# レポート問題

**1.** Let $\{X_i\}_{i=1}^{\infty}$ be the i.i.d. such that $P(X_i = 1) = p, P(X_i = 0) = 1-p$, where $0 < p < 1$. Let $S_n = \sum_{i=1}^{n} X_i$. Using the Stirling formula, prove that for any $p < a < b < 1$,

$$\lim_{n\to\infty} \frac{1}{n} \log P\left(\frac{S_n}{n} \in [a,b]\right) = -\inf_{x\in[a,b]} I(x),$$

where

$$I(x) = x\log\frac{x}{p} + (1-x)\log\frac{1-x}{1-p}.$$

**2.** (1) Let $\Omega$ be a set. For any subsets of $\Omega$, $\{A_i\}$, $\{B_i\}$, prove that

$$(\cup_{i=1}A_i) \bigtriangleup (\cup_{i=1}^{\infty}B_i) \subset \cup_{i=1}^{\infty}(A_i \triangle B_i).$$

Here $A\triangle B := (A\cap B^c)\cup(B\cap A^c)$.
(2) Let $(\Omega, \mathcal{F}, P)$ be a probability space. Prove that for any $A, B \in \mathcal{F}$,

$$|P(A) - P(B)| \le P(A\triangle B), \quad |P(A) - P(A\cap B)| \le P(A\triangle B).$$

**3.** Let $\{X_n\}_{n=1}^{\infty}$ be random variables. Let $\mathcal{B}_n = \sigma(X_n)$. Set

$$\begin{aligned}
A &= \left\{\omega \mid \lim_{n\to\infty} X_n(\omega) \text{ exists}\right\} \\
B_a &= \left\{\omega \mid \lim_{n\to\infty} X_n(\omega) = a\right\} \quad (a\in\mathbb{R}) \\
C &= \left\{\omega \mid \lim_{n\to\infty} \sum_{i=1}^{n} X_i(\omega) \text{ exists}\right\} \\
D &= \left\{\omega \mid \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} X_i(\omega) \text{ exists}\right\}.
\end{aligned}$$

Show that $B_a, C, D$ are tail events of $\{\mathcal{B}_n \mid n = 1, 2, \cdots\}$.

**4.** Let $X$ be a real-valued random variable. Let $M_X(\theta) = \log E[e^{\theta X}]$ and set $I_X(x) = \sup_{\theta\in\mathbb{R}}(x\theta - M(\theta))$. We assume that $M_X(\theta) < \infty$ for all $\theta \in \mathbb{R}$.

(1) Assume that $P(X = 1) = p, P(X = 0) = 1-p$, where $0 < p < 1$. Prove that $I(x) = x\log\frac{x}{p} + (1-x)\log\frac{1-x}{1-p}$ for $0 \le x \le 1$ and $I(x) = +\infty$ for $x > 1$ or $x < 0$.

(2) Let $X$ be the random variable whose law is the uniform distribution on $[0,1]$. Prove that $\lim_{x\to1-0} I_X(x) = \lim_{x\to+0} I_X(x) = +\infty$.

**5** Let $X$ be a metric space. Let $f_\lambda = f_\lambda(x)$ $(x \in X, \lambda \in \Lambda)$ be a family of continuous functions on $X$. Let $f(x) = \sup_{\lambda\in\Lambda} f_\lambda(x)$. Show that $f$ is a lower semi-continuous function on $X$.

**6** Let $f$ be a funcion on $X$ with values in $[-\infty, \infty]$. Prove that the following two statements are equivalent.

(1) $f$ is a lower semi-continuous function.

(2) For any $K \in \mathbb{R}$, $\{x \mid f(x) \leq K\}$ is a closed set.

**7.** Let $X = C([0,1] \to \mathbb{R}^d)$ be the set of continuous paths $x = x(t)$ $(0 \leq t \leq 1)$ with $x(0) = 0$. We define a norm on $X$ by $\|x\| = \max_t |x(t)|$. (Note: $(X, \| \ \|)$ is a separable Banach space). Let

$$H = \Big\{ h \in X \ \Big| \ h \text{ is an absolutely continuous function} \\ \text{and } h' \in L^2([0,1], dt) \Big\}.$$

Define

$$I(x) = \begin{cases} \frac{1}{2}\int_0^1 |x'(t)|^2 dt & \text{if } x \in H \\ +\infty & \text{if } x \notin H \end{cases}$$

Prove that $I$ satisfies the property of the rate function.

**8** Let $\mu$ be a probability measure on a metric space $(X, d)$, where $d$ denotes the distance function.

(1) Prove the existence of the largest open set $O_\mu$ of $X$ such that $\mu(O_\mu) = 0$. (Note: the empty set is an open set).

(2) Let us define supp $\mu := O_\mu^c$ (supp $\mu$ is called the topological support (if there are no confusion, support, in short) of the measure $\mu$). Then prove that

$$\text{supp } \mu = \{x \in X \mid \text{for any } \varepsilon > 0,\ \mu(B_\varepsilon(x)) > 0 \text{ holds}\},$$

where $B_\varepsilon(x) = \{y \in X \mid d(x,y) < \varepsilon\}$.

(3) It is trivial to see that supp$\mu$ is a closed set by the definition in (2). By the way, prove that supp $\mu$ is a closed set, using the expression

$$\text{supp } \mu = \{x \in X \mid \text{for any } \varepsilon > 0,\ \mu(B_\varepsilon(x)) > 0 \text{ holds }\}.$$

**9.** Let $X$ be a real-valued random variable. Let $M_X(\theta) = \log E[e^{\theta X}]$ and set $I_X(x) = \sup_{\theta\in\mathbb{R}}(x\theta - M(\theta))$. We assume that $M_X(\theta) < \infty$ for all $\theta \in \mathbb{R}$ and $X \neq$ const *a.s.*. Let $m = E[X]$. Prove that

(1) Let $x > m$. Then $I(x) \leq -\log P(X \geq x)$
(2) Let $x < m$. Then $I(x) \leq -\log P(X \leq x)$.

**Remark** This shows $I(x) < \infty$ for $x \in (\inf\ \mathrm{supp}\mu_X, \sup\ \mathrm{supp}\mu_X)$, where $\mu_X$ is the law of $X$. (In the class, we denote $r_{\mu_X} = \sup \mathrm{supp}\mu_X$). Actually, the above inequalities are the same as for $x \geq m$

$$P\left(\frac{S_n}{n} \geq x\right) \leq \exp\left(-nI(x)\right),$$

which we prove in the class and so on.

**10.** Let $\{X_i\}_{i=1}^{\infty}$ be independent integer-valued random variables. We assume that $X_i$ and $-X_i$ have the same law. Let

$$S_0 = 0, \qquad S_n = \sum_{i=1}^{n} X_i \quad (n \geq 1).$$

Then for any positive integers $k$ and $N$, we have

$$P\left(\max_{0 \leq n \leq N} S_n \geq k\right) = 2P(S_N > k) + P(S_N = k) \qquad (*)$$

Prove $(*)$ following the next argument.

(1) Let $A_n = \{\omega \in \Omega \mid S_1(\omega) < k, \ldots, S_{n-1}(\omega) < k, S_n(\omega) = k\}$ $(1 \leq n \leq N)$. Show that $P\left(\{S_N - S_n > 0\} \cap A_n\right) = P\left(\{S_N - S_n < 0\} \cap A_n\right)$ for all $1 \leq n \leq N$. Summing both sides in this identity, show that

$$P\left(\{S_N > k\} \cap \{\max_{1 \leq n \leq N} S_n \geq k\}\right) = P\left(\{S_N < k\} \cap \{\max_{1 \leq n \leq N} S_n \geq k\}\right).$$

(2) Prove the identity $(*)$.

**11.** Let $\{a_\alpha\}_{\alpha \in A} \subset \mathbb{R}^d$ and $\{b_\alpha\}_{\alpha \in A} \subset \mathbb{R}$. Define

$$\varphi(x) = \sup_{\alpha \in A} \left\{(a_\alpha, x) + b_\alpha\right\}, \qquad x \in \mathbb{R}^d$$

where $(\cdot, \cdot)$ is the inner product in $\mathbb{R}^d$. Suppose that $\varphi(x) < \infty$ for all $x$. Prove that $\varphi(x)$ is a lower semi-continuous convex function.

**12.** Let $\mu$ be a probability measure on $\mathbb{R}^d$. Let $F$ be the support of $\mu$. Assume the following (i), (ii):

(i) $\int_{\mathbb{R}^d} \|x\| d\mu(x) < \infty$,

(ii) The smallest affine subspace including $F$ is $\mathbb{R}^d$.

Let $m := \int_{\mathbb{R}^d} x d\mu(x)$ be the mean value of the probability distribution $\mu$. Let Conv $F$ be the smallest convex set containing $F$. Prove that $m$ is an interior point of Conv $F$.

**13.** Let $\{X_i\}$ be $\mathbb{R}^d$-valued i.i.d. We write $S_n = \sum_{i=1}^n X_i$. Assume that $E[\exp(\|X_i\|)] < \infty$, where $\| \ \|$ denotes the Euclidean norm. Using the Chebyshev inequality, prove that for any $L > 0$, there exists $R_L > 0$ such that

$$\limsup_{n\to\infty} \frac{1}{n} \log P\left(\left\|\frac{S_n}{n}\right\| \geq R_L\right) \leq -L.$$

**14.** Let $f = f(x)$ $(a \leq x \leq b)$ be a non-negative bounded Borel measurable function. Prove that

$$\lim_{p\to\infty} \left(\int_a^b f(x)^p dx\right)^{1/p} = \|f\|_\infty,$$

where $\|f\|_\infty = \inf\{\alpha \mid f(x) \leq \alpha \ dx - \text{a.s.on } [a,b]\}$ and $dx$ denotes the Lebesgue measure.

**15.** Let $I = I(x)$ be a rate function on a separable metric space $X$. Let $\Phi$ be a bounded continuous function on $X$ and set $\Psi(x) = I(x) + \Phi(x)$. Prove that there exists $x_0 \in X$ such that

$$\Psi(x_0) = \inf_{x\in X} \Psi(x).$$

**16.** Let $f : \mathbb{N} \to [0, +\infty]$ be a subadditive function. That is, $f$ satisfies

$$f(n+m) \leq f(n) + f(m) \qquad \text{for any } n, m \in \mathbb{N}.$$

Assume that there exists $N \in \mathbb{N}$ such that $f(n) < \infty$ for all $n \geq N$.

Under these assumptions, prove

$$\lim_{n\to\infty} \frac{f(n)}{n} = \inf_{n\geq 1} \frac{f(n)}{n} < \infty.$$

**17.** Let $X_i$ be i.i.d. with values in $\mathbb{R}^d$. Assume that $E[e^{(\theta, X_i)}] < \infty$ for all $\theta \in \mathbb{R}^d$. Let $S_n = \sum_{i=1}^n X_i$. Let $x \in \mathbb{R}^d$ and $B_\varepsilon(x) = \{y \in \mathbb{R}^d \mid \|x - y\| < \varepsilon\}$. Define

$$l(B_\varepsilon(x)) = \lim_{n\to\infty} \frac{1}{n} \log P\left(\frac{S_n}{n} \in B_\varepsilon(x)\right)$$

and set $\lambda(x) = \lim_{\varepsilon\to 0}(-l(B_\varepsilon(x)))$. Prove that $\lambda : \mathbb{R}^d \to [0, +\infty]$ is a convex function. Hint: First prove that $\lambda\left(\frac{x+y}{2}\right) \leq \frac{1}{2}(\lambda(x) + \lambda(y))$. Next, using the lower semi-continuity of $\lambda$, prove that

$$\lambda(tx + (1-t)y) \leq t\lambda(x) + (1-t)\lambda(y) \qquad \text{for all } 0 < t < 1.$$