Asymptotics of spectral gaps on loop spaces over a class of Riemannian manifolds *

Shigeki Aida
Mathematical Institute
Tohoku University, Sendai, 980-8578, JAPAN
e-mail: aida@math.tohoku.ac.jp

Abstract

We prove the existence of spectral gaps of Ornstein-Uhlenbeck operators on loop spaces over a class of Riemannian manifolds which include hyperbolic spaces. This is an alternative proof and an extension of a result in Chen-Li-Wu in J. Funct. Anal. 259 (2010), 1421-1442. Further, we study the asymptotic behavior of the spectral gap.

Keywords : spectral gap, loop space, logarithmic Sobolev inequality, semi-classical limit

1 Introduction

Let E be a smooth non-negative function on a Riemannian manifold X. Let λ be a positive number and consider a weighted probability measure $d\nu^\lambda(x) = Z_\lambda^{-1}e^{-\lambda E(x)}dx$ on X, where Z_λ denotes the normalized constant and dx denotes the Riemannian volume. We consider a Dirichlet form on $L^2(X, d\nu^\lambda)$ such that

$$\mathcal{E}^\lambda(F,F) = \int_X |\nabla F(x)|^2 d\nu^\lambda(x),$$

where ∇ denotes the Levi-Civita covariant derivative. Under mild assumptions on E and the Riemannian metric, $1 \in \mathrm{D}(\mathcal{E}^\lambda)$ and the corresponding lowest eigenvalue e_1^λ of the generator of the Dirichlet form is 0. The spectral gap e_2^λ of $\mathcal{E}^\lambda$ is defined by

$$e_2^\lambda = \inf\Big\{\mathcal{E}^\lambda(F,F) \;\Big|\; \|F\|_{L^2(\nu^\lambda)} = 1, \int_X F(x)d\nu^\lambda(x) = 0\Big\}. \tag{1.1}$$

The study on the estimate and the asymptotic behavior of e_2^λ as $\lambda \to \infty$ is an interesting and important subject. In this problem, one of the simplest cases is the following:

(i) E has a unique minimum point c_0 and there are no critical points other than c_0,

(ii) the Hessian of E at c_0 is non-degenerate.

*Accepted for publication in Journal of Functional Analysis, doi:10.1016/j.jfa.2015.09.023

In this case, under some additional technical assumptions, it holds that $\lim_{\lambda\to\infty}\frac{e_2^\lambda}{\lambda}=\sigma_1$, where σ_1 is the lowest eigenvalue of the Hessian of E at c_0. When $X=\mathbb{R}^N$ and $E(x)=\frac{|x|^2}{2}$, the generator of the Dirichlet form is called the Ornstein-Uhlenbeck operator(=OU operator) and the spectral set is completely known.

We are interested in the case where X is an "infinite dimensional Riemannian manifold" and ν^λ is a probability measure on it. Let us explain our model. Let (M,g) be an n-dimensional complete Riemannian manifold. Let $x_0, y_0 \in M$ and consider a space of continuous paths $P_{x_0}(M)=C([0,1]\to M \mid \gamma(0)=x_0)$ and its subset $P_{x_0,y_0}(M)=\{\gamma\in P_{x_0}(M)\mid \gamma(1)=y_0\}$. Our X is $P_{x_0}(M)$ or $P_{x_0,y_0}(M)$ and ν^λ is the (pinned) Brownian motion measure. The transition probability of the Brownian motion is given by $p(t/\lambda,x,y)$, where $p(t,x,y)$ denotes the heat kernel of the diffusion semigroup $e^{t\Delta/2}$ and Δ is the Laplace-Bertlami operator. In many problems, we use the following heuristically appealing path integral expression,

$$d\nu^\lambda(\gamma)=\frac{1}{Z_\lambda}\exp\left(-\lambda E(\gamma)\right)d\gamma,$$

where $E(\gamma)$ is the energy of path γ and $d\gamma$ is the "infinite dimensional Riemannian measure". Needless to say, the energy function cannot be defined on the continuous path spaces on which the (pinned) Brownian motion measures exist and there do not exist the "Riemannian measures" on the infinite dimensional spaces. We refer the reader to [9, 45, 41] for some rigorous study of the path integral. On the other hand, by using an H-derivative D on X (see the definition in Section 3), we can define a Dirichlet form $\mathcal{E}^\lambda$ on $L^2(X,d\nu^\lambda)$. Our interest is in the study of the spectral gap of $\mathcal{E}^\lambda$. Since the triple $(X,\nu^\lambda,\mathcal{E}^\lambda)$ is formally an infinite dimensional analogue of the finite dimensional one, we may conjecture some results on the asymptotics of the spectral gap.

In the case where $X=P_{x_0}(M)$, the critical point of E on the subset of H^1 paths is just a constant path and this problem corresponds to the simplest case which we explained. Fang [27] proved the existence of the spectral gap by establishing the COH(=Clark-Ocone-Haussmann) formula for functions on $X=P_{x_0}(M)$. Also it is not difficult to prove that $\lim_{\lambda\to\infty}\frac{e_2^\lambda}{\lambda}=1$ by using the COH formula. We prove this in Section 3. Here note that the Hessian of E at the constant path is identity. On the other hand, if X is the pinned space $P_{x_0,y_0}(M)$, the set of critical points of the functional E on the set of H^1 paths of P_{x_0,y_0} is the set of geodesics. Therefore, by an analogy of finite dimensional cases, one may expect that the asymptotic behavior of the low-lying spectrum of the generator of $\mathcal{E}_\lambda$ is related to the set of the geodesics in this case. However, it is not even easy to find examples of Riemannian manifolds on which loop spaces the spectral gaps exist. In fact, Eberle [21] gave an example of a Riemannian manifold which is diffeomorphic to a sphere over which there is no spectral gap on the loop space. At the moment, there are no examples of loop spaces over simply connected compact Riemannian manifold for which the spectral gap exists.

If M is a Riemannian manifold with a pole y_0, the situation is simpler. In this case, the function E defined on the H^1 subset of $P_{x_0,y_0}(M)$ satisfies the above mentioned assumptions (i) and (ii). The author proved the existence of spectral gap in that case under additional strong assumptions on the Riemannian metric in [5]. Unfortunately, the assumption is not valid for hyperbolic spaces. The existence of the spectral gap on loop spaces over hyperbolic spaces was proved by Chen-Li-Wu [13] for the first time (see [14] also). They used results in [3, 12]. We give an alternative proof of their result and prove that $\lim_{\lambda\to\infty}\frac{e_2^\lambda}{\lambda}=\sigma_1$, where σ_1 is the spectrum bottom of the Hessian of E at the unique geodesic for a certain class of Riemannian manifolds.

Now let us recall a rough idea how to prove the asymptotic behavior of e_2^λ under the assumptions (i), (ii) when X is a finite dimensional space. By the unitary transformation $M_\lambda : F(\in L^2(d\nu^\lambda)) \mapsto F\left(Z_\lambda^{-1}e^{-\lambda E}\right)^{1/2}(\in L^2(dx))$, the problem is changed to determine the limit of the gap of spectrum of a Schrödinger operator. In this context, $\lambda \to \infty$ corresponds to the semi-classical limit of a physical system. In a small neighborhood of c_0, the Schrödinger operator can be approximated by a harmonic oscillator and we obtain the main term of the divergence of e_2^λ. As for outside the neighborhood, the potential function is very large and it has nothing to do with low energy part of the operator. In the present infinite dimensional problems, we cannot use the unitary transformation since there does not exist Riemannian volume measure and the function E cannot be defined on the whole space X. Moreover, there are difficulties in the proof of each parts, (a) Local estimate in a neighborhood $U(c_0)$ of the minimizer, (b) Estimate outside $U(c_0)$. In the problem (a), one may think that the problem can be reduced to a Gaussian measure case by a certain "local diffeomorphism". A natural candidate of the local diffeomorphism is an Itô map. Certainly, the mapping is measure preserving but the derivative of the mapping does not behave well because of the irregularity of the Brownian paths [20, 17, 24]. In problem (b), it is not clear how to use "the potential function is big" outside $U(c_0)$. To solve these problems, we use COH formula and a logarithmic Sobolev inequality on X. Clearly, it is more interesting to consider the cases where there are two or more local minimum points of E. We refer the reader to [35, 34] and references therein for finite dimensional cases. Also we note that Eberle [22, 23] studied such a problem on certain approximate spaces of loop spaces.

The paper is organized as follows. We already explained a rough idea of a proof of the asymptotic behavior of e_2^λ. In Section 2, we give a different proof based on a log-Sobolev inequality. Our proof for loop spaces is a modification of the proof. Also we explain the difficulty of the proof in the case of loop spaces.

In Section 3, we prepare necessary definitions and lemmas and explain our main theorems for $P_{x_0,y_0}(M)$. In this case, the minima c_0 is the minimal geodesic c_{x_0,y_0} between x_0 and y_0. As we explained, we need local analysis in a neighborhood of c_{x_0,y_0} of the generators of Dirichlet forms. Thus we consider an OU operator with Dirichlet boundary condition on a small neighborhood $\mathcal{D}$ of the minimal geodesic in a loop space over a Riemannian manifold. We define the generalized second lowest eigenvalue $e_{Dir,2,\mathcal{D}}^\lambda$ of the Dirichlet Laplacian and determine the asymptotic behavior of $e_{Dir,2,\mathcal{D}}^\lambda$ in our first main theorem (Theorem 3.2).

In the second main theorem (Theorem 3.6), we consider a rotationally symmetric Riemannian manifold M with a pole y_0 and a loop space $P_{x_0,y_0}(M)$, where x_0 is an arbitrary point of M. Under certain assumptions on the Riemannian metric, we prove the existence of the spectral gap and determine the asymptotic behavior of e_2^λ. The class of Riemannian manifolds includes the hyperbolic spaces. Actually, the same result as in the second main theorem holds true under the validity of a certain log-Sobolev inequality and a tail estimate of a certain random variable describing the size of γ. The log-Sobolev inequality can be proved by a COH-formula on $P_{x_0,y_0}(M)$. The diffusion coefficient of the Dirichlet form in the log-Sobolev inequality is unbounded and it is still an open problem whether a log-Sobolev inequality with a bounded coefficient holds on a loop space over a hyperbolic space .

In this paper, the COH formula plays a crucial role. Let us recall what COH formula is. Let F be an L^2 random variable on $P_{x_0}(M)$. By the Itô theorem, $F - E^{\nu^\lambda}[F]$ can be represented as a stochastic integral with respect to the Brownian motion b which is obtained as an anti-stochastic development of γ to $\mathbb{R}^n$ (see [36]). The COH formula gives an explicit form of the integrand as a conditional expectation of the H-derivative DF. As we noted, Fang proved the COH formula on $P_{x_0}(M)$ when M is a compact Riemannian manifold. But it is not difficult to prove the same formula for more general Riemannian manifold (see Lemma 3.8). In the case of $P_{x_0,y_0}(M)$, it is necessary to consider a Brownian motion w under the pinned measure which is obtained by adding a singular drift to b. The singular drift is defined by a logarithmic derivative of $p(t, y_0, z)$. For this, see Lemma 3.10 and [3, 7]. In both cases of $P_{x_0}(M)$ and $P_{x_0,y_0}(M)$, the integrand in the COH formula is the conditional expectation of the quantity $A(\gamma)_\lambda(DF')$, where $A(\gamma)_\lambda$ is a certain bounded linear operator depending on the path γ and λ. $A(\gamma)_\lambda$ for $P_{x_0}(M)$ is defined by the Ricci curvature and the operator norm is uniformly bounded for large λ. On the other hand, in the case of $P_{x_0,y_0}(M)$, the definition of $A(\gamma)_\lambda$ contains the Hessian of the heat kernel, $\nabla_z^2 \log p(t/\lambda, y_0, z)$ $(0 < t \le 1)$ because the stochastic differential equation of γ contains the singular drift term of the logarithmic derivative of the heat kernel. To control this term, we need results for a short time behavior of $\lim_{t\to 0} \nabla_z^2 \log p(t, x, z)$ which were studied for the first time by Malliavin and Stroock [49] (see (3.18) and Lemma 3.9). In view of this, it is easier to study the spectral gap for $P_{x_0}(M)$ than that for $P_{x_0,y_0}(M)$. In the final part of this section, we prove $\lim_{\lambda\to\infty} \frac{e_2^\lambda}{\lambda} = 1$ for $P_{x_0}(M)$.

In order to show the precise asymptotics of $e^\lambda_{Dir,2,\mathcal{D}}$ and e_2^λ, we need to identify $A(c_{x_0,y_0})_\infty = \lim_{\lambda\to\infty} A(c_{x_0,y_0})_\lambda$. This is necessary for local analysis near c_{x_0,y_0}. In Section 4, first we formally show that $A(c_{x_0,y_0})_\infty$ is an operator which is defined by the Hessian of the square of the distance function $k(z) = \frac{d(z,y_0)^2}{2}$. After that we prove a key relation between the Hessian of the energy function E at c_{x_0,y_0} and $A(c_{x_0,y_0})_\infty$. In that proof, Jacobi fields along the geodesic play an important role.

In Section 5, we prove Theorem 3.2. The proof LHS $\le$ RHS in (3.11) relies on a representation (5.2) of $e^\lambda_{Dir,2,\mathcal{D}}$ by the unique eigenfunction (ground state) Ψ_λ associated with the first eigenvalue of the Dirichlet Laplacian. By using this representation and a trial function, we prove the upper bound. The trial function is closely related with "eigenfunctions" associated with the bottom of the spectrum of the Hessian of the energy function E at c_{x_0,y_0}.

As already mentioned, we need to study $A(c_{x_0,y_0})_\infty$. In addition, we need to show that $A(\gamma)_\lambda$ can be approximated by $A(c_{x_0,y_0})_\infty$ when γ is close to c_{x_0,y_0} and λ is large. This is correct but not trivial because $A(\gamma)_\lambda$ is defined by solutions of Itô's stochastic differential equations driven by b and the solution mappings are not continuous in usual topology such as the uniform convergence topology. Actually the solution mappings are continuous in the topology of rough paths. Thus, we need to apply rough path analysis to our problem. Note that the law of b under the pinned measure is singular with respect to the Brownian motion measure. However, the probability distribution of b does not charge the slim sets in the sense of Malliavin. Hence, we need to consider Brownian rough paths for all Brownian paths except a slim set as in ([8]). After preparation of necessary estimates from rough paths (see Lemma 5.3), we prove Theorem 3.2.

In Section 6, we prove the existence of the spectral gap in a certain general setting as in [13]. This third main theorem (Theorem 6.2) implies the first half of the statement in Theorem 3.6. In Section 7, we complete the proof of Theorem 3.6.

2 A proof in $\mathbb{R}^N$ and some remarks

In this section, we show a proof of the asymptotics $\lim_{\lambda\to\infty} \frac{e_2^\lambda}{\lambda} = \sigma_1$ on $\mathbb{R}^N$ under the validity of a log-Sobolev inequality. Our proof for $P_{x_0,y_0}(M)$ is a suitable modification of this proof. In this section, D stands for the usual Fréchet derivative on $\mathbb{R}^N$.

Let E be a non-negative C^∞ function on $\mathbb{R}^N$ and suppose the following (1), (2), (3), (4).

(1) $E(0) = 0$ and 0 is the unique minimum point and $D^2E(0) > 0$. Further $\liminf_{|x|\to\infty} E(x) > 0$.

(2) Let $\lambda > 0$. Suppose that $e^{-\lambda E(x)}$ is an integrable function and define a probability measure,

$$\nu^\lambda(dx) = Z_\lambda^{-1} e^{-\lambda E(x)} dx, \tag{2.1}$$

where $Z_\lambda = \int_{\mathbb{R}^N} e^{-\lambda E(x)} dx$.

(3) Let $\mathcal{E}^\lambda(F, F) = \int_{\mathbb{R}^N} |DF(x)|^2 d\nu^\lambda(x)$, where $F \in C_0^\infty(\mathbb{R}^N)$. Also let $\mathcal{E}^\lambda$ denote the Dirichlet form which is the closure of the closable form. It holds that $|x|, 1 \in \mathrm{D}(\mathcal{E}^\lambda)$ and $\mathcal{E}^\lambda(1, 1) = 0$ for all $\lambda > 0$. The notation $|\cdot|$ denotes the usual Euclidean norm.

(4) There exists a constant $C > 0$ such that the following log-Sobolev inequality holds:

$$\int_{\mathbb{R}^N} F(x)^2 \log\left(F(x)^2/\|F\|^2_{L^2(\nu_\lambda)}\right) d\nu^\lambda(x) \le \frac{C}{\lambda}\mathcal{E}^\lambda(F, F), \quad F \in \mathrm{D}(\mathcal{E}^\lambda). \tag{2.2}$$

Clearly the spectral bottom e_1^λ of the Dirichlet form $\mathcal{E}^\lambda$ is 0. Under the above assumptions, we prove that

Theorem 2.1. *Let e_2^λ be the spectral gap of $\mathcal{E}^\lambda$. Then*

$$\lim_{\lambda\to\infty} \frac{e_2^\lambda}{\lambda} = \sigma_1, \tag{2.3}$$

where σ_1 denotes the smallest eigenvalue of the matrix $D^2E(0)$.

The log-Sobolev inequality (2.2) implies the bound $e_2^\lambda \ge 2\lambda/C$ for all λ. Thus it holds that $C\sigma_1 \ge 2$. Note that the assumption in the above is very strong and we cannot say the result is "nice".

Proof. We prove the lower bound estimate $\liminf_{\lambda\to\infty} \frac{e_2^\lambda}{\lambda} \ge \sigma_1$. By the assumptions (1) and (2), we have for any $r > 0$ there exists K_r and M_r such that

$$\nu^\lambda\left(|x| \ge r\right) \le K_r e^{-\lambda M_r} \qquad \text{for all } \lambda \ge 1 \tag{2.4}$$

and

$$\lim_{\lambda\to\infty} \left(\frac{\lambda}{2\pi}\right)^{N/2} Z_\lambda = \det\left(D^2E(0)\right)^{-1/2}. \tag{2.5}$$

The estimate (2.5) can be proved by Laplace's method. From now on, we always assume $\lambda \ge 1$. The log-Sobolev inequality (2.2) implies that for any bounded measurable function V, it holds that

$$\mathcal{E}^\lambda(F, F) + \int_{\mathbb{R}^N} V(x)F(x)^2 d\nu^\lambda(x) \ge -\frac{\lambda}{C}\log\left(\int_{\mathbb{R}^N} e^{-\frac{C}{\lambda}V} d\nu^\lambda\right)\|F\|^2_{L^2(\nu^\lambda)}, \tag{2.6}$$

where the constant C is the same number as in (2.2). We refer the reader to [33] for this estimate. Let χ_0 be a smooth function with $\chi_0(u) = 1$ for $|u| \le 1$ and $\chi_0(u) = 0$ for $|u| \ge 2$. Let $\kappa > 0$ be a small number and set $\chi_{0,\kappa}(x) = \chi_0(\kappa^{-1}|x|)$ and $\chi_{1,\kappa}(x) = \sqrt{1 - \chi_{0,\kappa}^2(x)}$. Let $F \in \mathrm{D}(\mathcal{E}^\lambda)$ and assume $\|F\|_{L^2(\nu^\lambda)} = 1$ and $\int_{\mathbb{R}^N} F(x)d\nu^\lambda(x) = 0$. By an elementary calculation,

$$\begin{aligned}\mathcal{E}^\lambda(F, F) &= \mathcal{E}^\lambda(F\chi_{0,\kappa}, F\chi_{0,\kappa}) + \mathcal{E}^\lambda(F\chi_{1,\kappa}, F\chi_{1,\kappa}) \\ &\quad - \int_{\mathbb{R}^N} \left(|D\chi_{0,\kappa}|^2 + |D\chi_{1,\kappa}|^2\right) F(x)^2 d\nu^\lambda(x). \qquad (2.7)\end{aligned}$$

This identity is called the IMS localization formula ([52]). We have $|D\chi_{0,\kappa}|^2 + |D\chi_{1,\kappa}|^2 \le C'\kappa^{-2}$. By applying (2.6),

$$\begin{aligned}\mathcal{E}^\lambda(F\chi_{1,\kappa}, F\chi_{1,\kappa}) &= \mathcal{E}^\lambda(F\chi_{1,\kappa}, F\chi_{1,\kappa}) - \int_{\mathbb{R}^N} \delta\lambda^2 (F\chi_{1,\kappa})^2 1_{|x|\ge\kappa} d\nu^\lambda \\ &\quad + \int_{\mathbb{R}^N} \delta\lambda^2 (F\chi_{1,\kappa})^2 1_{|x|\ge\kappa} d\nu^\lambda \\ &\ge -\frac{\lambda}{C} \log\left(\int_{\mathbb{R}^N} e^{\delta C\lambda 1_{|x|\ge\kappa}} d\nu^\lambda\right) \|F\chi_{1,\kappa}\|_{L^2(\nu^\lambda)}^2 + \delta\lambda^2 \|F\chi_{1,\kappa}\|_{L^2(\nu^\lambda)}^2 \\ &\ge \left\{-\frac{\lambda}{C} \log\left(1 + K_\kappa e^{\delta C\lambda - M_\kappa\lambda}\right) + \delta\lambda^2\right\} \|F\chi_{1,\kappa}\|_{L^2(\nu^\lambda)}^2 \\ &\ge \left\{-\frac{\lambda}{C} K_\kappa e^{(\delta C - M_\kappa)\lambda} + \delta\lambda^2\right\} \|F\chi_{1,\kappa}\|_{L^2(\nu^\lambda)}^2, \qquad (2.8)\end{aligned}$$

where we have used (2.4). Thus, by choosing δ so that $\delta C < M_\kappa$, there exists $\delta' > 0$ such that for large λ,

$$\mathcal{E}^\lambda(F\chi_{1,\kappa}, F\chi_{1,\kappa}) \ge \delta'\lambda^2 \|F\chi_{1,\kappa}\|_{L^2(\nu^\lambda)}^2. \qquad (2.9)$$

We estimate $\mathcal{E}^\lambda(F\chi_{0,\kappa}, F\chi_{0,\kappa})$. Note that the support of $F\chi_{0,\kappa}$ is included in $\{x \mid |x| \le 2\kappa\}$. Let $V = \{x \mid |x| < 3\kappa\}$. For small κ, by the Morse lemma, there exists an open neighborhood U of 0 and a C^∞-diffeomorphism $\Phi : y(\in U) \mapsto x(\in V)$ such that $\Phi(0) = 0$ and $E(\Phi(y)) = \frac{1}{2}|y|^2$ for all $y \in U$. We write $m^\lambda(dy) = \left(\frac{\lambda}{2\pi}\right)^{N/2} e^{-\lambda|y|^2/2} dy$. By using this coordinate, we have

$$\begin{aligned}\mathcal{E}^\lambda(F\chi_{0,\kappa}, F\chi_{0,\kappa}) &= \int_V |D(F\chi_{0,\kappa})(x)|^2 e^{-\lambda E(x)} Z_\lambda^{-1} dx \\ &= \int_U |D(F\chi_{0,\kappa})(\Phi(y))|^2 e^{-\frac{\lambda}{2}|y|^2} Z_\lambda^{-1} |\det(D\Phi(y))| dy \\ &= \int_U |\{(D\Phi(y))^*\}^{-1} D\{(F\chi_{0,\kappa})(\Phi(y))\}|^2 e^{-\frac{\lambda}{2}|y|^2} Z_\lambda^{-1} |\det(D\Phi(y))| dy. \qquad (2.10)\end{aligned}$$

We may assume that the mappings $y \mapsto \{(D\Phi(y))^*\}^{-1}$ and $y \mapsto |\det(D\Phi(y))|$ are Lipschitz continuous. Let $\tilde{\sigma}_1$ be the smallest eigenvalue of $(D\Phi(0))^{-1}\{(D\Phi(0))^*\}^{-1}$. Then there exists a positive function $\varepsilon(\kappa)$ satisfying $\lim_{\kappa\to 0}\varepsilon(\kappa)=0$ such that

$$\begin{aligned}\mathcal{E}^\lambda(F\chi_{0,\kappa},F\chi_{0,\kappa}) &\geq (1-\varepsilon(\kappa))\tilde{\sigma}_1|\det D\Phi(0)|Z_\lambda^{-1}\left(\frac{\lambda}{2\pi}\right)^{-N/2}\int_U |D\left\{(F\chi_{0,\kappa})(\Phi(y))\right\}|^2 dm^\lambda(y)\\ &\geq (1-\varepsilon(\kappa))\tilde{\sigma}_1|\det D\Phi(0)|Z_\lambda^{-1}\left(\frac{\lambda}{2\pi}\right)^{-N/2}\\ &\quad\times\lambda\left\{\int_{\mathbb{R}^N}(F\chi_{0,\kappa})^2(\Phi(y))dm^\lambda(y)-\left(\int_{\mathbb{R}^N}(F\chi_{0,\kappa})(\Phi(y))dm^\lambda(y)\right)^2\right\},\end{aligned} \tag{2.11}$$

where we have used the spectral gap of the generator of the Dirichlet form $\int_{\mathbb{R}^N}|DF(y)|^2 dm^\lambda(y)$ is λ. We have

$$\begin{aligned}&|\det D\Phi(0)|Z_\lambda^{-1}\left(\frac{\lambda}{2\pi}\right)^{-N/2}\int_{\mathbb{R}^N}(F\chi_{0,\kappa})^2(\Phi(y))dm^\lambda(y)\\ &=|\det D\Phi(0)|Z_\lambda^{-1}\left(\frac{\lambda}{2\pi}\right)^{-N/2}\int_U(F\chi_{0,\kappa})^2(\Phi(y))dm^\lambda(y)\\ &=|\det D\Phi(0)|Z_\lambda^{-1}\left(\frac{\lambda}{2\pi}\right)^{-N/2}\int_V(F\chi_{0,\kappa})^2(x)e^{-\lambda E(x)}\left(\frac{\lambda}{2\pi}\right)^{N/2}\left|\det(D(\Phi^{-1})(x))\right|dx\\ &\geq (1-\varepsilon(\kappa))Z_\lambda^{-1}\int_V(F\chi_{0,\kappa})^2(x)e^{-\lambda E(x)}dx\\ &=(1-\varepsilon(\kappa))Z_\lambda^{-1}\int_{\mathbb{R}^N}(F\chi_{0,\kappa})^2(x)e^{-\lambda E(x)}dx\end{aligned} \tag{2.12}$$

and

$$\begin{aligned}&\int_{\mathbb{R}^N}(F\chi_{0,\kappa})(\Phi(y))dm^\lambda(y)\\ &\quad=\int_U(F\chi_{0,\kappa})(\Phi(y))dm^\lambda(y)\\ &\quad=\int_V(F\chi_{0,\kappa})(x)|\det(D(\Phi^{-1})(x))|\left(\frac{\lambda}{2\pi}\right)^{N/2}Z_\lambda d\nu^\lambda(x)\\ &\quad=\int_V(F\chi_{0,\kappa})(x)\left(|\det(D(\Phi^{-1})(x))|-|\det(D(\Phi^{-1})(0))|\right)\left(\frac{\lambda}{2\pi}\right)^{N/2}Z_\lambda d\nu^\lambda(x)\\ &\qquad+|\det(D(\Phi^{-1})(0))|\int_V(F\chi_{0,\kappa})(x)\left(\frac{\lambda}{2\pi}\right)^{N/2}Z_\lambda d\nu^\lambda(x)\\ &\quad=: I_1+I_2.\end{aligned} \tag{2.13}$$

Here

$$|I_1|\leq\varepsilon(\kappa)\left(\frac{\lambda}{2\pi}\right)^{N/2}Z_\lambda\|F\chi_{0,\kappa}\|_{L^2(\nu^\lambda)} \tag{2.14}$$

and by the Schwarz inequality,

$$
\begin{aligned}
&|I_2| \\
&\le |\det(D(\Phi^{-1})(0))| \left\{ \left| \int_{\mathbb{R}^N} F(x) d\nu^\lambda(x) \right| + \left| \int_{\mathbb{R}^N} F(y) \left(\chi_{0,\kappa}(x) - 1\right) d\nu^\lambda(x) \right| \right\} \left(\frac{\lambda}{2\pi}\right)^{N/2} Z_\lambda \\
&\le |\det(D(\Phi^{-1})(0))| \, \nu^\lambda\left(|x| \ge \kappa\right)^{1/2} \left(\frac{\lambda}{2\pi}\right)^{N/2} Z_\lambda \\
&\le |\det(D(\Phi^{-1})(0))| \sqrt{K_\kappa} e^{-\lambda M_\kappa/2} \left(\frac{\lambda}{2\pi}\right)^{N/2} Z_\lambda.
\end{aligned}
\tag{2.15}
$$

By the definition of Φ, we have $D^2E(0) = \{(D\Phi(0))^*\}^{-1}(D\Phi(0))^{-1}$. Since the set of eigenvalues of $(D\Phi(0))^{-1}\{(D\Phi(0))^*\}^{-1}$ and $\{(D\Phi(0))^*\}^{-1}(D\Phi(0))^{-1}$ are the same, we obtain $\tilde{\sigma}_1 = \sigma_1$. Thus, we get

$$
\begin{aligned}
&\mathcal{E}^\lambda(F\chi_{0,\kappa}, F\chi_{0,\kappa}) \\
&\ge \lambda(1 - \varepsilon(\kappa))\sigma_1 \|F\chi_{0,\kappa}\|^2_{L^2(\nu^\lambda)} \\
&\quad - \lambda Z_\lambda^2 \left(\frac{\lambda}{2\pi}\right)^N \left\{ \varepsilon(\kappa) \|F\chi_{0,\kappa}\|_{L^2(\nu^\lambda)} + |\det(D(\Phi^{-1})(0))| \sqrt{K_\kappa} e^{-\lambda M_\kappa/2} \right\}^2.
\end{aligned}
\tag{2.16}
$$

By (2.7), (2.9), (2.16) and $\chi_{0,\kappa}^2(x) + \chi_{1,\kappa}^2(x) = 1$ for all x, we complete the proof of the lower bound. The upper bound $\limsup_{\lambda\to\infty} \frac{e_2^\lambda}{\lambda} \le \sigma_1$ can be proved in a standard way. Let v be a unit eigenvector such that $D^2E(0)v = \sigma_1 v$. For this v, let $F^\lambda(x) = \sqrt{\lambda\sigma_1}(x, v)$. Then we have $\frac{\mathcal{E}^\lambda(F^\lambda, F^\lambda)}{\lambda} = \sigma_1$, $\lim_{\lambda\to\infty} \int_{\mathbb{R}^N} F^\lambda(x) d\nu^\lambda(x) = 0$ and $\lim_{\lambda\to\infty} \|F^\lambda\|_{L^2(\nu^\lambda)} = 1$ which imply the upper bound. □

Remark 2.2. (1) In the estimate of $\mathcal{E}^\lambda(F\chi_{0,\kappa}, F\chi_{0,\kappa})$, we reduce the problem to Gaussian case with the help of the Morse lemma. The Itô map is a measure preserving map between $P_{x_0}(M)$ with the Brownian motion measure and the Wiener space. However, the derivative of the Itô map is not a bounded linear operator between two tangent spaces ([20, 17, 24]). In the study of the asymptotic behavior of the lowest eigenvalue of a Schrödinger operator on $P_{x_0}(M)$ in [6], the author reduced the local analysis to the analysis in Wiener spaces by using the Itô map and a ground state transformation. At the moment, it is not clear that similar consideration can be applied to the local analysis in the present problem. In this paper, instead, we use the COH formula in Lemma 3.10.
(2) Let us consider a Dirichlet form

$$
\mathcal{E}^{A,\lambda}(F, F) = \int_{\mathbb{R}^N} |A(x)DF(x)|^2 d\nu^\lambda(x), \tag{2.17}
$$

where $A(x)$ is an $N \times N$ regular matrix-valued continuous mapping on $\mathbb{R}^N$ satisfying that there exists a positive number $C > 1$ such that $C^{-1}|\xi|^2 \le (A(x)\xi, \xi) \le C|\xi|^2$ for all x, ξ. Suppose $\mathcal{E}^{A,\lambda}$ satisfies the above assumption (3) and (4). Then, for the asymptotic behavior of the spectral gap of $\mathcal{E}^{A,\lambda}$, the same result as in Theorem 2.1 holds replacing σ_1 by the lowest eigenvalue of the Hessian of E with respect to the Riemannian metric defined by $g_A(x)(\xi, \xi) = |A(x)^{-1}\xi|^2$. In that proof, we use the continuity of the map $x \mapsto A(x)$. In the case of $P_{x_0,y_0}(M)$, a local Poincaré inequality (3.32) and a log-Sobolev inequality (3.34) holds. However the mapping $\gamma \mapsto A(\gamma)_\lambda$ is not a continuous mapping in the uniform convergence topology and just a continuous mapping in the topology of rough paths. In this sense, we need the result in rough paths. Moreover, in that case, the operator norm of $A(\gamma)_\lambda$ is not uniformly bounded in γ. Hence the argument is not so simple as in the above case. Note that $A(\gamma)_\lambda$ depends on λ. Hence we need to estimate $A(\gamma)_\lambda$ for large λ. In this calculation, we use the short time behavior of the Hessian of the logarithm of the heat kernel.

3 Preliminary and Statement of results

Let (M, g) be an n-dimensional complete Riemannian manifold. Let $d(x, y)$ denote the Riemannian distance between x and y. Let $p(t, x, y)$ be the heat kernel of the diffusion semigroup $e^{t\Delta/2}$ defined by the Laplace-Bertlami operator Δ. We refer the readers to [36, 38] for stochastic analysis on manifolds. The following assumption is natural for analysis on Riemannian manifolds.

Assumption A. (1) *There exist positive constants C, C' such that for any $0 < t \le 1$, $x, y \in M$,*

$$p(t, x, y) \le Ct^{-n/2}e^{-C'd(x,y)^2/t}. \tag{3.1}$$

(2) *The Ricci curvature of M is bounded, i.e., $\|\mathrm{Ric}\|_\infty < \infty$.*

The condition (2) implies $\int_M p(t, x, y)dy = 1$ holds for all $t > 0$ and $x \in M$, where dy denotes the Riemannian volume. In second main theorem (Theorem 3.6), we consider rotationally symmetric Riemannian metrics. We prove the above assumption holds true in such a case by using the following observation in Lemma 3.5. Assumption A (1) holds true if the Ricci curvature is bounded from below and the volume of small balls have uniform lower bound ([44]). That is, there exist $C > 0$ and $l_0 > 0$ such that $\mathrm{vol}(B_l(x)) \ge Cl^n$ for all $0 < l < l_0$ and any $x \in M$. Here $\mathrm{vol}(B_l(x))$ denotes the volume of the open metric ball $B_l(x)$ centered at x with radius l.

In order to define (pinned) Brownian motion measure, we assume M satisfies Assumption A. Let $x_0 \in M$. The probability measure $\nu^\lambda_{x_0}$ on $P_{x_0}(M)$ satisfying the following is called the Brownian motion measure starting at x_0:
For any Borel measurable subsets $A_k \subset M$ $(1 \le k \le m)$ and $0 = t_0 < t_1 < \cdots < t_m \le 1$,

$$\begin{aligned} &\nu^\lambda_{x_0}\left(\{\gamma \mid \gamma(t_1) \in A_1, \dots, \gamma(t_m) \in A_m\}\right) \\ &\quad = \int_{M^m} \prod_{k=1}^m p\left((t_k - t_{k-1})/\lambda, x_{k-1}, x_k\right) 1_{A_k}(x_k)dx_1 \cdots dx_m. \end{aligned} \tag{3.2}$$

The process $\gamma(t)$ under $\nu^\lambda_{x_0}$ is a semimartingale. When $M = \mathbb{R}^n$, $\gamma(t)$ is the ordinary Brownian motion whose covariance matrix is equal to tI/λ. Let $\pi : O(M) \to M$ be the orthonormal frame bundle with the Levi-Civita connection. We fix a frame $u_0 = \{\varepsilon_i\}_{i=1}^n \in \pi^{-1}(x_0)$. By the mapping $u_0 : \mathbb{R}^n \to T_{x_0}M$, we identify $\mathbb{R}^n$ with $T_{x_0}M$. Let $\tau(\gamma)_t : T_{x_0}M \to T_{\gamma(t)}M$ denote the stochastic parallel translation along γ. For smooth cylindrical function $F(\gamma) = f(\gamma(t_1), \dots, \gamma(t_m)) \in \mathcal{FC}^\infty_b(P_{x_0}(M))$ $(0 < t_1 < \cdots < t_m \le 1)$, the H-derivative $DF(\gamma)$ is defined by

$$DF(\gamma)_t = \sum_{i=1}^m u_0^{-1}\tau(\gamma)_{t_i}^{-1}(\nabla_i f)(\gamma(t_1), \dots, \gamma(t_m))t \wedge t_i, \tag{3.3}$$

where $\nabla_i f$ denotes the derivative of f with respect to the i-th variable. Note that $DF(\gamma) \in \mathrm{H} := H^1([0,1] \to \mathbb{R}^n \mid h(0) = 0)$. Under Assumption A, the symmetric form

$$\mathcal{E}^\lambda(F,F) = \int_{P_{x_0}(M)} |DF(\gamma)|_{\mathrm{H}}^2 d\nu_{x_0}^\lambda(\gamma), \qquad F \in \mathcal{FC}_b^\infty(P_{x_0}(M)) \tag{3.4}$$

is closable. We refer the reader to [18, 36, 37] for the closability. The Dirichlet form of the smallest closed extension is denoted by the same notation and the the generator $-L_\lambda$ is a natural generalization of OU operators in Gaussian cases.

We now consider the pinned case. It is elementary fact that regular conditional probability (pinned Brownian motion measure) $\nu_{x_0,y}^\lambda(\cdot) = \nu_{x_0}^\lambda(\cdot \mid \gamma(1) = y)$ exists on $P_{x_0,y}(M)$ for $p(1,x_0,y)dy$-almost all y. However, it is necessary for us to define $\nu_{x_0,y}^\lambda$ for all $y \in M$. Actually, under Assumption A (1) and (2), one can prove that the regular conditional probability $\nu_{x_0,y}^\lambda$ on $P_{x_0,y}(M)$ exists for all $y \in M$. This can be checked by using the volume comparison theorem and the Kolmogorov criterion (see [3, 36, 19]). Moreover, the pinned Brownian motion measure is equivalent to the Brownian motion measure up to any time $t < 1$ with respect to the natural σ-field generated by the paths. This implies that the pinned Brownian motion is a semimartingale for $t < 1$. Hence the stochatsic parallel translation is well defined and one can define the H-derivative of a smooth $F(\gamma) = f(\gamma(t_1), \ldots, \gamma(t_m)) \in \mathcal{FC}_b^\infty(P_{x_0,y_0}(M))$ $(t_m < 1)$ by $D_0F(\gamma)_t = P_0\left(DF(\gamma)\right)_t$, where P_0 is the projection operator on H onto the subspace $\mathrm{H}_0 := \{h \in \mathrm{H} \mid h(1) = 0\}$. Using D_0 on $\mathcal{FC}_b^\infty(P_{x_0,y_0(M)})$, we can define a symmetric bilinear form $\mathcal{E}^\lambda$ similarly to non-pinned case. However, we need additional assumption on the Riemannian manifold M to prove the closability since M may be non-compact. Hence we consider the following assumption.

Assumption B. $(\mathcal{E}^\lambda, \mathcal{FC}_b^\infty(P_{x_0,y_0}(M)))$ *is closable.*

We explain the reason why we need additional assumption. Let $b(t) = \int_0^t u_0^{-1}\tau(\gamma)_s^{-1} \circ d\gamma(s)$ $(0 \le t \le 1)$, where $\circ d$ means Stratonovich integral. The process $b(t)$ is anti-stochastic development of $\gamma(t)$. Under the law $\nu_{x_0}^\lambda$, $b(t)$ is the ordinary Brownian motion with variance $1/\lambda$. We will discuss $b(t)$ later again in the explanation of the COH formula. Note that the law of $\{b(t)\}_{0 \le t \le 1}$ is singular with respect to the Brownian motion measure under ν_{x_0,y_0}^λ. This is related to the singularity of the pinned Brownian motion itself. The closability of $\mathcal{E}^\lambda$ can be proved by using the integration by parts(=IBP) formula for D, D_0. The formula contains stochastic integrals with respect to $b(t)$ and the integrability of the stochastic integrals when t converges to 1 is the main issue to establish the formula for the pinned measure. See [19, 3, 25, 26, 36, 31] for this problem. If either (i) M is compact, or (ii) M is diffeomorphic to $\mathbb{R}^n$ and the metric is flat outside a certain bounded set, holds, by applying the Malliavin's quasi-sure analysis, we can prove the integrability of the stochastic integrals and we obtain the IBP formula and the closability. Also, under the condition,

There exists a positive constant C such that for any $0 < t \le 1$ and $z \in M$,

$$|\nabla_z \log p(t, y_0, z)| \le C\frac{d(y_0, z)}{t} + \frac{C}{\sqrt{t}}, \tag{3.5}$$

the IBP formula and the closability hold. This inequality holds for any compact Riemannian manifolds ([36]). For rotationally symmetric Riemannian manifolds, we will give a sufficient condition for this. See Assumption C and Lemma 3.9 (2).

We now define a Dirichlet Laplacian on a certain domain $\mathcal{D}$ in $P_{x_0,y_0}(M)$.

Definition 3.1. *Let l be a positive number with $l > d(x_0, y_0)$. Let $B_l(y_0)$ denote the open ball centered at y_0 with radius l. Define*

$$\mathcal{D}_l = \{\gamma \in P_{x_0,y_0}(M) \mid \gamma(t) \in B_l(y_0) \text{ for all } 0 \le t \le 1\}. \tag{3.6}$$

For $l = +\infty$, we set $\mathcal{D}_\infty = P_{x_0,y_0}(M)$.

We may omit the subscript l for simplicity. In order to define the H^1-Sobolev spaces, we assume Assumption B for the moment. Let $H^{1,2}(P_{x_0,y_0}(M), \nu^\lambda_{x_0,y_0})$ denote the H^1-Sobolev space which is the closure of $\mathcal{FC}^\infty_b(P_{x_0,y_0}(M))$ with respect to the norm $\|F\|_{H^1} = \left(\|F\|^2_{L^2(\nu^\lambda_{x_0,y_0})} + \mathcal{E}^\lambda(F, F)\right)^{1/2}$. Let

$$H^{1,2}_0(\mathcal{D}, \nu^\lambda_{x_0,y_0}) = \left\{F \in H^{1,2}(P_{x_0,y_0}(M), \nu^\lambda_{x_0,y_0}) \mid F = 0 \ \nu^\lambda_{x_0,y_0}\text{-a.s. outside } \mathcal{D}\right\} \tag{3.7}$$

which is a closed linear subspace of $H^{1,2}(P_{x_0,y_0}(M), \nu^\lambda_{x_0,y_0})$.

The non-positive generator L_λ corresponding to the densely defined closed form

$$\mathcal{E}^\lambda(F, F), \quad F \in H^{1,2}_0(\mathcal{D}, \nu^\lambda_{x_0,y_0})$$

in the Hilbert space $L^2(\mathcal{D}, \nu^\lambda_{x_0,y_0})$ is the Dirichlet Laplacian on $\mathcal{D}$. Let

$$e^\lambda_{Dir,1,\mathcal{D}} = \inf_{F(\neq 0)\in H^{1,2}_0(\mathcal{D})} \frac{\int_{\mathcal{D}} |D_0F|^2 d\nu^\lambda_{x_0,y_0}}{\|F\|^2_{L^2(\nu^\lambda_{x_0,y_0})}}. \tag{3.8}$$

This is equal to $\inf \sigma(-L_\lambda)$, where $\sigma(-L_\lambda)$ denotes the spectral set of $-L_\lambda$. We next introduce

$$e^\lambda_{Dir,2,\mathcal{D}} = \sup_{G(\neq 0)\in L^2(\nu^\lambda_{x_0,y_0})} \inf\left\{ \frac{\int_{\mathcal{D}} |D_0F|^2 d\nu^\lambda_{x_0,y_0}}{\|F\|^2_{L^2(\nu^\lambda_{x_0,y_0})}} \;\middle|\; F \in H^{1,2}_0(\mathcal{D}), \quad (F, G)_{L^2(\nu^\lambda_{x_0,y_0})} = 0 \right\}. \tag{3.9}$$

This is the generalized second lowest eigenvalue of $-L_\lambda$. When $l = +\infty$, $e^\lambda_{Dir,1,\mathcal{D}} = 0$ and $e^\lambda_{Dir,2,\mathcal{D}}$ is equal to the spectral gap of $-L_\lambda$ on the whole space $P_{x_0,y_0}(M)$. We use the notations e^λ_1 and e^λ_2 instead of $e^\lambda_{Dir,1,\mathcal{D}}$ and $e^\lambda_{Dir,2,\mathcal{D}}$ respectively in this case. To state our first main theorem, let us define the energy of H^1 path γ belonging to $P_{x_0,y_0}(M)$,

$$E(\gamma) = \frac{1}{2}\int_0^1 |\gamma'(t)|^2_{T_{\gamma(t)}M}\, dt. \tag{3.10}$$

We use the same notation D_0 for the derivative of the smooth function on the Hilbert manifold of the H^1 subset of $P_{x_0,y_0}(M)$. Note that $D_0^2 E(c_{x_0,y_0})$ is a symmetric bounded linear operator on H_0. See Lemma 4.1 for the explicit form. The following is our first main theorem.

Theorem 3.2. *Assume M satisfies Assumption* A *and Assumption* B. *Let $0 < l < \infty$. Assume that l satisfies the following.*
(1) l is smaller than the injectivity radius at y_0. In particular, there are no intersection of the closure of $B_l(y_0)$ and Cut(y_0), *where* Cut(y_0) *denotes the cut-locus of y_0.*
(2) The Hessian of $k(z) = \frac{1}{2}d(z, y_0)^2$ satisfies that $\inf_{z\in B_l(y_0)} \nabla^2 k(z) > 1/2$.

Then we have

$$\lim_{\lambda\to\infty}\frac{e^{\lambda}_{Dir,2,\mathcal{D}}}{\lambda}=\sigma_1, \tag{3.11}$$

where $\sigma_1=\inf\sigma((D_0^2E)(c_{x_0,y_0}))$.

Since $\nabla_z^2 k(z)|_{z=y_0}=I_{T_{y_0}M}$, the above conditions (1), (2) hold true for small l. Also, if M is negatively curved manifold, the condition (2) holds for all l. We need condition (2) to prove a COH formula by applying Lemma 3.1 in [3] although this may be just a technical condition. Under the above condition, clearly the minimal geodesic $c_{x_0,y_0}=c_{x_0,y_0}(t)$ $(0\le t\le 1)$ $(c_{x_0,y_0}(0)=x_0, c_{x_0,y_0}(1)=y_0)$ belongs to $\mathcal{D}$. Further, $\lim_{\lambda\to\infty}\nu^{\lambda}_{x_0,y_0}(\mathcal{D})=1$ holds true by a large deviation result (see Section 5).

For a certain class of Riemannian manifolds M, the same result holds for $P_{x_0,y_0}(M)$. It is the second main theorem. Let M be a Riemannain manifold with a pole y_0. That is, the exponential map $\exp_{y_0}: T_{y_0}M\to M$ is a diffeomorphism. We pick an orthonormal frame $\tilde{u}_0$ of $T_{y_0}M$. Let S^{n-1} be the unit sphere centered at the origin in $\mathbb{R}^n$. We identify $\mathbb{R}^n\setminus\{0\}$ with $(0,+\infty)\times S^{n-1}$ by $(r,\Theta)(\in(0,+\infty)\times S^{n-1})\mapsto r\Theta\in(\mathbb{R}^n\setminus\{0\})$. Let us define $\Psi:(0,+\infty)\times S^{n-1}\to M$ by $x=\Psi(r,\Theta)=\exp_{y_0}(\tilde{u}_0(r\Theta))$. Then $r=d(y_0,x)$ holds. The Riemannian metric g is said to be rotationally symmetric at y_0 if the pull back of g by Ψ can be expressed as

$$\Psi^*g=dr^2+f(r)^2d\Theta^2, \tag{3.12}$$

where $d\Theta^2$ denotes the standard Riemannian metric on the sphere. Note that if g is a smooth Riemannian metric on M, $f(r)$ is a C^∞ function on $[0,\infty)$ satisfying $f(0)=0$ and $f'(0)=1$. We consider the following assumption on f.

Assumption C. *Let* $\varphi(r)=\log\frac{f(r)}{r}$. *The function* φ *satisfies the following.*
(1) φ *is a* C^∞ *function on* $[0,\infty)$. *The* k*-th derivative* $\varphi^{(k)}(r)$ *is bounded function on* $[0,\infty)$ *for all* $1\le k\le 4$.
(2) *There exists a* C^∞ *function* ϕ *on* $[0,\infty)$ *such that* $\varphi(r)=\phi(r^2)$.
(3) $\inf_{r>0} r\varphi'(r)>-\frac{1}{2}$.

By Lemma A.2 in [16], it is easy to deduce that for any smooth function f on $[0,\infty)$ satisfying $f(0)=0, f'(0)=1$ and Assumption C (2), the Riemannian metric $dr^2+f(r)^2d\Theta^2$ on $\mathbb{R}^n\setminus\{0\}$ can be extended to a smooth Riemannian metric on $\mathbb{R}^n$. The above condition on φ appeared in [5]. In [5], we assume all derivatives $\varphi^{(k)}$ are bounded. However we see that it is enough to assume the boundedness for $1\le k\le 4$ by checking the calculations there. We give examples of φ which satisfies the above assumption.

Example 3.3. For the hyperbolic space with the sectional curvature $K=-a$, $\varphi_a(r)=\log\frac{\sinh\sqrt{a}r}{\sqrt{a}r}$. This satisfies Assumption C. Actually $\varphi_a'(r)\ge 0$ for all r. Clearly, small perturbations of $\varphi_a(r)$ satisfy the assumption. Also if φ_i $(1\le i\le n)$ satisfy the assumption, then so do the function $\sum_{i=1}^n p_i\varphi_i$ for any positive numbers $\{p_i\}$ with $\sum_{i=1}^n p_i=1$.

The function f satisfies the Jacobi equation $f''(r)+K(r)f(r)=0$, where K is the radial curvature function. It is natural to put the assumptions on K instead of f. In fact, it is proved in [51] that necessary all estimates for the validity of our second main theorem (Theorem 3.6) hold true under some assumptions on K. Further related work is in progress.

The quantity $r\varphi'(r)$ is related to the second derivative of the squared distance function as in the following lemma ([32, 5]).

Lemma 3.4. *For $r = d(y_0, z)$, we have*

$$\nabla_z^2 \left(\frac{r^2}{2} \right) = I_{T_zM} + r\varphi'(r)P_z^{\perp}, \tag{3.13}$$

where $v_z \in T_zM$ is the element such that $\exp_z(v_z) = y_0$ and $P_z^{\perp}$ denotes the orthogonal projection onto the orthogonal complement of the 1 dimensional subspace spanned by $v_z \in T_zM$.

By this lemma, we see that Assumption C (3) implies the condition (2) in Theorem 3.2 with $l = +\infty$.

Lemma 3.5. *Suppose f satisfies Assumption* C (1), (2) *and* $\inf_{r>1} f(r) > 0$. *Then Assumption* A *and Assumption* B *hold.*

Proof. By Lemma 1.21 in [16] (see also Proposition 9.106 in [10]), it is easy to see the boundedness of the Ricci curvature under the Assumption C (1), (2). To prove the Gaussian upper bound in Assumption A (1), it suffices to prove that there exists $C > 0$ such that $\inf_{x\in M} \mathrm{vol}(B_l(x)) \geq Cl^n$ for small $l > 0$ because the Ricci curvature is bounded. Also under the assumption $\|\varphi'\|_\infty < \infty$, we obtain there exist positive constants $C(\varepsilon, R)$ and $c(\varepsilon, R)$ for any $\varepsilon > 0$ and $R > 0$ such that

$$c(\varepsilon, R) \leq \frac{f(r')}{f(r)} \leq C(\varepsilon, R) \qquad \text{for any } r, r' \text{ with } r, r' \geq R, |r - r'| \leq \varepsilon \tag{3.14}$$

and $\lim_{\varepsilon\to 0} c(\varepsilon, R) = \lim_{\varepsilon\to 0} C(\varepsilon, R) = 1$. By using this and $\inf_{r\geq 1} f(r) > 0$, it is not difficult to show the uniform lower boundedness of the volume by this estimate. Assumption B follows from the estimate of $\nabla_z \log p(t, y_0, z)$ in (3.19). □

We note that $\inf_{r>0} r\varphi'(r) > -1$ implies $\inf_{r>1} f(r) > 0$.

The following is our second main theorem. We prove the positivity of e_2^λ in more general setting in Theorem 6.2.

Theorem 3.6. *Let M be a rotationally symmetric Riemannian manifold with a pole y_0. Suppose f in* (3.12) *satisfies Assumption* C. *Then $e_2^\lambda > 0$ holds for all $\lambda > 0$ and*

$$\lim_{\lambda\to\infty} \frac{e_2^\lambda}{\lambda} = \sigma_1, \tag{3.15}$$

where σ_1 is the same number as in Theorem 3.2.

We make remarks on Theorem 3.2 and Theorem 3.6.

Remark 3.7. (1) It is not clear whether the same result as in Theorem 3.6 holds or not for $P_{x_0,y}(M)$ ($y \neq y_0$) under Assumption C. It is more interesting to study non-rotationally general cases.

(2) By checking the proof, the same results as in Theorem 3.6 hold if the following are satisfied,

(i) $d(x_0, y_0)$ is smaller than l which satisfies Theorem 3.2 (1) and (2),

(ii) the log-Sobolev inequality (3.34) holds,

(iii) the tail estimate (7.1) holds.

(3) If the sectional curvature along the geodesic c_{x_0,y_0} is positive, then $\inf\sigma(D_0^2E(c_{x_0,y_0}))<1$ and the bottom of the spectrum is an eigenvalue of $D_0^2E(c_{x_0,y_0})$ and is not an essential spectrum. While the curvature is strictly negative, $\inf\sigma(D_0^2E(c_{x_0,y_0}))=1$ and 1 is not an eigenvalue and belongs to essential spectrum. This suggests that the second lowest eigenvalue, or more generally, some low-lying spectrum of the OU operator (with Dirichlet boundary condition) on $\mathcal{D}$ or $P_{x_0,y_0}(M)$ over a positively curved manifold belongs to the discrete spectrum, while the second lowest eigenvalue is embedded in the essential spectrum in the case of negatively curved manifolds. In fact, in the proof of upper bound in the main theorems, we use "approximate second eigenfunctions" which are defined by the eigenfunction which achieves the value $\inf\sigma(D_0^2E(c_{x_0,y_0}))$ approximately. If some isometry group acts on M with the fixed points x_0 and y_0, we may expect the discrete spectrum have some multiplicities. We show these kind of results in the case where M is a compact Lie group in a forthcoming paper.

As mentioned in the Introduction, the spectral gap e_2^λ for $P_{x_0}(M)$ is defined similarly and $e_2^\lambda>0$ for all λ. This is due to Fang. He established a COH formula and proved the existence of the spectral gap in the case where M is compact and $\lambda=1$. However, it is obvious that the same result holds true on a complete Riemannian manifold with bounded Ricci curvature for all $\lambda>0$. See also [11, 30, 4, 2]. The variant of the COH formula in the loop space case is important in our case also. To explain the COH formula, we need some preparations. Let $\mathfrak{F}_t=\sigma\left(\gamma(s),0\le s\le t\right)\vee\mathcal{N}$, where $\mathcal{N}$ is the set of all null sets with respect to $\nu_{x_0}^\lambda$. Then $b(t)=\int_0^t u_0^{-1}\tau(\gamma)_s^{-1}\circ d\gamma(s)$ is an $\mathfrak{F}_t$-Brownian motion with the covariance $E^{\nu_{x_0}^\lambda}[(b(t),u)(b(s),v)]=(u,v)\frac{t\wedge s}{\lambda}$ $(u,v\in\mathbb{R}^n)$ on $\mathbb{R}^n$ under $\nu_{x_0}^\lambda$. We simply say $b(t)$ is a Brownian motion with variance $1/\lambda$ in this paper. We recall the notion of the trivialization. Let $T\in\Gamma(TM\otimes T^*M))$ be a $(1,1)$-tensor on M, that is, T is a linear transformation on each tangent space. We write

$$\overline{T(\gamma)_t}=u_0^{-1}\tau(\gamma)_t^{-1}T(\gamma(t))\tau(\gamma)_tu_0\in L(\mathbb{R}^n,\mathbb{R}^n). \tag{3.16}$$

The definition for general $T\in\Gamma\left((\otimes^pTM)\otimes(\otimes^qT^*M)\right)$ is similar.

We now state COH formula on $P_{x_0}(M)$. Below, we use the notation $\mathrm{L}^2:=L^2([0,1]\to\mathbb{R}^n,dt)$.

Lemma 3.8. *Assume $\|\mathrm{Ric}\|_\infty<\infty$. Let $F\in H^1(P_{x_0}(M),\nu_{x_0}^\lambda)$. Then*

$$F(\gamma)-E^{\nu_{x_0}^\lambda}[F]=\int_0^1\left(E\left[\left\{\left((I+R_{0,\lambda}(\gamma))^{-1}\right)^*(DF)(\gamma)'\right\}_t\mid\mathfrak{F}_t\right],db(t)\right), \tag{3.17}$$

where $\left(R_{0,\lambda}(\gamma)\varphi\right)(t)=\frac{1}{2\lambda}\overline{\mathrm{Ric}(\gamma)_t}\int_0^t\varphi(s)ds$, $*$ indicates the adjoint operator on L^2 and $DF(\gamma)'_t=\frac{d}{dt}DF(\gamma)_t$. Also I denotes the identity operator on L^2.

The second derivative of $\log p(t,x,y)$ is related to the COH formula on $P_{x_0,y_0}(M)$. Under Assumption C, we have a good estimate on the first and second derivatives of $\log p(t,y_0,z)$ with respect to z. Similar estimates of the heat kernel hold in a compact set outside cut-locus when M is a compact Riemannian manifold. This is studied by Stroock [54], Malliavin and Stroock [49] and Gong-Ma [30]. Their results clearly can be extended to non-compact $\mathbb{R}^n$ with a nice Riemannian metric which coincides with the Euclidean metric outside a bounded set. The estimates are as follows.

Assumption D. *For any compact subset $F\subset\mathrm{Cut}(y_0)^c$ and $0<t\le1$ there exists $C_F>0$ such that*

$$\sup_{z\in F}\left|t\nabla_z^2\log p(t,y_0,z)+\nabla_z^2\left(\frac{1}{2}d(y_0,z)^2\right)\right|\le C_Ft^{1/2}. \tag{3.18}$$

The following (1) and (2) can be found in [5] and [30] respectively.

Lemma 3.9. (1) *Let M be a compact Riemannian manifold or $\mathbb{R}^n$ with a Riemannian metric which coincides with the Euclidean metric outside a bounded set. Then Assumption* D *is satisfied.*
(2) *Suppose Assumption* C (1) *and* (2)*. Then Assumption* D *is satisfied. Actually the following stronger inequalities are valid:*

Let $T > 0$. There exist positive constants C_1, C_2 which may depend on T such that for all $0 < t \le T$,

$$\sup_{z\in M} |t\nabla_z \log p(t, y_0, z) - v_z| \le C_1 t, \tag{3.19}$$

$$\sup_{z\in M} \left|t\nabla_z^2 \log p(t, y_0, z) + I_{T_zM} + d(y_0, z)\varphi'(d(y_0, z))P_z^{\perp}\right| \le C_2 t, \tag{3.20}$$

where v_z and $P_z^{\perp}$ are defined in Lemma 3.4*.*

The important point in the estimate (3.20) is that the norm of the second derivative of $t \log p(t, y_0, z)$ is bounded from above by a linear function of $d(y_0, z)$. Probably, the estimates (3.19) and (3.20) hold under weaker assumptions on φ. It is natural and interesting to study non-rotationally symmetric general cases.

Our Dirichlet Laplacian is defined on the set of paths which are restricted in the small ball. Therefore, even if we vary the Riemannian metric outside the ball, the spectral property of the operator would not change. We explain this reasoning more precisely. Let (M, g) and (M', g') be Riemannian manifolds satisfying Assumption B. Let $y_0 \in M, y_0' \in M'$ and $B_l(y_0) \subset M, B_l(y_0') \subset M'$ be open metric balls. Let $x_0 \in B_l(y_0)$. Let $l_* > l$. Assume that l_* is smaller than the injectivity radius at y_0. We assume that there exists a Riemannian isometry $\Phi : B_{l_*}(y_0) \to B_{l_*}(y_0')$. Then $\Phi(B_l(y_0)) = B_l(y_0')$. Let $x_0' = \Phi(x_0)$. Let $\nu^{\lambda}_{M,x_0,y_0}$ and $\nu^{\lambda}_{M',x_0',y_0'}$ denote the pinned measures on each manifold. We write

$$\mathcal{D} = \{\gamma \in P_{x_0,y_0}(M) \mid \gamma(t) \in B_l(y_0) \text{ for all } 0 \le t \le 1\},$$
$$\mathcal{D}' = \{\gamma \in P_{x_0',y_0'}(M') \mid \gamma(t) \in B_l(y_0') \text{ for all } 0 \le t \le 1\}.$$

Let $A \subset \mathcal{D}$ be a Borel measurable subset. Define $\Phi : \mathcal{D} \to \mathcal{D}'$ by $\Phi(\gamma)(t) = \Phi(\gamma(t))$. Let $p^M(t, x, y)$ and $p^{M'}(t, x', y')$ denote the heat kernels on M and M'. Note that $p^M(t, x, y) \ne p^{M'}(t, \Phi(x), \Phi(y))$ $x, y \in B_{l_*}(y_0)$ generally. However, by the uniqueness of the solution of stochastic differential equations, we have

$$\frac{\nu^{\lambda}_{M,x_0,y_0}(A)}{\nu^{\lambda}_{M,x_0,y_0}(\mathcal{D})} = \frac{\nu^{\lambda}_{M',x_0',y_0'}(\Phi(A))}{\nu^{\lambda}_{M',x_0',y_0'}(\mathcal{D}')}. \tag{3.21}$$

By this, for any bounded Borel measurable function F on $\mathcal{D}'$,

$$\int_{\mathcal{D}} F(\Phi(\gamma)) \frac{d\nu^{\lambda}_{M,x_0,y_0}(\gamma)}{\nu^{\lambda}_{M,x_0,y_0}(\mathcal{D})} = \int_{\mathcal{D}'} F(\gamma) \frac{d\nu^{\lambda}_{M',x_0',y_0'}(\gamma)}{\nu^{\lambda}_{M',x_0',y_0'}(\mathcal{D}')}. \tag{3.22}$$

Let $F \in H^{1,2}(P_{x_0',y_0'}(M'))$. If $F \in H^{1,2}_0(\mathcal{D}', \nu^{\lambda}_{x_0',y_0'})$, then

$$\tilde{F}(\gamma) := F(\Phi(\gamma))\, \chi\left(\sup_{0\le t\le 1} d'(\Phi(\gamma)(t), y_0')\right) \in H^{1,2}_0(\mathcal{D}, \nu^{\lambda}_{x_0,y_0}),$$

where $\chi = \chi(t)$ is a non-negative smooth function such that $\chi(t) = 1$ for $t \le \frac{l+l_*}{2}$ and $\chi(t) = 0$ for $t \ge \frac{l+2l_*}{3}$. Moreover $\|D_0 F\|_{L^2(\nu^\lambda_{M',x_0',y_0'}/\nu^\lambda_{M',x_0',y_0'}(\mathcal{D}'))} = \|D_0 \tilde{F}\|_{L^2(\nu^\lambda_{M,x_0,y_0}/\nu^\lambda_{M,x_0,y_0}(\mathcal{D}))}$. To prove these results, we need $\sup_{0\le t\le 1} d(\gamma(t), \Phi^{-1}(y_0')) \in H^{1,2}(P_{x_0,y_0}(M))$ which can be found in Lemma 2.2 and Remark 2.4 in [3].

The above argument implies that

$$e^\lambda_{Dir,2,\mathcal{D}} = e^\lambda_{Dir,2,\mathcal{D}'}.$$

Hence, in the proof of Theorem 3.2, we may assume that M is diffeomorphic to $\mathbb{R}^n$ and the Riemannian metric is flat outside a certain bounded subset and Assumption D is satisfied. The key ingredient of the proof of Theorem 3.2 is a version of the COH formula in [7] which can be extended to the above non-compact $\mathbb{R}^n$ case with a nice Riemannian metric. Since the COH formula is strongly related to the heat kernel $p(t,x,y)$ on M itself, the above observation is important. We need some preparation to explain COH formula on $P_{x_0,y_0}(M)$. Let $V^\lambda_{y_0}(t,z) = \mathrm{grad}_z \log p\left(\frac{1-t}{\lambda}, y_0, z\right)$ $(0 \le t < 1)$. We write

$$\overline{V^\lambda_{y_0}(t,\gamma)}_t = u_0^{-1}\tau(\gamma)_t^{-1} V^\lambda_{y_0}(t,\gamma(t)) \in \mathbb{R}^n.$$

Also $\overline{\nabla V^\lambda_{y_0}(t,\gamma)}_t$ denotes an $n \times n$ matrix. More explicitly,

$$\overline{\nabla V^\lambda_{y_0}(t,\gamma)}_t = u_0^{-1}\tau(\gamma)_t^{-1}\nabla_z \mathrm{grad}_z \log p\left(\frac{1-t}{\lambda}, y_0, z\right)\Bigg|_{z=\gamma(t)} \tau(\gamma)_t u_0. \tag{3.23}$$

Let $w(t) = b(t) - \frac{1}{\lambda}\int_0^t \overline{V^\lambda_{y_0}(s,\gamma)}_s ds$. This process is defined for $t < 1$ and it is not difficult to check that this can be extended continuously up to $t = 1$. Let $\mathcal{N}^{x_0,y_0,t}$ be the set of all null sets of $\nu^\lambda_{x_0,y_0}|_{\mathfrak{F}_t}$ and set $\mathfrak{G}_t = \mathfrak{F}_t \vee \mathcal{N}^{x_0,y_0,1}$. Then w is an $\mathfrak{G}_t$-adapted Brownian motion for $0 \le t \le 1$ such that $E^{\nu^\lambda_{x_0,y_0}}[(w(t),u)\,(w(s),v)] = \frac{t\wedge s}{\lambda}(u,v)$ for any $u, v \in \mathbb{R}^n$. Let

$$K(\gamma)_{\lambda,t} = -\frac{1}{2\lambda}\overline{\mathrm{Ric}(\gamma)}_t + \frac{1}{\lambda}\overline{\nabla V^\lambda_{y_0}(t,\gamma)}_t. \tag{3.24}$$

Let $M(\gamma)_{\lambda,t}$ be the linear mapping on $\mathbb{R}^n$ satisfying the differential equation:

$$M(\gamma)'_{\lambda,t} = K(\gamma)_{\lambda,t} M(\gamma)_{\lambda,t} \quad 0 \le t < 1, \tag{3.25}$$

$$M(\gamma)_{\lambda,0} = I. \tag{3.26}$$

Using M and K, we define for a bounded measurable function φ with $\mathrm{supp}\,\varphi \subset [0,1)$,

$$J(\gamma)_\lambda \varphi(t) = (M(\gamma)^*_{\lambda,t})^{-1}\int_t^1 M(\gamma)^*_{\lambda,s} K(\gamma)_{\lambda,s}\varphi(s)ds. \tag{3.27}$$

Also let

$$A(\gamma)_\lambda = I + J(\gamma)_\lambda. \tag{3.28}$$

The operator $\left((I + R_{0,\lambda}(\gamma))^{-1}\right)^*$ in the COH formula in Lemma 3.8 coincides with $A(\gamma)_\lambda$ which is obtained by setting $K(\gamma)_\lambda = -\frac{1}{2\lambda}\overline{\mathrm{Ric}(\gamma)_t}$ in the above.

We are ready to state our COH formula for functions on $P_{x_0,y_0}(M)$ and its immediate consequences.

Lemma 3.10. (1) *Assume M is diffeomorphic to $\mathbb{R}^n$ and the Riemannian metric is flat outside a bounded subset. Let $0 < l < \infty$. Suppose $\mathcal{D}(= \mathcal{D}_l)$ satisfies conditions* (1), (2) *in Theorem* 3.2. *Let $F \in H_0^{1,2}(\mathcal{D})$.*

(i) *It holds that $D_0F(\gamma) = 0$ for $\nu^\lambda_{x_0,y_0}$-almost all $\gamma \in \mathcal{D}^c$.*

(ii) *There exists $\lambda_* > 0$ such that $A(\gamma)_\lambda$ can be extended to a bounded linear operator on L^2 for each γ for all $\lambda \geq \lambda_*$. Let $a(\lambda) = \operatorname{esssup}\left\{\|A(\gamma)_\lambda\|^2_{op} \mid \gamma \in \mathcal{D}\right\}$. Here $\|\cdot\|_{op}$ denotes the operator norm. $\sup_{\lambda \geq \lambda_*} a(\lambda) < \infty$ holds and for $\lambda \geq \lambda_*$, the following COH formula holds:*

$$E^{\nu^\lambda_{x_0,y_0}}[F|\mathfrak{G}_t] = E^{\nu^\lambda_{x_0,y_0}}[F] + \int_0^t (H(s,\gamma), dw(s)), \quad 0 \leq t \leq 1, \tag{3.29}$$

where

$$H(s,\gamma) = E^{\nu^\lambda_{x_0,y_0}}\left[A(\gamma)_\lambda(D_0F(\gamma)')(s)|\mathfrak{G}_s\right]. \tag{3.30}$$

and $D_0F(\gamma)'_t = \frac{d}{dt}(D_0F)(\gamma)_t$. Moreover the following inequalities hold for $\lambda \geq \lambda_*$.

$$E^{\nu^\lambda_{x_0,y_0}}\left[F^2 \log\left(F^2/\|F\|^2_{L^2(\nu^\lambda_{x_0,y_0})}\right)\right] \leq \frac{2a(\lambda)^2}{\lambda} E^{\nu^\lambda_{x_0,y_0}}\left[|D_0F|^2\right], \tag{3.31}$$

$$\lambda E^{\nu^\lambda_{x_0,y_0}}\left[\left(F - E^{\nu^\lambda_{x_0,y_0}}[F]\right)^2\right] \leq E^{\nu^\lambda_{x_0,y_0}}\left[|A(\gamma)_\lambda D_0F|^2\right]. \tag{3.32}$$

(2) *Assume M is a rotationally symmetric Riemannian manifold with a pole y_0. Suppose Assumption* C.

(i) *The operator $A(\gamma)_\lambda$ can be extended to a bounded linear operator on L^2 for each γ for all $\lambda > 0$. Moreover for each $\lambda_0 > 0$, there exists a positive constant C_0 which depends only on φ and λ_0 such that for all $\lambda \geq \lambda_0$,*

$$\|A(\gamma)_\lambda\|_{op} \leq C_0\rho_{y_0}(\gamma) \qquad \text{for any } \gamma, \tag{3.33}$$

where $\rho_{y_0}(\gamma) = 1 + \max_{0\leq t\leq 1} d(y_0, \gamma(t))$.

(ii) *For $F \in H^{1,2}(P_{x_0,y_0}(M), \nu^\lambda_{x_0,y_0})$, the COH formula* (3.29), (3.30) *hold.*

(iii) *For each $\lambda_0 > 0$, there exists a positive constant C_1 which depends only on φ and λ_0 such that for any $\lambda \geq \lambda_0$ and $F \in \mathcal{FC}^\infty_b(P_{x_0,y_0}(M))$,*

$$\begin{aligned}&\int_{P_{x_0,y_0}(M)} F^2(\gamma) \log\left(F(\gamma)^2/\|F\|^2_{L^2(\nu^\lambda_{x_0,y_0})}\right) d\nu^\lambda_{x_0,y_0}(\gamma)\\ &\leq \int_{P_{x_0,y_0}(M)} \frac{C_1}{\lambda}\rho_{y_0}(\gamma)^2|D_0F(\gamma)|^2 d\nu^\lambda_{x_0,y_0}(\gamma).\end{aligned} \tag{3.34}$$

Proof. The proof of (1) is similar to that in [7]. (2) follows from Lemma 3.2 and Theorem 3.3 in [3] and Lemma 2.3 in [7]. In the present case, we have

$$K(\gamma)_{\lambda,t} = \frac{1}{1-t}(-\alpha + C_1(t)) + C_2(t), \tag{3.35}$$

where

$$\alpha = \min\left(1, 1 + \inf_{r>0} r\varphi'(r)\right) > 1/2,$$
$$C_1(t) = \left\{\alpha - 1 - d(y_0, \gamma(t))\varphi'\Big(d(y_0, \gamma(t))\Big)\right\} \overline{P^{\perp}(\gamma)}_{\gamma(t)} + (\alpha - 1)\, \overline{P(\gamma)}_{\gamma(t)},$$
$$|C_2(t)| \leq \frac{C}{\lambda}$$

and C is a positive constant. The case $\lambda = 1$ is considered in [3] and the estimate in the hyperbolic space case with general λ can be found in Remark 2.4 in [7]. The proof of general cases are similar to them. □

Under the assumption in the lemma above, $A(\gamma)_\lambda$ is a bounded linear operator on L^2 for $\nu^{\lambda}_{x_0,y_0}$ almost all γ. However, we cannot expect the usual continuity property of the mapping $\gamma \mapsto A(\gamma)_\lambda$ because they are defined by using Itô's stochastic integrals. The inequality (3.32) implies that $\liminf_{\lambda\to\infty} \frac{e^{\lambda}_{Dir,2,\mathcal{D}}}{\lambda} > 0$. On the other hand, we cannot conclude $e^{\lambda}_2 > 0$ for $P_{x_0,y_0}(M)$ by the log-Sobolev inequality (3.34) because the operator norm $A(\gamma)_\lambda$ is not uniformly bounded.

As mentioned in the Introduction, the same result as in Theorem 3.6 holds for $P_{x_0}(M)$. We prove it as a warm up before proving our main theorems. For simplicity, we assume M is compact. After the proof, we explain different points of the proof in the loop space case.

Theorem 3.11. *Let M be a compact Riemannian manifold. Let e^{λ}_2 be the spectral gap of the Dirichlet form $\mathcal{E}^{\lambda}$ on $P_{x_0}(M)$ with $\nu^{\lambda}_{x_0}$. Then $e^{\lambda}_2 > 0$ for all $\lambda > 0$ and*

$$\lim_{\lambda\to\infty} \frac{e^{\lambda}_2}{\lambda} = 1. \tag{3.36}$$

Proof. We use the COH formula (3.17). By using

$$\|(I + R_{0,\lambda}(\gamma))^{-1}\|_{op} \leq 1 + \frac{C}{\lambda} \quad \text{for any } \lambda \geq \lambda_0 > 0, \tag{3.37}$$

we get

$$E^{\nu^{\lambda}_{x_0}}\left[(F - E^{\nu^{\lambda}_{x_0}}[F])^2\right] \leq \frac{1}{\lambda}\left(1 + \frac{C}{\lambda}\right)^2 E\left[|DF(\gamma)|^2\right]. \tag{3.38}$$

Here C depends on λ_0. Since $e^{\lambda}_1 = 0$ and the corresponding eigenfunction is a constant function, we have $e^{\lambda}_2 \geq \lambda(1 + \frac{C}{\lambda})^{-2}$ which proves that $\liminf_{\lambda\to\infty} \frac{e^{\lambda}_2}{\lambda} \geq 1$. We prove converse estimate. To this end, we consider a candidate of approximate second (generalized) eigenfunction. Let $\varphi \in \mathrm{L}^2$ and assume $\|\varphi\|_{\mathrm{L}^2} = 1$. Let $F(\gamma) = \sqrt{\lambda}\int_0^1 (\varphi(t), db(t))$. Then $E^{\nu^{\lambda}_{x_0}}[F] = 0$ and $E^{\nu^{\lambda}_{x_0}}[F^2] = 1$. We have $F \in \mathrm{D}(\mathcal{E})$ and

$$D_h \int_0^1 (\varphi(t), db(t)) = \int_0^1 \big(\varphi(t), h'(t)\big)\, dt + \int_0^1 \left\langle \varphi(t), \int_0^t \Big(\overline{R(\gamma)}_s(h(s), \circ db(s))(\circ db(t))\Big) \right\rangle,$$

where $\overline{R(\gamma)}_t$ is the trivialization of the Riemannian curvature tensor and $\langle\cdot,\cdot\rangle$ denotes the inner product in $\mathbb{R}^n$. The readers are referred to [17, 1] for this formula. See [18, 42] also.

Since

$$
\begin{aligned}
&\int_0^1 \Big\langle \varphi(t), \int_0^t \Big(\overline{R(\gamma)}_s(h(s), \circ db(s))(\circ db(t))\Big) \Big\rangle \\
&= \int_0^1 \varphi^j(t) \Big\langle \varepsilon_j, \int_0^t \Big(\overline{R(\gamma)}_s(h(s), \circ db(s))(\circ db(t))\Big) \Big\rangle \\
&= \int_0^1 \int_0^t \Big\langle \overline{R(\gamma)}_s(h(s), \circ db(s))(\varepsilon_j), \circ d \int_t^1 \varphi^j(s) db(s) \Big\rangle \\
&= \int_0^1 \Big\langle \overline{R(\gamma)}_t(\circ db(t), h(t))(\varepsilon_j), \int_t^1 \varphi^j(s) db(s) \Big\rangle \\
&= \int_0^1 \Big\langle \overline{R(\gamma)}_t(\varepsilon_j, \int_t^1 \varphi^j(s) db(s))(\circ db(t)), h(t) \Big\rangle \\
&= \int_0^1 \Big\langle \overline{R(\gamma)}_t(\int_t^1 \varphi(s) db^i(s), \varepsilon_i)(\circ db(t)), h(t) \Big\rangle \\
&= \int_0^1 \Big\langle \int_t^1 \overline{R(\gamma)}_u \left(\int_u^1 \varphi(s) db^i(s), \varepsilon_i \right) (\circ db(u)), h'(t) \Big\rangle dt, \qquad (3.39)
\end{aligned}
$$

we obtain

$$
\begin{aligned}
DF(\gamma)'_t &= \sqrt{\lambda}\varphi(t) + \sqrt{\lambda} \int_t^1 \overline{R(\gamma)}_u \left(\int_u^1 \varphi(s) db^i(s), \varepsilon_i \right) (\circ db(u)) \\
&= \sqrt{\lambda}\varphi(t) + \sqrt{\lambda} \int_t^1 \overline{R(\gamma)}_u (\varepsilon_j, \varepsilon_i) (\circ db(u)) \int_0^1 \varphi^j(s) db^i(s) \\
&\quad - \sqrt{\lambda} \int_t^1 \overline{R(\gamma)}_u \left(\int_0^u \varphi(s) db^i(s), \varepsilon_i \right) (\circ db(u)). \qquad (3.40)
\end{aligned}
$$

By a standard calculation, we have

$$
\int_0^1 E\left[|DF(\gamma)'|_t^2\right] dt \le \lambda + C. \qquad (3.41)
$$

This implies (3.36). □

As in the proof above, the COH formula and the estimate $\lim_{\lambda\to\infty} \|I + R_{0,\lambda}(\gamma)\|_{op} = 1$ immediately implies the lower bound of the limit. In the loop space case, $A(\gamma)_\lambda$ is not uniformly bounded in γ and the existence of the spectral gap is not obvious. This difficulty can be solved by using the log-Sobolev inequality (3.34). In order to obtain precise asymptotics of the spectral gap, we need continuity theorem in rough path analysis. For this purpose, we need to consider the operator $A(c_{x_0,y_0})_\infty = \lim_{\lambda\to\infty} A(c_{x_0,y_0})_\lambda$. In the next section, we study some relations between $A(c_{x_0,y_0})_\infty$ and the Hessian of the energy function E at c_{x_0,y_0}.

4 Square root of Hessian of the energy function and Jacobi fields

In this section, we assume $d(x_0, y_0)$ is smaller than the injectivity radius at y_0. We begin by determining $\lim_{\lambda\to\infty} K(c_{x_0,y_0})_{\lambda,t}$. By using (3.18), we have

$$\begin{aligned}\lim_{\lambda\to\infty} K(c_{x_0,y_0})_{\lambda,t} &= -\lim_{\lambda\to\infty}\frac{1}{2\lambda}\overline{R(c_{x_0,y_0})_t} + \lim_{\lambda\to\infty}\frac{1}{\lambda}\overline{V_{y_0}^{\lambda}(t, c_{x_0,y_0})_t} \\ &= \frac{1}{1-t}\lim_{\lambda\to\infty}\frac{1-t}{\lambda}\overline{V_{y_0}^{\lambda}(t, c_{x_0,y_0})_t} \\ &= -\frac{1}{1-t}\overline{\nabla^2 k(c_{x_0,y_0})_t}. \end{aligned} \tag{4.1}$$

We write

$$K(t) = -\frac{1}{1-t}\overline{\nabla^2 k(c_{x_0,y_0})_t}. \tag{4.2}$$

It is natural to conjecture that $A(c_{x_0,y_0})_\infty$ is equal to the operator in L^2 given by

$$\varphi(t) \mapsto \varphi(t) + M(t)^* \int_t^1 M(s)^* K(s)\varphi(s)ds, \tag{4.3}$$

where $M(t)$ is the solution to

$$M(t)' = K(t)M(t) \qquad 0 \le t < 1, \tag{4.4}$$

$$M(0) = I. \tag{4.5}$$

In fact, this is true and we prove it later in more general form in Lemma 5.2. We study the relation between the operator of (4.3) and $D^2E(c_{x_0,y_0})$. First, recall that we fix an frame $u_0 \in O(M)$ at x_0. Let us choose $\xi \in \mathbb{R}^n$ so that $\exp_{x_0}(tu_0(\xi)) = c_{x_0,y_0}(t)$ $(0 \le t \le 1)$, where $\exp_{x_0}$ stands for the exponential mapping at x_0. Clearly it holds that $d(x_0, y_0) = |\xi|$. Let $c_{y_0,x_0}(t) = c_{x_0,y_0}(1-t)$ denote the reverse geodesic path from y_0 to x_0. In order to obtain the explicit expression of the Hessian of $k(z)$ $(z \in c_{x_0,y_0})$, we recall the notion of Jacobi fields.

Let R be the curvature tensor and define $R(t) = \overline{R(c_{x_0,y_0})_t}(\cdot, \xi)(\xi)$ which is a linear mapping on $\mathbb{R}^n$. Also we define $R^{\leftarrow}(t) = R(1-t)$. Let $v \in \mathbb{R}^n$ and $W(t, v)$ be the solution to the following ODE:

$$W''(t, v) + R^{\leftarrow}(t)W(t, v) = 0 \quad 0 \le t \le 1, \quad W(0, v) = 0, \; W'(0, v) = v. \tag{4.6}$$

Since $t \mapsto W(t, v)$ is linear, let $W(t)$ denote the corresponding $n \times n$ matrix. Needless to say, $W(0) = 0, W'(0) = I$. Since $\mathrm{Cut}(y_0) \cap \{c_{y_0,x_0}(t) \mid 0 \le t \le 1\} = \emptyset$, $W(t)$ is an invertible linear mapping for all $0 < t \le 1$ and $\tilde{W}(t, v) = W(t)W(1)^{-1}v$ is the solution to

$$\tilde{W}''(t, v) + R^{\leftarrow}(t)\tilde{W}(t, v) = 0, \quad \tilde{W}(0, v) = 0, \; \tilde{W}(1, v) = v$$

and $(\nabla^2 k(c_{y_0,x_0}(1))(u_0 v, u_0 v) = (\tilde{W}'(1, v), \tilde{W}(1, v)) = (W'(1)W(1)^{-1}v, v)$. This result can be found in many standard books in differential geometry, *e.g.* [40]. Let $0 < T \le 1$. We can obtain an explicit form of the Jacobi field along $c_{y_0,x_0}(t)$ $(0 \le t \le T)$ with given terminal value at T using W. Let $\tilde{W}_T(t, v) = W(Tt)W(T)^{-1}v$. Then $\tilde{W}_T(t, v)$ $0 \le t \le 1$ satisfies the Jacobi equation

$$\tilde{W}_T''(t, v) + R^{\leftarrow}(tT)T^2\tilde{W}_T(t, v) = 0, \quad \tilde{W}_T(0, v) = 0, \; \tilde{W}_T(1, v) = v. \tag{4.7}$$

Hence $\nabla^2 k(c_{y_0,x_0}(t))\left(\tau(c_{x_0,y_0})_{1-t}u_0 v, \tau(c_{x_0,y_0})_{1-t}u_0 v\right) = t\left(W'(t)W(t)^{-1}v, v\right)$.

Next we prove that $A(t) := tW'(t)W(t)^{-1}$ is a symmetric matrix for $0 < t \le 1$. This can be checked by the following argument. Note that $W(t) = tI + \int_0^t\int_0^s\int_0^r W'''(u)du$. This follows from the equation of W. By this observation, if we extend $A = A(t)$ by setting $A(0) = I$, then $A(t)$ is continuously differentiable on $[0,1]$ and $A'(0) = 0$. We have

$$\begin{aligned} A'(t) &= W'(t)W(t)^{-1} + tW''(t)W(t)^{-1} - tW'(t)W(t)^{-1}W'(t)W(t)^{-1} \\ &= -tR^{\leftarrow}(t) - \frac{A(t)^2}{t} + \frac{A(t)}{t}. \end{aligned} \tag{4.8}$$

Let $B(t) = A(t) - A(t)^*$, where $A(t)^*$ denotes the transposed matrix. Since $R^{\leftarrow}(t)$ is a symmetric matrix, (4.8) implies

$$B(t) = \frac{1}{t}\int_0^t (I - A(s)^*)B(s)ds + \frac{1}{t}\int_0^t B(s)(I - A(s))ds, \qquad 0 < t \le 1. \tag{4.9}$$

Noting

$$\begin{aligned} &\frac{1}{t}\int_0^t (I - A(s)^*)B(s)ds \\ &= \frac{I - A(t)^*}{t}\int_0^t B(s)ds + \frac{1}{t}\int_0^t (A(s)^*)'\left(\int_0^s B(r)dr\right)ds \end{aligned} \tag{4.10}$$

and using Gronwall's inequality, we obtain $B(t) = 0$ for all t which implies the desired result.

Let $f(t) = W(1-t)$. Then f satisfies

$$f''(t) + R(t)f(t) = 0, \qquad 0 \le t \le 1, \qquad f(1) = 0, \quad f'(1) = -I. \tag{4.11}$$

Since $f'(t)f(t)^{-1}$ is a symmetric matrix, we have the following key relations:

$$\overline{\nabla^2 k(c_{x_0,y_0})}_t = -(1-t)f'(t)f(t)^{-1} \tag{4.12}$$

$$K(t) = -\frac{1}{1-t}\overline{\nabla^2 k(c_{x_0,y_0})}_t = f'(t)f(t)^{-1}. \tag{4.13}$$

Let

$$\tilde{K}(t) = K(t) + \frac{1}{1-t}. \tag{4.14}$$

Since $\tilde{K}(t) = \frac{I - A(1-t)}{1-t}$, we see that $\tilde{K}(t)$ $(0 \le t \le 1)$ is a matrix-valued continuous mapping. Let $N(t)$ be the solution to

$$N'(t) = \tilde{K}(t)N(t), \ N(0) = I.$$

Then $\sup_t(\|N(t)\|_{op} + \|N^{-1}(t)\|_{op}) < \infty$ and $M(t) = (1-t)N(t)$, where $M(t)$ is the solution to (4.4). Also we have $M(t) = f(t)f(0)^{-1}$.

We write $\mathrm{L}^2_0 = \{\varphi \in \mathrm{L}^2 \mid \int_0^1 \varphi(t)dt = 0\}$. Then $(U\varphi)(t) = \int_0^t \varphi(s)ds$ is a bijective linear isometry from L^2_0 to H_0. Also $U^{-1}h(t) = \dot{h}(t)$. Let us introduce an operator

$$(S\varphi)(t) = \varphi(t) - f'(t)f(t)^{-1}\int_0^t \varphi(s)ds, \tag{4.15}$$

$$\mathrm{D}(S) = \mathrm{L}^2_0. \tag{4.16}$$

By Hardy's inequality,

$$\int_0^1 \left| \frac{1}{1-t} \int_t^1 \varphi(s)ds \right|^2 dt \leq 4 \int_0^1 |\varphi(s)|^2 ds \qquad \text{for any } \varphi \in \mathrm{L}^2, \tag{4.17}$$

we see that S is a bounded linear operator from L_0^2 to L^2. The following lemma shows that S is a square root of the Hessian of the energy function E. This relation is key to identify the limit of $e^{\lambda}_{Dir,2,\mathcal{D}}$.

Lemma 4.1. *Let T be the bounded linear operator on L_0^2 such that*

$$(T\varphi)(t) = -\int_t^1 R(s) \left(\int_0^s \varphi(u)du \right) ds + \int_0^1 \left(\int_t^1 R(s) \Big(\int_0^s \varphi(u)du \Big) ds \right) dt. \tag{4.18}$$

Then T is a symmetric operator and for any $\varphi \in \mathrm{L}_0^2$,

$$\|S\varphi\|^2 = ((I+T)\varphi, \varphi), \tag{4.19}$$

where I denotes the identity operator on L_0^2. Moreover,

$$(D_0^2 E)(c_{x_0,y_0}) = U\,(I+T)\,U^{-1}, \tag{4.20}$$

where E is the energy function of the path (3.10)*.*

Proof. The symmetry of T follows from direct calculation. Using

$$\lim_{t\to 1} \frac{1}{1-t} \left| \int_t^1 \varphi(s)ds \right|^2 = 0, \quad f''(t) = -R(t)f(t), \quad f'(t)f(t)^{-1} \text{ is symmetric,}$$
$$(f(t)^{-1})' = -f(t)^{-1} f'(t) f(t)^{-1},$$

we have

$$\begin{aligned} \|S\varphi\|^2 &= \|\varphi\|^2 - 2\int_0^1 \left(f'(t)f(t)^{-1} \int_0^t \varphi(s)ds, \varphi(t) \right) dt \\ &\quad + \int_0^1 \left| f'(t)f(t)^{-1} \int_0^t \varphi(s)ds \right|^2 dt \\ &= \|\varphi\|^2 + \int_0^1 \left(\left(f'(t)f(t)^{-1}\right)' \int_0^t \varphi(s)ds, \int_0^t \varphi(s)ds \right) dt \\ &\quad + \int_0^1 \left| f'(t)f(t)^{-1} \int_0^t \varphi(s)ds \right|^2 dt \\ &= \|\varphi\|^2 - \int_0^1 \left(R(t) \int_0^t \varphi(s)ds, \int_0^t \varphi(s)ds \right) dt \\ &= ((I+T)\varphi, \varphi)\,. \end{aligned} \tag{4.21}$$

By the second variation formula of the energy function along geodesics ([40]), we have

$$(D_0^2 E)(c_{x_0,y_0})(U\varphi, U\varphi) = ((I+T)\varphi, \varphi)\,.$$

Thus the proof is completed. □

Let

$$(S_2\varphi)(t) = \varphi(t) + f'(t)\int_0^t f(s)^{-1}\varphi(s)ds. \tag{4.22}$$

Then again by Hardy's inequality S_2 is a bounded linear operator on L^2. Moreover, it is easy to see that $\mathrm{Image}(S_2) \subset \mathrm{L}_0^2$, $SS_2 = I_{\mathrm{L}^2}$ and $S_2S = I_{\mathrm{L}_0^2}$. Therefore, $S_2 = S^{-1}$ and $\mathrm{Image}(S) = \mathrm{L}^2$. Moreover we have $S^*S = I + T$ on L_0^2 by (4.19). Note that by identifying the dual space of a Hilbert space with the Hilbert space itself using Riesz's theorem, we view $S^* : (\mathrm{L}^2)^* \to (\mathrm{L}_0^2)^*$ as the operator from L^2 to L_0^2. We have the following explicit expression of S^{-1}, S^* and $(S^{-1})^*$.

Lemma 4.2. (1) *$S^{-1} : \mathrm{L}^2 \to \mathrm{L}_0^2$, $S^* : \mathrm{L}^2 \to \mathrm{L}_0^2$ are bijective linear maps and we have for any $\varphi \in \mathrm{L}^2$,*

$$\left(S^{-1}\varphi\right)(t) = \varphi(t) + f'(t)\int_0^t f(s)^{-1}\varphi(s)ds \tag{4.23}$$

$$\begin{aligned}(S^*\varphi)(t) &= \varphi(t) - \int_0^1 \varphi(t)dt + \int_0^t f'(s)f(s)^{-1}\varphi(s)ds \\ &\quad - \int_0^1 \left(\int_0^t f'(s)f(s)^{-1}\varphi(s)ds\right)dt. \end{aligned}\tag{4.24}$$

(2) *$(S^{-1})^*$ is a bijective linear map from L_0^2 to L^2. If we define $(S^{-1})^*$ is equal to 0 on the subset of constant functions, then for any $\varphi \in \mathrm{L}^2$,*

$$\left((S^{-1})^*\varphi\right)(t) = \varphi(t) + (f(t)^*)^{-1}\int_t^1 f(s)^*f'(s)f(s)^{-1}\varphi(s)ds. \tag{4.25}$$

Also $(S^{-1})^*\varphi$ can be written using $M(t)$ and $K(t)$ as

$$\left((S^{-1})^*\varphi\right)(t) = \varphi(t) + (M(t)^*)^{-1}\int_t^1 M(s)^*K(s)\varphi(s)ds. \tag{4.26}$$

Proof. All the calculation are almost similar and so we show how to calculate $(S^{-1})^*$ only. Using $(f'(t)f(t)^{-1})^* = f'(t)f(t)^{-1}$, we have for $\varphi \in \mathrm{L}^2$ and $\psi \in \mathrm{L}^2$,

$$\begin{aligned}&\left(S^{-1}\varphi, \psi\right)_{\mathrm{L}^2} \\ &= (\varphi,\psi) - \int_0^1 \left\langle \int_0^t f(s)^{-1}\varphi(s)ds, \left(\int_t^1 f(s)^*f'(s)f(s)^{-1}\psi(s)ds\right)' \right\rangle dt \\ &= (\varphi,\psi) + \int_0^1 \left\langle \varphi(t), \left(f(t)^{-1}\right)^* \int_t^1 f(s)^*f'(s)f(s)^{-1}\psi(s)ds\right\rangle dt. \end{aligned}\tag{4.27}$$

This shows (4.25) and $(S^{-1})^*\mathrm{const} = 0$. ☐

We summarize the relation between S and T in the proposition below.

Proposition 4.3. (1) *We have*

$$I + T = S^*S, \quad (S^{-1})^*(I+T) = S, \quad (I+T)^{-1} = S^{-1}(S^{-1})^*.$$

(2) *The following identities hold.*

$$\inf \sigma(I+T) = \inf\left\{\|S\varphi\|^2 \mid \|\varphi\|_{\mathrm{L}^2} = 1, \varphi \in \mathrm{L}_0^2\right\} = \frac{1}{\|(S^{-1})^*\|_{op}^2}. \tag{4.28}$$

Proof. $I+T=S^*S$ follows from Lemma 4.1. $(I+T)^{-1}=S^{-1}(S^{-1})^*$ follows from $(S^{-1})^* = (S^*)^{-1}$. (2) follows from (1). □

The identity $\|S\varphi\|^2=((I+T)\varphi,\varphi)$ $(\varphi\in \mathrm{L}_0^2)$ is used to prove the upper bound estimate, while the inequality $\inf\sigma((I+T))\le \frac{1}{\|(S^{-1})^*\|_{op}^2}$ is used for the proof of the lower bound estimate in Theorem 3.2. See (5.39) and (5.59).

5 Proof of Theorem 3.2

We prove Theorem 3.2. Hence we assume that $\mathcal{D}$ satisfies conditions (1), (2) in the theorem throughout this Section. As explained already, furthermore, we may assume M is diffeomorphic to $\mathbb{R}^n$ and the Riemannian metric is flat outside a compact set. Therefore, Assumptions A, B, D are satisfied.

We consider the ground state function of L_λ. Let $\tilde{\chi}_\delta(\gamma)=\chi_\delta\Big(\max_{0\le t\le 1} d(\gamma(t),c_{x_0,y_0}(t))\Big)$, where χ_δ is a non-negative smooth function such that $\chi_\delta(u)=1$ for $|u|\le\delta$ and $\chi_\delta(u)=0$ for $|u|\ge 2\delta$. Here δ is a sufficiently small positive number. Note that there exists $C_\delta>0$ such that $\nu^\lambda_{x_0,y_0}(\max_{0\le t\le 1} d(\gamma(t),c_{x_0,y_0}(t))\ge\delta)\le e^{-\lambda C_\delta}$. This can be proved by a large deviation result for solutions of SDE. Since the proof is similar to that of (5.32), we omit the proof.

Thus $\|\tilde{\chi}_\delta\|_{L^2(\nu^\lambda_{x_0,y_0})}\ge 1-Ce^{-C'\lambda}$. Also we have $\|D_0\tilde{\chi}_\delta\|_{L^2(\nu^\lambda_{x_0,y_0})}\le Ce^{-C'\lambda}$. Here we have used that the function $q(\gamma)=\max_{0\le t\le 1} d(\gamma(t),c_{x_0,y_0}(t))$ belongs to $\mathrm{D}(\mathcal{E}^\lambda)$ and $|D_0 q(\gamma)|\le 1$ $\nu^\lambda_{x_0,y_0}$-a.s. γ. This is proved in a similar way to Lemma 2.2 (2) in [3]. Hence

$$e^\lambda_{Dir,1,\mathcal{D}}\le Ce^{-\lambda C'}. \tag{5.1}$$

On the other hand, it is proved in [7] that $\liminf_{\lambda\to\infty}\frac{e^\lambda_{Dir,2,\mathcal{D}}}{\lambda}>0$. In [7], we studied the case of compact manifolds. However, the proof works as well as the present case by the assumption on M. These estimates imply that $e^\lambda_{Dir,1,\mathcal{D}}$ is a simple eigenvalue. Let Ψ_λ denote the normalized non-negative eigenfunction (ground state function). It is clear that $\Psi_\lambda\in H^{1,2}_0\left(\mathcal{D},\nu^\lambda_{x_0,y_0}\right)$. From (5.1), we obtain $\|D_0\Psi_\lambda\|_{L^2(\nu^\lambda_{x_0,y_0})}\le Ce^{-C'\lambda}$. It is plausible that Ψ_λ is strictly positive for $\nu^\lambda_{x_0,y_0}$ almost all γ which follows from the positivity improving property of the corresponding L^2-semigroup. However, we do not need such a property in this paper and we do not consider such a problem.

We use the following representation of $e^\lambda_{Dir,2,\mathcal{D}}$ to prove LHS $\le$ RHS in (3.11) in Theorem 3.2.

$$e^\lambda_{Dir,2,\mathcal{D}}=\inf\Biggl\{\frac{\int_{\mathcal{D}}|D_0(F-(\Psi_\lambda,F)\Psi_\lambda)|^2 d\nu^\lambda_{x_0,y_0}}{\|F-(\Psi_\lambda,F)\Psi_\lambda\|^2_{L^2(\nu^\lambda_{x_0,y_0})}}\ \Bigg|\ F\in H^{1,2}_0(\mathcal{D}) \text{ and } \|F-(\Psi_\lambda,F)\Psi_\lambda\|_{L^2(\nu^\lambda_{x_0,y_0})}\ne 0\Biggr\}. \tag{5.2}$$

The following estimate is necessary for the proof of Theorem 3.2.

Lemma 5.1. *We have*

$$\|\Psi_\lambda-1\|_{L^2(P_{x_0,y_0}(M),\nu^\lambda_{x_0,y_0})}\le Ce^{-C'\lambda}, \tag{5.3}$$

where C,C' are positive constants.

Proof. By the COH formula,

$$\|\Psi_\lambda - (\Psi_\lambda, 1)_{L^2(\nu^\lambda_{x,y})}\|_{L^2(\nu^\lambda_{x_0,y_0})} \le Ce^{-C'\lambda}.$$

This implies

$$1 - (\Psi_\lambda, 1)^2_{L^2(\nu^\lambda_{x,y})} = \Big(\Psi_\lambda, \Psi_\lambda - (\Psi_\lambda, 1)_{L^2(\nu^\lambda_{x,y})}\Big)_{L^2(\nu^\lambda_{x_0,y_0})} \le Ce^{-C'\lambda}$$

which shows $\|\Psi_\lambda - 1\|^2_{L^2(P_{x_0,y_0}(M),\nu^\lambda_{x_0,y_0})} \le 2Ce^{-C'\lambda}$. □

We need the following lemma to prove that $A(\gamma)_\lambda$ can be approximated by $A(c_{x_0,y_0})_\infty (= (S^{-1})^*)$ when γ is close to c_{x_0,y_0} and λ is large.

Lemma 5.2. *Recall that we have defined*

$$K(t) = -\frac{\overline{\nabla^2 k(c_{x_0,y_0})_t}}{1-t}. \tag{5.4}$$

We consider a perturbation of $K(t)$ such that

$$K_\varepsilon(t) = K(t) + \frac{C_\varepsilon(t)}{(1-t)^\delta},$$

where $0 < \delta < 1$ is a constant and $C_\varepsilon(t)$ $(0 \le \varepsilon \le 1)$ is a symmetric matrix-valued continuous function satisfying $\sup_t \|C_\varepsilon(t)\| \le \varepsilon$. Let $M_\varepsilon(t)$ be the solution to

$$M'_\varepsilon(t) = K_\varepsilon(t) M_\varepsilon(t) \quad 0 \le t < 1, \tag{5.5}$$

$$M_\varepsilon(0) = I. \tag{5.6}$$

Define

$$(J_\varepsilon \varphi)(t) = (M_\varepsilon(t)^*)^{-1} \int_t^1 M_\varepsilon(s)^* K_\varepsilon(s) \varphi(s) ds. \tag{5.7}$$

Then for sufficiently small ε, there exists a positive constant C which is independent of ε such that

$$\|J_\varepsilon - J_0\|_{op} \le C\varepsilon. \tag{5.8}$$

By Lemma 4.2, we see that $(S^{-1})^* = I + J_0$ holds.

Proof. As already mentioned, $\tilde{K}(t) = \frac{1}{1-t} + K(t)$ is a matrix-valued continuous mapping for $0 \le t \le 1$. Taking this into account, we rewrite

$$K_\varepsilon(t) = -\frac{1}{1-t} + \tilde{K}_\varepsilon(t),$$

where $\tilde{K}_\varepsilon(t) = \tilde{K}(t) + \frac{C_\varepsilon(t)}{(1-t)^\delta}$. Let $N_\varepsilon(t)$ be the solution to

$$N_\varepsilon(t)' = \tilde{K}_\varepsilon(t) N_\varepsilon(t) \quad 0 \le t < 1, \quad N_\varepsilon(0) = I. \tag{5.9}$$

Clearly, the solution to this equation exists. Moreover, $\lim_{t\to 1} N_\varepsilon(t)$ exists and $\sup_{0\le t<1}\|N_\varepsilon(t)\| < \infty$. To see this, we prove the continuity of $N_\varepsilon(t)$ with respect to t. Note that for $0 \le s \le t < 1$,

$$
\begin{aligned}
\|N_\varepsilon(t) - N_\varepsilon(s)\| &\le \int_s^t C\left(1 + \frac{1}{(1-u)^\delta}\right)\|N_\varepsilon(u)\| du \\
&\le \|N_\varepsilon(s)\| C\left((t-s) + \frac{(1-s)^{1-\delta} - (1-t)^{1-\delta}}{1-\delta}\right) \\
&\quad + \int_s^t C\left(1 + \frac{1}{(1-u)^\delta}\right)\|N_\varepsilon(u) - N_\varepsilon(s)\| du. \qquad (5.10)
\end{aligned}
$$

Hence by the Gronwall inequality, we have

$$
\begin{aligned}
\|N_\varepsilon(t) - N_\varepsilon(s)\| &\le \|N_\varepsilon(s)\| C\left((t-s) + \frac{(1-s)^{1-\delta} - (1-t)^{1-\delta}}{1-\delta}\right) \\
&\quad \times \exp\left\{C\left((t-s) + \frac{(1-s)^{1-\delta} - (1-t)^{1-\delta}}{1-\delta}\right)\right\} \qquad (5.11)
\end{aligned}
$$

which implies the desired result. Note that $\tilde{K}_0(t) = \tilde{K}(t)$, $N_0(t) = N(t)$ and $M_\varepsilon(t) = (1-t)N_\varepsilon(t)$. Also we have $N_\varepsilon(t)$ $(0 \le t < 1)$ is invertible and

$$
N_\varepsilon(s)N_\varepsilon(t)^{-1} = N_\varepsilon^t(s-t) \qquad 0 \le t \le s < 1,
$$

where $N_\varepsilon^t(u)$ $(0 \le u < 1-t)$ is the solution to the equation

$$
\partial_u N_\varepsilon^t(u) = \tilde{K}_\varepsilon(t+u)N_\varepsilon^t(u) \quad 0 \le u < 1-t, \qquad N_\varepsilon^t(0) = I.
$$

By a similar calculation to N_ε, we have $\sup_{\varepsilon,t,0\le u<1-t}\|N_\varepsilon^t(u)\| < \infty$. By the definition of J_ε, we have

$$
(J_\varepsilon\varphi)(t) = \frac{1}{1-t}\int_t^1 (1-s)N_\varepsilon^t(s-t)^* K_\varepsilon(s)\varphi(s) ds. \qquad (5.12)
$$

Hence by Hardy's inequality, in order to estimate $J_\varepsilon - J_0$, it suffices to estimate $N_\varepsilon^t - N_0^t$. Note that for $0 \le u < 1-t$,

$$
N_\varepsilon^t(u) = N_0^t(u)\left(I + \int_0^u N_0^t(\tau)^{-1}\frac{C_\varepsilon(t+\tau)}{(1-(t+\tau))^\delta}N_\varepsilon^t(\tau)d\tau\right).
$$

This and the estimate for C_ε and $N_\varepsilon^t(u)$ imply

$$
\sup_t |N_\varepsilon^t(u) - N_0^t(u)| \le C\varepsilon,
$$

which completes the proof of (5.8). □

Let us apply the lemma above in the case where $K_\varepsilon(t) = K(\gamma)_{\lambda,t}$. We have

$$
\begin{aligned}
K(\gamma)_{\lambda,t} &= K(t) + \frac{1}{1-t}\left(\frac{1-t}{\lambda}\overline{\nabla^2 \log p\left(\frac{1-t}{\lambda}, y_0, \gamma\right)}_t + \overline{\nabla^2 k(c_{x_0,y_0})}_t\right) - \frac{1}{2\lambda}\overline{\mathrm{Ric}(\gamma)}_t \\
&= K(t) + \frac{1}{1-t}\left(\frac{1-t}{\lambda}\overline{\nabla^2 \log p\left(\frac{1-t}{\lambda}, y_0, \gamma\right)}_t + \overline{\nabla^2 k(\gamma)}_t\right) \\
&\quad + \frac{1}{1-t}\left(\overline{\nabla^2 k(c_{x_0,y_0})}_t - \overline{\nabla^2 k(\gamma)}_t\right) - \frac{1}{2\lambda}\overline{\mathrm{Ric}(\gamma)}_t. \qquad (5.13)
\end{aligned}
$$

Therefore,

$$C_\varepsilon(t) = \frac{1}{(1-t)^{1-\delta}} \left(\frac{1-t}{\lambda} \overline{\nabla^2 \log p \left(\frac{1-t}{\lambda}, y_0, \gamma \right)}_t + \overline{\nabla^2 k(\gamma)}_t \right)$$
$$+ \frac{1}{(1-t)^{1-\delta}} \left(\overline{\nabla^2 k(c_{x_0,y_0})}_t - \overline{\nabla^2 k(\gamma)}_t \right)$$
$$- \frac{(1-t)^\delta}{2\lambda} \overline{\mathrm{Ric}(\gamma)}_t. \tag{5.14}$$

We need to show that if γ and c_{x_0,y_0} are close enough and λ is large, then $C_\varepsilon(t)$ is small. Then by Lemma 5.2, we obtain that $\|J(\gamma)_\lambda - (S^{-1})^*\|_{op}$ is small. Let us check each term of $C_\varepsilon(t)$. If $\gamma(t) \in B_l(y_0)$ for all $0 \le t \le 1$, the first term converges to 0 by Lemma 3.9 (1) as $\lambda \to \infty$ for $\delta > 1/2$. It is trivial to see that the third term goes to 0. Hence, it suffices to prove that if γ and c_{x_0,y_0} is close enough, then the difference $\overline{\nabla^2 k(c_{x_0,y_0})}_t - \overline{\nabla^2 k(\gamma)}_t$ is small. To this end, we use the results in rough path analysis.

Here, we summarize necessary results from rough path analysis. The readers are referred to [46, 48, 47, 29, 28] for rough path analysis. In Section 3, we define a Brownian motion b with variance $1/\lambda$ on $\mathbb{R}^n$ by using the stochastic parallel translation along γ and b is a functional of γ. Conversely, γ can be obtained by solving a stochastic differential equation driven by a Brownian motion $b(t)$. We may use notation b_t instead of $b(t)$. From now on, μ^λ denotes the Brownian motion measure with variance $1/\lambda$. We use the notation μ when $\lambda = 1$. Let $\{L_i\}_{i=1}^n$ be the canonical horizontal vector fields and consider an SDE on $O(M)$:

$$dr(t,u,b) = \sum_{i=1}^n L_i(r(t,u,b)) \circ db^i(t) \tag{5.15}$$
$$r(0,u,b) = u \in O(M). \tag{5.16}$$

Let $X(t,b) = \pi(r(t,u_0,b))$. Then the law of $X(\cdot,b)$ coincides with $\nu_{x_0}^\lambda$. Also it holds that

$$\overline{\nabla^2 k(X(b))}_t = r(t,u_0,b)^{-1}(\nabla^2 k)(X(t,b)) r(t,u_0,b) \qquad \mu^\lambda\text{-a.s. } b. \tag{5.17}$$

Note that if b is the anti-stochastic development of the Brownian motion $\gamma(t)$ on M, then it holds that $\tau(\gamma)_t = r(t,u_0,b)u_0^{-1}$ $\nu_{x_0}^\lambda$-a.s. γ. Since we assume M is diffeomorphic to $\mathbb{R}^n$, we have a global coordinate $x = (x^i) \in \mathbb{R}^n$ and the Riemannian metric $g(x) = (g_{ij}(x))$ on the tangent space T_xM which can be identified with $\mathbb{R}^n$. Then the SDE of $r(t,u_0,b) = (X^i(t,b), e_l^k(t,b))$ $(e(t,b) = (e_l^k(t,b)) \in GL(n,\mathbb{R}))$ can be written down explicitly (see [38, 36]) as

$$dX^i(t) = e_j^i(t) \circ db^j(t) \tag{5.18}$$
$$de_j^i(t) = -\sum_{k,l} \Gamma_{kl}^i(X(t)) e_j^l(t) \circ dX^k(t). \tag{5.19}$$

Moreover, the coefficients of the SDE are C_b^∞ because the Riemannian metric is flat outside a certain compact subset. Therefore we can apply rough path analysis and Malliavin calculus to the solution of the SDE. Now let us recall the definition of the Brownian rough path. Let $b(N)$ be the dyadic polygonal approximation of b such that $b(N)_{k2^{-N}} = b_{k2^{-N}}$ and $b(N)_t$ is linear for $k2^{-N} \le t \le (k+1)2^{-N}$ with $0 \le k \le 2^N - 1$. Define $b(N)^1_{s,t} = b(N)_t - b(N)_s$,

$b(N)^2_{s,t} = \int_s^t (b(N)_u - b(N)_s) \otimes db(N)_u$ for $0 \le s \le t \le 1$. Let Ω be all elements b belonging to the Wiener space W^n such that $b(N)^1_{s,t}$ and $b(N)^2_{s,t}$ converge in the Besov type norm $\|\cdot\|_{4m,\theta/2}$ and $\|\cdot\|_{2m,\theta}$ respectively ([8]). Here $2/3 < \theta < 1$ and m is a sufficiently large positive number. It is proved in [8] that Ω^c is a slim set in the sense of Malliavin with respect to the Brownian motion measure μ. However, it is easy to check that the same result holds for the Brownian motion measure μ^λ with variance $1/\lambda$ for any $\lambda > 0$. Moreover, if $b \in \Omega$, then $b + h \in \Omega$ for any element $h \in \mathrm{H}$. For $b \in \Omega$, we define $b^1_{s,t} = \lim_{N\to\infty} b(N)^1_{s,t}$ and $b^2_{s,t} = \lim_{N\to\infty} b(N)^2_{s,t}$. The triple $(1, b^1_{s,t}, b^2_{s,t})$ is a p-rough path $(2 < p = \frac{2}{\theta} < 3)$ and its control function is given by $\omega(s,t) = C(b)|t-s|$. $C(b)$ depends on the Besov norm of b^1 and b^2. For $h \in \mathrm{H}$, we have, $(b+h)^1_{s,t} = b^1_{s,t} + h^1_{s,t}$ and

$$(b+h)^2_{s,t} = b^2_{s,t} + h^2_{s,t} + \int_s^t (b_u - b_s) \otimes dh_u + \int_s^t (h_u - h_s) \otimes db_u.$$

Note that solutions of rough differential equations driven by geometric rough paths are smooth. See Definition 7.1.1 and Corollary 7.1.1 in [48]. Therefore, considering the composition of the two maps, $b(\in \Omega) \mapsto (1, b^1_{s,t}, b^2_{s,t})$ and the solution map between geometric rough paths, we obtain a smooth version $r(t, u_0, b)$ of the solution to (5.18) and (5.19). Here smooth means

1. the mapping $b(\in \Omega) \mapsto r(t, u_0, b)$ is differentiable in the H-direction and smooth in the sense of Malliavin,

2. the mapping $b(\in \Omega) \mapsto r(t, u_0, b)$ is ∞-quasi-continuous (See Theorem 3.2 in [8]).

In the terminology of Malliavin calculus, $r(t, u_0, b)$ is a version of redifinition of the solution to (5.15).

By the uniform ellipticity of (5.18), we have the following estimate for the Malliavin covariance matrix. For $p \ge 1$, there exists $p' > 0$ such that for large λ,

$$E[\{\det (DX(1,b)DX(1,b)^*)\}^{-p}] \le C\lambda^{p'}. \tag{5.20}$$

Thus the probability measure $d\mu^\lambda_{x_0,y_0} = \frac{\delta_{y_0}(X(1,b))d\mu^\lambda}{c(y_0)p(1/\lambda, x_0, y_0)}$ is well-defined, where $c(y_0) = \sqrt{\det(g_{ij}(y_0))}$ and δ_{y_0} denotes Dirac's delta function on $\mathbb{R}^n$ and $\delta_{y_0}(X(1,b))$ is a generalized Wiener functional ([55]). Note that $\mu^\lambda_{x_0,y_0}$ does not charge the slim sets. Thus the image measure $X_*\mu^\lambda_{x_0,y_0}$ is well-defined for smooth $X(b)$. Moreover, we have

$$\text{The joint law of } (b,\gamma) \text{ under } \nu^\lambda_{x_0,y_0} = \text{The joint law of } (b, X(b)) \text{ under } \mu^\lambda_{x_0,y_0}. \tag{5.21}$$

This observation implies that one can use estimates on integration with respect to (Brownian) rough paths to study the estimate on the stochastic integrals for the pinned Brownian motion. In the proof in Section 2, we use cut-off functions $\chi_{1,\kappa}, \chi_{2,\kappa}$. In our problem, the existence of such cut-off functions is not trivial. The existence of such an appropriate cut-off functions are proved in [6]. We use the following result in rough paths. Below, $r(t, u_0, b)$ may be denoted by $r(t, b)$ for simplicity.

Lemma 5.3. (1) *In this statement, we consider the smooth version $r(t,b)$ for $b \in \Omega$. By adopting this version, a version of $\overline{\nabla^2 k(X(b))}_t$ can be defined as $r(t,u_0,b)^{-1}(\nabla^2 k)(X(t,b))r(t,u_0,b)$ which is smooth in the above sense. Let $l_\xi(t) = t\xi$, where ξ is chosen as $\exp_{x_0}(u_0\xi) = y_0$. Let us define*

$$\Xi(b) = \|b^1\|^{4m}_{4m,\theta/2} + \|b^2\|^{2m}_{2m,\theta} \qquad b \in \Omega. \tag{5.22}$$

Then for any $\varepsilon > 0$, there exists $\varepsilon' > 0$ such that if $\Xi(b - l_\xi) \le \varepsilon'$ and $X(1, b) = y_0$,

$$|X(t,b) - c_{x_0,y_0}(t)| \le \varepsilon t^{\theta/2} \quad 0 \le t \le 1, \tag{5.23}$$

$$\left|\overline{\nabla^2 k(X(b))}_t - \overline{\nabla^2 k(X(l_\xi))}_t\right| \le \varepsilon(1-t)^{\theta/2} \quad 0 \le t \le 1, \tag{5.24}$$

$$\sup_{0\le t\le 1} \left|I^2(b)_{t,1} - I^2(l_\xi)_{t,1}\right| \le (\|\varphi\|_\infty + \|\varphi'\|_\infty)\varepsilon, \tag{5.25}$$

where

$$I^2(b)_{s,t} = \int_t^1 \overline{R(X(b))}_s \left(\int_s^1 \varphi(r)db^i(r), \varepsilon_i\right) \circ db(s) \tag{5.26}$$

and $\varphi \in C^1([0,1], \mathbb{R}^n)$. The integral is defined in the sense of rough paths.
(2) In this statement, let b be the Brownian motion which is obtained by the anti-stochastic development of the pinned Brownian motion γ. Let η be a C_b^1 function with compact support on $\mathbb{R}$. Let $\tilde{\eta}(\gamma) = \eta\left(\Xi(b - l_\xi)\right)$. Then there exists a constant $C > 0$ such that for all $\lambda \ge 1$

$$|D_0\tilde{\eta}(\gamma)|_{\mathrm{H}_0} \le C \quad \textit{for } \nu^\lambda_{x_0,y_0}\textit{-almost all } \gamma. \tag{5.27}$$

Proof. (1) (5.23) and (5.25) follow from the fact that $c_{x_0,y_0}(t) = X(t, l_\xi)$ and the continuity theorem for p-rough paths ($2 < p = \frac{2}{\theta} < 3$). We prove (5.24). We have

$$\begin{aligned}
&\overline{\nabla^2 k(X(b))}_t - \overline{\nabla^2 k(X(l_\xi))}_t \\
&= \left\{\overline{\nabla^2 k(X(b))}_t - \overline{\nabla^2 k(X(b))}_1\right\} - \left\{\overline{\nabla^2 k(X(l_\xi))}_t - \overline{\nabla^2 k(X(l_\xi))}_1\right\} \\
&\quad + \overline{\nabla^2 k(X(b))}_1 - \overline{\nabla^2 k(X(l_\xi))}_1 \\
&= \left\{\overline{\nabla^2 k(X(b))}_t - \overline{\nabla^2 k(X(b))}_1\right\} - \left\{\overline{\nabla^2 k(X(l_\xi))}_t - \overline{\nabla^2 k(X(l_\xi))}_1\right\},
\end{aligned} \tag{5.28}$$

where we have used $(\nabla^2 k)(y_0) = I_{T_{y_0}M}$ and $X(1,b) = c_{x_0,y_0}(1) = y_0$. Hence it suffices to apply the continuity theorem for p-rough paths ($2 < p = \frac{2}{\theta} < 3$).
(2) In the case of the derivative D, this immediately follows from Lemma 7.11 in [6]. The proof for D_0 is the same. Here we give a sketch of the proof. Recall that

$$(D_0)_h b(t) = h(t) + \int_0^t \int_0^s \overline{R(\gamma)}_u (h(u), \circ db(u))(\circ db(s)). \tag{5.29}$$

We already used this formula in the proof of Theorem 3.11 for the derivative D. From this formula, we see that $D_0\left(\Xi(b - l_\xi)\right)$ are given by iterated stochastic integrals of b and γ. By (5.21), we can apply estimates for integration with respect to the Brownian rough path for $b \in \Omega$. Thus, the iterated integrals of solutions of rough differential equations can be estimated by the control function of the Brownian rough path. Since the support of η is compact, this implies the desired estimate. □

Now, we are ready to prove our first main theorem.

Proof of Theorem 3.2. First we prove the upper bound estimate. This will be done by using (5.2) and choosing appropriate functions F below. For that purpose, we prepare a large deviation estimate. Below, several constants depending on parameters κ, ε appear. We use the notation

$M(x)$ to denote positive functions of x which may diverge as $x \to 0$. On the other hand, we use the notation $C(x)$ to denote positive functions of x which converge to 0 as $x \to 0$. $M(x)$ and $C(x)$ may change line by line. Let η be a non-negative smooth function such that $\eta(u) = 1$ for $u \le 1$ and $\eta(u) = 0$ for $u \ge 2$. Let $0 < \kappa < 1$ and set

$$\eta_{1,\kappa}(\gamma) = \eta\left(\kappa^{-1}\Xi(b - l_\xi)\right), \quad \eta_{2,\kappa}(\gamma) = \left\{1 - \eta_{1,\kappa}(\gamma)^2\right\}^{1/2}. \tag{5.30}$$

By (5.29) and Lemma 5.3 (2), there exists a positive constant $M(\kappa)$ such that

$$|D_0\eta_{1,\kappa}(\gamma)| + |D_0\eta_{2,\kappa}(\gamma)| \le M(\kappa) \qquad \nu^\lambda_{x_0,y_0} - a.s.\gamma. \tag{5.31}$$

From (5.23), for any $\varepsilon > 0$, $\sup_{0\le t\le 1} |X(t, b) - c_{x_0,y_0}(t)| \le \varepsilon$ holds if κ is sufficiently small and $\eta_{1,\kappa}(\gamma) \ne 0$. Hence $\eta_{1,\kappa} \in H^{1,2}_0(\mathcal{D})$. Let ψ be a smooth non-negative function on $\mathbb{R}$ satisfying $\psi(u) = 0$ for $u \le \delta_1$ and $\psi(u) = 1$ for $u \ge \delta_2$, where $0 < \delta_1 < \delta_2$. Then there exist $C, C' > 0$ which depend on ψ such that for large λ

$$E^{\nu^\lambda_{x_0,y_0}}\left[\psi\left(\Xi(b - l_\xi)\right)\right] \le Ce^{-C'\lambda}. \tag{5.32}$$

We prove this estimate. Let B be a standard Brownian motion on $\mathbb{R}^n$. Since the Wiener functional $B \mapsto X\left(1, \frac{B}{\sqrt{\lambda}}\right)$ is non-degenerate, by using the integration by parts formula (see [50, 53]),

$$\begin{aligned}
&E^{\nu^\lambda_{x_0,y_0}}\left[\psi\left(\Xi(b - l_\xi)\right)\right] \\
&= (c(y_0)p(1/\lambda, x_0, y_0))^{-1} E\left[\psi\left(\Xi\left(\frac{B}{\sqrt{\lambda}} - l_\xi\right)\right)\delta_{y_0}\left(X\left(1, \frac{B}{\sqrt{\lambda}}\right)\right)\right] \\
&= (c(y_0)p(1/\lambda, x_0, y_0))^{-1} E\left[\tilde{\psi}\left(\Xi\left(\frac{B}{\sqrt{\lambda}} - l_\xi\right)\right)G(\varepsilon, \lambda, B)\phi_\varepsilon\left(X\left(1, \frac{B}{\sqrt{\lambda}}\right) - y_0\right)\right],
\end{aligned} \tag{5.33}$$

where $\tilde{\psi}, \phi_\varepsilon$ are bounded continuous functions on $\mathbb{R}$ and $\mathbb{R}^n$ respectively such that $\tilde{\psi} \subset [\delta_1, \infty)$ and $\operatorname{supp}\phi_\varepsilon \subset B_\varepsilon(0)$. Also the random variable $G(\lambda, \varepsilon, B)$ satisfies that for any $p > 1$

$$E\left[|G(\lambda, \varepsilon, B)|^p\right]^{1/p} \le C_{\varepsilon,p}(\lambda), \tag{5.34}$$

where $C_{\varepsilon,p}(\lambda)$ is a polynomial function of λ. Let $q = p/(p-1)$. By the Hölder inequality,

$$E^{\nu^\lambda_{x_0,y_0}}\left[\psi\left(\Xi(b - l_\xi)\right)\right] \le p(1/\lambda, x_0, y_0)^{-1}C_{\varepsilon,p}(\lambda)\mu(A_\varepsilon)^{1/q}, \tag{5.35}$$

where

$$A_\varepsilon = \left\{B \;\middle|\; \Xi\left(\frac{B}{\sqrt{\lambda}} - l_\xi\right) \ge \delta_1, \; \left|X\left(1, \frac{B}{\sqrt{\lambda}}\right) - y_0\right| \le \varepsilon\right\}. \tag{5.36}$$

By the large deviation estimate for Brownian rough path ([29, 39, 43]), we have

$$\limsup_{\lambda\to\infty} \frac{1}{\lambda}\log\mu(A_\varepsilon) \le -\frac{1}{2}\inf\left\{\|h\|^2_{\mathrm{H}} \mid \Xi(h - l_\xi) \ge \delta_1, \; |X(1, h) - y_0| \le \varepsilon\right\} =: J_\varepsilon. \tag{5.37}$$

For sufficiently small ε, it holds that $J_\varepsilon < -\frac{1}{2}d(x_0, y_0)^2$ which can be proved by a contradiction. Suppose there exists $h_\varepsilon \in \mathrm{H}$ such that $\lim_{\varepsilon\to 0}\|h_\varepsilon\|_{\mathrm{H}} \le d(x_0, y_0)$, $\Xi(h_\varepsilon - l_\xi) \ge \delta_1$ and

$|X(1, h_\varepsilon) - y_0| \le \varepsilon$. Let h_0 be a weak limit point of h_ε. Then $\|h_0\|_{\mathrm{H}} \le d(x_0, y_0)$. By Lemma 7.12 in [6], $\Xi\left(h_0 - l_\xi\right) = \lim_{\varepsilon\to 0} \Xi\left(h_\varepsilon - l_\xi\right) \ge \delta_1$ and $X(1, h_0) = \lim_{\varepsilon\to 0} X(1, h_\varepsilon) = y_0$. By the uniqueness of the minimal geodesic between x_0 and y_0, we have $h_0 = l_\xi$. This contradicts $\Xi\left(h_0 - l_\xi\right) \ge \delta_1$. Hence there exist $\varepsilon > 0$ and $\delta > 0$ such that

$$E^{\nu^\lambda_{x_0,y_0}}\left[\psi\left(\Xi(b - l_\xi)\right)\right] \le C_{\varepsilon,p}(\lambda) p(1/\lambda, x_0, y_0)^{-1} \exp\left\{-\lambda\left(\frac{d(x_0, y_0)^2 + \delta}{2q}\right)\right\}. \tag{5.38}$$

Since $\lim_{\lambda\to\infty} \frac{\lambda^{n/2}\exp\left(-\lambda d(x_0,y_0)^2/2\right)}{p(1/\lambda,x_0,y_0)}$ exists, by taking p sufficiently large, this proved the desired inequality.

We now apply (5.2) to prove the upper bound. Let us fix a positive number $\varepsilon > 0$ and choose $\varphi_\varepsilon \in \mathrm{L}^2_0 \cap C^1([0, 1], \mathbb{R}^n)$ with $\|\varphi_\varepsilon\| = 1$ such that

$$\sigma_1 \le \|S\varphi_\varepsilon\|^2 \le \|(I + T)\varphi_\varepsilon\| \le \sigma_1 + \varepsilon. \tag{5.39}$$

This is possible thanks to Lemma 4.1. Note that $\|\varphi'_\varepsilon\|_\infty$ may diverge when $\varepsilon \to 0$. Define

$$F_\varepsilon(\gamma) = \sqrt{\lambda}\left(\int_0^1 (\varphi_\varepsilon(t), db(t)) - \int_0^1 (\varphi_\varepsilon(t), \xi)dt\right). \tag{5.40}$$

Let $\tilde{F}_\varepsilon = F_\varepsilon \eta_{1,\kappa} \in H^{1,2}_0(\mathcal{D})$. We estimate the numerator of the ratio in (5.2) for $\tilde{F}_\varepsilon$. Since the Besov norm is stronger than the supremum norm, we have

$$|\tilde{F}_\varepsilon(\gamma)| \le C\sqrt{\lambda} M(\varepsilon) C(\kappa). \tag{5.41}$$

By (5.29)

$$\begin{aligned}
(D_0 F_\varepsilon(\gamma), h)_{\mathrm{H}_0} &= \sqrt{\lambda}\int_0^1 \left(\varphi_\varepsilon(t), h'(t)\right) dt + \sqrt{\lambda}\int_0^1 \left(\varphi_\varepsilon(t), \int_0^t \overline{R(\gamma)}_u (h(u), \circ db(u)) \circ db(t)\right) \\
&= \sqrt{\lambda}\int_0^1 \left(\varphi_\varepsilon(t), h'(t)\right) dt \\
&\quad + \int_0^1 \left(\int_t^1 \overline{R(\gamma)}_s \left(\int_s^1 \varphi_\varepsilon(u) db^i(u), \varepsilon_i\right) \circ db(t), h'(t)\right) dt
\end{aligned} \tag{5.42}$$

and so we have

$$\begin{aligned}
D_0 F_\varepsilon(\gamma)'_t &= \sqrt{\lambda}\varphi_\varepsilon(t) + \sqrt{\lambda}\int_t^1 \overline{R(\gamma)}_s \left(\int_s^1 \varphi_\varepsilon(r) db^i(r), \varepsilon_i\right) \circ db(s) \\
&\quad - \sqrt{\lambda}\int_0^1 \int_t^1 \overline{R(\gamma)}_s \left(\int_s^1 \varphi_\varepsilon(r) db^i(r), \circ\varepsilon_i\right) (\circ db(s)) dt \\
&= \sqrt{\lambda}\varphi_\varepsilon(t) - \sqrt{\lambda}\int_t^1 R(s)\left(\int_0^s \varphi_\varepsilon(u)du\right) ds \\
&\quad + \sqrt{\lambda}\int_0^1 \int_t^1 R(s)\left(\int_0^s \varphi_\varepsilon(u)du\right) ds dt + I(\lambda)_t \\
&= \sqrt{\lambda}(I + T)(\varphi_\varepsilon)(t) + I(\lambda)_t,
\end{aligned} \tag{5.43}$$

where $R(s) = \overline{R(c_{x_0,y_0})_s}(\cdot,\xi)(\xi)$ and $I(\lambda)_t = (D_0F)(X(\cdot,b))'_t - (D_0F)(X(\cdot,l_\xi))'_t$. Note that we have used $\varphi_\varepsilon \in \mathrm{L}_0^2$ in the above. By (5.25), we have

$$\sup_{0\le t\le 1} |I(\lambda)_t| \le \sqrt{\lambda}C(\kappa)M(\varepsilon) \quad \text{if } \eta_{1,\kappa}(\gamma) \ne 0. \tag{5.44}$$

Thus we have

$$\begin{aligned}|D_0\tilde{F}_\varepsilon(\gamma)|^2 &= \lambda|(I+T)\varphi_\varepsilon|^2\eta_{1,\kappa}^2 + |I(\lambda)|^2\eta_{1,\kappa}^2 + 2\sqrt{\lambda}((I+T)\varphi_\varepsilon, I(\lambda))\eta_{1,\kappa}^2 \\ &\quad + F_\varepsilon^2|D_0\eta_{1,\kappa}|^2 + 2(D_0F_\varepsilon, D_0\eta_{1,\kappa})\eta_{1,\kappa}.\end{aligned} \tag{5.45}$$

By (5.32) and (5.44), we get

$$\|D_0\tilde{F}_\varepsilon\|^2_{L^2(\nu^\lambda_{x_0,y_0})} \le \lambda\|(I+T)\varphi_\varepsilon\|^2_{L^2} + \lambda C(\kappa)M(\varepsilon) + \lambda CM(\varepsilon)M(\kappa)e^{-C(\kappa)\lambda}. \tag{5.46}$$

Combining $\|D_0\Psi_\lambda\| \le Ce^{-C'\lambda}$, we obtain

$$\|D_0\tilde{F}_\varepsilon - \left(\tilde{F}_\varepsilon, \Psi_\lambda\right) D_0\Psi_\lambda\|^2_{L^2(\nu^\lambda_{x_0,y_0})} \le \lambda\|(I+T)\varphi_\varepsilon\|^2_{L^2} + \lambda C(\kappa)M(\varepsilon) + \lambda M(\varepsilon)M(\kappa)e^{-C(\kappa)\lambda}. \tag{5.47}$$

We next turn to the estimate of the denominator in (5.2) for $\tilde{F}_\varepsilon$. To do so, we use COH formula. For large $\lambda > 0$, by taking κ sufficiently small and combining Lemma 5.2 and Lemma 5.3 (1), we have

$$|(J(\gamma)_\lambda - J_0)(D_0\tilde{F}_\varepsilon)(\gamma)'|_{L^2(0,1)} \le \varepsilon|D_0\tilde{F}_\varepsilon(\gamma)'|_{L^2(0,1)}. \tag{5.48}$$

Therefore, using $A(\gamma)_\lambda = I + J(\gamma)_\lambda$, $(S^{-1})^* = I + J_0$ and $(S^{-1})^*(I+T) = S$, we have

$$\begin{aligned}&A(\gamma)_\lambda(D_0\tilde{F}_\varepsilon(\gamma)')_t \\ &= \left(S^{-1}\right)^*\left(D_0\tilde{F}_\varepsilon(\gamma)'\right)_t + (J(\gamma)_\lambda - J_0)(D_0\tilde{F}_\varepsilon(\gamma)')_t \\ &= \sqrt{\lambda}\left(S^{-1}\right)^*(I+T)\varphi_\varepsilon(t)\eta_{1,\kappa} + (S^{-1})^*I(\lambda)_t\eta_{1,\kappa} + F_\varepsilon(\gamma)(S^{-1})^*\left(D_0\eta_{1,\kappa}\right)'_t \\ &\quad + (J(\gamma)_\lambda - J_0)(D_0\tilde{F}_\varepsilon(\gamma)')_t \\ &= \sqrt{\lambda}S\varphi_\varepsilon(t) + I_2(\lambda),\end{aligned} \tag{5.49}$$

and

$$\begin{aligned}\|I_2(\lambda)\|_{L^2(\nu^\lambda_{x_0,y_0})} &\le \sqrt{\lambda}M(\varepsilon)e^{-C(\kappa)\lambda} + \sqrt{\lambda}C(\kappa)M(\varepsilon) + \sqrt{\lambda}M(\varepsilon)M(\kappa)e^{-C(\kappa)\lambda} \\ &\quad + \varepsilon\sqrt{\lambda}\left(C + C(\kappa)M(\varepsilon) + M(\varepsilon)M(\kappa)e^{-C(\kappa)\lambda}\right).\end{aligned} \tag{5.50}$$

Since $S\varphi_\varepsilon(t)$ is a non-random function, from (5.49) and (5.50) and the COH formula (3.29), we obtain

$$\begin{aligned}\|\tilde{F}_\varepsilon - E^{\nu^\lambda_{x_0,y_0}}[\tilde{F}_\varepsilon]\|^2_{L^2(\nu^\lambda_{x_0,y_0})} &\ge \|S\varphi_\varepsilon\|^2 - C(\kappa)M(\varepsilon) - M(\varepsilon)M(\kappa)e^{-C(\kappa)\lambda} \\ &\quad - \varepsilon\left(C + C(\kappa)M(\varepsilon) + M(\varepsilon)M(\kappa)e^{-C(\kappa)\lambda}\right).\end{aligned} \tag{5.51}$$

Using Lemma 5.1,

$$\begin{aligned}\|\tilde{F}_\varepsilon - \left(\tilde{F}_\varepsilon, \Psi_\lambda\right) \Psi_\lambda\|^2_{L^2(\nu^\lambda_{x_0,y_0})} &= \|\tilde{F}_\varepsilon - (\tilde{F}_\varepsilon, 1)\|^2_{L^2} - 2\left(\tilde{F}_\varepsilon - (\tilde{F}_\varepsilon, \Psi_\lambda), (\tilde{F}_\varepsilon, \Psi_\lambda)(\Psi_\lambda - 1)\right) \\ &\quad + (\tilde{F}_\varepsilon, 1 - \Psi_\lambda)^2 + (\tilde{F}_\varepsilon, \Psi_\lambda)^2 \|1 - \Psi_\lambda\|^2 \\ &\geq \|\tilde{F}_\varepsilon - (\tilde{F}_\varepsilon, 1)\|^2_{L^2} - M(\varepsilon)M(\kappa)e^{-C'\lambda}. \end{aligned} \tag{5.52}$$

Now we set ε sufficiently small and next κ sufficiently small. By using the estimates (5.47), (5.51), (5.52) and (5.39), we obtain for large λ,

$$\begin{aligned}\frac{\|D_0\tilde{F}_\varepsilon - (\tilde{F}_\varepsilon, \Psi_\lambda)D_0\Psi_\lambda\|^2_{L^2(\nu^\lambda_{x_0,y_0})}}{\|\tilde{F}_\varepsilon - (\tilde{F}_\varepsilon, \Psi_\lambda)\Psi_\lambda\|^2_{L^2(\nu^\lambda_{x_0,y_0})}} &\leq \frac{\lambda\|(I+T)\varphi_\varepsilon\|^2_{L^2} + \lambda\varepsilon + \lambda M(\varepsilon)M(\kappa)e^{-C(\kappa)\lambda}}{\|S\varphi_\varepsilon\|^2_{L^2} - C\varepsilon - M(\varepsilon)M(\kappa)e^{-C(\kappa)\lambda}} \\ &\leq \frac{\lambda(\sigma_1+\varepsilon)^2 + \lambda\varepsilon + \lambda M(\varepsilon)M(\kappa)e^{-C(\kappa)\lambda}}{\sigma_1 - C\varepsilon - M(\varepsilon)M(\kappa)e^{-C(\kappa)\lambda}}. \end{aligned} \tag{5.53}$$

This completes the proof of the upper bound.

We next prove lower bound estimate. Take $F \in H^{1,2}_0(\mathcal{D})$ such that $\|F\|_{L^2(\nu^\lambda_{x_0,y_0})} = 1$ and $(F, \eta_{1,\kappa}) = 0$. By the IMS localization formula,

$$\mathcal{E}(F,F) = \sum_{i=1,2} \mathcal{E}(F\eta_{i,\kappa}, F\eta_{i,\kappa}) - \sum_{i=1,2} E^{\nu^\lambda_{x_0,y_0}}[|D_0\eta_{i,\kappa}|^2 F^2]. \tag{5.54}$$

For any $\varepsilon > 0$, by taking κ sufficiently small and large λ, by Lemma 3.10 (1), Lemma 5.2, Lemma 5.3,

$$\begin{aligned}&\|F\eta_{1,\kappa} - E^{\nu^\lambda_{x_0,y_0}}[F\eta_{1,\kappa}]\|^2_{L^2(\nu^\lambda_{x_0,y_0})} \\ &\quad\leq \frac{\left(\|(S^{-1})^*\|_{op} + C\varepsilon\right)^2}{\lambda} E^{\nu^\lambda_{x_0,y_0}}\left[|D_0(F\eta_{1,\kappa})|^2\right]. \end{aligned} \tag{5.55}$$

Thus we have

$$\|F\eta_{1,\kappa}\|^2_{L^2(\nu^\lambda_{x_0,y_0})} \leq \frac{\left(\|(S^{-1})^*\|_{op} + C\varepsilon\right)^2}{\lambda} E^{\nu^\lambda_{x_0,y_0}}\left[|D_0(F\eta_{1,\kappa})|^2\right]. \tag{5.56}$$

Now we estimate the Dirichlet norm of $F\eta_{2,\kappa}$. The log-Sobolev inequality (3.31) implies that there exists a positive constant C such that for any $F \in H^{1,2}_0(\mathcal{D})$ and bounded measurable function V on $P_{x_0,y_0}(M)$,

$$\mathcal{E}(F,F) + E^{\nu^\lambda_{x_0,y_0}}\left[\lambda^2 V F^2\right] \geq -\frac{\lambda}{C}\log E^{\nu^\lambda_{x_0,y_0}}\left[e^{-C\lambda V}\right]\|F\|^2_{L^2(\nu^\lambda_{x_0,y_0})}. \tag{5.57}$$

See Theorem 7 in [33]. Also see Lemma 6.1 in the present paper. Let δ be a sufficiently small positive number and define $V(\gamma) = \delta 1_{\eta_{2,\kappa}\neq 0}(\gamma)$, where 1_A denotes the indicator function of a set

A. By (5.57), there exists $\delta' > 0$ such that

$$\begin{aligned}
&\mathcal{E}(F\eta_{2,\kappa}, F\eta_{2,\kappa}) \\
&\quad = \mathcal{E}(F\eta_{2,\kappa}, F\eta_{2,\kappa}) - \lambda^2 E^{\nu^{\lambda}_{x_0,y_0}}\left[V(F\eta_{2,\kappa})^2\right] + \lambda^2 E^{\nu^{\lambda}_{x_0,y_0}}\left[V(F\eta_{2,\kappa})^2\right] \\
&\quad \geq -\frac{\lambda}{C}\log E^{\nu^{\lambda}_{x_0,y_0}}\left[e^{C\lambda V}\right]\|F\eta_{2,\kappa}\|^2_{L^2(\nu^{\lambda}_{x_0,y_0})} + \lambda^2\delta\|F\eta_{2,\kappa}\|^2_{L^2(\nu^{\lambda}_{x_0,y_0})} \\
&\quad \geq -\frac{\lambda}{C}\log\left(1 + e^{-\lambda\delta'}\right)\|F\eta_{2,\kappa}\|^2_{L^2(\nu^{\lambda}_{x_0,y_0})} + \lambda^2\delta\|F\eta_{2,\kappa}\|^2_{L^2(\nu^{\lambda}_{x_0,y_0})} \\
&\quad \geq (\lambda^2\delta - C\lambda e^{-\lambda\delta'})\|F\eta_{2,\kappa}\|^2_{L^2(\nu^{\lambda}_{x_0,y_0})},
\end{aligned} \tag{5.58}$$

where in the third inequality we have used the estimate (5.32).

By the estimates (5.31), (5.56), (5.58) and the fact that $\|F\eta_{1,\kappa}\|^2_{L^2(\nu^{\lambda}_{x_0,y_0})} + \|F\eta_{2,\kappa}\|^2_{L^2(\nu^{\lambda}_{x_0,y_0})} = 1$, we get

$$\mathcal{E}^{\lambda}(F,F) \geq \lambda \min\left(\left(\|(S^{-1})^*\|_{op} + C\varepsilon\right)^{-2}, \lambda\delta - Ce^{-\lambda\delta'}\right) - M(\kappa). \tag{5.59}$$

By the definition of $e^{\lambda}_{Dir,2,\mathcal{D}}$, this completes the proof. □

Remark 5.4. Eberle [23] defined a local spectral gap on $\mathcal{D}$ by

$$e^{\lambda}_{E} = \inf_{F(\neq 0)\in H^{1,2}_0(\mathcal{D})} \frac{\int_{\mathcal{D}} |D_0 F|^2 d\nu^{\lambda}_{x_0,y_0}}{\int_{\mathcal{D}}\left(F - \frac{1}{\nu^{\lambda}_{x_0,y_0}(\mathcal{D})}\int_{\mathcal{D}} F d\nu^{\lambda}_{x_0,y_0}\right)^2 d\nu^{\lambda}_{x_0,y_0}}. \tag{5.60}$$

When $\mathcal{D}$ satisfies conditions (1), (2) in Theorem 3.2, the above proof shows also that

$$\lim_{\lambda\to\infty}\frac{e^{\lambda}_{E}}{\lambda} = \sigma_1. \tag{5.61}$$

Actually, e^{λ}_{E} is more related to e^{λ}_{2} than $e^{\lambda}_{Dir,2,\mathcal{D}}$. We cannot expect the existence of the spectral gap in general as we already mentioned. However, the weak Poincaré inequality does hold on the loop space over a simply connected compact Riemannian manifold. We refer the reader to [15] and references therein for the weak Poincaré inequality.

6 A proof of existence of spectral gap

We consider the following setting. Let $(\Omega, \mathfrak{F}, \nu)$ be a probability space and consider a Dirichlet form $(\mathcal{E}, \mathcal{F})$ defined on $L^2(\Omega,\nu)$. We assume the existence of square field operator Γ such that

$$\mathcal{E}(F,F) = \int_{\Omega}\Gamma(F,F)d\nu, \qquad F \in \mathcal{F}.$$

Also we assume $1 \in \mathcal{F}$ and the diffusion property. That is, for any $\varphi \in C^1_b(\mathbb{R})$ and $F \in \mathcal{F}$, it holds that $\varphi(F) \in \mathcal{F}$ and

$$\Gamma(\varphi(F), \varphi(F)) = \Gamma(F,F)\varphi'(F)^2. \tag{6.1}$$

We write $\Gamma(F) = \Gamma(F,F)$. We already used the following well known estimate ([33]).

Lemma 6.1. *Suppose that for any $F \in \mathcal{F}$,*

$$\int_\Omega F(w)^2 \log(F(w)^2/\|F\|_{L^2(\nu)}^2) d\nu \le \alpha \mathcal{E}(F,F). \tag{6.2}$$

Then for any bounded measurable function V, we have

$$\mathcal{E}(F,F) + \int_\Omega V(w)F(w)^2 d\nu(w) \ge -\frac{1}{\alpha}\log\left(\int_\Omega e^{-\alpha V(\omega)} d\nu(\omega)\right) \|F\|_{L^2(\nu)}^2 \qquad \text{for any } F \in \mathcal{F}. \tag{6.3}$$

Note that in the above lemma, $(\mathcal{E},\mathcal{F})$ is not necessarily a closed form and the lemma holds for any bilinear form $(\mathcal{E},\mathcal{F})$ satisfying the logarithmic Sobolev inequality (6.2). The spectral gap e_2 is defined by

$$e_2 = \inf\left\{\mathcal{E}(F,F) \;\middle|\; \|F\|_{L^2(\nu)} = 1, \int_\Omega F(w)d\nu(w) = 0, F \in \mathcal{F}\right\}.$$

Theorem 6.2. *Let $\mathcal{F}_0$ be a dense linear subset of $\mathcal{F}$ with respect to $\mathcal{E}_1$-norm. Suppose that there exist positive numbers α, β, r_0 and $\rho \in \mathcal{F}$ such that $\Gamma(\rho)(w) \le 1$ ν-a.s. w and*

$$\int_\Omega F(w)^2 \log(F(w)^2/\|F\|_{L^2(\nu)}^2) d\nu \le \alpha \int_\Omega \rho(w)^2 \Gamma(F,F)(w) d\nu(w), \quad \text{for all } F \in \mathcal{F}_0, \tag{6.4}$$

$$\nu(\rho \ge r) \le e^{-\beta r^2}, \qquad \text{for all } r \ge r_0. \tag{6.5}$$

Then

$$e_2 \ge \frac{1}{4}\min\left(\frac{1}{8\alpha R(\alpha,\beta,r_0)^2},\ \frac{\beta}{36\alpha}\right), \tag{6.6}$$

where

$$R(\alpha,\beta,r_0) = \max\left(\sqrt{\frac{2}{\beta}},\ \frac{192\alpha}{\sqrt{\beta}},\ 48\sqrt{\frac{\alpha}{\beta}},\ r_0\right). \tag{6.7}$$

Proof. Let $R \ge r_0$. We consider the partition of unity $\{\chi_k\}_{k\ge 0}$ on $[0,\infty)$ such that

(i) χ_k is a C^1 function,

(ii) $\chi_0(u) = 1$ for $0 \le u \le R$ and $\chi_0(u) = 0$ for $u \ge 2R$,

(iii) $\operatorname{supp}\chi_k \subset [Rk, R(k+2)] \qquad (k \ge 1)$,

(iv) $\sum_{k=0}^\infty \chi_k(u)^2 = 1$ for all $u \ge 0$.

(v) $\sup_{k,u} |\chi_k'(u)| \le \frac{2}{R}$,

Define $\tilde\chi_k(w) = \chi_k(\rho(w))$. Let $F \in \mathcal{F}_0$ and assume $\|F\|_{L^2(\nu)} = 1$ and $\int_\Omega F(w)d\nu(w) = 0$. By the IMS localization formula, we have

$$\mathcal{E}(F,F) = \sum_{k=0}^\infty \mathcal{E}(F\tilde\chi_k, F\tilde\chi_k) - \sum_{k=0}^\infty \int_\Omega \Gamma(\tilde\chi_k)F^2 d\nu \ge \sum_{k=0}^\infty \mathcal{E}(F\tilde\chi_k, F\tilde\chi_k) - \frac{8}{R^2}\int_{\rho \ge R} F^2 d\nu. \tag{6.8}$$

We estimate each term $\mathcal{E}(F\tilde{\chi}_k, F\tilde{\chi}_k)$. First, we estimate $\mathcal{E}(F\tilde{\chi}_0, F\tilde{\chi}_0)$. We have

$$\left|\int_\Omega F(w)\tilde{\chi}_0(w)d\nu(w)\right| = \left|\int_\Omega F(w)(\tilde{\chi}_0(w)-1)d\nu(w)\right| \le \nu\left(\rho \ge R\right)^{1/2} \le e^{-\beta R^2/2}. \tag{6.9}$$

The log-Sobolev inequality implies the Poincaré inequality and we have

$$\mathcal{E}(F\tilde{\chi}_0, F\tilde{\chi}_0) \ge \frac{1}{2\alpha R^2}\left(\|F\tilde{\chi}_0\|^2_{L^2(\nu)} - e^{-\beta R^2}\right). \tag{6.10}$$

Next we estimate $\mathcal{E}(F\tilde{\chi}_k, F\tilde{\chi}_k)$ for $k \ge 1$. Let $\phi_k(w) = 1_{[Rk,R(k+2)]}(\rho(w))$ and $\delta > 0$. Then by (6.4) and Lemma 6.1,

$$\begin{aligned}\mathcal{E}(F\tilde{\chi}_k, F\tilde{\chi}_k) &= \mathcal{E}(F\tilde{\chi}_k, F\tilde{\chi}_k) - \int_\Omega \delta\phi_k(w)(F\tilde{\chi}_k)^2(w)d\nu(w) + \int_\Omega \delta\phi_k(w)(F\tilde{\chi}_k)^2(w)d\nu(w)\\ &\ge -\frac{1}{\alpha R^2(k+2)^2}\log\left(\int_\Omega e^{\alpha\delta R^2(k+2)^2\phi_k(w)}d\nu(w)\right)\|F\tilde{\chi}_k\|^2_{L^2(\nu)}\\ &\quad + \delta\|F\tilde{\chi}_k\|^2_{L^2(\nu)}.\end{aligned} \tag{6.11}$$

By the tail estimate of ρ, we have

$$\int_\Omega e^{\alpha\delta R^2(k+2)^2\phi_k(w)}d\nu(w) \le 1 + e^{\alpha\delta R^2(k+2)^2 - \beta(Rk)^2}. \tag{6.12}$$

Hence

$$\mathcal{E}(F\tilde{\chi}_k, F\tilde{\chi}_k) \ge \left(\delta - \frac{\exp\left\{\left(\alpha\delta(k+2)^2 - \beta k^2\right)R^2\right\}}{\alpha R^2(k+2)^2}\right)\|F\tilde{\chi}_k\|^2_{L^2(\nu)}. \tag{6.13}$$

For simplicity, we write

$$G(\delta,\alpha,\beta,R) = \delta - \sup_{k\ge 1}\frac{\exp\left\{\left(\alpha\delta(k+2)^2 - \beta k^2\right)R^2\right\}}{\alpha R^2(k+2)^2}. \tag{6.14}$$

Summing the both sides in the inequalities (6.10), (6.13) and by using the property (iv), we obtain the following inequality

$$\mathcal{E}(F,F) \ge \min\left(\frac{1}{2\alpha R^2}, G(\delta,\alpha,\beta,R)\right) - \frac{e^{-\beta R^2}}{2\alpha R^2} - \frac{8}{R^2}\int_{\rho\ge R} F^2 d\nu \tag{6.15}$$

which is denoted by $I(\delta,\alpha,\beta,R)$. If $\frac{1}{2\alpha} > 8$, this inequality with large R and small δ implies the existence of spectral gap. In general, we need more considerations. Since $\sum_{k=1}^\infty \|F\tilde{\chi}_k\|^2_{L^2(\nu)} \ge \int_{\rho\ge 2R} F(w)^2 d\nu(w)$, by (6.8) and (6.13),

$$\mathcal{E}(F,F) \ge G(\delta,\alpha,\beta,R)\int_{\rho\ge 2R} F^2(w)d\nu(w) - \frac{8}{R^2}. \tag{6.16}$$

Let $0 \le \varepsilon \le 1$. Multiplying both sides on the inequality $I(\delta,\alpha,\beta,2R)$ by $1-\varepsilon$ and the both sides on (6.16) by ε and taking summation, we obtain

$$\begin{aligned}\mathcal{E}(F,F) &\ge (1-\varepsilon)\min\left(\frac{1}{8\alpha R^2}, G(\delta,\alpha,\beta,2R)\right)\\ &\quad - \frac{(1-\varepsilon)e^{-4\beta R^2}}{8\alpha R^2} - \frac{8\varepsilon}{R^2} + \left(\varepsilon G(\delta,\alpha,\beta,R) - \frac{2(1-\varepsilon)}{R^2}\right)\int_{\rho\ge 2R} F^2(w)d\nu(w).\end{aligned} \tag{6.17}$$

Now let $\delta = \frac{\beta}{18\alpha}$. Then by an elementary calculation,

$$G(\delta, \alpha, \beta, R) \geq \frac{\beta}{18\alpha} - \frac{e^{-\beta R^2/2}}{9\alpha R^2}. \tag{6.18}$$

Hence, if

$$\frac{\beta}{18\alpha} \geq \frac{e^{-\beta R^2/2}}{9\alpha R^2} + \frac{2(1-\varepsilon)}{R^2 \varepsilon}, \tag{6.19}$$

then

$$\mathcal{E}(F,F) \geq (1-\varepsilon) \min\left(\frac{1}{8\alpha R^2}, \frac{\beta}{18\alpha} - \frac{e^{-2\beta R^2}}{36\alpha R^2}\right) - \frac{(1-\varepsilon)e^{-4\beta R^2}}{8\alpha R^2} - \frac{8\varepsilon}{R^2} \tag{6.20}$$

By choosing ε, R appropriately, we give a lower bound for $\mathcal{E}(F,F)$. First, let us choose ε such that

$$\varepsilon = \min\left(\frac{1}{2},\ \frac{1}{512\alpha}\right). \tag{6.21}$$

We next choose R such that

$$\max\left(\frac{e^{-\beta R^2/2}}{9\alpha R^2},\ \frac{2}{R^2\varepsilon}\right) \leq \frac{\beta}{36\alpha}. \tag{6.22}$$

This condition is equivalent to

$$e^{-\beta R^2/2} \leq \frac{1}{4}\beta R^2, \quad R^2 \geq \frac{72\alpha}{\beta\varepsilon}. \tag{6.23}$$

Under this condition, the inequality (6.19) holds and by using (6.20), we have

$$\mathcal{E}(F,F) \geq \frac{1}{2} \min\left(\frac{1}{8\alpha R^2},\ \frac{\beta}{36\alpha}\right) - \frac{e^{-4\beta R^2}}{8\alpha R^2} - \frac{8\varepsilon}{R^2}. \tag{6.24}$$

Furthermore, we restrict R so that

$$\max\left(\frac{e^{-4\beta R^2}}{8\alpha R^2},\ \frac{8\varepsilon}{R^2}\right) \leq \frac{1}{8} \min\left(\frac{1}{8\alpha R^2},\ \frac{\beta}{36\alpha}\right). \tag{6.25}$$

This condition is equivalent to

$$e^{-2\beta R^2} \leq \frac{1}{8}, \quad e^{-4\beta R^2} \leq \frac{\beta R^2}{36}, \quad \varepsilon \leq \frac{1}{512\alpha}, \quad R^2 \geq \frac{48^2\alpha}{\beta}\varepsilon. \tag{6.26}$$

Thus, (6.22) and (6.25) hold if

$$R \geq \max\left(\sqrt{\frac{2}{\beta}},\ 48\sqrt{\frac{\alpha}{\beta}},\ \sqrt{\frac{72\alpha}{\beta\varepsilon}}\right). \tag{6.27}$$

Combining the inequalities (6.24) and (6.25), we obtain the desired estimate. □

7 Proof of Theorem 3.6

We prove Theorem 3.6 by using the argument in the proof of Theorem 3.2 and Theorem 6.2. To this end, we need a tail estimate of $\rho_{y_0}(\gamma)$.

Lemma 7.1. *Let M be an n-dimensional rotationally symmetric Riemannian manifold with a pole y_0. Suppose $\|\varphi'\|_\infty < \infty$ and Assumption A is satisfied. Let $\lambda_0 > 0$. Let $\rho_{y_0}(\gamma) = 1 + \max_{0\le t\le 1} d(y_0, \gamma(t))$. Then there exists a positive constant r_0 which depends on φ, λ_0, $d(x_0, y_0)$ and the dimension n and a positive constant C_2 which depends only on n such that*

$$\nu^{\lambda}_{x_0,y_0}\left(\rho_{y_0}(\gamma) \ge r\right) \le e^{-C_2\lambda r^2} \quad \text{for all } r \ge r_0 \text{ and } \lambda \ge \lambda_0. \tag{7.1}$$

Proof. Let z_0 be a point either x_0 or y_0. Let X_t be the Brownian motion starting at z_0 on M whose generator is $\Delta/(2\lambda)$. First, we give a tail estimate for ρ_{y_0} with respect to $\nu^{\lambda}_{z_0}$. Let $Y_t = d(X_t, y_0)$. Note that $\Delta_x d(x, y_0) = (n-1)\left(\frac{1}{d(x,y_0)} + \varphi'(d(x, y_0))\right)$ and $|\nabla_x d(x, y_0)| = 1$. By the Itô formula, we have

$$Y_t = d(z_0, y_0) + \frac{1}{\sqrt{\lambda}}B_t + \int_0^t \frac{n-1}{2\lambda}\left(\frac{1}{Y_s} + \varphi'(Y_s)\right) ds. \tag{7.2}$$

Here B_t is 1-dimensional standard Brownian motion. We can rewrite this equation as

$$\begin{aligned}\sqrt{\lambda}Y_t = \sqrt{\lambda}d(z_0, y_0) + B_t + \frac{n-1}{2\sqrt{\lambda}}\|\varphi'\|_\infty t + \int_0^t \frac{n-1}{2\sqrt{\lambda}Y_s} ds \\ + \int_0^t \frac{n-1}{2\sqrt{\lambda}}\left(\varphi'(Y_s) - \|\varphi'\|_\infty\right) ds.\end{aligned} \tag{7.3}$$

Let $\tilde{Z}_t$ be the strong solution to the SDE:

$$\tilde{Z}_t = \sqrt{\lambda}d(z_0, y_0) + B_t + \frac{n-1}{2\sqrt{\lambda}}\|\varphi'\|_\infty t + \int_0^t \frac{n-1}{2\tilde{Z}_s} ds, \tag{7.4}$$

where B_t is the same Brownian motion as in (7.3). Then by the comparison theorem of 1 dimensional SDE (see Chapter VI in [38]), we see

$$\sqrt{\lambda}Y_t \le \tilde{Z}_t \qquad t \ge 0. \tag{7.5}$$

Let us define $\hat{Z}_t = \tilde{Z}_t - \frac{n-1}{2\sqrt{\lambda}}\|\varphi'\|_\infty t$. Then $\hat{Z}_t$ satisfies the SDE

$$\hat{Z}_t = \sqrt{\lambda}d(z_0, y_0) + \int_0^t \frac{n-1}{2} \frac{1}{\hat{Z}_s + \frac{n-1}{2\sqrt{\lambda}}\|\varphi'\| s} ds + B_t. \tag{7.6}$$

Now consider the $n-1$ dimensional Bessel process Z_t as the strong solution of the SDE:

$$Z_t = \sqrt{\lambda}d(z_0, y_0) + \int_0^t \frac{n-1}{2}\frac{1}{Z_s} ds + B_t. \tag{7.7}$$

Again by the comparison theorem, we have

$$\hat{Z}_t \le Z_t \qquad t \ge 0. \tag{7.8}$$

The law of $\{Z_t\}_{t\ge 0}$ is the same as the law of $\{|B_t^{(n)}+\sqrt{\lambda}d(z_0,y_0)\mathbf{e}|\}$, where $B^{(n)}$ is the standard Brownian motion starting at 0 and $\mathbf{e}$ is the unit vector in $\mathbb{R}^n$. Thus, for any $r>0$, we have

$$\begin{aligned} P\left(\max_{0\le t\le 1} Y_t \ge r\right) &\le P\left(\max_{0\le t\le 1}|B_t^{(n)}+\sqrt{\lambda}d(z_0,y_0)\mathbf{e}|+\frac{n-1}{2\sqrt{\lambda}}\|\varphi'\|_\infty \ge \sqrt{\lambda}r\right) \\ &\le P\left(\max_{0\le t\le 1}|B_t^{(n)}| \ge \sqrt{\lambda}\left(r-d(z_0,y_0)-\frac{n-1}{2\lambda}\|\varphi'\|_\infty\right)\right). \end{aligned} \tag{7.9}$$

Let $C_n=E[\max_{0\le t\le 1}|B_t^{(n)}|]$. Then there exists $C>0$ such that for any $r>C_n$,

$$P\left(\max_{0\le t\le 1}|B_t^{(n)}|\ge r\right)\le C\exp\left(-\frac{1}{2}(r-C_n)^2\right). \tag{7.10}$$

Hence, if $r>d(z_0,y_0)+\frac{n-1}{2\lambda}\|\varphi'\|_\infty+\frac{C_n}{\sqrt{\lambda}}$, then

$$P\left(\max_{0\le t\le 1} Y_t\ge r\right)\le C\exp\left[-\frac{\lambda}{2}\left(r-d(z_0,y_0)-\frac{n-1}{2\lambda}\|\varphi'\|_\infty-\frac{C_n}{\sqrt{\lambda}}\right)^2\right]. \tag{7.11}$$

This shows that there exists $r_0>0$ which depends only on $d(z_0,y_0),\lambda_0$ and a positive constant C such that

$$\nu_{z_0}^\lambda\left(\rho_{y_0}(\gamma)\ge r\right)\le e^{-\lambda C r^2}\quad \text{for all } r\ge r_0. \tag{7.12}$$

The tail estimate for ν_{x_0,y_0}^λ can be proved by using the absolute continuity of ν_{x_0,y_0}^λ with respect to $\nu_{x_0}^\lambda$ up to time $t<1$. The density is given by

$$\frac{d\nu_{x_0,y_0}^\lambda}{d\nu_{x_0}^\lambda}(\gamma)\Big|_{\mathfrak{F}_t}=\frac{p\left(\frac{1-t}{\lambda},y_0,\gamma(t)\right)}{p\left(\frac{1}{\lambda},y_0,x_0\right)}=\varphi_{x_0,y_0}(t,\gamma). \tag{7.13}$$

Recall that Gaussian upper bound holds for all $0<t\le 1$ and $x,y\in M$,

$$p(t,x,y)\le C't^{-n/2}e^{-C''d(x,y)^2/t}. \tag{7.14}$$

By Varadhan's heat kernel estimate, for any $\varepsilon>0$, we have for sufficiently large λ,

$$p(1/\lambda,y_0,x_0)\ge e^{-\lambda\frac{d(y_0,x_0)^2+\varepsilon}{2}}. \tag{7.15}$$

By using these estimates, we obtain

$$\varphi_{x_0,y_0}\left(\frac{1}{2},\gamma\right)\le C'\lambda^{n/2}e^{\frac{\lambda}{2}\left(d(x_0,y_0)^2+\varepsilon\right)}. \tag{7.16}$$

This estimate and (7.12) implies that

$$\nu_{x_0,y_0}^\lambda\left(1+\max_{0\le t\le 1/2}d(y_0,\gamma(t))\ge r\right)\le C'\lambda^{n/2}e^{\frac{\lambda}{2}\left(d(x_0,y_0)^2+\varepsilon\right)-\lambda C r^2}\quad \text{for all } r\ge r_0 \tag{7.17}$$

Since

$$\nu_{x_0,y_0}^\lambda\left(1+\max_{1/2\le t\le 1}d(y_0,\gamma(t))\ge r\right)=\nu_{y_0,x_0}^\lambda\left(1+\max_{0\le t\le 1/2}d(y_0,\gamma(t))\ge r\right), \tag{7.18}$$

using (7.12) with $z_0=y_0$, similarly, we obtain the desired tail estimate for ρ_{y_0} under ν_{x_0,y_0}^λ. $\square$

Proof of Theorem 3.6. Let $\lambda_0 > 0$ and consider a positive number $\lambda \geq \lambda_0$. By Lemma 7.1, the assumptions in Theorem 6.2 are valid for $\rho = \rho_{y_0}$, $\alpha = C_1/\lambda, \beta = C_2\lambda$ and r_0. Hence Theorem 6.2 implies $e_2^\lambda > 0$ for all $\lambda > 0$. We need to prove the asymptotic behavior (3.15). We argue similarly to the proof of Theorem 6.2. That is, we use the same functions there and choose R, δ, ε which were defined there. Let $F \in \mathcal{FC}_b^\infty(P_{x_0,y_0}(M))$ and assume $\|F\|_{L^2(\nu^\lambda_{x_0,y_0})} = 1$ and $E^{\nu^\lambda_{x_0,y_0}}[F] = 0$. Then by the IMS localization formula (6.8), we get

$$\mathcal{E}^\lambda(F,F) \geq \mathcal{E}^\lambda(F\tilde{\chi}_0, F\tilde{\chi}_0) + (C\lambda^2 - C'\lambda)\sum_{k=1}^\infty \|F\tilde{\chi}_k\|^2_{L^2} - \frac{8}{R^2}. \tag{7.19}$$

Next we estimate $\mathcal{E}^\lambda(F\tilde{\chi}_0, F\tilde{\chi}_0)$. Since this is a local estimate, we may vary the Riemannian metric so that the metric is flat outside certain compact subset. Take the same function $\eta_{1,\kappa}, \eta_{2,\kappa}$ as in the proof of the lower bound estimate in Theorem 3.2. Then by the estimate (5.32), $|E^{\nu^\lambda_{x_0,y_0}}[F\tilde{\chi}_0\eta_{1,\kappa}]| \leq Ce^{-\lambda C}$. In a similar way to the proof of the lower bound in Theorem 3.2, we obtain

$$\mathcal{E}^\lambda(F\tilde{\chi}_0, F\tilde{\chi}_0) \geq \lambda \min\left(\left(\left(\|(S^{-1})^*\|_{op} + C\varepsilon\right)^{-2}, \lambda\delta - Ce^{-\lambda\delta'}\right)\right) \|F\tilde{\chi}_0\|^2_{L^2(\nu^\lambda_{x_0,y_0})} - Ce^{-\lambda C} - M(\kappa).$$

Combining the above, we complete the proof of the lower bound. The upper bound estimate immediately follows from the estimate (5.46) and (5.51). □

Acknowledgement
This research was partially supported by Grant-in-Aid for Scientific Research (B) No.24340023. The author would like to thank referees for their valuable comments and suggestions which improve the quality of the paper.

References

[1] S. Aida, On the irreducibility of certain Dirichlet forms on loop spaces over compact homogeneous spaces. New trends in stochastic analysis (Charingworth, 1994), 3—42, World Sci. Publ., River Edge, NJ, 1997.

[2] S. Aida, Gradient estimates of harmonic functions and the asymptotics of spectral gaps on path spaces, Interdisciplinary Information Sciences, Vol.2, No.1, 75–84 (1996).

[3] S. Aida, Logarithmic derivatives of heat kernels and logarithmic Sobolev inequalities with unbounded diffusion coefficients on loop spaces. J. Funct. Anal. 174 (2000), no. 2, 430–477.

[4] S. Aida, Semiclassical limit of the lowest eigenvalue of a Schrödinger operator on a Wiener space, J. Funct. Anal. 203 (2003), no.2, 401–424.

[5] S. Aida, Precise Gaussian estimates of heat kernels on asymptotically flat Riemannian manifolds with poles, in "Recent developments in stochastic analysis and related topics", Proceedings of the First Sino-German conference on stochatsic analysis 1–19, 2004.

[6] S. Aida, Semi-classical limit of the bottom of spectrum of a Schrödinger operator on a path space over a compact Riemannian manifold. J. Funct. Anal. 251 (2007), no. 1, 59–121.

[7] S. Aida, COH formula and Dirichlet Laplacians on small domains of pinned path spaces, Contemporary Mathematics, 545, Amer.Math. Soc.,Providence, RI, 2011, 1-12.

[8] S. Aida, Vanishing of one dimensional L^2-cohomologies of loop groups, J.Funct.Anal. 261 (2011), no.8, 2164-2213.

[9] L. Andersson and B. Driver, Finite-dimensional approximations to Wiener measure and path integral formulas on manifolds. J. Funct. Anal. 165 (1999), no. 2, 430—498.

[10] A. Besse, Einstein manifolds. Ergebnisse der Mathematik und ihrer Grenzgebiete, 10. Springer-Verlag, Berlin, 1987.

[11] M. Capitaine, E. Hsu and M. Ledoux, Martingale representation and a simple proof of logarithmic Sobolev inequalities on path spaces. Electron. Comm. Probab. 2 (1997), 71–81

[12] P. Cattiaux, I. Gentil and A. Guillin, Weak logarithmic Sobolev inequalities and entropic convergence. Probab. Theory Related Fields 139 (2007), no. 3-4, 563–603.

[13] X. Chen, X.-M. Li and B. Wu, A Poincaré inequality on loop spaces. J. Funct. Anal. 259 (2010), no. 6, 1421—1442.

[14] X. Chen, X.-M. Li and B. Wu, A spectral gap for the Brownian bridge measure on hyperbolic spaces. Progress in analysis and its applications, 398–404, World Sci. Publ., 2010.

[15] X. Chen, X.-M. Li and B. Wu, A concrete estimate for the weak Poincaré inequality on loop space, Probab.Theory Relat. Fields 151 (2011), no.3-4, 559-590.

[16] B. Chow and S-C, Chu, *etal.*, The Ricci flow: techniques and applications. Part I. Geometric aspects. Mathematical Surveys and Monographs, 135. American Mathematical Society, Providence, RI, 2007.

[17] A.B. Cruzeiro and P. Malliavin, Renormalized differential geometry on path space: structural equation, curvature. J. Funct. Anal. 139 (1996), no. 1, 119—181.

[18] B.K. Driver, A Cameron-Martin type quasi-invariance theorem for Brownian motion on a compact Riemannian manifold. J. Funct. Anal. 110 (1992), no. 2, 272—376.

[19] B.K. Driver, A Cameron-Martin type quasi-invariance theorem for pinned Brownian motion on a compact Riemannian manifold, Trans. Amer. Math. Soc. 342 (1994), no. 1, 375—395.

[20] B.K. Driver, The non-equivalence of Dirichlet forms on path spaces. Stochastic analysis on infinite-dimensional spaces (Baton Rouge, LA, 1994), 75—87, Pitman Res. Notes Math. Ser., 310, Longman Sci. Tech., Harlow, 1994.

[21] A. Eberle, Absence of spectral gaps on a class of loop spaces. J. Math. Pures Appl. (9) 81 (2002), no. 10, 915–955.

[22] A. Eberle, Spectral gaps on discretized loop spaces. Infin. Dimens. Anal. Quantum Probab. Relat. Top. 6 (2003), no. 2, 265–300.

[23] A. Eberle, Local spectral gaps on loop spaces. J. Math. Pures Appl. (9) 82 (2003), no. 3, 313–365.

[24] K.D. Elworthy and Xue-Mei Li, Itô maps and analysis on path spaces. Math. Z. 257 (2007), no. 3, 643—706.

[25] O. Enchev and D.W. Stroock, Integration by parts for pinned Brownian motion. Math. Res. Lett. 2 (1995), no. 2, 161—169.

[26] O. Enchev and D.W. Stroock, Pinned Brownian motion and its perturbations. Adv. Math. 119 (1996), no. 2, 127—154.

[27] S. Fang, Inégalité du type de Poincaré sur l'espace des chemins riemanniens. C. R. Acad. Sci. Paris Sér. I Math. 318 (1994), no. 3, 257—260.

[28] P.Friz and M. Hairer, A course on rough paths. With an introduction to regularity structures. Universitext. Springer, 2014.

[29] P. Friz and N. Victoir, Multidimensional stochastic processes as rough paths. Theory and applications. Cambridge Studies in Advanced Mathematics, 120. Cambridge University Press, Cambridge, 2010.

[30] F. Gong and Z. Ma, The log-Sobolev inequality on loop space over a compact Riemannian manifold. J. Funct. Anal. 157 (1998), no. 2, 599–623.

[31] M. Gordina, Quasi-invariance for the pinned Brownian motion on a Lie group, Stochastic Process. Appl. Vol. 104 (2003), 243–257.

[32] R.E. Greene and H. Wu, Function thorey on manifolds which possess a pole, Lecture Notes in Mathematics, 699 (1979), Springer, Berlin.

[33] L. Gross, Logarithmic Sobolev inequalities. Amer. J. Math. 97 (1975), no. 4, 1061—1083.

[34] B. Helffer and F. Nier, Quantitative analysis of metastability in reversible diffusion processes via a Witten complex approach: the case with boundary. Mm. Soc. Math. Fr. (N.S.) No. 105 (2006),

[35] R. Holley, S. Kusuoka and D.W. Stroock, Asymptotics of the spectral gap with applications to the theory of simulated annealing. J. Funct. Anal. 83 (1989), no. 2, 333—347.

[36] E. Hsu, Stochastic analysis on manifolds. Graduate Studies in Mathematics, 38. American Mathematical Society, Providence, RI, 2002.

[37] E. Hsu, Quasi-invariance of the Wiener measure on path spaces: noncompact case, J. Funct. Anal. 193 (2002), no. 2, 278—290.

[38] N. Ikeda and S. Watanabe, Stochastic differential equations and diffusion processes. Second edition. North-Holland Mathematical Library, 24, 1989.

[39] Y. Inahama, Large deviation principle of Freidlin-Wentzell type for pinned diffusion processes, to appear in Trans. Amer. Math. Soc., 35 pages. arXiv:1203.5177

[40] J. Jost, Riemannian geometry and geometric analysis, second edition, Springer, 1998.

[41] T. Laetsch, An approximation to Wiener measure and quantization of the Hamiltonian on manifolds with non-positive sectional curvature. J. Funct. Anal. 265 (2013), no. 8, 1667–1727.

[42] R. Léandre, Integration by parts formulas and rotationally invariant Sobolev calculus on free loop spaces, J. Geom. Phys. 11 (1993), no. 1-4, 517–528.

[43] M. Ledoux, Z. Qian and T. Zhang, Large deviations and support theorem for diffusions via rough paths, Stochastic proccess and their applications, 102, No.2 (2002), 265–283.

[44] P. Li and S.T. Yau, On the parabolic kernel of the Schrödinger operator. Acta Math. 156 (1986), no. 3-4, 153—201.

[45] A. Lim, Path integrals on a compact manifold with non-negative curvature, Reviews in Mathematical Physics, Vol. 19, No. 9 (2007), 967–1044.

[46] T. Lyons, Differential equations driven by rough signals, Rev.Mat.Iberoamer., 14 (1998), 215-310.

[47] T. Lyons, M. Caruana and T. Lévy, Difefrential equations driven by rough paths, Ecole d'Eté de Probabilités de Saint-Flour XXXIV-2004, Lecture Notes in Mathematics, 1908, Springer-Verlag Berlin Heiderberg 2007.

[48] T. Lyons and Z. Qian, System control and rough paths, Oxford Mathematical Monographs, 2002.

[49] P. Malliavin and D.W. Stroock, Short time behavior of the heat kernel and its logarithmic derivatives. J. Differential Geom. 44 (1996), no. 3, 550–570.

[50] D. Nualart, The Malliavin calculus and related topics, Second edition. Probability and its Applications, Springer-Verlag, Berlin, 2006.

[51] T. Sasamori, On estimates of heat kernels on Riemannian manifolds with poles, Master Thesis in 2015, March.

[52] B. Simon, Semiclassical Analysis of Low Lying Eigenvalues I. Nondegenerate Minima: Asymptotic Expansions, Ann. Inst. Henri Poincaré, Section A, Vol. XXXVIII, no. 4, (1983), 295–308.

[53] I. Shigekawa, Stochastic analysis, Translations of Mathematical Monographs, 224. Iwanami Series in Modern Mathematics, American Mathematical Society, Providence, RI, 2004.

[54] D.W. Stroock, An estimate on the Hessian of the heat kernel. Itô's stochastic calculus and probability theory, 355–371, Springer, Tokyo, 1996.

[55] S. Watanabe, Analysis of Wiener functionals (Malliavin calculus) and its applications to heat kernels, Ann. of Probab. Vol. 15, No.1, (1987), 1–39.