

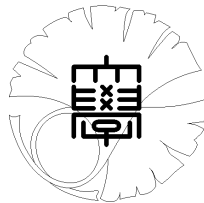
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On fatou and julia sets of foliations

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ABSTRACT. The Fatou–Julia decomposition is significant in the study of iterations of holomorphic mappings. Such a decomposition can be also considered for foliations in a unified manner [9], [10], [2], [3]. Although the decomposition will be fundamental in the study, it is not easy to decide the decomposition. In this article, we give a sufficient condition for open sets to be contained in Fatou sets. We also discuss relations between Fatou–Julia decompositions and minimal sets.

INTRODUCTION

The Fatou–Julia decomposition is significant in the study of iterations of holomorphic mappings and semigroups generated by rational mappings. Such decompositions are also possible for transversely holomorphic foliations of complex codimension one in a unified manner [9], [10], [2], [3]. Dynamics of foliations on Fatou sets are expected to be tame. For example, Fatou sets of foliations are known to admit transverse invariant metrics [2, Theorem 4.21], [3, Theorem 5.5]. However, as in the classical case, it is difficult to decide Fatou sets. In this article, we give a criterion in terms of transverse invariant metrics. The basic idea is to use a partial converse to the above-mentioned result [2, Lemma 2.16], namely, if regular foliations of compact manifolds admit transverse invariant metrics, then Julia sets are empty, where we consider Julia sets in the sense of [2]: if we consider Julia sets in the sense of [9] or [10], then there are foliations which admits transverse invariant metrics and of which the Fatou sets are empty [9, Example 8.6]. A simple example shows that existence of transverse invariant metrics are not sufficient to find Fatou sets (see Remark 3.10). We will introduce a notion of compact approximations which is a slight generalization of approximations of open sets by compact sets (Definition 3.6) and show the following

Theorem 3.9. *Let \mathcal{F} be a transversely holomorphic foliation of a compact manifold M , of complex codimension one. Let U be an \mathcal{F} -invariant open set. Suppose that*

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- 1) *There exists a transverse Hermitian metric on U invariant under the holonomies and bounded from below.*
- 2) *The open set U is compactly approximated.*

Then, U is contained in the Fatou set of \mathcal{F} .

We will also show that both metrics and compact approximations are necessary.

When studying foliations, minimal sets are significant. In the theory of secondary characteristic classes for foliations, some similarities between minimal sets and Julia sets are known [2, Section 6]. We will discuss relations between minimal sets and Julia sets from dynamical point of view.

This article is organized as follows. First we recall definitions of foliations and their singularities. Next, we introduce Fatou and Julia sets after [3] in Section 2. Relations between Fatou sets and transverse invariant metrics are discussed in Section 3, where the main result will be shown. Finally, minimal sets are discussed in Section 4.

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1. FOLIATIONS

Throughout this article, we work on the C^∞ or holomorphic category. In view of [6] and [1], we introduce the following

Definition 1.1 ([1], cf. [6]). Let M be a manifold. A partition $\mathcal{F} = \{L_\lambda\}$ of M into immersed manifold is called a *singular foliation* of M if the following condition is satisfied: for any $p \in M$, there exists an open neighborhood U_p of p such that there is a finite number of vector fields, say X_1, \dots, X_r , on U_p such that $[X_i, X_j] \in \langle X_1, \dots, X_r \rangle$ and that $T_q L_\lambda = \langle X_1(q), \dots, X_r(q) \rangle$ for $q \in U_p$, where $\langle X_1, \dots, X_r \rangle$ is the submodule of sections to TM . The pair of such vector fields X_1, \dots, X_r is called a *local generator* of \mathcal{F} . Submanifolds L_λ are called the *leaves* of \mathcal{F} . A leaf which contains $p \in M$ is said to be the leaf which passes p and denoted by L_p . If M is a complex manifold and if X_i 's are holomorphic, then \mathcal{F} is said to be *holomorphic*.

It is easy to show the following

Lemma 1.2. *The mapping $p \mapsto \dim L_p$ is lower semi-continuous.*

Definition 1.3. Let \mathcal{F} be a foliation of M . The maximal value of $\{\dim L_p \mid p \in M\}$ is said to be the *dimension* of \mathcal{F} and denoted by $\dim \mathcal{F}$. If $\dim M = m$, then $m - \dim \mathcal{F}$ is called the *codimension* of \mathcal{F} and denoted by $\text{codim } \mathcal{F}$. We

set

$$\text{Sing } \mathcal{F} = \{p \in M \mid \dim L_p < \dim \mathcal{F}\}.$$

The restriction of \mathcal{F} to $M \setminus \text{Sing } \mathcal{F}$ is called the *regular part* of \mathcal{F} and denoted by \mathcal{F}^{reg} . If $\text{Sing } \mathcal{F} = \emptyset$, then \mathcal{F} is said to be *regular* or *non-singular*.

Definition 1.4. A foliation \mathcal{F} of M is said to be *transversely holomorphic* if \mathcal{F}^{reg} is transversely holomorphic. That is, \mathcal{F}^{reg} admits a transversal complex structure invariant by holonomies.

A holomorphic foliation is a transversely holomorphic foliation. It is well-known what we may assume that the complex codimension of $\text{Sing } \mathcal{F}$ is greater than one if \mathcal{F} is holomorphic. We will assume this condition when holomorphic foliations are considered.

2. FATOU AND JULIA SETS

We briefly recall the definition of the Fatou sets for foliations in the sense of [3]. Let \mathcal{F} be a transversely holomorphic foliation of a closed manifold M , of complex codimension one. Let \mathcal{F}^{reg} be the regular part of \mathcal{F} , namely, the restriction of \mathcal{F} to $M \setminus \text{Sing } \mathcal{F}$. Let T be a complete transversal for \mathcal{F}^{reg} , namely, we assume that T meets every leaf of \mathcal{F}^{reg} (so that T is quite possibly disconnected). We may moreover assume that T is biholomorphic to a disjoint union of discs in \mathbb{C} , where the complex structure of T is induced by the transversal holomorphic structure of \mathcal{F}^{reg} . Let Γ be the holonomy pseudogroup of \mathcal{F}^{reg} on T . We have then a Fatou–Julia decomposition of T [3, Definitions 2.2 and 2.10]. Roughly speaking, the Fatou set is defined as follows. Let \mathcal{T} be the set of relatively compact open subsets of T . Let $T' \in \mathcal{T}$ and $\Gamma_{T'}$ the *restriction* of Γ to T' , namely, we set

$$\Gamma_{T'} = \{\gamma \in \Gamma \mid \text{dom } \gamma \subset T' \text{ and range } \gamma \subset T'\},$$

where $\text{dom } \gamma$ and $\text{range } \gamma$ denote the domain and range of γ , respectively. Note that $\Gamma_{T'}$ is a pseudogroup on T' . An open connected subset U of T' is said to be an F-open set if every germ of elements of $\Gamma_{T'}$ at a point in U is represented by an element of T (not T' in general) defined on U , where the letter ‘F’ stands for ‘Fatou’. We then define $F^*(\Gamma_{T'})$ to be the union of F-open sets and $J^*(\Gamma_{T'})$ its complement in T' . Finally, the Julia set of (Γ, T) is defined by

$$J(\Gamma) = \overline{\bigcup_{T' \in \mathcal{T}} J^*(\Gamma_{T'})},$$

and $F(\Gamma) = T \setminus J(\Gamma)$.

Remark 2.1. *The Fatou and Julia sets $F(\Gamma)$ and $J(\Gamma)$ in this article is referred as $F_{\text{pg}}(\Gamma)$ and $J_{\text{pg}}(\Gamma)$ in [3]. That is, we can consider pseudosemigroups generated by pseudogroups, and $F(\Gamma)$ and $J(\Gamma)$ in [3] mention the Fatou and Julia sets of Γ as pseudosemigroups. If the pseudogroup is compactly generated, then these coincide but in general not.*

Definition 2.2 ([3, Definition 5.3]). The saturation of $F(\Gamma)$ is called the *Fatou set* of \mathcal{F} and denoted by $F(\mathcal{F})$. The complement of $F(\mathcal{F})$ in M is called the *Julia set* of \mathcal{F} and denoted by $J(\mathcal{F})$.

Note that $J(\mathcal{F})$ is the union on the saturation of $J(\Gamma)$ and $\text{Sing } \mathcal{F}$.

Definition 2.2 makes a sense. Indeed, we have the following

Lemma 2.3 ([3, Lemma 2.18]). *Both $F(\Gamma)$ and $J(\Gamma)$ are invariant under Γ .*

Definition 2.4. A subset $X \subset M$ is said to be \mathcal{F} -invariant if $p \in X$, then $L_p \subset X$, where L_p denotes the leaf which contains p .

The following fundamental property is now clear from definitions.

Lemma 2.5. *Both $F(\mathcal{F})$ and $J(\mathcal{F})$ are \mathcal{F} -invariant.*

The Fatou and Julia sets do not depend on the choice of realizations of holonomy pseudogroups. More precisely, there is a notion of *equivalence* between pseudogroups. Roughly speaking, an equivalence from (Γ_1, T_1) to (Γ_2, T_2) is a certain family of mappings from open sets of T_1 to T_2 which conjugates elements of Γ_1 and Γ_2 . Pseudogroups (Γ_1, T_1) and (Γ_2, T_2) are equivalent if they are associated with the same foliation. For the details of equivalence, we refer readers to [10]. See also [3, Definition 1.22]. We have the following

Theorem 2.6 ([3, Theorem 2.19]). *Let (Γ_1, T_1) and (Γ_2, T_2) be pseudogroups and $\Phi: \Gamma_1 \rightarrow \Gamma_2$ an equivalence. Then, we have $\Phi(F(\Gamma_1)) = F(\Gamma_2)$ and $\Phi(J(\Gamma_1)) = J(\Gamma_2)$.*

Lemma 2.7. *The Fatou and Julia sets $F(\mathcal{F})$ and $J(\mathcal{F})$ do not depend on the choice of realizations of the holonomy pseudogroup of \mathcal{F}^{reg} .*

Proof. By Theorem 2.6, the saturation of $F(\Gamma)$ is independent of the choice of (Γ, T) . Therefore $F(\mathcal{F})$ is also. \square

Remark 2.8. *The Fatou-Julia decomposition for foliations is firstly introduced in [9] and refined in [10]. These definitions pay attention to deformations of foliations while the definition in [3] follows a rather classical definition in terms of normal families. It is known that the Julia sets in the sense of [9] and [10] are contained in those of [3]. The inclusion can be either strict or not.*

Note also that a Fatou–Julia decomposition of singular foliations of a complex surface with Poincaré type singularities are introduced in [9, Example 8.1]. The Fatou–Julia decomposition given by Definition 2.2 of this article differs from it in general. See 2) of Example 3.11.

We need the following

Definition 2.9 ([10, 1.3] cf. [3, Definition 3.1]). A pseudogroup (Γ, T) is *compactly generated* if there is a relatively compact open set T' of T , and a finite collection of elements $\{\gamma_1, \dots, \gamma_r\}$ of Γ of which the domains and the ranges are contained in T' such that

- 1) the family $\{\gamma_1, \dots, \gamma_r\}$ generates $\Gamma_{T'}$, where $\Gamma_{T'}$ is the restriction of Γ to T' .
- 2) for each γ_i , there exists an element $\tilde{\gamma}_i$ of Γ such that $\text{dom } \tilde{\gamma}_i$ contains the closure of $\text{dom } \gamma_i$ and that $\tilde{\gamma}_i|_{\text{dom } \gamma_i} = \gamma_i$.
- 3) the inclusion of T' into T induces an equivalence from $\Gamma_{T'}$ to Γ .

$(\Gamma_{T'}, T')$ is called a *reduction* of (Γ, T) .

It is known that if (Γ, T) is compactly generated and if (Γ', T') is equivalent to (Γ, T) , then (Γ', T') is also compactly generated.

Example 2.10. If (Γ, T) is a holonomy pseudogroup associated with a regular foliation of a closed manifold M , then (Γ, T) is compactly generated. Also, if \mathcal{F} is a complex foliation of a complex surface and if every singularity of \mathcal{F} is of Poincaré type, then the holonomy pseudogroup of \mathcal{F}^{reg} is compactly generated. A basic example of such kind is given by a vector field $\sum_{i=1}^n \lambda_i z_i \frac{\partial}{\partial z_i}$ such that the convex hull of $\{\lambda_1, \dots, \lambda_n\}$ does not contain the origin.

3. FATOU SETS AND TRANSVERSE METRICS

The following is known.

Theorem 3.1 ([3, Theorem 5.5], [2, Theorem 4.21]). *The Fatou set $F(\mathcal{F})$ admits a transverse Hermitian metric transversely of class $C_{\text{loc}}^{\text{Lip}}$. If in addition Γ is compactly generated, then there is such a metric transversely of class C^ω .*

We will show a partial converse to Theorem 3.1 to find Fatou sets.

Definition 3.2. Let $U \subset \mathbb{C}$ be an open set. Let h_1 and h_2 be Hermitian metrics on U . We say that $h_1 \geq ch_2$ if $h_1(v, v) \geq ch_2(v, v)$ holds for any $v \in TU$, where $c \in \mathbb{R}$. We say that h_1 and h_2 are *equivalent* if $\frac{1}{c}h_1 \leq h_2 \leq ch_1$ holds for some $c \geq 1$.

Definition 3.3. Let U be a complex manifold and we fix a Hermitian metric, say h_0 , on U . A Hermitian metric h on U is said to be *bounded from below* if there exists $c > 0$ such that $h \geq ch_0$ holds on U .

In general, there is no canonical choice of h_0 so that we introduce the following

Assumption 3.4. Let \mathcal{F} be a transversely holomorphic foliation of M , and (Γ, T) the holonomy pseudogroup of \mathcal{F}^{reg} . We fix a Hermitian metric g on M and a realization of (Γ, T) by choosing a complete transversal for \mathcal{F}^{reg} . Then let h_0 be the restriction of g to T .

Remark 3.5. If M is compact, then h_0 obtained as above are mutually equivalent. There are several cases where we have a natural choice of g and hence h_0 as above. For example, if $M = \mathbb{C}P^n$, then we can choose g as the Fubini–Study metric. If $M = T^{2n} = \mathbb{C}^n/\Gamma$, where Γ is a lattice isomorphic to \mathbb{Z}^{2n} , then it is natural to g as the one induced from the standard Hermitian metric on \mathbb{C}^n . In what follows, we consider the Fubini–Study metric when foliations of $\mathbb{C}P^2$ are discussed.

Definition 3.6. Let U be an open subset of $M \setminus \text{Sing } \mathcal{F}$. A family $\{K_n\}_{n \in \mathbb{N}}$ of closed subset of U is called a *compact approximation* if the following conditions are satisfied, namely,

- i) Each K_n is a closed subset of U with boundary of class C^1 , and $K_n \subsetneq U$.
- ii) Each K_n is either saturated by the leaves of \mathcal{F}^{reg} or ∂K is transversal to \mathcal{F}^{reg} .
- iii) The holonomy pseudogroup of the foliation obtained by restricting \mathcal{F}^{reg} to K_n is compactly generated.
- iv) For each n , we have $K_n \subset \text{Int } K_{n+1}$, where $\text{Int } K_{n+1}$ denotes the interior of K_{n+1} .
- v) We have $U = \bigcup_{n \in \mathbb{N}} K_n$.

We say also that U is *compactly approximated* by $\{K_n\}_{n \in \mathbb{N}}$.

In practice, the index n may begin by an arbitrary integer.

Remark 3.7. There are some typical cases where the condition iii) in Definition 3.6 is satisfied:

- 1) Each K_n is compact.
- 2) For each n , ∂K_n is tangent to \mathcal{F} and there exists a compact subset, say K'_n , of K_n with the following properties:
 - i) $\partial K'_n \setminus \partial K_n$ is of class C^1 and transversal to \mathcal{F} .
 - ii) The restriction of \mathcal{F} to $K_n \setminus \text{Int } K'_n$ is a product foliation.

We will actually make use of this fact in Example 3.11.

We give some basic examples of compact approximations.

Example 3.8. Let (z, w) be the standard coordinates for \mathbb{C}^2 . Let $\omega = \mu w dz - \lambda z dw$, where $\lambda, \mu \in \mathbb{C} \setminus \{0\}$. We set $\alpha = \lambda/\mu$ and denote by \mathcal{F}_α the foliation of \mathbb{C}^2 defined by ω .

- 1) Suppose that $\alpha \notin \mathbb{R}_{\leq 0}$. Let $K_n = \{(z, w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 \geq 1/n^2\}$ for $n \geq 1$. Then, $\{K_n\}_{n \geq 1}$ is a compact approximation of $\mathbb{C}^2 \setminus \{(0, 0)\} = M \setminus \text{Sing } \mathcal{F}_\alpha$ such that ∂K_n is transversal to \mathcal{F}_α for each n .
- 2) Suppose that $\alpha \in \mathbb{R}_{< 0}$. Let $f(z, w) = |z||w|^{-\alpha}$. If we set $K_n = \{(z, w) \in \mathbb{C}^2 \mid f(z, w) \geq 1/n\}$ for $n \geq 1$, then $\{K_n\}$ is a compact approximation of U such that ∂K_n is tangent to \mathcal{F}_α for each n .
- 3) In general, suppose that $\dim M = 2$ and that $\text{Sing } \mathcal{F}$ is a finite set. If moreover each singularity is of Poincaré type, then $M \setminus \text{Sing } \mathcal{F}$ admits a compact approximation. Indeed, we fix a metric on M and set $K_n = \{p \in M \mid \text{dist}(p, \text{Sing } \mathcal{F}) \geq 1/n\}$. If $N \in \mathbb{N}$ is large enough, then $\{K_n\}_{n \geq N}$ is a compact approximation of $M \setminus \text{Sing } \mathcal{F}$. For example, if $\alpha \notin \mathbb{R}$ in the case 1), then \mathcal{F}_α is extended to $\mathbb{C}P^2$ with $\text{Sing } \mathcal{F}_\alpha = \{[0 : 0 : 1], [0 : 1 : 0], [1 : 0 : 0]\}$, where $[z_0 : z_1 : z_2]$ denotes the standard homogeneous coordinates. A compact approximation for $\mathbb{C}P^2 \setminus \text{Sing } \mathcal{F}_\alpha$ is given by setting $K_n = \mathbb{C}P^2 \setminus (\{[z_0 : z_1 : 1] \mid |z_0|^2 + |z_1|^2 < 1/n^2\} \cup \{[z_0 : 1 : z_2] \mid |z_0|^2 + |z_2|^2 < 1/n^2\} \cup \{[1 : z_1 : z_2] \mid |z_1|^2 + |z_2|^2 < 1/n^2\})$.

Now we will show the following

Theorem 3.9. *Let \mathcal{F} be a transversely holomorphic foliation of a compact manifold M , of complex codimension one. Let U be an \mathcal{F} -invariant open set. Suppose that*

- 1) *There exists a transverse Hermitian metric on U invariant under the holonomies and bounded from below.*
- 2) *The open set U admits a compact approximation.*

Then, U is contained in the Fatou set of \mathcal{F} .

Proof. Let (Γ, T) be the holonomy pseudogroup of \mathcal{F}^{reg} . The proof is basically parallel to the case where Γ is compactly generated, we need however additional observations. We denote by \mathcal{T} the set of relatively compact subsets of T . Let $T' = \{T'_i\} \in \mathcal{T}$ and $\Gamma_{T'}$ the restriction of Γ to T' , where T'_i denotes the connected components of T' . Let $\{K_n\}$ be a compact approximation of U . We will show that $K_n \cap T' \subset F^*(\Gamma_{T'})$ for any n . Once this is established, $U \cap T' \subset F^*(\Gamma_{T'})$ so that $U \cap J^*(\Gamma_{T'}) = \emptyset$ for any T' . It follows that

$U \cap \left(\bigcup_{T' \in \mathcal{T}} J^*(\Gamma_{T'}) \right) = \emptyset$. Since U is open, $U \cap J(\Gamma) = U \cap \overline{\bigcup_{T' \in \mathcal{T}} J^*(\Gamma_{T'})} = \emptyset$. This will complete the proof. In what follows, we assume for simplicity that T is contained in \mathbb{C} and is equipped with the standard Hermitian metric which we denote by h_0 . We may moreover assume that T is a disjoint union of relatively compact discs. We do not lose generality because M is compact so that Hermitian metrics on T are equivalent (see Remark 3.5).

Let now h be a transverse Hermitian metric on U as in the statement. We denote by Γ_n the holonomy pseudogroup of $\mathcal{F}|_{K_n}$ associated with $K_n \cap T'$. As T' is relatively compact, we can find a finite set $\{\gamma_i\}$ of generators of Γ_n . Therefore, there are $\delta > 0$ and $C > 0$ such that the germ of any γ_i at a point, say p , in $K_n \cap T'$ is represented by an element of Γ , actually of Γ_{n+1} , defined on the δ -ball $B_\delta(p) \subset T$ centered at p and $|(\gamma'_i)_p| \leq C$. Note that we may assume that $C \geq 1$. On the other hand, we have the following, namely, let $B'_\delta(p)$ be the δ -ball with respect to h centered at p . By the assumption, h is bounded from below so that we have $h \geq c^2 h_0$ for some $c > 0$. We have then

$$\forall p \in K_n \cap T', \forall \delta > 0, B'_\delta(p) \subset T' \Rightarrow B'_\delta(p) \subset B_{\delta/c}(p).$$

We now set $\delta' = \delta c / 2C$. By decreasing δ' if necessary, we assume that $B_{\delta'}(p) \subset U$. We claim then that the germ of any element of Γ_n at any $p \in K_n \cap T'$ is represented by an element of Γ_{n+1} defined on $B'_{\delta'}(p)$. This is shown as follows. Let $\Gamma_n(k)$ be the subset of Γ_n which consists of elements presented by composition of at most k generators, where $\Gamma_n(0)$ is generated by $\{\text{id}_{K_n \cap T'}\}$, and let $\Gamma_n(k)_p$ be the set of germs at p of elements of $\Gamma_n(k)$. We have $\Gamma_n = \bigcup_{k=0}^{+\infty} \Gamma_n(k)$. If $\gamma_p \in \Gamma_n(1)_p$, then $B'_{\delta'}(p) \subset B_{\delta/2C}(p) \subset B_\delta(p)$ so that the claim holds. Assume by induction that $\gamma_p \in \Gamma_n(k)_p$ is represented by an element of Γ_{n+1} defined on $B'_{\delta'}(p)$. Let $\zeta_p \in \Gamma_n(k+1)_p$. Then, ζ_p is represented by an element of Γ_n of the form $\gamma_i \circ \gamma$, where $\gamma \in \Gamma_n(k)_p$ and γ_i is one of the generators. We may assume that γ is well-defined on $B'_{\delta'}(p)$ as an element of Γ_{n+1} . We have $\gamma(B'_{\delta'}(p)) = B'_{\delta'}(\gamma(p)) \subset B_{\delta/2C}(\gamma(p)) \subset B_\delta(\gamma(p))$ because γ is an isometry on U . As $\gamma(p) \in T'$, γ_i is well-defined on $B'_{\delta'}(\gamma(p))$ as an element of Γ_{n+1} . It follows that $\gamma_i \circ \gamma$ is also well-defined on $B'_{\delta'}(p)$ as an element of Γ_{n+1} . Since T is assumed to be a disjoint union of relatively compact discs in \mathbb{C} , the family

$$\Gamma_{n+1}(U) = \{\gamma \in \Gamma_{n+1} \mid \text{dom } \gamma = U, \gamma(U) \cap T' \neq \emptyset\}$$

which consists of elements of Γ_{n+1} obtained by extension as above, is a normal family. This directly verifies that $B'_{\delta'}(p)$ has the property (wF) [3]. Let now $\gamma \in \Gamma_n$ and $\text{dom } \gamma \subset B'_{\delta'}(p)$. Since $\gamma(B'_{\delta'}(p)) = B'_{\delta'}(\gamma(p))$, range γ itself is again a wF-open sets. Thus $B'_{\delta'}(p)$ is an F-open set so that $p \in F^*(\Gamma_{T'})$. \square

- Remark 3.10.** 1) *The fact that a wF -open set is an F -open set always holds if Γ is a pseudogroup. These will differ if we study pseudosemi-groups. See [3] for the details.*
- 2) *The induction in the proof is taken from the proof of [8, Lemme 2.2].*
- 3) *If Γ is compactly generated, then we can choose T' in the above proof so that $(\Gamma_{T'}, T')$ is equivalent to (Γ, T) and that the arguments can be simplified.*

Example 3.11 (cf. Example 3.8 and [3, Example 5.11]). Let $\omega = \mu y dx - \lambda x dy$ be a holomorphic 1-form on \mathbb{C}^2 , where $\lambda, \mu \neq 0$. We set $\alpha = \lambda/\mu$ and let \mathcal{F}_α be the foliation of \mathbb{C}^2 defined by ω . We denote by \mathcal{G}_α the natural extension of \mathcal{F}_α to $\mathbb{C}P^2$. Let $[z_0 : z_1 : z_2]$ be the standard homogeneous coordinates for $\mathbb{C}P^2$, where $x = z_0/z_2$ and $y = z_1/z_2$. We set $a = z_0/z_1, b = z_2/z_1$ if $z_1 \neq 0$, and $u = z_1/z_0, v = z_2/z_0$ if $z_0 \neq 0$. We set $\mathbb{C}^2(x, y) = \{[x : y : 1] \in \mathbb{C}P^2\}$. Similarly we define $\mathbb{C}^2(a, b)$ and $\mathbb{C}^2(u, v)$.

- 1) Suppose that $\alpha \notin \mathbb{R}$. Let $U = \mathbb{C}P^2 \setminus \{z_0 z_1 z_2 = 0\}$. It is known that the Fatou set $F(\mathcal{G}_\alpha)$ is equal to U . We define $f : \mathbb{C}P^2 \rightarrow \mathbb{R}$ by

$$f([z_0 : z_1 : z_2]) = \frac{|z_0|^2 |z_1|^2 |z_2|^2}{(|z_0|^2 + |z_1|^2 + |z_2|^2)^3}.$$

We have

$$f(x, y) = \frac{|x|^2 |y|^2}{(1 + |x|^2 + |y|^2)^3}$$

and

$$\begin{aligned} \frac{\partial f}{\partial x}(x, y) &= \frac{\bar{x}|y|^2(1 - 2|y|^2 + |z|^2)}{(1 + |x|^2 + |y|^2)^4}, \\ \frac{\partial f}{\partial y}(x, y) &= \frac{\bar{y}|x|^2(1 - 2|z|^2 + |w|^2)}{(1 + |x|^2 + |y|^2)^4}. \end{aligned}$$

Let, for $n \geq 28$,

$$K_n = \left\{ [z_0 : z_1 : z_2] \in \mathbb{C}P^2 \mid f([z_0 : z_1 : z_2]) \geq \frac{1}{n} \right\}.$$

Note that K_n is a compact subset contained in U . This can be seen by for example by the fact that $(1, 1)$ is the unique maximum of the function $(t, s) \mapsto (ts)/(1 + t + s)^3$, where $t, s > 0$. We will show that ∂K_n is transversal to \mathcal{G}_α . If we restrict ourselves to $U \cap \mathbb{C}^2(x, y) = \{(x, y) \in \mathbb{C}P^2 \mid xy \neq 0\}$, then by Lemma 3.14 below, ∂K_n is transversal to \mathcal{F}_α if and only if

$$(3.12) \quad \lambda(1 - 2|x|^2 + |y|^2) + \mu(1 - 2|y|^2 + |x|^2) \neq 0.$$

Suppose the contrary and let $\alpha = \mu/\lambda = a + \sqrt{-1}b$, where $a, b \in \mathbb{R}$. By the assumption $b \neq 0$ so that the equalities

$$\begin{aligned} (1+a) + (-2+a)|x|^2 + (1-2a)|y|^2 &= 0, \\ 1 - 2|y|^2 + |x|^2 &= 0 \end{aligned}$$

hold by (3.12). It follows that $3 - 3|y|^2 = 0$ and further that $|x| = |y| = 1$. As $f(1, 1) = 1/27$, we never have $(|x|, |y|) = (1, 1)$ for $(x, y) \in \partial K_n \cap \mathbb{C}^2(x, y)$. Since K_n is contained in $\mathbb{C}^2(x, y)$, we see that ∂K_n is transversal to \mathcal{G}_α . Therefore $\{K_n\}_{n \geq 28}$ is a compact approximation of U . We now set $\omega' = \frac{1}{\lambda} \frac{dz}{z} - \frac{1}{\mu} \frac{dw}{w}$. Then, $d\omega' = 0$ and ω' also defines \mathcal{G}_α on U . Therefore, an invariant metric on U is defined by setting $h = \omega' \otimes \overline{\omega'}$. The metric h is bounded from below so that U is contained in the Fatou set of \mathcal{G}_α . In this case, it is also known that $F(\mathcal{F}_\alpha) = U \cap \mathbb{C}^2$. The family $\{K_n\}_{n \geq 28}$ is a compact approximation of $U \cap \mathbb{C}^2$ with respect to \mathcal{F}_α .

2) If $\alpha \in \mathbb{R}$, then the Fatou–Julia decomposition of $F(\mathcal{F}_\alpha)$ and that of $F(\mathcal{G}_\alpha)$ are known to be different [3, Example 5.11]. This is also seen by Theorem 3.9.

i) First we study \mathcal{F}_α .

a) Assume that $\alpha > 0$. Then, $F(\mathcal{F}_\alpha) = \mathbb{C}^2 \setminus \{(0, 0)\}$ and a transverse invariant metric, say h , on $F(\mathcal{F}_\alpha)$ is given by $h = \eta_\alpha \otimes \overline{\eta_\alpha}$, where

$$\eta_\alpha = \frac{\alpha y dx - x dy}{|x|^{\alpha+1} + |y|^{(\alpha+1)/\alpha}}.$$

Note that h is bounded from below. If we set $K_n = \{(x, y) \in \mathbb{C}^2 \mid 1/n \leq |x|^2 + |y|^2 \leq n\}$, then $\{K_n\}_{n \geq 1}$ is a compact approximation of $\mathbb{C}^2 \setminus \{(0, 0)\}$.

b) Assume that $\alpha < 0$. Then, $F(\mathcal{F}_\alpha) = \mathbb{C}^2 \setminus \{(x, y) \in \mathbb{C}^2 \mid xy = 0\}$ and a transverse invariant metric h on $F(\mathcal{F}_\alpha)$ is given by $h = \nu_\alpha \otimes \overline{\nu_\alpha}$, where

$$\nu_\alpha = \alpha \frac{dx}{x} - \frac{dy}{y}.$$

The metric h is bounded from below. A compact approximation of $F(\mathcal{F}_\alpha)$ is given by $\{K_n\}$ with $K_n = \{(x, y) \in \mathbb{C}^2 \mid 1/n \leq |x||y| \leq n\}$.

ii) Next we study \mathcal{G}_α .

a) Assume that $\alpha > 0$. By exchanging z_0 and z_1 if necessary, we may assume that $0 < \alpha < 1$. We have $F(\mathcal{G}_\alpha) = \mathbb{C}P^2 \setminus \{[z_0 : z_1 : z_2] \in \mathbb{C}P^2 \mid z_1 z_2 = 0\}$. Note that we have $F(\mathcal{G}_\alpha) \cap \mathbb{C}(x, y) = \{(x, y) \in \mathbb{C}^2 \mid y \neq 0\}$ while we have $F(\mathcal{F}_\alpha) = \mathbb{C}^2 \setminus \{(0, 0)\}$.

This is because \mathcal{G}_α is isomorphic to $\mathcal{F}_{\alpha/(\alpha-1)}$ on $\mathbb{C}^2(a, b)$ and to $\mathcal{F}_{1/(1-\alpha)}$ on $\mathbb{C}^2(u, v)$. As $0 < \alpha < 1$, we have $\alpha/(\alpha-1) < 0$ so that we are in the same situation as in the case i)-b). Namely, the singularity $(0, 0)$ on $\mathbb{C}^2(a, b)$ is of Siegel type (not of Poincaré type) so that the both a -axis and b -axis are contained in the Julia set $J(\mathcal{G}_\alpha)$ of \mathcal{G}_α . Therefore the y -axis and the u -axis are contained in $J(\mathcal{G}_\alpha)$. It follows that $J(\mathcal{F}_\alpha) \neq \mathbb{C}^2 \cap J(\mathcal{G}_\alpha)$. Note that this shows that the Julia sets in the sense of Definition 2.2 and those of [9, Example 8.1] are different in general. Let

$$\gamma_\alpha = \frac{\alpha y dx - x dy}{|x|^k(|x|^{\alpha l} + |y|^l)},$$

where $k + \alpha l = 1 + \alpha$. We have

$$\begin{aligned} |\gamma_\alpha| &= \frac{|\alpha b d a - (\alpha - 1) a d b|}{|a|^k |b|^{3-k-l} (|a|^{\alpha l} |b|^{(1-\alpha)l} + 1)} \\ &= \frac{|(1-\alpha) u d v - v d u|}{|v|^{3-k-l} (|u|^l + |v|^{(1-\alpha)l})}. \end{aligned}$$

Then, $h = \gamma_\alpha \otimes \overline{\gamma_\alpha}$ gives an invariant metric on $F(\mathcal{G}_\alpha)$. If we set $k = l = 1$, then h is bounded from below. A compact approximation of $F(\mathcal{G}_\alpha)$ is given by $\{K_n\}_{n \geq 1}$ with $K_n = \{[z_0 : z_1 : z_2] \mid |z_0|^{1-\alpha} |z_2|^\alpha \geq |z_1|/n\}$. We have $K_n \cap \mathbb{C}^2(x, y) = \{(x, y) \mid |x|^{1-\alpha} \geq |y|/n\}$ and $K_n \cap \mathbb{C}^2(a, b) = \{(a, b) \mid |a|^{1-\alpha} |b|^\alpha \geq 1/n\}$.

b) If $\alpha = 1$, then \mathcal{G}_1 is transversal to the line at infinity $\{[z_0 : z_1 : 0]\}$ and $\text{Sing } \mathcal{G}_1 = \{[0 : 0 : 1]\}$. We have $F(\mathcal{G}_1) = \mathbb{C}P^2 \setminus \{[0 : 0 : 1]\}$. Note that $F(\mathcal{F}_1) = F(\mathcal{G}_1) \cap \mathbb{C}(x, y)$. If we set

$$\begin{aligned} K_n &= \{[z_0 : z_1 : z_2] \mid |z_0|^2 + |z_1|^2 \geq 1/n |z_2|\} \\ &= \{(x, y) \in \mathbb{C}(x, y) \mid |x|^2 + |y|^2 \geq 1/n\} \cup \{[z_0 : z_1 : 0]\}, \end{aligned}$$

then $\{K_n\}_{n \geq 1}$ is a compact approximation of $\mathbb{C}P^2 \setminus \{[0 : 0 : 1]\}$.

An invariant metric on $\mathbb{C}P^2 \setminus \{[0 : 0 : 1]\}$ is given by $\eta_1 \otimes \overline{\eta_1}$.

c) If $\alpha < 0$, then $[0 : 1 : 0]$ and $[1 : 0 : 0]$ are of Poincaré type so that we have the case a) again.

Remark 3.13. *We need both a metric and a compact approximation in Theorem 3.9. Let \mathcal{F}_α be as in Example 3.11.*

- 1) *If $\alpha \notin \mathbb{R}$, then $\mathbb{C}^2 \setminus \{(0, 0)\}$ admits a compact approximation with respect to \mathcal{F}_α however there are no invariant metrics on U . Indeed, the dynamics along the z -axis and the w -axis are contracting-repelling.*
- 2) *If $\alpha = -1$, then $\mathbb{C}^2 \setminus \{(0, 0)\}$ admits an invariant metric. Indeed, if we set $\eta' = y dx + x dy$, when $\eta' \otimes \overline{\eta'}$ gives an invariant metric. However,*

$\mathbb{C}^2 \setminus \{(0, 0)\}$ does not admit a compact approximation. Indeed, if $\{K_n\}$ is a compact approximation, then the restriction of \mathcal{F}^{reg} to K_n is compactly generated so that it cannot contain the x -axis and y -axis at the same time. Note that $\eta' \otimes \bar{\eta}'$ is not bounded from below.

- 3) Let again $\alpha = -1$, and set $\eta' = ydx + xdy$. If we set $U = \{(x, y) \in \mathbb{C}^2 \mid y \neq 0\}$, then U admits a compact approximation $\{K_n\}$, where $K_n = \{(x, y) \in \mathbb{C}^2 \mid |x| \geq 1/n\}$. The metric $\eta' \otimes \bar{\eta}'$ is certainly invariant but not bounded from below. As the y -axis is contained in $J(\mathcal{F}_{-1})$, U is not contained in $F(\mathcal{F}_{-1})$.

The following lemma is well-known but we give a proof for completeness.

Lemma 3.14. *Let $U \subset \mathbb{C}^n$ be an open subset and $g: U \rightarrow \mathbb{R}$ a smooth function. Let $M = g^{-1}(c)$, where $c \in g(U)$ is assumed to be a regular value. Finally let $X = \sum_{i=1}^n f_i \frac{\partial}{\partial z_i}$ be a holomorphic vector field on U , where (z_1, \dots, z_n) are the standard coordinates for \mathbb{C}^n . Then, X is transversal to M at $p \in M$ if and only if*

$$\sum_{i=1}^n f_i(p) \frac{\partial g}{\partial z_i}(p) \neq 0$$

holds, where X is said to be transversal to M at p if and only if the integral curve of X and M transversally intersects at p .

Proof. First note the X and M is transversal at p if and only if X is not tangent to M for the dimensional reason. We identify \mathbb{C}^n with \mathbb{R}^{2n} and equip \mathbb{C}^n with the standard Euclidean metric. Let x_i, y_i be the real and imaginary parts of z_i , respectively. Then, the normal direction of $T_p M$ is given by $\sum_{i=1}^n \left(\frac{\partial g}{\partial x_i}(p) \frac{\partial}{\partial x_{i_p}} + \frac{\partial g}{\partial y_j}(p) \frac{\partial}{\partial y_{i_p}} \right)$. On the other hand, the tangent space of the integral curve of X at p is spanned by

$$\begin{aligned} & \sum_{i=1}^n \left(a_i(p) \frac{\partial}{\partial x_{i_p}} + b_i(p) \frac{\partial}{\partial y_{i_p}} \right) \text{ and} \\ & \sum_{i=1}^n \left(-b_i(p) \frac{\partial}{\partial x_{i_p}} + a_i(p) \frac{\partial}{\partial y_{i_p}} \right). \end{aligned}$$

Therefore, $X(p)$ is tangent to $T_p M$ if and only if the both

$$\begin{aligned} & \sum_{i=1}^n \left(a_i(p) \frac{\partial g}{\partial x_i}(p) + b_i(p) \frac{\partial g}{\partial y_i}(p) \right) = 0, \\ & \sum_{i=1}^n \left(-b_i(p) \frac{\partial g}{\partial x_i}(p) + a_i(p) \frac{\partial g}{\partial y_i}(p) \right) = 0 \end{aligned}$$

hold. This is equivalent to

$$\begin{aligned} & \sum_{i=1}^n f_i(p) \frac{\partial g}{\partial z_i}(p) \\ &= \frac{1}{2} \sum_{i=1}^n \left(a_i(p) \frac{\partial g}{\partial x_i}(p) + b_i(p) \frac{\partial g}{\partial y_i}(p) \right) + \frac{\sqrt{-1}}{2} \sum_{i=1}^n \left(b_i(p) \frac{\partial g}{\partial x_i}(p) - a_i(p) \frac{\partial g}{\partial y_i}(p) \right) \\ &= 0. \end{aligned} \quad \square$$

4. JULIA SETS AND MINIMAL SETS

We recall the following classical

Definition 4.1. Let \mathcal{F} be a foliation of a manifold M . A subset \mathcal{M} of M is said to be *minimal* if

- 1) \mathcal{M} is non-empty and closed.
- 2) \mathcal{M} is minimal with respect to inclusions.
- 3) \mathcal{M} is saturated by leaves of \mathcal{F} , namely, if $p \in \mathcal{M}$, then the leaf which passes p is contained in \mathcal{M} .

Definition 4.2. Let \mathcal{M} be a minimal set.

- 1) We say that \mathcal{M} is *trivial* if it consists of a point in $\text{Sing } \mathcal{F}$.
- 2) We say that \mathcal{M} is *proper* if it consists of a closed leaf of \mathcal{F}^{reg} .
- 3) We say that \mathcal{M} is *exceptional* if it is non-trivial, non-proper and not equal to the whole M .

Remark 4.3. Let \mathcal{M} be a minimal set.

- 1) If \mathcal{F} is singular, then \mathcal{M} cannot be equal to M .
- 2) It is well-known that foliations of $\mathbb{C}P^n$ do not admit closed leaf in \mathcal{F}^{reg} (cf. [5, Theorem 2]). Therefore, non-trivial minimal sets of $\mathbb{C}P^n$ are exceptional.
- 3) The classification of minimal sets in Definition 4.2 is known to work well for real codimension-one regular foliations [7]. On the other hand, even in the complex codimension-one case, it is not sufficient. For example, let us consider a suspension of an action of a torsion-free Kleinian group on $\mathbb{C}P^1$. In this case, \mathcal{M} is contained in $J(\mathcal{F})$ which coincides with the suspension of the limit set [2]. On the other hand, let \mathcal{F} be a foliation of $S^3 \subset \mathbb{C}^2$ induced from the flow of a vector field $z \frac{\partial}{\partial z} + \alpha w \frac{\partial}{\partial w}$ with $\alpha \in \mathbb{R}_{>0}$. Then, \mathcal{F} is always transversely Hermitian (cf. 2) of Example 3.11). Suppose that $\alpha \notin \mathbb{Q}$ and let L be a leaf which does not belong to the Hopf link. Then, closure of L form a minimal set which

is diffeomorphic to a 2-torus as a submanifold of S^3 . This means that exceptional minimal sets should be more precised.

If foliations of $\mathbb{C}P^n$ are considered, then it is known that an exceptional minimal set contains a hyperbolic holonomy [4, Théorème]. That is, there is a loop on a leaf contained in the minimal set such that associated holonomy, in other words, the first return map or the Poincaré map, is of modulus not equal to one.

This implies the following

Theorem 4.4. *The Fatou set of a foliation of $\mathbb{C}P^n$, of codimension one, does not contain any exceptional minimal sets.*

Proof. The Fatou set admits an invariant transverse Hermitian metric by Theorem 3.1. By [5, Theorem 2], we can find a hyperbolic holonomy in the Fatou set. This is impossible because the holonomy should be an isometry for the transverse Hermitian metric. \square

Note that foliations of $\mathbb{C}P^2$ have unique minimal sets [5, Theorem 1]. Such a minimal sets are contained in the Julia sets by Theorem 4.4.

An immediate consequence is the following

Proposition 4.5. *Let \mathcal{F} be a foliation of $\mathbb{C}P^2$ and $\mathbb{C}P^2 = F(\mathcal{F}) \cup J(\mathcal{F})$ the Fatou–Julia decomposition. Then,*

- 1) *We have $J(\mathcal{F}) = \text{Sing } \mathcal{F}$, and \mathcal{F} admits no exceptional minimal set.*
- 2) *We have $\text{Sing } \mathcal{F} \subsetneq J(\mathcal{F}) \subsetneq \mathbb{C}P^2$. If \mathcal{F} admits an exceptional minimal set, say \mathcal{M} , then either $\mathcal{M} \subset \partial F(\mathcal{F}) \setminus \text{Sing } \mathcal{F}$ or $\mathcal{M} \subset J(\mathcal{F}) \setminus (\partial F(\mathcal{F}) \cup \text{Sing } \mathcal{F})$. In the latter case, the closure of any leaf in $\partial F(\mathcal{F})$ meets $\text{Sing } \mathcal{F}$.*
- 3) *We have $\mathbb{C}P^2 = J(\mathcal{F})$. If \mathcal{F} admits an exceptional minimal set, say \mathcal{M} , then $\mathcal{M} \subset J(\mathcal{F}) \setminus \text{Sing } \mathcal{F}$.*

Proof. Let \mathcal{M} be an exceptional minimal set, which is contained in $J(\mathcal{F}) \setminus \text{Sing } \mathcal{F}$ by Theorem 4.4. Therefore, if $J(\mathcal{F}) = \text{Sing } \mathcal{F}$ then such an \mathcal{M} does not exist. Suppose that $\text{Sing } \mathcal{F} \subsetneq J(\mathcal{F})$. If $F(\mathcal{F}) \neq \emptyset$, then $\partial F(\mathcal{F}) \subset J(\mathcal{F})$ is a non-empty invariant closed subset. If $\partial F(\mathcal{F}) = \text{Sing } \mathcal{F}$, then we have $F(\mathcal{F}) = \mathbb{C}P^2 \setminus \text{Sing } \mathcal{F}$ because $\text{Sing } \mathcal{F}$ consists of points. This implies that $J(\mathcal{F}) = \text{Sing } \mathcal{F}$ and contradicts the assumption. Since \mathcal{M} is unique, \mathcal{M} is contained in exactly one of $\partial F(\mathcal{F})$ or $J(\mathcal{F}) \setminus \partial F(\mathcal{F})$. Suppose that $\mathcal{M} \subset J(\mathcal{F}) \setminus \partial F(\mathcal{F})$ and L be a leaf in $\partial F(\mathcal{F})$. If $\partial L \neq \emptyset$, then it contains a minimal set, which should be trivial. Therefore $\partial L \subset \text{Sing } \mathcal{F}$. The remaining possibility is the last case. \square

We introduce the following in view of [4, IV].

Definition 4.6. Let M be a complex manifold and \mathcal{F} a holomorphic foliation of M , of codimension one. We say that \mathcal{F} satisfies the condition (H) if there exists a meromorphic 1-form on M which is not identically zero and which defines \mathcal{F} .

Definition 4.7. We denote by $\text{Sing } \omega$ the union of zeroes and poles of ω .

Note that $\text{Sing } \mathcal{F} \subset \text{Sing } \omega$.

In a quite particular case, we can find a large Fatou set. Suppose that \mathcal{F} satisfies the condition (H) and that ω has no zeroes. This occurs for example on $M = \mathbb{C}P^2$, or almost equivalently, on \mathbb{C}^2 . Let $\omega = Pdx + Qdy$ a polynomial 1-form on \mathbb{C}^2 . If we set $\omega' = \frac{dx}{Q} + \frac{dy}{P}$, then ω' also defines \mathcal{F} on $\mathbb{C}^2 \setminus \text{Pole}(\omega')$, where $\text{Pole}(\omega') = \{(x, y) \in \mathbb{C}^2 \mid P(x, y) = 0 \text{ or } Q(x, y) = 0\}$. Then ω' has no zeroes.

Assume still that ω has no zeroes. If moreover we can find a compact approximation of $M \setminus \text{Pole}(\omega)$, then we have the following

Theorem 4.8. *Let \mathcal{F} be a holomorphic foliation of a compact complex manifold M , of codimension one. Suppose that \mathcal{F} satisfies the condition (H) and let ω be a meromorphic 1-form which defines \mathcal{F} . Suppose that the following conditions are satisfied:*

- 1) *The 1-form ω is closed and has no zeroes.*
- 2) *The complement $M \setminus \text{Pole}(\omega)$ admits a compact approximation.*

Then we have $F(\mathcal{F}) \supset M \setminus \text{Pole}(\omega)$.

Proof. Let $h = \omega \otimes \bar{\omega}$ and $U = M \setminus \text{Pole}(\omega)$. As $d\omega = 0$, h determines an invariant Hermitian metric on U . Moreover, singularities of h are poles so that h is bounded from below. Then by Theorem 3.9, U is contained in the Fatou set of \mathcal{F} . \square

Note that as ω is closed, there are no exceptional minimal sets. The assertion $F(\mathcal{F}) \supset M \setminus \text{Pole}(\omega)$ can be seen as a reproduction of this fact by Theorem 4.4. Note also that a typical example is a linear foliation of $\mathbb{C}P^2$ discussed in Example 3.11.

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