Convolution algebras and a new proof of Kazhdan-Lusztig formula

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※ Convolution algebra

Toy model

- X : finite set

$\mathcal{F}(X) = \{\mathbb{C}\text{-valued functions on } X\}$

$\mathcal{F}(X \times X)$ = algebra w.r.t. the convolution

$\mathcal{F}(X)$ = its module

$f, g \in \mathcal{F}(X\times X)$

$$(f * g)(x_1, x_3) = \sum_{x_2 \in X} f(x_1, x_2)\, g(x_2, x_3)$$

$$\# X = n \quad \Rightarrow \quad \begin{aligned} \mathcal{F}(X \times X) &\simeq \mathrm{Mat}_{n\times n}(\mathbb{C}) \\ \mathcal{F}(X) &\simeq \mathbb{C}^n \end{aligned}$$

variant

- $G \curvearrowright X$ (finite grp) G-invariant functions on $X \times X$

$\Rightarrow$ more interesting noncommutative algebras and their modules

e.g. group ring, Hecke algebra etc

☆ Convolution on cohomology of topological spaces

trivial examples

- X : compact oriented manifold

$$H^*(X \times X) \underset{\uparrow}{\cong} H^*(X) \otimes H^*(X)$$

$\mathbb{C}$-coeff.

$$\cong H^*(X) \otimes H_*(X) \cong \mathrm{End}(H^*(X))$$

$$\left(\begin{array}{l} H^*(X \times X) \cong \mathrm{Mat}_{n \times n}(\mathbb{C}) \\ H^*(X) \cong \mathbb{C}^n \end{array} \right.$$

- $H^*(X)$: a (graded) commutative ring by $\cup$

More interesting algebras are constructed as follows:

variants

- $M \xrightarrow{\pi} X$ proper e.g. resolution of singularities

mfd
not nec. cpt

$$Z = M \times_X M = \{(m_1, m_2) \mid \pi(m_1) = \pi(m_2)\}$$

$$H_*^{(BM)}(Z) = \underbrace{H^*(M \times M ; (M \times M) \setminus Z)}_{\text{relative cohomology}}$$

This is an algebra under the convolution

as $Z \circ Z = Z$ $\quad (x_1, x_2, x_3)$

$$\underbrace{(x_1, x_2)}_{Z}, \quad \underbrace{(x_2, x_3)}_{Z} \Rightarrow \underbrace{(x_1, x_3)}_{Z}$$

Similarly $x \in X$

$$H_*(\overset{\pi^{-1}(x)}{\overset{\shortparallel}{M_x}}) : \text{module of } H_*(Z)$$

Slogan: $H_*(Z)$ reflects topology of π.

<u>more variant</u>

- equivariant BM homology $G \curvearrowright M \xrightarrow{\pi} X$
  (G: Lie group)

Then $H_*^G(Z)$: algebra over $H_G^*(pt) = A$

e.g. $G = \mathbb{C}^\times$ $H_G^*(pt) = \mathbb{C}[a]$ polynomial ring

More generally $G = T$ torus
$\Rightarrow H_T^*(pt) \cong \mathbb{C}[t]$ $t = \mathrm{Lie}\,T$

A is often the center of $H_*^G(Z)$

L : simple module of $H_*^G(Z)$
$\Rightarrow A \ni a$ acts by scalar
$\therefore \exists\, \chi : A \longrightarrow \mathbb{C}$ multiplicative character
s.t. A acts on L via χ
(Schur's Lemma)

$\therefore$ L is a module of a specialized algebra $H_*^G(Z) \otimes_{A \to \chi} \mathbb{C}$

$G = T \Rightarrow \chi \in \mathrm{Lie}\,T$ (χ is an evaluation at χ)

$\exp(t\chi) \in T \curvearrowright X$, M $\quad X^\chi = \{x \in X \mid \exp t\chi \cdot x = x \ \forall t\}$ fixed pt set

$x \in X^\chi$ $\quad H_*^T(M_x) \otimes_{A \to \chi} \mathbb{C} \cong H_*(M_x^\chi)$

localization thm in equivariant homology

$H_*^T(Z) \otimes_A \mathbb{C} \cong H_*(Z^\chi)$

Study of representation theory of $H_*^T(Z)$
algebra
is related to homology of fibers of
topology $\pi^x : M^x \longrightarrow X^x$

- If we further assume
$M \xrightarrow{\pi} X$: algebraic variety $/\mathbb{C}$ or $/\overline{\mathbb{F}_q}$
$\circlearrowleft_T \circlearrowleft$
$\Longrightarrow M^x \xrightarrow{\pi^x} X^x$ algebraic

Then powerful tools for
(étale) cohomology can be used to analyze
$H_*^T(Z)$. Lusztig, Ginzburg, $\cdots$

$\rightsquigarrow$ classification of simple modules
characters of simple modules etc

In particular, representation theory of

$\left\{\begin{array}{l} \text{affine Hecke algebras } (\text{Springer resol. } \underset{M}{\underset{\shortparallel}{T^*\mathcal{B}}} \to \underset{X}{\underset{\shortparallel}{\mathcal{N}}}) \\ \text{quantum affine algebras} \end{array}\right.$
(via quiver varieties)

can be analyzed in this way

Braverman - Finkelberg - N 2020 (In fact, this work was done basically ~2014)

— consider the case when we only have $X \curvearrowleft T$.
No (natural) resolution M

powerful tools are still available.
- intersection cohomology
- hyperbolic localization functor

— A new example:

(slightly modified version of)
$U(\mathfrak{g})$ universal enveloping algebra of a cpx simple Lie algebra $\mathfrak{g}$
is constructed

via zastava spaces
= moduli space of based quasimaps $\mathbb{P}^1 \longrightarrow$ flag mfd for $\mathfrak{g}^\vee$

We get a formula of characters of simple modules in terms of intersection cohomology of fixed pt set in zastava space
= Schubert variety

$B \curvearrowright \mathcal{B}$ $\overline{\text{orbit}}$

$\longrightarrow$ new proof of Kazhdan-Lusztig conjecture

original proof by Beilinson-Bernstein
1980 Brylinski-柏原

Dimensions of IH^*
are calculated by Kazhdan-Lusztig polynomials
↑ defined by a combinatorial algorithm

Therefore ch(simple modules)
= combinatorial algorithm

Original proof:

representations of $U(\mathfrak{g})$ $\longleftrightarrow$ D-modules on flag mfd for $\mathfrak{g}$ $\longleftrightarrow$ intersection cohomology of Schubert variety for $\mathfrak{g}$ $\longleftrightarrow$ KL poly.

Riemann-Hilbert correspondences

This part is replaced by equiv. intersection cohomology

slightly changed as
$\mathfrak{g} \rightsquigarrow \mathfrak{g}^\vee$
(but combinatorics remain the same.)

<u>Remark</u> Much more different and deeper new proof was given by Elias-Williamson
2012

Hope

much more examples!

e.g. • zastava spaces for affine Lie algebras
$\rightsquigarrow$ new proof of KL conjecture for affine Lie algebras originally proved by 柏原-谷崎

• IH_T^*(instanton moduli spaces on $\mathbb{R}^4$)
= rep. of W-algebras [BFN 16]
(conjectured by AGT)
$\rightsquigarrow$ new proof of 荒川's character formula

• $H_*^{T\times\mathbb{C}^*}$(variety of BFN triples)
= quantized Coulomb branches [BFN18]
$\supset$ truncated shifted Yangian [BFN+KKWW]
cyclotomic DAHA [小寺-N]

• IH_T^*(bow variety) = rep. of coset VOA
(higher level AGT) [Muthiah-N]

⋮

These realization of noncommutative algebras are related to theoretical physics

mathematically nonrigorous, but powerful tools are available

☆ Zastava space and Schubert varieties

$G^\vee \supset B^\vee \supset T^\vee$

$\mathcal{B} = G^\vee / B^\vee$ flag variety $\ni \infty_{\mathcal{B}}$

$\alpha \in \Lambda_+$ = semigroup of positive coroots (for $G^\vee$)

$\cap$

$\Lambda \cong H_2(\mathcal{B}, \mathbb{Z})$

$\overset{\circ}{Z}{}^\alpha := \{ f : \mathbb{P}^1 \xrightarrow[\text{holomorphic map}]{} \mathcal{B} \mid \deg f = \alpha,\ f(\infty_{\mathbb{P}^1}) = \infty_{\mathcal{B}} \}$

moduli space of based maps

Z^α : zastava   partial compactification of $\overset{\circ}{Z}{}^\alpha$

e.g. $G^\vee = SL_2$, $\mathcal{B} = \mathbb{P}^1$

$\overset{\circ}{Z}{}^d = \{ [f_0(z) : f_1(z)] \mid f_0(z) = z^d + a_1 z^{d-1} + \cdots + a_d,\ f_1(z) = b_1 z^{d-1} + \cdots + b_d$ no common zero $\}$

$\cap$

Z^d = ~~no common zero~~ $\cong \mathbb{C}^{2d}$

$Z^\alpha \subset Z^\beta$ if $\alpha \le \beta$

$\mathbb{T}^\vee := \mathbb{C}^\times \times T^\vee \curvearrowright Z^\alpha$

$\mathbb{C}^\times$: acting on $\mathbb{P}^1$; $T^\vee$: acting on $\mathcal{B}$

Th. $\bigoplus_\alpha IH^*_{\mathbb{T}^\vee}(\mathcal{Z}^\alpha)$ is the universal Verma module for $\mathcal{U}_\hbar(\mathfrak{g}) \otimes_{S_\hbar(\mathfrak{t})^W} S_\hbar(\mathfrak{t})$

$XY - YX = \hbar[X,Y]$

s.t. $H^*_{\mathbb{T}^\vee}(pt) \cong S_\hbar(\mathfrak{t})$

$f \in \mathring{Z}^\alpha$ is fixed by $x = d\rho$

$\rho : \mathbb{C}^\times \longrightarrow \mathbb{T}^\vee = \mathbb{C}^\times \times T^\vee$
$t \mapsto (t^m, t^\lambda)$

$\Leftrightarrow \quad t^\lambda f(t^{-m} z) = f(z) \quad \forall t \in \mathbb{C}^\times$

Then f is determined by $f(1)$, a point in $\mathcal{B}$
(assume $m \neq 0$)

and $t^\lambda f(0) = f(0) \quad \therefore f(0) \in \mathcal{B}^{t^\lambda} \cong \underset{\text{grp}}{\text{Weyl}}$

assume λ : regular

Get
$\rightsquigarrow$ Schubert cell

Rem. λ : nonregular $\Longrightarrow$ very similar analysis is possible
partial flag variety naturally appears.