Rough path analysis : An Introduction

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1 Introduction

Let $x : [0, T] \to \mathbb{R}^d$ be a C^1 path with $x_0 = 0$. Let $f \in C_b^\infty(\mathbb{R}^d, L(\mathbb{R}^d \to \mathbb{R}^m))$ *. Then the limit

$$I_{0,T}(x) = \int_0^T f(x_u)dx_u = \lim_{|D|\to 0} \sum_{i=1}^N f(x_{s_{i-1}})\left(x_{t_i} - x_{t_{i-1}}\right) \in \mathbb{R}^m, \qquad t_{i-1} \le s_{i-1} \le t_i, \tag{1.1}$$

exists. Here $D = \{0 = t_0 < \cdots < t_N = T\}$ and $|D| = \max_{1\le i\le N}(t_i - t_{i-1})$. Moreover the functional $x \to I_{0,T}(x)$ is continuous in the topology of C^1. Let $p \ge 1$ and define the p-variation norm † of x by

$$\|x\|_p := \left\{\sup_D \sum_{i=1}^N |x_{t_i} - x_{t_{i-1}}|^p\right\}^{1/p}.$$

We denote by $B_{p,T}(\mathbb{R}^d)$ the Banach space which consists of continuous paths starting at 0 with finite p-variation norm $\| \ \|_p$. The 1-variation norm of x is the same as the total variation of x. The p-variation norm defines a weaker topology of the C^1-path space with $x_0 = 0$. Actually it is proved that $I_{0,T}(x)$ is a continuous functional of x in the p-variation topology for any $1 \le p < 2$. However it is not a continuous function in general with respect to p-variation norm for $p \ge 2$ if $d \ge 2$. Terry Lyons [15] proved the following continuity results:

Let $2 \le p < 3$ and x and y be C^1-path starting from 0. If the p-variation norm of $x - y$ is small and the $p/2$-variation norm of the difference X^2 and Y^2 (see the next section with respect to the definition of $p/2$-variation norm of them) is also small, then $I_{0,T}(x) - I_{0,T}(y)$ is small. Here $X_{s,t}^2 = \left(X^{2,ij}(s,t)\right)_{1\le i\le j\le d}$ and

$$\begin{aligned} X_{s,t}^{2,ij} &= \int_s^t (x_u^i - x_s^i)dx_u^j, \qquad 0 \le s \le t \le T, \\ x_t &= (x_t^i)_{1\le i\le d}. \end{aligned}$$

The iterated integral $X_{s,t}^2$ naturally appears when we approximate the integral $I_{0,T}(x)$ using Taylor's expansion. This observation leads to a notion of p-rough path ($p \ge 2$) which is an extended notion of smooth (or p-bounded variation ($1 \le p < 2$)) path. The aim of this talk is to introduce the audiences to rough path analysis. I recommend [17] as well as [16] as good references for rough path analysis.

*C_b^m denotes the set of functions which are m-times continuously differentiable and themselves and all derivatives are bounded. The norm is given by $\sum_{i=0}^m \|\nabla^i f\|_\infty$. $\| \ \|_\infty$ denotes the supremum norm.

†If we do not fix the starting point, we need to add $|x_0|$ or the supremum norm $\|x\|_\infty$ to make $\| \ \|_p$ to be a norm.

2 A continuity theorem of line integrals as a functional of paths

Let $f(x)=(f^i_j(x))_{1\le i\le m,1\le j\le d}$ $(x\in\mathbb{R}^d)$ be a (m,d)matrices valued C^∞_b function. For a C^1 path $x:[0,T]\to\mathbb{R}^d$, define the Riemann-Stieltjes integral

$$I_{s,t}(x):=\int_s^t f(x_u)dx_u=\left(\int_s^t f^i_j(x_u)dx^j_u\right)_i,\quad (s,t\in[0,T]) \tag{2.1}$$

which gives a C^1 path on $\mathbb{R}^m$.

We consider two quantities which are approximations of $I_{s,t}(x)$. To this end we introduce the following notation.

Definition 2.1 *Let $x:[0,T]\to\mathbb{R}^d$ be a C^1 path on $\mathbb{R}^d$. Define continuous mappings from $\Delta_T=\{(s,t)\mid 0\le s\le t\le T\}$ to $\mathbb{R}^d$ and $\mathbb{R}^d\otimes\mathbb{R}^d$ by*

$$X^1_{s,t} = \sum_{i=1}^d X^{1,i}_{s,t}e_i := x_t-x_s=\sum_{i=1}^d(x^i_t-x^i_s)e_i\in\mathbb{R}^d \tag{2.2}$$

$$\begin{aligned}X^2_{s,t} &= \sum_{1\le i,j\le d} X^{2,ij}_{s,t}e_i\otimes e_j\\ &:= \int_s^t (x_u-x_s)\otimes dx_u\\ &= \sum_{1\le i,j\le d}\left(\int_s^t(x^i_u-x^i_s)dx^j_u\right)e_i\otimes e_j\in\mathbb{R}^d\otimes\mathbb{R}^d,\end{aligned} \tag{2.3}$$

where $e_i={}^t(0,\ldots,\overset{i}{1},\ldots,0)$ denotes the orthonormal basis of $\mathbb{R}^d$. $\mathbb{R}^d\otimes\mathbb{R}^d$ denotes the tensor product and $\{e_i\otimes e_j\}_{1\le i,j\le d}$ is an orthonormal basis of it.

Using these notation, we define

$$\tilde{I}_{s,t}(x) := f(x_s)X^1_{s,t}, \tag{2.4}$$

$$J_{s,t}(x) := f(x_s)X^1_{s,t}+(\nabla f)(x_s)\left(X^2_{s,t}\right). \tag{2.5}$$

We use the convention:

$$[(\nabla f)(x)(a\otimes b)]^i = \sum_{1\le j,k\le d}\frac{\partial f^i_j}{\partial x_k}(x)a^kb^j, \tag{2.6}$$

$$\left[(\nabla f)(x_s)(X^2_{s,t})\right]^i = \sum_{1\le j,k\le d}\frac{\partial f^i_j}{\partial x_k}(x_s)\int_s^t(x^k_u-x^k_s)dx^j_u \tag{2.7}$$

$$\left[(\nabla^2 f)(x)(a\otimes b\otimes c)\right]^i = \sum_{1\le k,l\le d}\frac{\partial^2 f^i_j}{\partial x_l\partial x_k}(x)a^lb^kc^j, \tag{2.8}$$

where $a=\sum_{i=1}^d a^ie_i, b=\sum_{i=1}^d b^ie_i, c=\sum_{i=1}^d c^ie_i$ and $[\cdot]^i$ denotes the i-th element.

One may say that $\tilde{I}_{s,t}(x)$is the first approximation of $I_{s,t}(x)$ and $J_{s,t}(x)$ is the second approximation because

$$\begin{aligned}
I_{s,t}(x) &= \int_s^t \left[f(x_s) + \left\{ \int_0^1 (\nabla f)(x_s + \theta(x_u - x_s))d\theta \right\} (x_u - x_s) \right] dx_u \\
&= \int_s^t \left[f(x_s) + (\nabla f)(x_s)(x_u - x_s) \right] dx_u \\
&\quad + \int_s^t \left\{ \int_0^1 (\nabla f)(x_s + \theta(x_u - x_s))d\theta - (\nabla f)(x_s) \right\} (x_u - x_s)\, dx_u \\
&= J_{s,t}(x) + \int_s^t \left\{ \int_0^1 \left(\int_0^\theta (\nabla^2 f)(x_s + r(x_u - x_s))dr \right) d\theta \right\} \\
&\qquad\qquad [(x_u - x_s) \otimes (x_u - x_s) \otimes dx_u] \\
&=: J_{s,t}(x) + R_{s,t}(x) \qquad (2.9)
\end{aligned}$$

and

$$|R_{s,t}(x)| \le C \int_s^t |x_u - x_s|^2 |\dot{x}_u| du. \qquad (2.10)$$

Let $D := \{s = t_0 < t_1 < \cdots < t_N = t\}$ be a partition of $[s,t]$. Let

$$\tilde{I}_{s,t}(x, D) := \sum_{i=1}^N \tilde{I}_{t_{i-1},t_i}(x) \qquad (2.11)$$

$$J_{s,t}(x, D) := \sum_{i=1}^N J_{t_{i-1},t_i}(x) \qquad (2.12)$$

It is trivial that

$$I_{s,t}(x) = \sum_{i=1}^N I_{t_{i-1},t_i}(x) \qquad (2.13)$$

$$I_{s,t}(x) = \lim_{|D|\to 0} \tilde{I}_{s,t}(x, D) \qquad (2.14)$$

$$I_{s,t}(x) = \lim_{|D|\to 0} J_{s,t}(x, D), \qquad (2.15)$$

where $|D| = \max_{1\le i\le N}(t_i - t_{i-1})$. The limit of (2.14) is nothing but the definition of Stieltjes integral. Actually this limit exists for any continuous path x with finite p-variation with $1 \le p < 2$. This is proved by L.C. Young [24]. The approximation of (2.15) is crucial to prove the continuity theorem which is mentioned in the Introduction. We define q-variation norms for continuous mappings from Δ_T to V.

Definition 2.2 *Let $(V, \| \ \|)$ be a normed linear space. For a continuous mapping $\psi : \Delta_T \to V$ and $q \ge 1$, we define*

$$\|\psi\|_q = \sup_D \left\{ \sum_{i=1}^N |\psi_{t_{i-1},t_i}|^q \right\}^{1/q}, \qquad (2.16)$$

where $D := \{0 = t_0 < t_1 < \cdots < t_N = T\}$ *is a partition. Typical examples are* $\psi = X^1, X^2$ *which are defined by a* C^1 *path.* $\|X^1\|_p$ *coincides with the p-variation norm of* $x_t - x_0$ *which is defined in the Introduction. Also we denote* $\|\psi\|_{q,[s,t]} = \sup\Big\{\{\sum_{i=1}^N |\psi_{t_{i-1},t_i}|^q\}^{1/q} \;\Big|\; D = \{s = t_0 < \ldots < t_N = t\}\Big\}$.

For a C^1-path y, we define Y^1, Y^2 by y in the same way as in Definition 2.1. We can prove the following.

Theorem 2.3 *Let* $1 \le p < 2$. *Let* x *and* y *be* C^1 *paths on* $\mathbb{R}^d$ *with* $x_0 = y_0$. *Assume that*

$$\max\left\{\|X^1\|_p, \|Y^1\|_p\right\} \quad \le \quad R < \infty \tag{2.17}$$

$$\max\left\{\|X^1 - Y^1\|_p\right\} \le \varepsilon. \tag{2.18}$$

Then for all $0 \le s \le t \le T$,

$$|I_{s,t}(x) - I_{s,t}(y)| \le \varepsilon C, \tag{2.19}$$

where C *is a constant which depends on* $p, R, \|\nabla^i f\|_\infty \quad (i = 0, 1, 2)$.

Theorem 2.4 *Let* $2 \le p < 3$ *and* x, y *be* C^1 *paths on* $\mathbb{R}^d$ *with* $x_0 = y_0$. *Assume that*

$$\max\left\{\|X^1\|_p, \|Y^1\|_p, \|X^2\|_{p/2}, \|Y^2\|_{p/2}\right\} \quad \le \quad R < \infty \tag{2.20}$$

$$\max\left\{\|X^1 - Y^1\|_p, \|X^2 - Y^2\|_{p/2}\right\} \le \varepsilon. \tag{2.21}$$

Then for all $0 \le s \le t \le T$,

$$|I_{s,t}(x) - I_{s,t}(y)| \le \varepsilon \cdot C, \tag{2.22}$$

where C *is a constant which depends on* $p, R, \|\nabla^i f\|_\infty \quad (i = 0, 1, 2, 3)$.

In the theorem above, the starting point of x and y is the same. For the case where $x_0 \neq y_0$, we have the following.

Theorem 2.5 *Let* $y_t = x_t + \xi$ $(0 \le t \le T)$. *Then the following estimates hold, where* C, C' *are polynomial functions of* $\|X^1\|_p$ *and* $\|\nabla^i f\|$ $(i = 0, 1, 2)$, $\|X^1\|_p, \|X^2\|_{p/2}$ *and* $\|\nabla^i f\|$ $(i = 0, 1, 2, 3)$ *respectively.*
(1) *Let* $1 \le p < 2$. *It holds that* $|I_{s,t}(x) - I_{s,t}(y)| \le C|\xi|$.
(2) *Let* $2 \le p < 3$. *It holds that* $|I_{s,t}(x) - I_{s,t}(y)| \le C'|\xi|$.

Let μ be the Wiener measure on $\Theta^d = C([0,T] \to \mathbb{R}^d \mid w_0 = 0)$. That is μ is the unique probability measure such that for any increasing sequence $0 = t_0 < t_1 < \ldots < t_l = T$ and Borel subsets $A_i \subset \mathbb{R}^d$ $(1 \le i \le l)$ it holds that

$$\mu\left(\left\{w \in \Theta^d \;\Big|\; w_{t_i} \in A_i, 1 \le i \le l\right\}\right) = \int_{\mathbb{R}^d} dx_1 \cdots \int_{\mathbb{R}^d} dx_l \prod_{i=1}^{l} p(t_i - t_{i-1}, x_{i-1}, x_i) 1_{A_i}(x_i),$$

where $p(t, x, y) = \frac{1}{\sqrt{2\pi}^d} \exp\left(-\frac{|x-y|^2}{2t}\right)$ and $x_0 = 0$. The continuous stochastic process $X(t, w) = w_t$ $(0 \le t \le T, w \in \Theta^d)$ defined on a probability space $(\Theta^d, \mathcal{B}(\Theta^d), \mu)$ is a realization of Brownian motion starting at 0. For μ-almost all w, $\|w\|_2 = \infty$ although $\|w\|_p < \infty$ for all $p > 2$. See [22]. Therefore we cannot define the integral $\int_0^T f(w_t) dw_t$ as the Young integral. This integral is typical stochastic integral and is defined as an "Itô integral" or "Stratonovich integral" . However, note that Theorem 2.4 and the following theorem gives a meaning of this integral.

Theorem 2.6 *Let $w(n)$ be the dyadic polygonal approximation of w such that $w(n)_t = w_t$ for $t = \frac{k}{2^n}T$, $0 \le k \le 2^n$ and $w(n)$ is a linear function on each small interval $\left[\frac{k}{2^n}T, \frac{k+1}{2^n}T\right]$. We define $W(n)^1, W(n)^2$ by $w(n)$ in the same way as in Definition 2.1. Let*

$$\tilde{\Theta}^d = \left\{ w \in \Theta^d \;\middle|\; \lim_{n,m\to\infty} \max\{\|W(n)^1 - W(m)^1\|_p, \|W(n)^2 - W(m)^2\|_{p/2}\} = 0 \right\}. \tag{2.23}$$

Then $\mu(\tilde{\Theta}^d) = 1$.

By Theorem 2.4, for all $w \in X$, $\lim_{n\to\infty} I_{s,t}(w(n))$ exist. This is almost surely equal to the Stratonovich integral. That is,

Theorem 2.7 *For μ-almost all w,*

$$\lim_{n\to\infty} I_{s,t}(w(n)) = \int_s^t f(w_u) \circ dw_u = \sum_{i=1}^d \left(\int_s^t f_j^i(w_u) \circ dw_u^j \right) e_i \tag{2.24}$$

$$\lim_{n\to\infty} W(n)^2_{s,t} = \sum_{1\le i,j\le d} \left(\int_s^t (w_u^i - w_s^i) \circ dw_u^j \right) e_i \otimes e_j, \tag{2.25}$$

The integral of the right-hand side is the Stratonovich integral.

Remark 2.8 Stratonovich integral is defined by

$$\int_s^t f_j^i(w_u) \circ dw_u^j = \lim_{|D|\to 0} \sum_{i=1}^N \frac{f_j^i\left(w_{t_{i-1}}\right) + f_j^i\left(w_{t_i}\right)}{2} \left(w_{t_i}^j - w_{t_{i-1}}^j \right).$$

See [10]. Here $D = \{s = t_0 < \cdots < t_N = t\}$ is a partition of $[s,t]$. The limit exists in L^2. It is not trivial that this limit equals to the limit of the left-hand side of (2.24). However the proof is not difficult. For $w \in X$, we define $I_{s,t}(w)$ and $W^2_{s,t}$ by the limit of $I_{s,t}(w(n))$ and $W(n)^2_{s,t}$. The functional $w \to I_{s,t}(w)$ is a version of the Stratonovich integral and satisfies the continuity property as in Theorem 2.4.

3 Proof of continuity theorem

To prove Theorem 2.4, we introduce the following.

Definition 3.1 *A continuous function $\omega(\cdot,\cdot) : \Delta_T \to [0,\infty)$ is called a control function if it holds that for any $0 \le s \le u \le t \le T$,*

$$\omega(s,u) + \omega(u,t) \le \omega(s,t). \tag{3.1}$$

Let ω_1 and ω_2 be control functions. Then $(\omega_1^r + \omega_2^r)^{1/r}$ $(0 < r \le 1)$ is also a control function but $\omega(s,t) = \max\{\omega_1(s,t), \omega_2(s,t)\}$ may not be a control function. We give examples of control functions.

Example 3.2 (1) *Let $p \geq 1$ and $x \in B_{p,T}(\mathbb{R}^d)$. Let $\omega(s,t) = \|X^1\|^p_{p,[s,t]}$. Then $\omega(s,t)$ is a control function.*
(2) *For a C^1 path x and $p \geq 2$, define*

$$\omega(s,t) = \|X^1\|^p_{p,[s,t]} + \|X^2\|^{p/2}_{p/2,[s,t]}. \tag{3.2}$$

First we prove the following.

Theorem 3.3 *Let x be a C^1 path on $\mathbb{R}^d$.*
(1) *Let $1 \leq p < 2$. Assume that there exists a control function ω such that for all $0 \leq s \leq t \leq T$*

$$|X^1_{s,t}| \leq \omega(s,t)^{1/p}, \tag{3.3}$$

Then

$$|I_{s,t}(x)| = \left|\int_s^t f(x_u)dx_u\right| \leq C\left(\omega(s,t)^{1/p} + \omega(s,t)^{2/p}\right), \tag{3.4}$$

where C denotes a constant which depends on p, $\|\nabla^i f\|_\infty$ $(0 \leq i \leq 1)$.
(2) *Let $2 \leq p < 3$. Assume that there exists a control function ω such that for all $0 \leq s \leq t \leq T$*

$$|X^1_{s,t}| \leq \omega(s,t)^{1/p}, \tag{3.5}$$
$$|X^2_{s,t}| \leq \omega(s,t)^{2/p}. \tag{3.6}$$

Then

$$|I_{s,t}(x)| = \left|\int_s^t f(x_u)dx_u\right| \leq C\left(\omega(s,t)^{1/p} + \omega(s,t)^{2/p} + \omega(s,t)^{3/p}\right), \tag{3.7}$$

where C denotes a constant which depends on p, $\|\nabla^i f\|_\infty$ $(0 \leq i \leq 2)$.

Remark 3.4 (1) We give the proof for (2) only because the proof of (1) is almost similar to (1) and much easier. But here is a important remark. Assume that the continuous path x is of finite p-variation $(1 \leq p < 2)$ only. In this case, $\omega(s,t) = \|X^1\|^p_{[s,t],p}$ is a control function of X^1. As noted already, the limit $\lim_{|D|\to 0} \tilde{I}_{s,t}(x,D)$ exists and is called the Young integral. The convergence can be proved by the idea of the proof of this theorem and the estimate in (3.4) holds. See Remark 3.7.
(2) In the above theorem, we assume the boundedness of f and its derivatives. However, it is easy to check that the same theorem holds for C^2 function (or C^1 function in the case of (1)) without assuming the boundedness on $\nabla^i f$ $(0 \leq i \leq 2)$, for example, in the case of (2), replacing the supremum norm of $\nabla^i f$ on $\mathbb{R}^d$ by

$$\sup\left\{\sum_{i=0}^2 \|\nabla^i f(x)\| \mid |x| \leq |x_0| + \omega(0,T)^{1/p}\right\}.$$

For the proof, we use the following simple two lemmas.

Lemma 3.5 *Let $N \geq 2$ be a natural number. For a partition $D = \{s = t_0 < t_1 < \cdots < t_N = t\}$, there exists t_i such that*

$$\omega(t_{i-1}, t_{i+1}) \leq \frac{2\omega(s,t)}{N-1} \tag{3.8}$$

Proof.

$$\begin{aligned}
(N-1)\min_i \omega(t_{i-1}, t_{i+1}) &\leq \sum_{i=1}^{N-1} \omega(t_{i-1}, t_{i+1}) \\
&= \sum_{j\geq 0, 2j+2\leq N} \omega(t_{2j}, t_{2j+2}) + \sum_{l\geq 0, 2l+3\leq N} \omega(t_{2l+1}, t_{2l+3}) \\
&\leq 2\omega(s,t). \qquad (3.9)
\end{aligned}$$

■

Lemma 3.6 (Chen's identity) *$X^1_{s,t}, X^2_{s,t}$ satisfies the following algebraic relations. For any $0 \leq s < u < t \leq T$,*

$$\begin{aligned}
X^1_{s,t} &= X^1_{s,u} + X^1_{u,t} \qquad (3.10) \\
X^2_{s,t} &= X^2_{s,u} + X^2_{u,t} + X^1_{s,u} \otimes X^1_{u,t}. \qquad (3.11)
\end{aligned}$$

Proof. (3.10) is trivial. We prove (3.11).

$$\begin{aligned}
X^2_{s,t} &= \int_s^t (x_r - x_s) \otimes dx_r \\
&= \int_s^u (x_r - x_s) \otimes dx_r + \int_u^t (x_r - x_s) \otimes dx_r \\
&= \int_s^u (x_r - x_s) \otimes dx_r + \int_u^t (x_r - x_u) \otimes dx_r + (x_u - x_s) \otimes (x_t - x_u) \\
&= X^2_{s,u} + X^2_{u,t} + X^1_{s,u} \otimes X^1_{u,t}.
\end{aligned}$$

■

Proof of Theorem 3.3 (2) Let $D = \{s = t_0 < \cdots < t_N = t\}$ and assume that $N \geq 2$. Take a division point i satisfying (3.8). Set $D_{-1} := D \setminus \{t_i\}$. We estimate $J_{s,t}(x, D) - J_{s,t}(x, D_{-1})$.

Using (3.10), we have

$$\begin{aligned}
&J_{s,t}(x, D) - J_{s,t}(x, D_{-1}) \\
&\quad = J_{t_{i-1},t_i}(x) + J_{t_i,t_{i+1}}(x) - J_{t_{i-1},t_{i+1}}(x) \\
&\quad = f(x_{t_{i-1}})X^1_{t_{i-1},t_i} + f(x_{t_i})X^1_{t_i,t_{i+1}} - f(x_{t_{i-1}})X^1_{t_{i-1},t_{i+1}} \\
&\quad +\nabla f(x_{t_{i-1}})X^2_{t_{i-1},t_i} + \nabla f(x_{t_i})X^2_{t_i,t_{i+1}} - \nabla f(x_{t_{i-1}})X^2_{t_{i-1},t_{i+1}} \\
&\quad = \left(f(x_{t_i}) - f(x_{t_{i-1}})\right) X^1_{t_i,t_{i+1}} \\
&\qquad +\nabla f(x_{t_{i-1}})X^2_{t_{i-1},t_i} + \nabla f(x_{t_i})X^2_{t_i,t_{i+1}} - \nabla f(x_{t_{i-1}})X^2_{t_{i-1},t_{i+1}}. \qquad (3.12)
\end{aligned}$$

Next using (3.11) and Taylor's theorem,

$$
\begin{aligned}
&J_{s,t}(x,D)-J_{s,t}(x,D_{-1})\\
&=\left[\int_0^1\Big\{(\nabla f)\left(x_{t_{i-1}}+\theta(x_{t_i}-x_{t_{i-1}})\right)-(\nabla f)(x_{t_{i-1}})\Big\}d\theta\right]X^1_{t_{i-1},t_i}\otimes X^1_{t_i,t_{i+1}}\\
&+\nabla f(x_{t_{i-1}})X^1_{t_{i-1},t_i}\otimes X^1_{t_i,t_{i+1}}\\
&+\nabla f(x_{t_{i-1}})X^2_{t_{i-1},t_i}+\nabla f(x_{t_i})X^2_{t_i,t_{i+1}}-\nabla f(x_{t_{i-1}})X^2_{t_{i-1},t_{i+1}}\\
&=\left[\int_0^1\Big\{(\nabla f)\left(x_{t_{i-1}}+\theta(x_{t_i}-x_{t_{i-1}})\right)-(\nabla f)(x_{t_{i-1}})\Big\}d\theta\right]X^1_{t_{i-1},t_i}\otimes X^1_{t_i,t_{i+1}}\\
&+\left((\nabla f)(x_{t_i})-(\nabla f)(x_{t_{i-1}})\right)X^2_{t_i,t_{i+1}} \qquad (3.13)\\
&=R(f,x,t_{i-1},t_i)\left[X^1_{t_{i-1},t_i}\otimes X^1_{t_{i-1},t_i}\otimes X^1_{t_i,t_{i+1}}\right]\\
&+S(f,x,t_{i-1},t_i)\left[X^1_{t_{i-1},t_i}\otimes X^2_{t_i,t_{i+1}}\right], \qquad (3.14)
\end{aligned}
$$

where

$$
\begin{aligned}
R(f,x,t_{i-1},t_i) &= \int_0^1\left(\int_0^\theta(\nabla^2 f)\left(x_{t_{i-1}}+\tau(x_{t_i}-x_{t_{i-1}})\right)d\tau\right)d\theta\\
S(f,x,t_{i-1},t_i) &= \int_0^1(\nabla^2 f)\Big(x_{t_{i-1}}+\theta(x_{t_i}-x_{t_{i-1}})\Big)d\theta.
\end{aligned}
$$

By the assumption on the division point t_i,

$$
\begin{aligned}
|J_{s,t}(x,D)-J_{s,t}(x,D_{-1})| &\le C\cdot\|\nabla^2 f\|_\infty\left\{\left(\frac{2\omega(s,t)}{N-1}\right)^{3/p}+\left(\frac{2\omega(s,t)}{N-1}\right)^{1/p}\left(\frac{2\omega(s,t)}{N-1}\right)^{2/p}\right\}\\
&\le C\left(\frac{2\omega(s,t)}{N-1}\right)^{3/p}\|\nabla^2 f\|_\infty. \qquad (3.15)
\end{aligned}
$$

Next, we choose a division point t_i' from the partition $D\setminus\{t_i\}$ so that (3.8) holds replacing $N-1$ by $N-2$. Repeating this procedure $N-1$ times, we obtain

$$
\begin{aligned}
\left|J_{s,t}(x,D)-\left(f(x_s)X^1_{s,t}+\nabla f(x_s)X^2_{s,t}\right)\right| &\le C\cdot\sum_{k=2}^N\left(\frac{2\omega(s,t)}{k-1}\right)^{3/p}\|\nabla^2 f\|_\infty\\
&\le 2^{3/p}C\zeta\left(\frac{3}{p}\right)\|\nabla^2 f\|_\infty\omega(s,t)^{3/p},
\end{aligned}
$$

where $\zeta(s)=\sum_{n=1}^\infty\frac{1}{n^s}$. By the assumption that $p<3$, $\zeta\left(\frac{3}{p}\right)<\infty$. Thus,

$$
|J_{s,t}(x,D)|\le\|f\|_\infty\omega(s,t)^{1/p}+\|\nabla f\|_\infty\omega(s,t)^{2/p}+C\|\nabla^2 f\|_\infty\omega(s,t)^{3/p}.
$$

Since $\lim_{|D|\to0}J_{s,t}(x,D)=I_{s,t}(x)$, the proof is completed. ■

Remark 3.7 (1) The idea of the estimate of the integral choosing the minimum point is due to Young [24]. However, the use of the notion of control function seems to be due to Lyons [15].

(2) In the above proof, we use the elementary fact that $\lim_{|D|\to 0} J_{s,t}(x,D)$ converges. But this elementary fact actually can be checked by the argument above. Let D' be another partition of $[s,t]$. Let D'' be the common refinement of these two partitions. Let $s=t_0<\dots t_N=t$ be the division point of D. Let $D''(i)=\{t_{i-1}=s_0^i<\dots<s_{n(i)}^i=t_i\}$ be the division point in $[t_{i-1},t_i]$ of D''. Note that $J_{s,t}(x,D'')=\sum_{i=1}^N J_{t_{i-1},t_i}(x,D''(i))$ and $J_{t_{i-1},t_i}(x,D'')=\sum_{i=0}^{n(i)-1} J_{s_j^i,s_{j+1}^i}(x)$. Then using the above argument,

$$|J_{t_{i-1},t_i}(x)-J_{t_{i-1},t_i}(x,D'')| \quad\le\quad 2^{3/p}C\zeta\left(\frac{3}{p}\right)\|\nabla^2 f\|_\infty\omega(t_{i-1},t_i)^{3/p}$$

Therefore

$$\begin{aligned}|J_{s,t}(x,D)-J_{s,t}(x,D'')| &\le \sum_{i=1}^N |J_{t_{i-1},t_i}(x)-J_{t_{i-1},t_i}(x,D'')|\\ &\le 2^{3/p}C\zeta\left(\frac{3}{p}\right)\|\nabla^2 f\|_\infty \max_{1\le i\le N}\omega(t_{i-1},t_i)^{\frac{3}{p}-1}\omega(s,t). \qquad (3.16)\end{aligned}$$

This shows the convergence of $\lim_{|D|\to 0} J_{s,t}(x,D)$. This proof can be extended to the case of classical Young's integral.

We prove Theorem 2.4. We state more general results using control functions.

Theorem 3.8 *Let x and y be C^1 paths on $\mathbb{R}^d$ with $x_0=y_0$ and $2\le p<3$. Assume that there exists a control function ω such that for all $0\le s\le t\le T$,*

$$\begin{aligned}\max\left\{|X^1_{s,t}|,|Y^1_{s,t}|\right\} &\le \omega(s,t)^{1/p} \qquad (3.17)\\ \max\left\{|X^2_{s,t}|,|Y^2_{s,t}|\right\} &\le \omega(s,t)^{2/p} \qquad (3.18)\\ |X^1_{s,t}-Y^1_{s,t}| &\le \varepsilon\omega(s,t)^{1/p} \qquad (3.19)\\ |X^2_{s,t}-Y^2_{s,t}| &\le \varepsilon\omega(s,t)^{2/p}. \qquad (3.20)\end{aligned}$$

Then

$$|I_{s,t}(x)-I_{s,t}(y)|\le\varepsilon C\omega(s,t)^{1/p}. \qquad (3.21)$$

C is a constant which depends on $\omega(0,T), p$ and $\|\nabla^i f\|_\infty$ $(0\le i\le 3)$.

Proof. Let $N\ge 2$. We choose the sequence of division points as in (3.8) repeatedly from $D=\{s=t_0<\cdots<t_N=t\}$. We denote the sequence of partitions by D_{-k} $(1\le k\le N-1)$. Then

$$\begin{aligned}&|J_{s,t}(x,D)-J_{s,t}(y,D)|\\ &\quad\le\sum_{k=0}^{N-2}|\{J_{s,t}(x,D_{-k})-J_{s,t}(x,D_{-k-1})\}-\{J_{s,t}(y,D_{-k})-J_{s,t}(y,D_{-k-1})\}|\\ &\quad+|J_{s,t}(x)-J_{s,t}(y)|. \qquad (3.22)\end{aligned}$$

By (3.14) and the assumption,

$$
\begin{aligned}
&|\{J_{s,t}(x,D_{-k})-J_{s,t}(x,D_{-k-1})\}-\{J_{s,t}(y,D_{-k})-J_{s,t}(y,D_{-k-1})\}| \\
&\quad \le C\cdot\varepsilon\left(\frac{2\omega(s,t)}{N-k-1}\right)^{3/p}\left(\|\nabla^2 f\|_\infty+\|\nabla^3 f\|_\infty\right). \qquad (3.23)
\end{aligned}
$$

Also

$$
|J_{s,t}(x)-J_{s,t}(y)|\le\varepsilon\left(\|f\|_\infty+\|\nabla f\|_\infty(\omega(0,s)^{1/p}+\omega(s,t)^{1/p})+\|\nabla^2 f\|_\infty\omega(0,s)^{1/p}\right)\omega(s,t)^{1/p}.
$$

Taking the sum and the limit $|D|\to 0$, we get the conclusion.■

We prove Theorem 2.4 using Theorem 3.8.

Proof of Theorem 2.4 Let us define a control function ω by

$$
\begin{aligned}
\omega(s,t) \;=\;& \|X^1\|_{p,[s,t]}^p+\|Y^1\|_{p,[s,t]}^p+\|X^2\|_{p/2,[s,t]}^{p/2}+\|Y^2\|_{p/2,[s,t]}^{p/2} \\
&+\left(\varepsilon^{-1}\|X^1-Y^1\|_{p,[s,t]}\right)^p+\left(\varepsilon^{-1}\|X^2-Y^2\|_{p/2,[s,t]}\right)^{p/2}. \qquad (3.24)
\end{aligned}
$$

Then all assumptions of Theorem 3.8 hold. (3.21) implies the conclusion. ■

Remark 3.9 (1) Let $x,y:[0,T]\to\mathbb{R}$ be continuous paths with finite p-variation $(1\le p<2)$. Then we proved that $\int_0^T x_t dy_t$ can be defined as a limit of Riemann sums. Actually Young proved that for x and y with $\|x\|_p<\infty$ and $\|y\|_q<\infty$, where $\frac{1}{p}+\frac{1}{q}>1$, $\int_0^T x_t dy_t$ converges. Actually the above method proves this result too.
(2) Let us consider the case where $d=1$. In this case $I_{s,t}(x)=\int_s^t f(x_u)dx_u$ is a continuous functional in the uniform convergence topology. Actually in this case, iterated integral reads

$$
\int_s^t (x_u-x_s)dx_u=\frac{1}{2}(x_t-x_s)^2
$$

and this is a continuous functional of $X^1_{s,t}$. Also it is not difficult to see that the functional $x\to y$ which is obtained by solving the integral equation below is continuous in the uniform convergence topology:

$$
y_t \;=\; \xi+\int_0^t \sigma\left(y_s+\xi\right)dx_t \qquad (3.25)
$$

where σ is a bounded Lipschitz continuous function. In higher dimensional cases (x is a path on $\mathbb{R}^d$, $\sigma\in C_b^1(\mathbb{R}^m,L(\mathbb{R}^d,\mathbb{R}^m))$) too, we have continuity property of the map in the uniform convergence topology under the assumptions that the vector fields $\{\sigma(x)e_i;1\le i\le d\}$ commute. See [5, 10]. In this case, we do not need the iterated integrals to solve the equation.

4 The notion of rough path

We can abstract the notion of rough paths from the discussion in the previous sections. Let us denote $\mathbb{R}^d$ and $\mathbb{R}^m$ by V and V' respectively. But note that rough path theory can be formulated for Banach spaces V, V'.

Definition 4.1 (Multiplicative functional) *Let $T(V) = \oplus_{k=0}^{\infty} V^{\otimes k}$, where where $V^0 = \mathbb{R}$. This space is a non-commutative algebra by the product*

$$(a_i)_{i=0}^{\infty} \otimes (b_i)_{i=0}^{\infty} = (c_i)_{i=0}^{\infty},$$

where $c_i = \sum_{j=0}^{i} a_j \otimes b_{i-j}$ and the natural sum. Also let $T^{(n)}(V) = \oplus_{k=0}^{n} V^{\otimes k}$ on which the multiplication is defined by $(a_i)_{i=0}^{n} \otimes (b_i)_{i=0}^{n} = (c_i)_{i=0}^{n}$, where $c_i = \sum_{j=0}^{i} a_j \otimes b_{i-j}$. We give natural inner product on $V^{\otimes k}$ using the orthonormal basis $\{e_{i_1} \otimes \cdots \otimes e_{i_k}\}$ and the norm on $T^{(n)}(V)$ by $|a| = \sum_{i=0}^{n} |a_i|$ for $a = (a_0, \ldots, a_n)$. A map X from Δ_T to $T^{(n)}(V)$ can be written as $X = (X_{s,t}^0, \ldots, X_{s,t}^n)$ $((s,t) \in \Delta_T)$. We denote the all continuous mapping X from Δ_T to $T^{(n)}(V)$ with $X_{s,t}^0 \equiv 1$ by $C_0(\Delta_T, T^{(n)}(V))$. $X \in C_0(\Delta_T, T^{(n)}(V))$ is said to have finite total p-variation if $\|X^i\|_{p/i} < \infty$ for all $1 \le i \le n$. Let us denote by $C_{0,p}(\Delta_T, T^{(n)}(V))$ the subset consisting of all continuous maps of finite total p-variation. $X \in C_0(\Delta_T, T^{(n)}(V))$ is called a multiplicative functional of degree n if for any $0 \le s \le u \le t \le T$

$$X_{s,u} \otimes X_{u,t} = X_{s,t}. \tag{4.1}$$

(4.1) *is called Chen's identity.*

Remark 4.2 (1) Let X be a multiplicative functional. Then $X_{s,t}^1 = X_{s,u}^1 + X_{u,t}^1$ for all $0 \le s \le u \le t \le T$. Hence $X_{s,t}^1 = X_{0,t}^1 - X_{0,s}^1$ for $0 \le s \le t \le T$. We say that the multiplicative functional X is a lift of a path $\xi + X_{0,t}^1$ $(\xi \in V)$. The lift is not unique. See Remark 4.5 (3).
(2) Let $X \in C_0(\Delta_T, T^{(n)}(V))$ be a multiplicative functional. Then $X \in C_{0,p}(\Delta_T, T^{(n)}(V))$ is equivalent to that $\omega(s,t) = \sum_{i=1}^{n} \|X^i\|_{p/i,[s,t]}^{p/i}$ is a control function. In [16], $X \in C_0(\Delta_T, T^{(n)}(V))$ is said to be of finite p-variation if and only if there exists a control function ω such that

$$|X_{s,t}^i| \le \omega(s,t)^{i/p}, \forall i = 1, \ldots, n, \quad \forall (s,t) \in \Delta_T.$$

Thus two properties (i) "finite total p-variation", (ii) "finite p-variation are the same properties for the multiplicative functionals.

Definition 4.3 (Signature of path) *Let $x : [0,T] \to V$ be a continuous bounded variation path[‡] For $(s,t) \in \Delta_T$, set $X_{s,t}^1 = x_t - x_s$ and*

$$X_{s,t}^{i+1} = \int_s^t X_{s,u}^i \otimes dx_u \in V^{\otimes(i+1)} \qquad (i \ge 1)$$

inductively. Let $S(x)_{[s,t]} = (1, X_{s,t}^1, X_{s,t}^2, \ldots)$. $S(x)_{[0,T]}$ is called the signature of the path x. The first $n+1$ element of $S(x)_{[s,t]}$,

$$(1, X_{s,t}^1, \ldots, X_{s,t}^n), \qquad ((s,t) \in \Delta_T)$$

is an example of multiplicative functional of degree n.

[‡]In previous sections, we assume that x is a C^1-path. But the continuity and the bounded variation are enough in the arguments.

Definition 4.4 (p-rough path, geometric p-rough path) *Let $p \geq 1$. A multiplicative functional of degree $[p] =$ (the largest integer less than or equal to p) with finite p-variation is called a p-rough path and we denote the set of all p-rough paths by $\Omega_p(V)$. (It is proved that $\Omega_p(V)$ is a complete metric space with the distance function:*

$$d_p(X,Y) = \max_{1 \leq i \leq [p]} \|X^i - Y^i\|_{p/i}.$$

) The first $n+1$-component of the signature of a continuous bounded variation path is an example of p-rough path for all $p \geq 1$ and is called a smooth rough path. The closure of all smooth rough paths in $\Omega_p(V)$ is denoted by $G\Omega_p(V)$ and the element is called a p-geometric rough path.

Remark 4.5 (1) In Remark 2.8, we define $W^2_{s,t}$ for $w \in \tilde{\Theta}^d \subset \Theta^d$. $(1, W^1_{s,t}, W^2_{s,t})$ ($W^1_{s,t} = w(t) - w(s)$) is an example of p-geometric rough path, where $2 < p < 3$. This geometric rough path is called the Brownian rough path.
(2) For simplicity, let $2 \leq p < 3$. For a geometric p-rough path $X = (1, X^1_{s,t}, X^2_{s,t})$, we denote $X^1_{s,t} = \sum_{i=1}^d X^{1,i}_{s,t} e_i$, $X^2_{s,t} = \sum_{1 \leq i,j \leq d} X^{2,ij}_{s,t} e_i \otimes e_j$. Then X satisfies the following relations: for any $1 \leq i,j \leq d$ and $0 \leq s \leq t \leq T$,

$$X^{2,ij}_{s,t} + X^{2,ji}_{s,t} = X^{1,i}_{s,t} + X^{1,j}_{s,t} \tag{4.2}$$

This implies the symmetric part of X^2 is uniquely determined by X^1. This suggests us another definition of subset of p-rough path. A p-rough path which satisfies the relation (4.2) is called a weakly geometric p-rough path. We denote the set which consists of weakly geometric p-rough paths by $WG\Omega_p(V)$. Therefore we may define a weakly geometric rough path is a continuous mapping which takes the values in $G^{(2)}(\mathbb{R}^d) = \mathbb{R}^d \times so(d)$ which is the free nilpotent Lie group of step 2. The group multiplication is defined as follows: for $x = (x^i, x^{jk}), y = (y^i, y^{jk}) \in G^{(2)}(\mathbb{R}^d)$,

$$x \cdot y = \left(x^i + y^i, x^{ij} + y^{ij} + \frac{1}{2}\left(x^i y^j - x^j y^i\right) \right).$$

Anti-symmetric part of X^2 is $A^{ij}_{s,t} = \frac{1}{2}\left(X^{2,ij}_{s,t} - X^{2,ji}_{s,t}\right)$. Using the multiplication of $G^{(2)}(\mathbb{R}^d)$, we see that $(X^1_{s,t}, A_{s,t}) = (X^1_{0,s}, A_{0,s})^{-1} \cdot (X^1_{0,t}, A_{0,t})$ holds. A weakly geometric rough path can be viewed as a continuous map from $[0,T]$ to $G^{(2)}(\mathbb{R}^d)$. See [15], [9].

$A^{ij}_{s,t}$ has a simple geometric meaning if X is a smooth rough path defined by a continuous bonded variation path x. In this case,

$$A^{ij}_{s,t} = \frac{1}{2}\left(\int_s^t \left(x^i_u - x^i_s\right) dx^j_u - \int_s^t \left(x^j_u - x^j_s\right) dx^i_u \right)$$

holds. That is $A^{ij}_{s,t}$ is the signed area enclosed by the closed curve which is defined by the chord $\overline{x_s x_t}$ and the curve $u \in [s,t] \to x_u$ on $\mathbb{R}^2$, where $x_u = x^i_u e_i + x^j_u e_j$. If the path is the Brownian path, this is called the Lévy's stochastic area. The functional $x \in B_{p,T}(\mathbb{R}^d) \to A^{ij}_{0,T}$ is not continuous for $i \neq j, p = 2$. Here is an example for $p > 2$. Let $x(n)_t = (x^1(n)_t, x^2(n)_t)$ and $x^1(n)_t = \frac{\cos(n^2 t)}{n}, x^2(n)(t) = \frac{\sin(n^2 t)}{n}$, where $T = 2\pi$ and $d = 2$. Then the limit of the associated smooth rough path $X(n) = (1, X(n)^1, X(n)^2)$ in $G\Omega_p(\mathbb{R}^2)$ $(2 < p < 3)$ is

$$(1, 0, \frac{1}{2}A(t-s)), \quad A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

This also implies the non-uniqueness of the extension $x \to X$. Also it is proved that the functional is not continuous with respect to any topology which are defined by measurable norms on the Cameron-Martin subspace of the classical Wiener space in [21].
(3) Again for simplicity, let $2 \le p < 3$. Let $\phi \in B_{p/2,T}(V \otimes V)$. Let $X = (1, X^1_{s,t}, X^2_{s,t})$ be a p-rough path then $X' = (1, X^1_{s,t}, X^2_{s,t} + \phi(t) - \phi(s))$ is also a p-rough path. The necessary and sufficient condition for that X' is a weakly geometric rough path is $\phi(t)$ is a skew-symmetric matrix for all $t \in [0, T]$. Hence we see strict inclusion $G\Omega_p(V) \subset \Omega_p(V)$. Actually $WG\Omega_p(V) \subset G\Omega_p(V)$ is also a strict inclusion. This follows that $B_{p/2,T}(V \otimes V)$ is not a separable space. The following example is due to [11]. Let $q > 1$ and $f_\varepsilon(t) = \sum_{k=1}^{\infty} \varepsilon_k 2^{-k/q} \sin\left(2^k \pi t\right) \quad (0 \le t \le 1)$. Here $\varepsilon = (\varepsilon_k)$ is a sequence such that $\varepsilon_k = 1$ or -1. Then $f_\varepsilon \in B_{q,1}(\mathbb{R})$. Also if $\varepsilon \ne \varepsilon'$, $\|f_\varepsilon - f_{\varepsilon'}\|_q > 2$ which shows non-separability of $B_{p,1}(\mathbb{R})$. On the other hand, $G\Omega_p(V)$ is separable since

$$\overline{\{C^1\text{-smooth rough paths}\}}^{d_p} = \overline{\{\text{smooth rough path}\}}^{d_p}.$$

Note that C^1-smooth rough path is a smooth rough path defined by a C^1-path (this terminology is not common).

In the definition of p-rough path, the reason why it does not contain tensor elements in $V^{\otimes k}$ for $k > [p]$ is in the following theorem.

Theorem 4.6 (Extension theorem) *Let X be a p-rough path. Then for any $n \ge [p]$, there exists a unique multiplicative functional $\tilde{X} = (1, \tilde{X}^1, \ldots, \tilde{X}^n)$ such that $\tilde{X}^i_{s,t} = X^i_{s,t}$ for all $(s,t) \in \Delta_T$ and $1 \le i \le [p]$. We denote it by the same notation. Also the map $X \to (X^0, X^1, \ldots, X^{[p]}, X^{[p]+1}, \ldots)$ is a continuous map in the p-variation topology. More precisely, let X and Y be two p-rough paths. Let β be a constant such that*

$$\beta \ge 2p^2 \left(1 + \sum_{r=3}^{\infty} \left(\frac{2}{r-2}\right)^{([p]+1)/p}\right).$$

Let $\varepsilon > 0$. If ω is a control function such that

$$\max\left(|X^i_{s,t}|, |Y^i_{s,t}|\right) \le \frac{\omega(s,t)^{i/p}}{\beta(i/p)!} \qquad 1 \le i \le [p], \quad (s,t) \in \Delta_T \tag{4.3}$$

$$|X^i_{s,t} - Y^i_{s,t}| \le \varepsilon \frac{\omega(s,t)^{i/p}}{\beta(i/p)!} \qquad 1 \le i \le [p], \quad (s,t) \in \Delta_T, \tag{4.4}$$

then (4.3) and (4.4) hold for all $i \ge [p]+1$.

The following inequality is used to prove Theorem 4.6.

Lemma 4.7 (Neo-classical inequality) *For any $p \in [1, \infty], n \in \mathbb{N}$ and $s, t \ge 0$,*

$$\frac{1}{p^2} \sum_{i=0}^{n} \frac{s^{\frac{i}{p}} t^{\frac{n-i}{p}}}{\left(\frac{i}{p}\right)! \left(\frac{n-i}{p}\right)!} \le \frac{(s+t)^{\frac{n}{p}}}{\left(\frac{n}{p}\right)!}.$$

In previous sections, we define a line integral along C^1-path x_t by the limit $\lim_{|D|\to 0} J_{s,t}(x,D)$. Note that we use the algebraic relations ((3.10) and (3.11)) and estimates ((3.5) and (3.6)) only to prove the convergence. By this we arrive at the notion of integral against a rough path. However, an integration of a one form against a rough path should be also a rough path. This is important when we solve a differential equation driven by rough path because we use Picard's iteration procedure to solve the equation.

Definition 4.8 (Line integral along rough path) *Let $2 \le p < 3$ and $X = (1, X^1, X^2)$ be a p-rough path. Let $X_t = \xi + X^1_{0,t}$*[§]*. Let $f \in C^2_b(V, L(V,W))$ ($V = \mathbb{R}^d, W = \mathbb{R}^m$) and set*

$$\begin{aligned} \tilde{X}^1_{s,t} &:= f(X_s)X^1_{s,t} + (\nabla f)(X_s)X^2_{s,t} && (4.5)\\ \tilde{X}^2_{s,t} &:= f(X_s)\otimes f(X_s)\left(X^2_{s,t}\right). && (4.6)\end{aligned}$$

It can be proved that the following limits exist

$$\begin{aligned} Z^1_{s,t} &= \lim_{|D|\to 0}\sum_{i=1}^N \tilde{X}^1_{t_{i-1},t_i} && (4.7)\\ Z^2_{s,t} &= \lim_{|D|\to 0}\left\{\sum_{i=1}^N \tilde{X}^2_{t_{i-1},t_i} + \sum_{i=1}^N Z^1_{s,t_{i-1}}\otimes Z^1_{t_{i-1},t_i}\right\}, && (4.8)\end{aligned}$$

and $(1, Z^1, Z^2)$ is also a p-rough path, where $D = \{s = t_0 < \ldots < t_N = t\}$. We denote $\int_s^t f(X_u)dX^1_u = Z^1_{s,t}$, $\int_s^t f(X_s)dX^2_u = Z^2_{s,t}$ and

$$\int_s^t f(X_u)dX_u = \left(1, \int_s^t f(X_u)dX^1_u, \int_s^t f(X_s)dX^2_u\right) \in \Omega_p(W). \tag{4.9}$$

Remark 4.9 (1) Let $X = (1, X^1, X^2)$ be a smooth rough path. Then

$$\begin{aligned} Z^2_{s,t} &\simeq \int_s^t \tilde{X}^1_{s,u}\otimes d_u\tilde{X}^1_{s,u}\\ &= \int_{\{s<u_1<u_2<t\}} \dot{\tilde{X}}^1_{s,u_1}\otimes \dot{\tilde{X}}^1_{s,u_2}du_1du_2\\ &= \int_{\{s<u_1<u_2<t\}} \left(f(X_s)\dot{X}^1_{s,u_1} + (\nabla f)(X_s)\dot{X}^2_{s,u_1}\right)\otimes\\ &\qquad \left(f(X_s)\dot{X}^1_{s,u_2} + (\nabla f)(X_s)\dot{X}^2_{s,u_2}\right)du_1du_2\\ &\simeq f(X_s)\otimes f(X_s)\left(\int_{\{s<u_1<u_2<t\}}\dot{X}^1_{s,u_1}\otimes\dot{X}^1_{s,u_2}du_1du_2\right)\\ &= f(X_s)\otimes f(X_s)X^2_{s,t}\end{aligned}$$

(2) $(1, \tilde{X}^1, \tilde{X}^2)$ does not satisfies the Chen identity. But it belongs to $C_{0,p}(\Delta_T, T^{(2)}(V))$ and satisfies the following relation: there exists a constant C such that for any $s < u < t$

$$\begin{aligned} |\tilde{X}^1_{s,u} + \tilde{X}^1_{u,t} - \tilde{X}^1_{s,t}| &\le C\omega(s,t)^\theta && (4.10)\\ |\tilde{X}^2_{s,u} + \tilde{X}^2_{u,t} + \tilde{X}^1_{s,u}\otimes\tilde{X}^1_{u,t} - \tilde{X}^2_{s,t}| &\le C\omega(s,t)^\theta, && (4.11)\end{aligned}$$

[§]Note that X_t is a continuous path on $\mathbb{R}^d$ starting at ξ. Line integral depends on the extra datum, the starting point ξ.

where $\theta > 1$ (in this case $\theta = \frac{3}{p} > 1$). (4.10) follows from the same argument as in (3.14). The existence of the limit of (4.8) also follows from the above estimates (4.10) and (4.11). The element of $C_{0,p}(\Delta_T, T^{(2)}(V))$ satisfying (4.10) and (4.11) is called a almost p-rough path.
(3) Let us consider the case where $p > 3$. Then the line integral for p-geometric rough path also can be defined.
(4) Let $X = (1, X^1, X^2)$ be a smooth rough path. Then rough path $\int f(X_u)dX_u$ is nothing but the smooth rough path which is defined by the continuous bounded variation path $\int_0^t f(X_u)dX_u$.

Theorem 3.8 can be extended to the following theorem.

Theorem 4.10 (Continuity theorem of line integral as a functional of rough path) *Let $f \in C_b^3(V, L(V, V'))$. Let $X = (1, X^1, X^2), Y = (1, Y^1, Y^2) \in \Omega_p(V)$. Assume that there exists a control function ω such that*

$$\begin{aligned} \max\left(|X_{s,t}^i|, |Y_{s,t}^i|\right) &\leq \omega(s,t)^{i/p}, \qquad i = 1, 2, \quad (s,t) \in \Delta_T \qquad (4.12) \\ |X_{s,t}^i - Y_{s,t}^i| &\leq \varepsilon\omega(s,t)^{i/p}, \qquad i = 1, 2, \quad (s,t) \in \Delta_T. \qquad (4.13) \end{aligned}$$

Also assume that $X_0 = Y_0$. Then there exists a constant C,C' which depend only on $\omega(0,T)$, $\|\nabla^i f\|_\infty$ $(i = 0, 1, 2, 3)$ such that for all $(s,t) \in \Delta_T$

$$\begin{aligned} \max\left\{\left|\int_s^t f(X_s)dX_s^i\right|, \left|\int_s^t f(Y_s)dY_s^i\right|\right\} &\leq C\omega(s,t)^{i/p}, \qquad i = 1, 2 \qquad (4.14) \\ \left|\int_s^t f(X_s)dX_s^i - \int_s^t f(Y_s)dY_s^i\right| &\leq \varepsilon C'\omega(s,t)^{i/p}, \qquad i = 1, 2 \qquad (4.15) \end{aligned}$$

Let us consider a differential equation driven by C^1-path $x : [0,T] \to V$. Let $f \in C_b^3(V', L(V, V'))$ and consider

$$\begin{aligned} \dot{y}_t &= f(\xi + y_t)\dot{x}_t \qquad (4.16) \\ y_0 &= 0. \qquad (4.17) \end{aligned}$$

We rewrite this equation as follows.

$$\begin{pmatrix} \dot{x}_t \\ \dot{y}_t \end{pmatrix} = \begin{pmatrix} I & 0 \\ f(\xi + y_t) & 0 \end{pmatrix} \begin{pmatrix} \dot{x}_t \\ \dot{y}_t \end{pmatrix}. \qquad (4.18)$$

Set

$$\hat{f}\,(x, y) = \begin{pmatrix} I & 0 \\ f(\xi + y) & 0 \end{pmatrix} \in L(V \oplus V', V \oplus V').$$

Then $z_t = {}^t(x_t, y_t) \in V \oplus V'$ is a solution of

$$z_t = \xi + \int_0^t \hat{f}(z_s)dz_s. \qquad (4.19)$$

Definition 4.11 (Differential equation driven by rough path) *Let $X = (1, X^1, X^2) \in \Omega_p(V)$ be a p-rough path $(2 \leq p < 3)$. Then p-rough path $Z \in \Omega_p(V \oplus V')$ is called a solution to the following differential equation:*

$$dY_t = f(Y_t)dX_t, \qquad Y_0 = \xi, \qquad (4.20)$$

if (1) $Z = \int \hat{f}(Z)dZ$, (2) $\pi_V(Z) = X$ *hold.*

Remark 4.12 Let $2 \le p < 3$. Let $A \in L(V, V')$ be a linear map between euclidean spaces. This induces a map from $T^{(2)}(V)$ to $T^{(2)}(V')$ by

$$A(v_1 \otimes \cdots \otimes v_n) = Av_1 \otimes \cdots \otimes Av_n,$$

where $v_1 \otimes \cdots \otimes v_n \in V^{\otimes n}$. This induces a map $A : \Omega_p(V) \to \Omega_p(V')$. π_V is a projection operator on $V \oplus W$ to V and this defines a map $\pi_V : \Omega_p(V \oplus W) \to \Omega_p(V)$.

The following theorem can be found in [16]. Actually, under weaker assumptions, the universal limit theorem holds but the continuity property of the solution is weaker than the following local Lipschitz continuity property.

Theorem 4.13 (Universal limit theorem) *Let $2 \le p < 3$. Let $f \in C_b^3(V', L(V, V'))$.*
(1) *Let $X = (1, X^1, X^2) \in \Omega_p(V)$. Then there exists a unique solution to* (4.20).
(2) *Let $X = (1, X^1, X^2), Y = (1, Y^1, Y^2) \in \Omega_p(V)$. Assume that there exists a control function ω such that*

$$\max\left(|X^i_{s,t}|, |Y^i_{s,t}|\right) \le \omega(s,t)^{i/p}, \qquad i = 1, 2, \quad (s,t) \in \Delta_T, \tag{4.21}$$

$$|X^i_{s,t} - Y^i_{s,t}| \le \varepsilon\omega(s,t)^{i/p}, \qquad i = 1, 2, \quad (s,t) \in \Delta_T. \tag{4.22}$$

Let $Z(X), Z(Y)$ be the solutions whose driving rough paths are X and Y with the same starting point. Then there exist constants C, C' which depends on $\omega(0,T)$, $\|\nabla^i f\|_\infty$ $(i = 0, 1, 2, 3)$ such that

$$\max\left(|Z(X)^i_{s,t}|, |Z(Y)^i_{s,t}|\right) \le C\omega(s,t)^{i/p}, \qquad i = 1, 2, \tag{4.23}$$

$$\varepsilon|Z(X)^i_{s,t} - Z(Y)^i_{s,t}| \le C'\omega(s,t)^{i/p}, \qquad i = 1, 2. \tag{4.24}$$

Remark 4.14 (1) Suppose that $X = (1, X^1, X^2)$ is a smooth rough path defined by a path $x \in B_{1,T}(V)$. Then there exists a unique solution to the classical integral equation:

$$z_t = \xi + \int_0^t \hat{f}(z_s) dz_s.$$

The solution $Z(X)$ in rough path analysis is the smooth rough path associated to this z.
(2) In terms of p-variation distance, the universal limit theorem implies that the map $X \to Z(X)$ is a continuous (actually locally Lipschits continuous under the above strong assumption) mapping.
(3) Note that we can define a line integral for a geometric p-rough path for $p \ge 3$. We have the similar kind of continuity theorem for solutions driven by geometric p-rough paths for $p \ge 3$.

5 Relation to stochastic processes

5.1 Brownian rough path

Let $f \in C_b^2(V', L(V, V'))$ and w be the standard Brownian motion on V.We consider Stratonovich SDE:

$$X(t, \xi, w) = \xi + \int_0^t f(X(s, \xi, w)) \circ dw(s).$$

Recall that $w(n)$ is the dyadic polygonal approximation of w.

Theorem 5.1 (Ikeda-Watanabe, Wong-Zakai)

$$\lim_{n\to\infty}\int_{\Theta^d}\sup_{0\le t\le T}|X(t,\xi,w(n))-X(t,\xi,w)|^2 d\mu(w)=0$$

By the remark in the previous section, $Z(w(n))^1_{s,t}=X(t,\xi,w(n))-X(s,\xi,w(n))$. Thus

$$X(t,\xi,w)=\xi+Z^1(W)_{0,t}\qquad \mu-a.s.\ w.$$

This shows that a version of the solution of SDE is a continuous functional on the space of Brownian rough path in p-variation distance.

Once, this is checked using the polygonal approximation, we see the following theorem by the universal limit theorem.

Theorem 5.2 *Let $\phi_n:[0,T]\times\Theta^d\to V$ $(n\in\mathbb{N})$ be a measurable map such that*
(i) $\phi_n(\cdot,w)\in B_{1,T}(V)$for all n,w.
(ii) Let W be the Brownian rough path and Φ_n be the smooth rough path associated to ϕ_n. It holds that

$$\lim_{n\to 0} d_p(\Phi_n,W)=0\quad \textit{in probability.}$$

Then for all $p>2$

$$\lim_{n\to\infty}\|X(\cdot,\xi,\phi_n(w))-X(\cdot,\xi,w)\|_p=0\quad \textit{in probability.}$$

Let $e_i={}^t(0,\dots,1,\dots,0)$ and $\{f_n\}_{n=1}^\infty$ be a orthonormal basis in $L^2([0,T],dx)$. Let $\{g_n\mid n=1,2,\dots\}=\{f_k e_i\mid k\ge 1, 1\le i\le d\}$. Then

$$\phi_n(t,w)=\sum_{k=1}^n\int_0^T(g_k(t),dw_t)\,g_k(t)$$

is an example. We refer this to [23] for Japanese audiences.

S. Watanabe, RIMS Kokyuroku 1032, 3rd Workshop on Stochastic Numerics, 1998.

5.2 Fractional Brownian motion

Actually, [16] discussed more general Gaussian processes. Here we explain fractional Brownain motion only.

Definition 5.3 A real valued mean 0 Gaussian process (w_t) is called fractional Brownian motion(fBm) with Hurst parameter h $(0<h<1)$ if

$$E[w_t w_s]=\frac{1}{2}\left(|t|^{2h}+|s|^{2h}-|t-s|^{2h}\right).$$

If (w_t) is fBm with h, then $E[|w_t-w_s|^m]=C_p|t-s|^{hm}$ $t,s\ge 0, m>0$.

- $h=1/2 \Longrightarrow (w_t)$ is standard Brownian motion.
- (w_t) is Hölder continuous of order $h-\varepsilon$, $\forall\varepsilon>0$.

- (w_t) is of finite $\frac{1}{h} + \delta$-variation $\forall \delta > 0$.

Let $\{w_t^i\}_{i=1}^d$ be independent 1-dimensional fBms and set $w_t = (w_t^1, \ldots, w_t^d)$. Then we see that

(i) If $h \neq 1/2$, then w_t is not a semimartingale.

(ii) If $h > 1/2$, then sample path is more regular than standard Bm and Young integral can be applied.

(iii) For $1/4 < h < 1/2$, one can lift the fBm to geometric p-rough path ($2 < p < 4$) canonically.

Theorem 5.4 *Let w_t be a d-dimensional fBm with $h > 1/4$. Let $X(n) = (1, X(n)^1_{s,t}, X(n)^2_{s,t}, X(n)^3_{s,t})$ be a smooth rough path associated to the dyadic polygonal approximation of w. Then $X(n)$ converges to the unique geometric rough path $(1, X^1, X^2, X^3)$ in p-variation distance both almost surely and in L^1, for any $p < 4$ such that $ph > 1$.*

Why $h \leq 1/4$? For any $p > 4$ and $m > n$,

$$E[\|X^2(m) - X^2(m+1)\|_{p/2}] \geq C \left(\frac{1}{2^n}\right)^{(p-4)/4}$$

which shows $X^2(n)$ is not a Cauchy sequence. On the other hand, Lyons-Victoir [18] proved that

Let $x \in B_{p,T}(V)$. Let $q > p$. Then there exists a geometric q-rough path $X = (1, X^1, \ldots, X^{[q]})$ such that $X^1_{0,t} = x_t$ for all t.

In Brownian motion case, the first level path of the rough path solution coincides with the solution of the Stratonovich stochastic differential equation. Stratonovich integral is defined for more general stochastic processes (including anticipating cases). See Nualart [20]. Thus, we can give a meaning to the solution driven by fractional Brownian motions using the Stratonovich integral if the solution exists. Hence, it is one of a basic problem to study the relation between two solutions.

Coutin, Friz, Victoir [4] proved the following:

Suppose that the driving process is fBm with $h > 1/4$. Then the first level path $X^1_{0,t}$ of the rough path solution is equal to the solution in the sense of Stratonovich differential equation.

Finally, I make a small remark on some works on Malliavin calculus (existence of density function of the law of the random variable) of fBms.

Consider differential equation driven by d-dimensional fBm:

$$\begin{aligned} dX(t,\xi,w) &= \sum_{i=1}^d f_i(X(t,\xi,w))dw^i(t) \\ X(t,\xi,w) &= \xi \end{aligned}$$

- $h > 1/2$: Assume $f_i \in C_b^\infty(\mathbb{R}^m, \mathbb{R}^m)$.

F. Baudoin and M. Hairer [3] proved that the existence of smooth density of $X(t)$ under Hörmander's condition.

- $1/4 < h \leq 1/2$

(1) Cass, T., Friz,P. and Victoir, N.: Non-degeneracy of Wiener functionals arising from rough differential equations, arXiv:0707.0154 (July, 2007)

This studies elliptic cases and prove the existence of the density function.

(2) Cass,T., Friz,P.: Densities for rough differential equations under Hörmander's condition, arXiv:0708.3730 (August,2007)

This proves the existence of density function under Hörmander's condition.

However the smoothness of the density function seems to be still open problems.

6 Applications

There are several applications of rough path analysis to problems in stochastic analysis.

We list some of them below.

(I) Freidlin-Wentzell type large deviation

(II) Support theorem

Also I note that universal limit theorem and estimates in rough path analysis were applied in the following problems:

(III) Weak Poincaré inequality [1]

(IV) Semiclassical approximation of Schrödinger operators on path sapces by the Schrödinger operators with quadratic potential finctions on Wiener spaces [2]

We explain (I) and (II). Ledoux, Qian and Zhang [13] are the first to study these problems using rough path analysis. Below, we put $T = 1$.
(I) Freidlin-Wentzell type large deviation

Theorem 6.1 (Schilder) *Let μ_ε be the law of εw, where w is the d-dimensional Bm. Let*

$$I(w) = \begin{cases} \frac{1}{2}\|w\|_{H^d}^2 & w \in H^d, \\ +\infty & w \notin H^d. \end{cases}$$

Then μ_ε satsifies the Large deviation principle with the good rate function I: for any $A \subset \Theta^d$,

$$-\inf_{w \in \overset{\circ}{A}} I(w) \le \lim_{\varepsilon \to 0} \varepsilon^2 \log \mu_\varepsilon(\overset{\circ}{A}) \le \lim_{\varepsilon \to 0} \varepsilon^2 \log \mu_\varepsilon(\bar{A}) \le -\inf_{w \in \bar{A}} I(w).$$

Now we consider the image measure of ν_ε by a measurable map $F : \Theta^d \to S$, where S is a Polish space. Suppose that $F : \Theta^d \to S$ be a continuous mapping. Let $\nu_\varepsilon = F_* \mu_\varepsilon$. Then

Theorem 6.2 ν_ε satisfies the LDP with the good rate function $J(\gamma) = \inf \{I(w) \mid F(w) = \gamma\}$.

The solution of $X(t, \xi, w)$ to

$$\begin{aligned} dX(t,\xi,w) &= f(X(t,\xi,w)) \circ dw_t \\ X(t,\xi,w) &= \xi \end{aligned}$$

is not continuous but $\nu_\varepsilon = X_*\mu_\varepsilon$ on $S = C([0,1] \to V')$ satsifies the LDP with the rate function: $J(\gamma) = \inf\{I(w) \mid X(\cdot,\xi,w) = \gamma\}$. (Freidlin-Wentzell)

This is usually proved by showing that $X(\cdot,\xi,w(n))$ is a exponentially good approximation of X. By the universal limit theorem, it is enough to show that $W(n)$ converges to the Brownian rough path W in exponentially good sense in p-variation distance.

To be explicit,

(1)For $\forall\delta > 0$,

$$\lim_{n\to\infty} \limsup_{\varepsilon\to 0} \varepsilon^2 \log \mu_\varepsilon \left(d_p(W(n), W) \geq \delta\right) = -\infty$$

(2) For all $\alpha > 0$,

$$\lim_{n\to\infty} \sup_{\|w\|_{H^d} \leq \alpha} d_p\left(W(n), W\right) = 0.$$

The above application is due to [13]. Here is some extension of it. One can consider the same problem in more general cases. In fact, in the case of fBm with $h > 1/4$, [19] proved that LDP holds with the similar rate function.

(II) Support theorem

Let H^d be the Cameron-Martin subspace of Θ^d.

Theorem 6.3 *Let S be a separable metric space. Let $F : \Theta^d \to S$ be a continuous map. Then*

$$\mathrm{supp}(F_*\mu) = \overline{\{F(h) \mid h \in H^d\}},$$

where H^d is the Cameron-Martin subspace and the closure is taken w.r.t. the topology of S.

Proof. $\subset$: This follows from $\lim_{n\to\infty} F(w(n)) = F(w)$ for $w \in X^d$. $\supset$: We show $F(h) \in \mathrm{supp}(F_*\mu)$ for all $h \in H^d$. Since $F : \Theta^d \to S$ is continuous, for any $\varepsilon > 0$, there exists $\delta > 0$ such that $F(B_\delta(h)) \subset B_\varepsilon(F(h))$. Hence

$$\left(F_*\mu\right)\left(B_\varepsilon(F(h))\right) \geq \mu\left(B_\delta(h)\right) > 0.$$

■

We have used that (i) $H^d \subset \mathrm{supp}\mu$, (ii) F is a continuous mapping from Θ^d to S.

Let $X(t,\xi,w)$ be the solution to

$$\begin{aligned} dX(t,\xi,w) &= f(X(t,\xi,w)) \circ dw(t), \\ X(0,\xi,w) &= \xi \in V'. \end{aligned}$$

We have a measurable map $X(\cdot,\xi,\cdot) : \Theta^d \to B_p(V')$.

Theorem 6.4

$$\mathrm{supp} X_*\mu = \overline{\{X(\cdot,\xi,h) \mid h \in H^d\}}^{\|\ \|_p}.$$

Let H be the smooth rough path associated to $h \in H^d$ and set

$$U_\varepsilon(h) = \left\{w \in \tilde{\Theta}^d \mid d_p(W,H) < \varepsilon\right\}.$$

Then (see also [1]),

Lemma 6.5 ([13]) *For any $\varepsilon > 0, h \in H^d$, it holds that $\mu(U_\varepsilon(h)) > 0$.*

This lemma and the universal limit theorem imply that $X(\cdot,\xi,h) \in \mathrm{supp}\ X_*\mu$.

There are extension of the above diffusion cases to fBms.

• $h > 1/3$: Feyel, D. and de La Pradelli,A [6]

• $1/4 < h < 1/3$: Coutin, Friz, Victoir [4]

References

[1] S. Aida, Weak Poincare inequalities on domains defined by Brownian rough paths, The Annals of Probability, 2004, Vol.32, No.4, 3116–3137.

[2] S. Aida, Semi-classical limit of the bottom of spectrum of a Schrödinger operator on a path space over a compact Riemannian manifold, J. Funct. Anal. Volume 251, Issue 1, 59–121, (2007).

[3] F. Baudoin and M. Hairer, A version of Hörmander's theorem for the fractional Brownian motion, Probab. Theory Related Fields 139 (2007), no. 3-4, 373–395

[4] L. Coutin, P. Friz and N. Victoir, Good rough path sequences and applications to anticipating stochastic calculus, Ann. Probab. 35 (2007), no. 3, 1172–1193.

[5] H. Doss, Liens entre équations différentielles stochastiques et ordinaires, Ann. Inst. H. Poincaré, **13** (1977), 99–125.

[6] D. Feyel and A. de La Pradelle, Curvilinear integrals along enriched paths, Electron. J. Probab. 11 (2006), no. 34, 860–892 (electronic).

[7] P. Friz, Continuity of the Itô-Map for Hölder rough paths with applications to the support theorem in Hölder norm, Probability and partial differential equations in modern applied mathematics, 117–135, IMA Vol. Math. Anal., **140**, Springer, New York, 2005.

[8] P. Friz and N. Victoir, Approximations of the Brownian rough path with applications to stochastic analysis, Ann. Inst. H. Poincaré Probab. Statist. **41** (2005), no. 4, 703–724.

[9] P. Friz and N. Victoir, A note on the notion of geometric rough paths, Probab. Theory relat. fields **136**, 395–416, 2006.

[10] N. Ikeda and S. Watanabe, Stochastic differential equations and diffusion processes, Second edition, North-Holland Mathematical Library, **24** North-Holland Publishing Co., 1989

[11] N.C. Jain and D. Monrad, Gaussian measures in B_p, Annals of Probability, 1983, Vol. 11, No. 1, 46–57.

[12] M. Ledoux, T. Lyons and Z. Qian, Lévy area of Wiener processes in Banach spaces, Annals of Probability, **30.** (2002), No.2, 546–578.

[13] M. Ledoux, Z. Qian and T. Zhang, Large deviations and support theorem for diffusions via rough paths, Stochastic processes and their applications, **102** (2002), No.2, 265–283.

[14] A. Lejay, An Introduction to Rough Paths, Séminaire de probabilités XXXVII, Lecture Notes in Mathematics (Springer-Verlag), (2003).

[15] T. Lyons, Differential equations driven by rough signals, Rev.Mat.Iberoamer., **14** (1998), 215–310.

[16] T. Lyons and Z. Qian, System control and rough paths, (2002), Oxford Mathematical Monographs.

[17] T. Lyons, M. Caruana and T. Lévy, Differential equations driven by rough paths, Ecole d'Eté de Probabilités de Saint-Flour XXXIV-2004, Lecture Notes in Math., 1908, Springer, 2007.

[18] T. Lyons and N. Victoir, An extension theorem to rough paths, Ann. Inst. H. Poincaré Anal.Non Linéare **24** 2007, no. 5, 835–847.

[19] Millet, A and Sanz-Solé, M, Large deviations for rough paths of the fractional Brownian motion, Ann.Inst.H.Poincaré, Probab.Statist. 42 (2006), no.2

[20] D. Nualart, The Malliavin Calculus and Related Topics, Second edition. Probability and its applications, Springer-Verlag, Berlin, 2006.

[21] H. Sugita, Hu-Meyer's multiple Stratonovich integral and essential continuity of multiple Wiener integral, Bull.Sci.Math. **113** (1989), no. 4, 463–474.

[22] S.J. Taylor, Exact asymptotic estimates of Brownian path variation, Duke Math. J. **39** (1972), 219–241.

[23] S. Watanabe, RIMS Kokyuroku 1032, 3rd Workshop on Stochastic Numerics, 1998.

[24] L.C. Young, An inequality of Hölder type, connected with Stieltjes integration, Acta Math., **67**, 251–282, 1936.