in the case of loop groups, the other natural measures, heat kernel measures and ground state measures exist. We note that LSI holds for the heat kernel measure and PI holds for the ground state measure on the loop group respectively. See [17, 10].

Now we introduce a positive parameter λ and consider the pinned Brownian motion measure $\nu_{x,y}^{\lambda}$ on $P_{x,y}(M)$ which is formally written by

$$d\nu_{x,y}^{\lambda}(\gamma) = Z_{\lambda}^{-1} \exp\left(-\frac{\lambda}{2}\|\gamma\|_{H^1}^2\right) d\gamma,$$

where $d\gamma$ is the Riemannian volume on $P_{x,y}(M)$ and $\|\gamma\|_{H^1}^2 = \int_0^1 |\dot{\gamma}(t)|^2 dt$. The "zero variance limit" $\lambda \to \infty$ is a kind of semi-classical limit and it was proved in [4] that the asymptotic behavior of the lowest eigenvalue of Schrödinger operators on the non-pinned path space $P_x(M)$ is determined by the Hessian of the "potential function". In the pinned case, as to the OU operator, we have a few examples for which there exist spectral gaps above 0. It is interesting to study the semi-classical limit of the operator on $P_{x,y}(M)$. However there are difficulties to study the global problem. Instead of doing so, in this paper, we will study the following problem. Let $\mathcal{D}$ be an open subset of $P_{x,y}(M)$ and consider the OU operator $-L_{\lambda}$ with Dirichlet boundary condition. We are interested in the asymptotic behavior of the lowlying spectrum of $-L_{\lambda}$ when $\lambda \to \infty$. Now suppose that x is outside the cut-locus of y and $\mathcal{D}$ contains the minimal geodesic $\gamma_{x,y}$ between x and y and there are no other geodesics than minimal one in the closure of $\mathcal{D}$. Then the probability measure concentrates on the neighborhood of $\gamma_{x,y}$ and it is easy to check that the bottom of spectrum of L_{λ} converges to 0. We may conjecture that the asymptotic behavior of the lowlying spectrum of L_{λ} can be determined by the Hessian of the energy function $\gamma \to \|\gamma\|_{H^1}^2/2$ at $\gamma_{x,y}$ similarly to the finite dimensional cases. For example, the gap of spectrum of L_{λ} between the bottom and the above goes to infinity of the order λ. Actually, Eberle gave a lower bound estimate for it in [13] for a certain domain $\mathcal{D}$. In [13], more precise estimates were given in the case where $\mathcal{D}$ is a small domain of $P_{x,y}(M)$. The aim of this paper is to establish a COH formula for functions of the H^1-Sobolev spaces on $\mathcal{D}$ with the Dirichlet boundary condition and prove a lower bound estimate of the gap of the spectrum in the case of the small domain. It seems that we need more detailed analysis to obtain precise asymptotics of the spectrum of the Dirichlet Laplacian. We will study this problem in the case of pinned path group using LSI with a potential function in [5] in a forthcoming paper.

2 Results

Let (M, g) be an n-dimensional compact Riemannian manifold. We denote the Riemannian distance between two points $a, b \in M$ by $d(a, b)$ and denote by $B_r(a) = \{x \in M \mid d(a, x) < r\}$ the ball centered at a with the radius r. Let $\lambda > 0$ be a positive number and ν_x^{λ} be the Brownian motion measure on $P_x(M)$ and $\nu_{x,y}^{\lambda}$ be the pinned Brownian motion measure on $P_{x,y}(M)$. Explicitly,

$$\begin{aligned}
&\nu_{x,y}^{\lambda}\left(\{\gamma \in P_{x,y}(M) \mid \gamma(t_i) \in A_i, 1 \le i \le n\}\right) \\
&\quad = \int_{\{x_i \in A_i, 1 \le i \le n-1\}} p(1/\lambda, x, y)^{-1} \prod_{i=1}^{n} p\left((t_i - t_{i-1})/\lambda, x_{i-1}, x_i\right) dx_1 \cdots dx_{n-1},
\end{aligned}$$

where $0 = t_0 < \cdots < t_n = 1$, $x_0 = x, x_n = y$ and $p(t, x, y)$ denotes the heat kernel of $e^{t\Delta/2}$. H^1-Sobolev space $H^{1,2}(P_{x,y}(M), \nu_{x,y}^{\lambda})$ is defined by using the H-derivative D_0 on $P_{x,y}(M)$. Note that the H-derivative is defined by using the Levi-Civita connection. We also denote the H-derivative on $P_x(M)$ by D. Below we consider open sets of $P_{x,y}(M)$ and the Dirichlet Laplacians on them. Let r_0 be a positive number such that there are no cut-locus of y in the closure of $B_{r_0}(y)$ and the infimum of the eigenvalues of the Hessian of $E(z) = \frac{1}{2}d(z,y)^2$ satisfies that $\inf_{z\in B_{r_0}(y)} \nabla^2 E(z) > 1/2$. Note that $\nabla_z^2 E(z)|_{z=y} = I_{T_yM}$. Let $0 < r < r_0$. Now we assume that $x \in B_r(y)$. We consider paths restricted to $B_r(y)$ such that

$$P_{x,y}(B_r(y)) = \{\gamma \in P_{x,y}(M) \mid \gamma(t) \in B_r(y) \text{ for all } 0 \le t \le 1\}. \tag{2.1}$$

Following [13], let us define

$$\begin{aligned} &H_0^{1,2}(P_{x,y}(B_r(y)), \nu_{x,y}^\lambda) \\ &\quad = \left\{ F|_{P_{x,y}(B_r(y))} \;\middle|\; F \in H^{1,2}(P_{x,y}(M), \nu_{x,y}^\lambda) \text{ and } F = 0 \text{ on } P_{x,y}(B_r(y))^c \right\}. \end{aligned} \tag{2.2}$$

The non-positive generator L_λ corresponding to the dense closed subspace $H_0^{1,2}(P_{x,y}(B_r(y)), \nu_{x,y}^\lambda)$ of $L^2(P_{x,y}(B_r(y)), \nu_{x,y}^\lambda)$ is the Dirichlet Laplacian on $P_{x,y}(B_r(y))$. We denote the normalized probability $d\nu_{x,y}^\lambda/\nu_{x,y}^\lambda(P_{x,y}(B_r(y)))$ on $P_{x,y}(B_r(y))$ by $d\bar{\nu}_{x,y}^{\lambda,r}$. Eberle ([11, 12, 13]) defined the value

$$e_{P_{x,y}(B_r(y))}^\lambda = \inf_{F\in H_0^{1,2}(P_{x,y}(B_r(y)))} \frac{\int_{P_{x,y}(B_r(y))} |D_0F|^2 d\bar{\nu}_{x,y}^{\lambda,r}}{\mathrm{Var}(F, \bar{\nu}_{x,y}^{\lambda,r})} \tag{2.3}$$

where $\mathrm{Var}(F, \bar{\nu}_{x,y}^{\lambda,r})$ denotes the variance of F with respect to $\bar{\nu}_{x,y}^{\lambda,r}$ and proved that

$$\liminf_{\lambda\to\infty} \frac{e_{P_{x,y}(B_r(y))}^\lambda}{\lambda} > 0. \tag{2.4}$$

Let

$$e_{Dir,1,P_{x,y}(B_r(y))}^\lambda = \inf_{F(\neq 0)\in H_0^{1,2}(P_{x,y}(B_r(y)))} \frac{\int_{P_{x,y}(B_r(y))} |D_0F|^2 d\bar{\nu}_{x,y}^{\lambda,r}}{\|F\|_{L^2(\bar{\nu}_{x,y}^{\lambda,r})}^2}. \tag{2.5}$$

This is equal to $\inf \sigma(-L_\lambda)$, where $\sigma(-L_\lambda)$ denotes the spectral set of $-L_\lambda$. Let us define the value $e_{Dir,2,P_{x,y}(B_r(y))}^\lambda$ as follows: if $e_{Dir,1,P_{x,y}(B_r(y))}^\lambda$ is an eigenvalue with multiplicity 1, then

$$e_{Dir,2,P_{x,y}(B_r(y))}^\lambda = \inf\left(\sigma(-L_\lambda) \setminus \{e_{Dir,1,P_{x,y}(B_r(y))}^\lambda\}\right). \tag{2.6}$$

Otherwise, $e_{Dir,2,P_{x,y}(B_r(y))}^\lambda = e_{Dir,1,P_{x,y}(B_r(y))}^\lambda$. If $-L_\lambda$ has discrete spectrum only, this value is equal to the second smallest eigenvalue of $-L_\lambda$ counting multiplicity although we cannot expect the discreteness of the spectrum in the present case. Also, by the standard argument, we have

$$\begin{aligned} &e_{Dir,2,P_{x,y}(B_r(y))}^\lambda \\ &\quad = \sup_{G(\neq 0)\in L^2(\bar{\nu}_{x,y}^{\lambda,r})} \inf \left\{ \frac{\int_{P_{x,y}(B_r(y))} |D_0F|^2 d\bar{\nu}_{x,y}^{\lambda,r}}{\|F\|_{L^2(\bar{\nu}_{x,y}^{\lambda,r})}^2} \;\middle|\; F \in H_0^{1,2}(P_{x,y}(B_r(y))), (F,G)_{L^2(\bar{\nu}_{x,y}^{\lambda,r})} = 0 \right\}. \end{aligned} \tag{2.7}$$

Hence $e_{Dir,2,P_{x,y}(B_r(y))}^\lambda \ge e_{P_{x,y}(B_r(y))}^\lambda$ and (2.4) gives a lower bound estimate of the limit of $e_{Dir,2,P_{x,y}(B_r(y))}^\lambda$. We will give an estimate for $e_{P_{x,y}(B_r(y))}^\lambda$ using a COH formula. To this end, we recall basic results for pinned Brownian motion measures. Let $\mathfrak{F}_t = \sigma(\{\gamma(s) \mid 0 \le s \le t\})$. Let $\tau(\gamma)_t : T_xM \to T_{\gamma(t)}M$ be the stochastic parallel translation along the semi-martingale $\gamma(t)$ under ν_x^λ which is defined by the Levi-Civita connection. Then $b(t) = \int_0^t \tau(\gamma)_s^{-1} \circ d\gamma(s)$ is an $\mathfrak{F}_t$-Brownian motion with the covariance $E^{\nu_x^\lambda}[(b(t),u)(b(s),v)] = (u,v)\frac{t\wedge s}{\lambda}$ $(u,v \in T_xM)$ on T_xM under ν_x^λ. Also for $0 \le t < 1$, we define $V_y^\lambda(t,z) = \mathrm{grad}_z \log p\left(\frac{1-t}{\lambda}, y, z\right)$ and

$$\overline{V_y^\lambda(t,\gamma)}_t = \tau(\gamma)_t^{-1} V_y^\lambda(t,\gamma(t)) \in T_xM.$$

ν_x^λ and $\nu_{x,y}^\lambda$ are equivalent on $\mathfrak{F}_t$ for any $0 < t < 1$ and the density function $\rho(t,\gamma) = \frac{d\nu_{x,y}^\lambda}{d\nu_x^\lambda}|_{\mathfrak{F}_t}$ is given by an $\mathfrak{F}_t$-martingale:

$$\begin{aligned} \rho(t,\gamma) &= \frac{p\left(\frac{1-t}{\lambda}, y, \gamma(t)\right)}{p\left(\frac{1}{\lambda}, x, y\right)} \\ &= \exp\left(\int_0^t \left(\overline{V_y^\lambda(s,\gamma)}_s, db(s)\right) - \frac{1}{2\lambda}\int_0^t |\overline{V_y^\lambda(s,\gamma)}_s|^2 ds\right). \end{aligned} \tag{2.8}$$

Thus $\gamma(t)$ $(t < 1)$ is a semi-martingale under both probabilities ν_x^λ and $\nu_{x,y}^\lambda$. We denote $\overline{\nabla V_y^\lambda(t,\gamma)}_t = \tau(\gamma)_t^{-1}\nabla_z V_y^\lambda(t,z)|_{z=\gamma(t)}$. More explicitly,

$$\overline{\nabla V_y^\lambda(t,\gamma)}_t = \tau(\gamma)_t^{-1}\nabla_z \mathrm{grad}_z \log p\left(\frac{1-t}{\lambda}, y, z\right)\Bigg|_{z=\gamma(t)}.$$

Let $w(t) = b(t) - \frac{1}{\lambda}\int_0^t \overline{V_y^\lambda(s,\gamma)}_s ds$. This process is defined for $t < 1$ and it is not difficult to check that this can be extended continuously up to $t = 1$. Let $\mathcal{N}^{x,y,t}$ be the set of all null sets of $\nu_{x,y}|_{\mathfrak{F}_t}$ and set $\mathfrak{G}_t = \mathfrak{F}_t \vee \mathcal{N}^{x,y,1}$. Then w is an $\mathfrak{G}_t$-adapted Brownian motion for $0 \le t \le 1$ such that $E^{\nu_{x,y}^\lambda}[(w(t),u)_{T_xM}(w(s),v)_{T_xM}] = \frac{t\wedge s}{\lambda}(u,v)$ for any $u,v \in T_xM$. Consequently, $\gamma, b, \overline{V_y^\lambda(t,\gamma)}_t, \overline{\nabla V_y^\lambda(t,\gamma)}_t$ are $\mathfrak{G}_t$-semi-martingales with respect to $\nu_{x,y}^\lambda$ for $0 \le t \le 1$. The following integration by parts formula can be proved in a similar way to [2].

Lemma 2.1. *Let $0 < T < 1$. Let F be an $\mathfrak{F}_T$-measurable smooth cylindrical function. Let $\varphi(t,\gamma)$ be an $\mathfrak{F}_t$-progressively measurable process such that*

$$E^{\nu_{x,y}^\lambda}\left[\int_0^1 |\varphi(t,\gamma)|^2 dt\right] < \infty.$$

Then it holds that

$$E^{\nu_{x,y}^\lambda}\left[\int_0^T (DF(\gamma)_t^{\cdot}, \varphi(t,\gamma))dt\right] = \lambda E^{\nu_{x,y}^\lambda}\left[F(\gamma)\int_0^T (S(\gamma)_{\lambda,T}(\varphi)(t,\gamma), dw(t))\right], \tag{2.9}$$

where $S(\gamma)_{\lambda,T}$ is a pathwise bounded linear operator on $L^2([0,T] \to T_xM)$ such that

$$S(\gamma)_{\lambda,T}(\varphi)(t) = \varphi(t) - K(\gamma)_{\lambda,t}\int_0^t \varphi(s)ds, \tag{2.10}$$

and $K(\gamma)_{\lambda,t} = -\frac{1}{2\lambda}\overline{\mathrm{Ric}(\gamma)}_t + \frac{1}{\lambda}\overline{\nabla V_y^\lambda(t,\gamma)}_t$.

We introduce operators $M(\gamma)_{\lambda,t}$ on T_xM by

$$\begin{aligned} M(\gamma)^{\cdot}_{\lambda,t} &= K(\gamma)_{\lambda,t}M(\gamma)_{\lambda,t} \quad 0 \le t \le 1, &\quad (2.11)\\ M(\gamma)_{\lambda,0} &= I. &\quad (2.12)\end{aligned}$$

The inverse operator of $S(\gamma)_{\lambda,T}$ and its adjoint in $L^2([0,T],dt)$ is given by

$$\begin{aligned} \left(S(\gamma)^{-1}_{\lambda,T}\varphi\right)(t) &= \varphi(t) + K(\gamma)_{\lambda,t}M(\gamma)_{\lambda,t}\int_0^t M(\gamma)^{-1}_{\lambda,u}\varphi(t)dt\\ \left[(S(\gamma)^{-1}_{\lambda,T})^*\varphi\right](t) &= \varphi(t) + (M(\gamma)^*_{\lambda,t})^{-1}\int_t^T M(\gamma)^*_{\lambda,s}K(\gamma)_{\lambda,s}\varphi(s)ds.\end{aligned}$$

Since $\|K(\gamma)_{\lambda,t}\|_{op}$ is uniformly bounded for $\gamma, 0 \le t \le T$ for fixed T and λ, we have

$$E^{\nu^\lambda_{x,y}}\left[\int_0^T \left(\left[\left(S(\gamma)^{-1}_{\lambda,T}\right)^*\right] DF(\gamma)^{\cdot}, \varphi(t,\gamma)\right) dt\right] = \lambda E^{\nu^\lambda_{x,y}}\left[F(\gamma)\int_0^T (\varphi(t,\gamma), dw(t))\right]. \quad (2.13)$$

$\| \|_{op}$ denotes the operator norm. Let $\varphi \in L^2([0,1] \to T_xM, dt)$ and assume that the support of φ is in $[0,1)$. Then $J(\gamma)_\lambda\varphi$ which is given by

$$J(\gamma)_\lambda\varphi(t) = (M(\gamma)^*_{\lambda,t})^{-1}\int_t^1 M(\gamma)^*_{\lambda,s}K(\gamma)_{\lambda,s}\varphi(s)ds \quad (2.14)$$

is well-defined. Let $L(\gamma)_\lambda = I + J(\gamma)_\lambda$. We can prove that $J(\gamma)_\lambda$ can be extended to a bounded linear operator on L^2 for $\nu^\lambda_{x,y}$-almost all γ using Malliavin and Stroock's result and Lemma 3.2 in [2]. In the following, $\mathrm{Cut}(y)$ denotes the cut-locus of y.

Theorem 2.2 (Malliavin-Stroock [20]). *Let $z \in \mathrm{Cut}(y)^c$. Then*

$$\lim_{t\to 0} t\nabla_z^2 \log p(t,y,z) = -\nabla_z^2 E(z) \quad (2.15)$$

uniformly on any compact subset of $\mathrm{Cut}(y)^c$.

However the upper bound of $\|J(\gamma)\|_{op}$ which is obtained by using Lemma 3.2 in [2] does not satisfy a good integrability condition with respect to $\nu^\lambda_{x,y}$. For the small paths restricted on the neighborhood of y, we can get better estimates. Below, we improve Lemma 3.2 in [2] slightly.

Lemma 2.3. *Let $0 \le t < 1, \alpha > 1/2$. Let M_t be the solution to the $n \times n$-matrices valued ODE such that*

$$\begin{aligned} \dot{M}_t &= K_tM_t\\ M_0 &= I,\\ K_t &= \frac{1}{1-t}\left(-\alpha + C_1(t)\right) + C_2(t),\end{aligned}$$

where $C_i(t)$ are symmetric matrices valued continuous functions on $[0,1]$ such that $C_i(t)$ is a non-positive matrix for all t and $\sup_t \|C_i(t)\|_{op} \le \delta_i$ $(i = 1,2)$.
(1) *Let $N(t)$ be the solution to the following ODE.*

$$\begin{aligned} \dot{N}_t &= \left(\frac{C_1(t)}{1-t} + C_2(t)\right)N_t, &\quad (2.16)\\ N_0 &= I.\end{aligned}$$

Then

$$M_t = (1-t)^\alpha N_t \tag{2.17}$$

and

$$\sup_{0\le t\le s<1} \|N_s N_t^{-1}\|_{op} \le e^{\delta_2}. \tag{2.18}$$

(2) *Let*

$$(J\varphi)(t) = (M_t^*)^{-1} \int_t^1 M_s^* K_s \varphi(s) ds. \tag{2.19}$$

Then

$$\begin{aligned} &\|I+J\|_{op} \\ &\le \left(1 + \frac{4\alpha(1-\alpha)}{(2\alpha-1)^2} e^{2\delta_2} + \frac{8\alpha\delta_2 e^{2\delta_2}}{(2\alpha-1)^2}\right)^{1/2} + \frac{2e^{\delta_2}}{2\alpha-1}(\delta_1+\delta_2). \end{aligned} \tag{2.20}$$

Proof. (1) follows from Lemma 3.2 in [2]. We prove (2). Let φ be bounded measurable function such that supp φ is in $[0,1)$. Let us introduce the operator $\tilde{J}$ by

$$(\tilde{J}\varphi)(t) = -\alpha(1-t)^{-\alpha} \int_t^1 (1-s)^{\alpha-1} (N_s N_t^{-1})^* \varphi(s) ds.$$

Then

$$\begin{aligned} (I+J)\varphi(t) &= (I+\tilde{J})\varphi(t) \\ &\quad + \frac{1}{(1-t)^\alpha} \int_t^1 (1-s)^{\alpha-1} \left(N_s N_t^{-1}\right)^* \{C_1(s) + (1-s)C_2(s)\} \varphi(s) ds. \end{aligned} \tag{2.21}$$

By Lemma 3.1 in [2], we see that the L^2-norm of the second term of the right-hand side of (2.21) can be estimated by $\frac{2e^{\delta_2}}{2\alpha-1}(\delta_1+\delta_2)\|\varphi\|$. Hence we need only to estimate $\|(I+\tilde{J})\varphi\|_{L^2((0,1))}$. Using

$$\varphi(t) = -(1-t)^{1-\alpha} (N_t^*)^{-1} \frac{d}{dt} \int_t^1 (1-s)^{\alpha-1} N_s^* \varphi(s) ds,$$

we have

$$\begin{aligned} &\|(I+\tilde{J})\varphi\|_{L^2}^2 \\ &= \|\varphi\|_{L^2}^2 + \alpha^2 \int_0^1 (1-t)^{-2\alpha} \left|(N_t^*)^{-1} \int_t^1 (1-s)^{\alpha-1} N_s^* \varphi(s)\right|^2 ds \\ &\quad + 2\alpha \int_0^1 \left((1-t)^{1-2\alpha} N_t^{-1} (N_t^*)^{-1} \frac{d}{dt} \int_t^1 (1-s)^{\alpha-1} N_s^* \varphi(s) ds, \int_t^1 (1-s)^{\alpha-1} N_s^* \varphi(s) ds \right) dt. \end{aligned} \tag{2.22}$$

It is easy to see that

$$\frac{d}{dt}\left((1-t)M_t^{-1}(M_t^*)^{-1}\right) = (2\alpha-1)M_t^{-1}(M_t^*)^{-1} - 2M_t^{-1}\left(C_1(t) + (1-t)C_2(t)\right)(M_t^*)^{-1}. \tag{2.23}$$

Thus, the third term I_3 on the right-hand side of (2.22) reads

$$\begin{aligned}
I_3 &= \alpha \int_0^1 \frac{d}{dt}\left((1-t)^{1-2\alpha} N_t^{-1}(N_t^*)^{-1} \int_t^1 (1-s)^{\alpha-1} N_s^* \varphi(s)ds, \int_t^1 (1-s)^{\alpha-1} N_s^* \varphi(s)ds\right) dt \\
&\quad -\alpha \int_0^1 \left(\frac{d}{dt}\left((1-t)M_t^{-1}(M_t^*)^{-1}\right) \int_t^1 (1-s)^{\alpha-1} N_s^* \varphi(s)ds, \int_t^1 (1-s)^{\alpha-1} N_s^* \varphi(s)ds\right) dt \\
&= -\alpha \left|\int_0^1 (1-t)^{\alpha-1} N_t^* \varphi(t)dt\right|^2 \\
&\quad -\alpha(2\alpha-1) \int_0^1 \left|(M_t^*)^{-1} \int_t^1 (1-s)^{\alpha-1} N_s^* \varphi(s)ds\right|^2 dt \\
&\quad +2\alpha \int_0^1 \Bigg((C_1(t)+(1-t)C_2(t))(1-t)^{-\alpha} \int_t^1 (1-s)^{\alpha-1}(N_s N_t^{-1})^* \varphi(s)ds, \\
&\qquad\qquad (1-t)^{-\alpha} \int_t^1 (1-s)^{\alpha-1}(N_s N_t^{-1})^* \varphi(s)ds\Bigg) dt \\
&\leq -\alpha(2\alpha-1) \int_0^1 \left|(M_t^*)^{-1} \int_t^1 (1-s)^{\alpha-1} N_s^* \varphi(s)ds\right|^2 dt \\
&\quad +8\alpha\delta_2 \frac{e^{2\delta_2}}{(2\alpha-1)^2} \|\varphi\|_{L^2}^2.
\end{aligned} \tag{2.24}$$

Consequently,

$$\|(I+\tilde{J})\varphi\|_{L^2}^2 \leq \left(1 + \frac{4\alpha(1-\alpha)}{(2\alpha-1)^2} e^{2\delta_2} + \frac{8\alpha\delta_2 e^{2\delta_2}}{(2\alpha-1)^2}\right) \|\varphi\|_{L^2}^2 \tag{2.25}$$

which completes the proof. □

By this lemma, we get

Lemma 2.4. *There exists a positive constant C and $\lambda_0 > 0$ such that for any $\lambda > \lambda_0$ and $0 < r < r_0$,*

$$\mathrm{esssup}_{\gamma \in P_{x,y}(B_r(y))} \|L(\gamma)_\lambda\|_{op}^2 \leq (1+Cr^2)(1+\varepsilon(\lambda)) =: \xi_\lambda(r). \tag{2.26}$$

Here $\lim_{\lambda\to\infty} \varepsilon(\lambda) = 0$.

Proof. Let $\kappa_1(r)$ and $\kappa_2(r)$ be the infimum and supremum of the eigenvalues of the Hessian of E on $B_r(y)$. By Theorem 2.2, we see that there exists λ_0 such that for any $\lambda > \lambda_0$ and $\gamma \in P_{x,y}(B_r(y))$, $K(\gamma)_{\lambda,t}$ can be written as a sum of the operators

$$K(\gamma)_{\lambda,t} = \frac{1}{1-t}\Big(-(\kappa_1(r)-\varepsilon_1(\lambda)) + C_{1,\lambda}(\gamma)_t\Big) - \frac{1}{2\lambda}\overline{\mathrm{Ric}(\gamma)}_t \qquad \text{for all } 0 \leq t \leq 1, \tag{2.27}$$

where $\varepsilon_1(\lambda)$ is a small positive number less than κ_1 satisfying that $\lim_{\lambda\to\infty} \varepsilon_1(\lambda) = 0$ and $C_{1,\lambda}$ is a path dependent symmetric operator on T_xM such that

(i) all eigenvalues of $C_{1,\lambda}(\gamma)_t$ are non-positive,

(ii) the absolute values of all eigenvalues of $C_{1,\lambda}(\gamma)_t$ are less than $\kappa_2(r)-\kappa_1(r)+\varepsilon_2(\lambda)$, where $\lim_{\lambda\to\infty} \varepsilon_2(\lambda) = 0$.

Therefore by Lemma 2.3, for $\gamma \in P_{x,y}(B_r(y))$ and sufficiently large λ, $J(\gamma)_\lambda$ can be extended to a bounded linear operator on $L^2([0,1] \to T_xM)$. By the Hessian comparison theorem [19], we have $\max(|1-\kappa_1(r)|, |1-\kappa_2(r)|) = O(r^2)$ for small r. Hence applying Lemma 2.3, we complete the proof. $\square$

Now, we state our Clark-Ocone-Haussman formula for $F \in H_0^{1,2}(P_{x,y}(B_r(y)))$.

Theorem 2.5. *Let $F \in H_0^{1,2}(P_{x,y}(B_r(y)))$. Let λ be a sufficiently large positive number.*
(1) *The following COH formula holds:*

$$E^{\nu^\lambda_{x,y}}[F|\mathfrak{G}_t] = E^{\nu^\lambda_{x,y}}[F] + \int_0^t (H(s,\gamma), dw(s)), \quad 0 \le t \le 1,$$

where

$$H(s,\gamma) = E^{\nu^\lambda_{x,y}}\left[L(\gamma)_\lambda(D_0F(\gamma)^{\cdot})(s)|\mathfrak{G}_s\right]. \tag{2.28}$$

(2.28) *denotes the predictable projection.*
(2) *The following inequalities hold.*

$$E^{\nu^\lambda_{x,y}}\left[F^2\log\left(F^2/\|F\|^2_{L^2(\nu^\lambda_{x,y})}\right)\right] \le \frac{2\xi_\lambda(r)}{\lambda}E^{\nu^\lambda_{x,y}}\left[|D_0F|^2\right], \tag{2.29}$$

$$\frac{\lambda}{\xi_\lambda(r)}E^{\nu^\lambda_{x,y}}\left[\left(F - E^{\nu^\lambda_{x,y}}[F]\right)^2\right] \le E^{\nu^\lambda_{x,y}}\left[|D_0F|^2\right]. \tag{2.30}$$

(3) *We have*

$$\liminf_{\lambda\to\infty}\frac{e^\lambda_{P_{x,y}(B_r(y))}}{\lambda} \ge \left(1+Cr^2\right)^{-1}. \tag{2.31}$$

Remark 2.6. (1) Let λ be a sufficiently large number. Since the L^2-semigroup associated with the Dirichlet Laplacian $-L_\lambda$ is positivity preserving, by the result in [16], LSI (2.29) implies that $\inf\sigma(-L_\lambda)$ is an eigenvalue of $-L_\lambda$ with finite multiplicity. Note that we need to replace $\nu^\lambda_{x,y}$ by $\bar{\nu}^{\lambda,r}_{x,y}$ in this argument actually. In the present case, by (2.30), we see that the multiplicity is one for large λ because $\lim_{\lambda\to\infty} e^\lambda_{Dir,1,P_{x,y}(B_r(y))} = 0$. Of course, if the diffusion semigroup e^{tL_λ} satisfies the positivity improving property, we can conclude that the multiplicity is one and the eigenfunction is strictly positive.
(2) The lower bound like (2.31) is due to Eberle [13]. When r is small, this estimate is good in the sense that $\lim_{\lambda\to\infty}\frac{e^\lambda_{P_{x,y}(B_r(y))}}{\lambda} = 1$ for any r in the case where M is a euclidean space. In our approach, Lemma 3.2 in [2] gives a crude bound and the estimate in Lemma 2.3 in the present paper is necessary to obtain more precise estimate in (2.31). This is also a lower bound on $e^\lambda_{Dir,2,P_{x,y}(B_r(y))}$. In the case of pinned path groups, we can obtain more precise estimate for $e^\lambda_{Dir,2,\mathcal{D}}$ on more general domain $\mathcal{D}$ which includes only one geodesic using a different approach. This will be studied in a separate paper. (2.31) is used for the study in the case where $\mathcal{D}$ contains more than 2 local minimum geodesics in [13]. This result is applied to obtain a concrete estimate for weak Poincaré inequalities on loop spaces over positive Ricci curvature manifolds in [9].

Proof of Theorem 2.5. Let r_1 be a positive number such that $0 < r < r_1 < r_0$. Let η be a smooth non-negative function on $\mathbb{R}$ such that $\eta(t) = 1$ for $t < (r+r_1)/2$ and $\eta(t) = 0$ for

$t \geq (r+2r_1)/3$. For $0 < T \leq 1$, let $\chi_T(\gamma) = \eta\left(\sup_{0\leq t\leq T} d(\gamma(t), y)\right)$. For simplicity, we denote $\chi_1(\gamma)$ by $\chi(\gamma)$. Then it holds that

$$F(\gamma)\chi(\gamma) = F(\gamma) \qquad \gamma \in P_{x,y}(M).$$

Let $G(\gamma) = g(\gamma(t_1), \ldots, \gamma(t_n))$ be a smooth cylindrical function, where $t_n < 1$ and set $\tilde{G}(\gamma) = G(\gamma)\chi(\gamma)$ and $\tilde{G}_T(\gamma) = G(\gamma)\chi_T(\gamma)$. Then we have

$$\lim_{T\to 1} \|\tilde{G}_T - \tilde{G}\|_{H^{1,p}(\nu^\lambda_{x,y})} = 0 \tag{2.32}$$

for any $p > 2$. First, we establish a Clark-Ocone-Haussman formula for $\tilde{G}$. For any $0 \leq T \leq 1$, by the Itô representation theorem, there exists a $\mathfrak{G}_t$-predictable L^2-process $H_T(t,\gamma)$ $(0 \leq t \leq 1)$ such that

$$E^{\nu^\lambda_{x,y}}[\tilde{G}_T \mid \mathfrak{G}_t] = E^{\nu^\lambda_{x,y}}[\tilde{G}_T] + \int_0^t (H_T(s,\gamma), dw(s)), \quad 0 \leq t \leq 1. \tag{2.33}$$

This implies that

$$E^{\nu^\lambda_{x,y}}\left[\|H_T - H_1\|^2_{L^2([0,1],dt)}\right] = \lambda \mathrm{Var}^{\nu^\lambda_{x,y}}\left(\tilde{G}_T - \tilde{G}_{T'}\right). \tag{2.34}$$

Hence $\lim_{T\to 1} E^{\nu^\lambda_{x,y}}\left[\|H_T - H_1\|^2_{L^2([0,1],dt)}\right] = 0$. We will identify $H_T(t,\gamma)$. Now we assume that $t_n < T$. By approximating $\tilde{G}_T$ by smooth cylindrical functions and applying Lemma 2.1,

$$\begin{aligned} E^{\nu^\lambda_{x,y}}\left[\int_0^T \left((S(\gamma)^{-1}_{\lambda,T})^*(D\tilde{G}_T(\gamma)^\cdot)(t), \varphi(t,\gamma)\right) dt\right] &= \lambda E^{\nu^\lambda_{x,y}}\left[\tilde{G}_T(\gamma)\int_0^T (\varphi(t,\gamma), dw(t))\right] \\ &= E^{\nu^\lambda_{x,y}}\left[\int_0^T (H_T(t,\gamma), \varphi(t,\gamma))\, dt\right]. \end{aligned} \tag{2.35}$$

Noting that $\{(M(\gamma)_{\lambda,s})^*\}^\cdot = M(\gamma)^*_{\lambda,s}K(\gamma)_{\lambda,s}$ and $D\tilde{G}_T(\gamma)^\cdot(t) = D_0\tilde{G}_T(\gamma)^\cdot(t) + D\tilde{G}_T(\gamma)(1)$, we have

$$\begin{aligned} &(S(\gamma)^{-1}_{\lambda,T})^*(D\tilde{G}_T(\gamma)^\cdot)(t) \\ &\quad = D_0\tilde{G}_T(\gamma)^\cdot(t) + D\tilde{G}_T(\gamma)(1) + (M(\gamma)^*_{\lambda,t})^{-1}\int_t^T \left(M(\gamma)^*_{\lambda,s}\right)^\cdot \left(D_0\tilde{G}_T(\gamma)^\cdot(s) + D\tilde{G}_T(\gamma)(1)\right) ds \\ &\quad = \left(S(\gamma)^{-1}_{\lambda,T}\right)^* (D_0\tilde{G}_T(\gamma)^\cdot)(t) + (M(\gamma)^*_{\lambda,t})^{-1}M(\gamma)^*_{\lambda,T}D\tilde{G}_T(\gamma)_1. \end{aligned} \tag{2.36}$$

Let $\gamma^T(t) = \gamma(t\wedge T)$. If $D_0\tilde{G}_T(\gamma) \neq 0$, then $\sup_{0\leq t\leq T} d(\gamma(t), y) < r_1$. Therefore $\gamma^T \in P_{x,y}(B_{r_1}(y))$. Thus, $L(\gamma^T)_\lambda\left(D_0\tilde{G}_T(\gamma)\right) \in L^2(\nu^\lambda_{x,y})$. By (2.36), we have, for $0 \leq t \leq T$,

$$\begin{aligned} &L(\gamma)_\lambda\left(D_0\tilde{G}(\gamma)^\cdot\right)(t) - \left[S(\gamma)^{-1}_{\lambda,T}\right]^*\left(D\tilde{G}_T(\gamma)^\cdot\right)(t) \\ &\quad = L(\gamma)_\lambda\left(D_0\tilde{G}(\gamma)^\cdot\right)(t) - L(\gamma^T)_\lambda\left(D_0\tilde{G}_T(\gamma)^\cdot\right)(t) \\ &\quad +L(\gamma^T)_\lambda\left(D_0\tilde{G}_T(\gamma)^\cdot\right)(t) - \left[S(\gamma)^{-1}_{\lambda,T}\right]^*\left(D_0\tilde{G}_T(\gamma)^\cdot\right)(t) - \left(M(\gamma)^*_{\lambda,t}\right)^{-1}M(\gamma)^*_{\lambda,T}D\tilde{G}_T(\gamma)(1) \\ &\quad = L(\gamma)_\lambda\left(D_0\tilde{G}(\gamma)^\cdot\right)(t) - L(\gamma^T)_\lambda\left(D_0\tilde{G}_T(\gamma)^\cdot\right)(t) + J(\gamma^T)_\lambda\left(D\tilde{G}_T(\gamma)^\cdot 1_{[T,1]}\right)(t) \\ &\qquad - \left(M(\gamma)^*_{\lambda,t}\right)^{-1}M(\gamma)^*_{\lambda,T}D\tilde{G}_T(\gamma)(1) \end{aligned} \tag{2.37}$$

where $1_{[T,1]}$ is the indicator function of $[T,1]$. Therefore

$$
\begin{aligned}
&E^{\nu^\lambda_{x,y}}\left[\left|\int_0^T \Big((S(\gamma)^{-1}_{\lambda,T})^*(D\tilde{G}_T(\gamma)^{\cdot})(t),\varphi(t,\gamma)\Big)\,dt-\int_0^1\Big(L(\gamma)_\lambda(D_0\tilde{G}(\gamma)^{\cdot})(t),\varphi(t,\gamma)\Big)\,dt\right|\right]\\
&\le E^{\nu^\lambda_{x,y}}\left[\left\|L(\gamma)_\lambda\left(D_0\tilde{G}(\gamma)^{\cdot}\right)-L(\gamma^T)_\lambda\left(D_0\tilde{G}_T(\gamma)^{\cdot}\right)+J(\gamma^T)_\lambda\left(D\tilde{G}_T(\gamma)^{\cdot}1_{[T,1]}\right)\right\|^2_{L^2([0,T])}\right]^{1/2}\\
&\times E^{\nu^\lambda_{x,y}}\left[\|\varphi(\cdot,\gamma)\|^2_{L^2([0,T])}\right]^{1/2}\\
&+E^{\nu^\lambda_{x,y}}\left[\left\|L(\gamma)_\lambda(D_0\tilde{G}(\gamma)^{\cdot})\right\|^2_{L^2([T,1])}\right]^{1/2}E^{\nu^\lambda_{x,y}}\left[\|\varphi(\cdot,\gamma)\|^2_{L^2([T,1])}\right]^{1/2}\\
&+E^{\nu^\lambda_{x,y}}\left[\left\|(M(\gamma)^*_{\lambda,\cdot})^{-1}M(\gamma)^*_{\lambda,T}D\tilde{G}_T(\gamma)_1\right\|^2_{L^2([0,T])}\right]^{1/2}E^{\nu^\lambda_{x,y}}\left[\|\varphi(\cdot,\gamma)\|^2_{L^2([0,T])}\right]^{1/2}.
\end{aligned}
$$

By Lemma 3.2 in [5], we have for $0\le t\le T<1$

$$
\|M(\gamma)_{\lambda,T}M(\gamma)^{-1}_{\lambda,t}\|\le e^{\|\mathrm{Ric}\|_\infty/2\lambda}\left(\frac{1-T}{1-t}\right)^{(\kappa_1(r_1)-\varepsilon_1(\lambda))},
$$

where $\kappa_1(r_1)=\inf_{z\in B_{r_1}(y)}\nabla^2_z E(z)$. Consequently, letting $T\to 1$ in (2.13),

$$
E^{\nu^\lambda_{x,y}}\left[\int_0^1\Big(L(\gamma)_\lambda(D_0\tilde{G}(\gamma)^{\cdot})(t),\varphi(t,\gamma)\Big)\,dt\right]=E^{\nu^\lambda_{x,y}}\left[\int_0^1(H_1(t,\gamma),\varphi(t,\gamma))\,dt\right].
$$

This implies that

$$
H_1(t,\gamma)=E^{\nu^\lambda_{x,y}}[L(\gamma)_\lambda(D_0\tilde{G}(\gamma)^{\cdot})(t)|\mathfrak{G}_t]\qquad dt\otimes d\nu^\lambda_{x,y}\text{-}a.s.\ (t,\gamma)
$$

holds. Since, F is a limit of $\tilde{G}$, this completes the proof of (1). Since $E^{\nu^\lambda_{x,y}}[|H(s,\gamma)|^2]\le\|\|L\|_{op}\|^2_{L^\infty(\nu^\lambda_{x,y})}E^{\nu^\lambda_{x,y}}\left[|D_0F|^2\right]$ and Lemma 2.4 using (1) and Ito's formula, it is easy to check (2). See [14, 6, 2]. We prove (3). First we note that $D_0F=0$ on $P_{x,y}(B_r(y))^c$. Hence $E^{\nu^\lambda_{x,y}}[|D_0F|^2]=E^{\nu^\lambda_{x,y}}[|D_0F|^2;P_{x,y}(B_r(y))]$. Therefore by (2.30), noting

$$
\begin{aligned}
\nu^\lambda_{x,y}\left(P_{x,y}(B_r(y))\right)^{-1}E^{\nu^\lambda_{x,y}}\left[\left(F-E^{\nu^\lambda_{x,y}}[F]\right)^2\right]&\ge E^{\bar{\nu}^{\lambda,r}_{x,y}}\left[\left(F-E^{\nu^\lambda_{x,y}}[F]\right)^2\right]\\
&\ge \mathrm{Var}(F,\bar{\nu}^{\lambda,r}_{x,y}) \qquad (2.38)
\end{aligned}
$$

and $\lim_{\lambda\to\infty}\xi_\lambda(r)=1+Cr^2$, (3) can be proved. □

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