# Rigid Local Systems: an overview of Arithmetic Properties

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# Acknowledgements

**We thank**

the scientific committee for the kind invitation.

**We apologize**

for the confusion in the schedule: on my agenda I had written Wednesday 9pm Harvard time. As Mike Hopkins had suggested to go out for dinner on this evening, I checked with the time translator whether 9pm Eastern=10:00 Tokyo. It said 8pm=10:00. So no dinner. But then it also said 8pm=10:00(+1). So dinner saved!

**We wish**

Keiji OGUISO (Prof. Mr. -san....friend) a beautiful birthday week and more to come.

# Acknowledgement


**We thank**

*Michael Groechenig for the years of collaboration on the topic, also Johan de Jong for a more recent part of the work.*

# Rigid Local Systems (RLS)

• $X$ topological space, for us *complex algebraic variety*;
• irreducible *local system* (LS) $\stackrel{\text{dfn}}{=}$ irreducible $\rho : \pi_1(X, x) \to GL_r(\mathbb{C})$ up to gauge transformation;
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- $\rho$ *rigid*: deformation $\rho_t : \pi_1(X, x) \to GL_r(\mathbb{C}[[t]])$ gauge equivalent to $\rho_0$, i.e. $\exists g_t \in GL_r(\mathbb{C}[[t]])$ with $\rho_t = g_t \rho_0 g_t^{-1}$
- May fix determinant and conjugacy classes of local (quasi-unipotent) monodromy at $\infty$ for $X$ not proper.

# Rigid Local Systems (RLS)

• equivalently: has $M_B(X, r, \mathrm{det}, T_i) =: M$ moduli (defined over the number ring $\mathcal{O}$ containing the EV of the monodromies at infinity) of irreducible local systems in char. 0 (fixed torsion determinant and EV of quasi-unipotent monodromies at infinity) (and Zariski closure over $\mathcal{O}$).
• rigid: isolated point.

# Occurence

dim $(X) = 1$ Katz: $\exists$ RLS $\Longrightarrow$
• $X = \mathbb{P}^1 \setminus \{\text{finitely many points}\}$;
• moduli points have no multiplicity on moduli;
• all RLS *come from geometry*, i.e. 'like' (summand of)
$\mathcal{V}_{\rho,\tau\in\mathbb{P}^1\setminus\{\infty,\tau_1,\ldots,\tau_n\}} = H^1_c(Y_\tau), Y_\tau \subset \mathbb{A}^2 : \ y^N = (x-\tau)\prod_1^n (x-\tau_i)$.

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Shimura varieties of rank $\geq 2$: Margulis superrigidity $\Longrightarrow$
- all irreducible LS are rigid;
- isolated moduli points have no multiplicity on moduli;
- **but** we do not know whether they come from geometry.

# Simpson's Geometricity Conjecture

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Geometricity conjecture inaccessible, except in dim. 1 (Katz).
Instead study consequences.

# Consequences of Simpson's geometricity conjecture

### Integrality conjecture (Simpson 1990)

RLS are integral, i.e. $\rho : \pi_1(X, x) \to GL_r(\mathcal{O})$, $\mathcal{O}$ number ring.

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ii) $\mathbb{L}_{\mathfrak{p}}$ on $X_{\bar{\mathbb{Q}}_p}$ descends to $X_{\mathbb{Q}_q}$ and there is crystalline for $p$ large.

≪ Gauß-Manin local systems have all those properties. ≫

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## Remarks

• There are RLS with moduli point with higher multiplicity (de Jong-E-Groechenig). Only known examples have finite monodromy, so they are nonetheless integral.
• ii) OK for projective varieties, also for all Shimuras or rank $\geq 2$ ($\rightsquigarrow$ André-Oort conjecture by Pila-Shankar-Tsimerman).

## Remark

• epsilon less: for $p$ large, $F$-isocrystal on $X_{\mathbb{Z}_q}$ has an extra structure (Fontaine-Laffaille module) to which one (Faltings) associates a crystalline $p$-adic local system on $X_{\mathbb{Q}_q}$. As we go to $X_{\bar{\mathbb{Q}}_p}$ we have to identify those with the $\mathbb{L}_{\mathfrak{p}}$. Can do this in the non-proper case under an extra cohomological assumption which is fulfilled on Shimuras of real rank $\geq 2$: multiplicity 1 on the moduli without decoration at infinity.

# Methods

## Integrality

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## Crystallinity

new proof uses strong $p$-adic topology, the theory of Higgs-de Rham flows (Lan-Sheng-Zuo/Xu) on $X_{\mathbb{Z}_q}$ yielding a Fontaine-Laffaille module, Faltings's theorem assigning a $p$-adic crystalline local system to a Fontaine-Laffaille module. Proof ($p$-adic) local and no longer appeals to complex geometry.

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## Theorem (de Jong-E 2022)

The group $\Gamma_0$ generated by two elements $(a, b)$ with one relation $b^2 = a^2ba^{-2}$ is not the fundamental group of $X(\mathbb{C})$, $X$ smooth quasi-projective over $\mathbb{C}$.

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• the pronilpotent completion (Malčev completion) is endowed with a mixed Hodge structure.
• Simpson's non-abelian Hodge theory in general: yields obstruction for the finitely presented group to be the topological fundamental group of a smooth *projective complex variety, not quasi-projective.*

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**Theorem**

If $\exists\ \mathbb{L}_{\ell_0}$ $\ell_0$-adic point of $M$, then for all $\ell \neq \ell_0$, $\exists\ \mathbb{L}_\ell$ on $M$.

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## Example

Not fulfilled for $\ell = 2$. (Becker–Breuillard–Varjú).

# Happy Birthday

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FEATURE ARTICLE

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## Some Arithmetic Properties of Complex Local Systems

Hélène Esnault

*Communicated by* Notices *Associate Editor Han-Bom Moon*

*Le niveau uniforme du varech sur toutes les roches marquait la ligne de flottaison de la marée pleine et de la mer étale.*

*(The uniform level of kelp on all the rocks marked the waterline of full tide and slack [étale] sea.)*

—Victor Hugo, *Les travailleurs de la mer*, 1866, p. 257