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Asymptotic Stability of Small Oseen Type Navier-Stokes Flow under Three-Dimensional Large Perturbation

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Abstract: We consider the three-dimensional Navier-Stokes equations whose initial data may have infinite kinetic energy. We establish the unique existence of the mild solution to the Navier-Stokes equations for small initial data in the whole space $\mathbb{R}^3$ and a vertically periodic space $\mathbb{R}^2_h \times \mathbb{T}^1_v$ which may be constant in vertical direction so that it includes Oseen vortex. We further discuss its asymptotic stability under arbitrarily large three dimensional perturbation in $\mathbb{R}^2_h \times \mathbb{T}^1_v$.

Keywords: Navier-Stokes equations; Asymptotic stability; Oseen vortex; three-dimensional domain; Large perturbation; Mild solution.

1 Introduction

Let Ω be $\mathbb{R}^3$ or $\mathbb{R}^2 \times \mathbb{T}^1$, where $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$ is one dimensional flat torus. We consider the incompressible Navier-Stokes equations

$$\begin{cases} \partial_t u - \Delta u + u \cdot \nabla u + \nabla p = 0 & \text{in} \quad \Omega \times (0, \infty), \\ \mathrm{div} u = 0 & \text{in} \quad \Omega \times (0, \infty), \\ u(0) = u_0 & \text{in} \quad \Omega, \end{cases} \tag{1.1}$$

where $u = (u_1(x,t), u_2(x,t), u_3(x,t))$ and $p(x,t)$ respectively stand for an unknown velocity field and a pressure. The functions u_0 denote a given initial velocity. ∂_t, Δ denotes partial derivative in time and Laplace operator on the Euclidean space respectively. The differential operator $u \cdot \nabla$ denotes $\sum_{1 \le j \le 3} u_j \partial_j$.

Let us recall a special self-similar solution called the three dimensional Oseen vortex or Lamb-Oseen vortex:

$$\mathrm{Os}(x_h, x_v, t) = \frac{\Gamma}{2\pi} \frac{(-x_2, x_1, 0)}{|x_h|^2} (1 - \mathrm{e}^{-\frac{|x_h|^2}{4t}}), \quad x_h = (x_1, x_2), \ x_v = x_3, \tag{1.2}$$

where Γ is the total circulations. The two-dimensional Oseen vortex is the Navier-Stokes flow whose initial vorticity is a Dirac measure supported at the origin, and it stands for one of the simplest vortex. The three-dimensional Oseen vortex is an extension of two-dimensional one.

The goal of our paper has two fields;

(1) We construct a unique solutions with non-smooth and singular initial data so that the Oseen vortex is included as a three-dimensional flow,

(2) We discuss its asymptotic stability under large three-dimensional perturbation periodic in vertical direction.

There are many results on the existence of the solution to (1.1). It is well known that Leray [17] showed the existence of a global-in-time weak solution u in $\mathbb{R}^n$ to (1.1) satisfying the following energy estimate:

$$\|u(\tau)\|_{L^2}^2 + \int_0^t \|\nabla u(\tau)\|_{L^2}^2 d\tau \leq \|u_0\|_{L^2}^2$$

for initial data $u_0 \in L^2$. Unfortunately, the Oseen vortex is not a Leray's weak solution since the energy of the Oseen vortex is infinite, .

For non-L^2-initial data, Kato [11] proved that (1.1) is globally well-posed for small L^m-initial data in $\mathbb{R}^m$ with $m \geq 2$ by using iteration to the integral formulation of (1.1):

$$u(t) = \mathrm{e}^{t\Delta} u_0 - \int_0^t \mathrm{e}^{(t-\tau)\Delta} P(u(\tau) \cdot \nabla u(\tau)) d\tau, \tag{1.3}$$

where $\mathrm{e}^{t\Delta}$ and P are the heat kernel and the Helmholtz projection respectively. The choice of function space is related to the scaling transformation:

$$v(x,t) \to \lambda v(\lambda x, \lambda^2 t), \quad p(x,t) \to \lambda^2 p(\lambda x, \lambda^2 t),$$

which dose not change the equation. Scale-invariant function spaces are critical ones that iteration method works. In this case $L^m(\mathbb{R}^m)$ and $L_t^\infty L_x^m(\mathbb{R}^m \times (0,\infty))$ are scale-invariant function space under the above scaling transformation. Independently, Giga and Miyakawa [7] proved the existence of the solutions in $L^r(\mathbb{R}^r)$ in bounded domains with the Dirichlet boundary condition. The result of this paper was obtained even before [11] but it took long time to be published after the paper was accepted.

In three-dimensional case, $L^3(\mathbb{R}^3)$ is the critical Lebesgue space, but it does not include homogeneous functions like $\frac{1}{|x|}$. This means that $L^3(\mathbb{R}^3)$ is too restrictive to construct a self-similar solution. In this direction, Giga and Miyakawa [6] proved that the vorticity equations is well-posed for small initial data and there is a unique self-similar solution by taking initial vorticity in the Morrey space $M^{\frac{3}{2}}(\mathbb{R}^3)$. The Morrey space is scale-invariant under natural the above natural scaling and include homogeneous functions. Moreover, since $\mathrm{rotOs}(\cdot, 0) \in M^{\frac{3}{2}}$, the result of [6] provides generalized Navier-Stokes flows that contain the three dimensional Oseen vortex provided that Γ is sufficiently small. However, in [6], smoothness for initial data is needed to define $\mathrm{rot}u_0$. For instance, for a bounded function $\Theta(x)$ on the two dimensional unit sphere whose derivative is not a Radon measure, $\mathrm{rot}(\Theta(\frac{x}{|x|})\mathrm{Os}(\mathrm{x}, 0))$ is not in $M^{\frac{3}{2}}$. On the other hand, Kozono and Yamazaki [14] proved well-posedness for small initial data in weak-L^2 space in two-dimensional exterior domains. Since the two-dimensional Oseen vortex is in weak-L^2 space, the results of [14] provide its generalization. There is no restriction on smoothness of initial data in [14]. In Cannone [2] and Koch and Tataru [12], it was showed that (1.1) is globally well-posed for small initial data in the Besov spaces $B_{p,\infty}^{-1+\frac{n}{p}}(\mathbb{R}^n)$ $(1 < p < \infty)$ and $BMO^{-1}(\mathbb{R}^n)$ space respectively. The result of [12] is the most general on the well-posedness to (1.1).

Our second aim is to show asymptotic stability to the solution that is constructed in the first aim under large three-dimensional perturbation. Asymptotic stability for the Navier-Stokes equations has been widely studied. However, there are few the results on the asymptotic stability under large perturbation. In three-dimensional case, Schonbek [20] proved that 0 is asymptotically stable for $L^2 \cap L^1$-perturbation on $\mathbb{R}^3$. Subsequently, Miyakawa and Schonbek [19] study optimal decay rate. On the other hand, Kozono [13] proved asymptotic stability for the Leray's weak solution $u \in L_t^p L_x^q$ satisfying Serrin's condition [21] ($\frac{2}{p} + \frac{3}{q} = 1$ for $2 \leq p < \infty$ and $3 < q \leq \infty$) on uniformly C^3 domains. This result allows unbounded domains such as a exterior domain or a domain with non-compact boundbary. Karch, Pilarczyk and Schonbek [10] proved L^2-asymptotic stability for small mild solution $V \in \mathcal{X}_\sigma$, where $\mathcal{X}_\sigma$ is a function space of solenoidal vector fields satisfying $|\langle v \cdot \nabla V, w\rangle| \leq C(\sup_{t>0}\|V(t)\|_{\mathcal{X}_\sigma})\|\nabla v\|_{L^2}\|\nabla w\|_{L^2}$ for all $v, w \in L_t^\infty L_x^2 \cap L_t^2 \dot{H}_x^1$. This result allows many function spaces. For instance, weak L^3 space satisfies above estimate, and then it is a subspace of $\mathcal{X}_\sigma$. The decay rate to $L^{3,\infty}$-mild solutions was also studied by [8]. Although [10] is the most comprehensive result for the asymptotic stability of small mild solutions to (1.1), the three dimensional Oseen vortex is not included in this result.

In the two-dimensional case, Maekawa [18] proved asymptotic stability for the solutions obtained by [14] under $\overline{C_0^\infty}^{L^{2,\infty}}$-large perturbation in the whole space and the exterior domain. This result give us asymptotic stability to the small two-dimensional Oseen vortex.

Let us consider our two problems in more detail. For the first problem, since the two-dimensional Oseen vortex is in $L^{2,\infty}$ and three dimensional Oseen vortex is independent of x_v variable, it is good idea to construct mild solution in an anisotropic function space $Y^2 := L^\infty_v L^{2,\infty}_h$ with the norm $\|f\|_{Y^2} = \|\|f(x_h, x_v)\|_{L^{2,\infty}_h}\|_{L^\infty_v}$. Note the three dimensional Oseen vortex is in Y^2 at fixed time. Moreover, Y^2 is scale-invariant under the natural scaling and does not require any smoothness. In fact, we are able to construct a mild solution in this space for small initial data by using iteration. To this end it is needed to establish some L^p-L^q-like estimates for the heat kernel and the composite operator. It is well known that usual L^p-L^q estimate are hold for the heat kernel and the composite operator, but they are less known on the anisotropic space. For that reason we first show L^p-L^q-like estimates, after that, we construct mild solution to (1.1). Althought the method is almost the same as [6] and [14], the choice of function space is new. Moreover, it is possible to construct mild solution to initial data which is not covered by [6] such as highly oscillating one.

Our second aim is to show asymptotic stability of mild solutions obtained in the first aim under arbitrarily large perturbation $v_0 \in L^\infty_v \overline{C^\infty_{0,h}}(\mathbb{R}^2_h \times \mathbb{T}^1_v)$. We call the mild solution constructed in the above procedure the basic flow with initial data b_0. To prove asymptotic stability, there are several step. We first decompose initial perturbation $v_0 \in L^\infty_v \overline{C^\infty_{0,h}}^{L^{2\infty}}(\mathbb{R}^2_h \times \mathbb{T}^1_v)$ into two parts;

$$v_0 = \tilde{v}_0 + b_{0,\epsilon},$$

where $\tilde{v}_0 \in L^\infty_v C_{0,h}(\mathbb{R}^2_h \times \mathbb{T}^1_v)$ and $b_{0,\epsilon} \in Y^2(\mathbb{R}^2_h \times \mathbb{T}^1_v)$ with $\|b_{0,\epsilon}\|_{Y^2(\mathbb{R}^2_h \times \mathbb{T}^1_v)} < \epsilon$ for arbitrarily small $\epsilon > 0$. For the basic flow b with initial data b_0, we can construct a new basic flow $\tilde{b}$ with initial data $\tilde{b}_0 = b_0 + b_{0,\epsilon}$ so that the difference $\|\tilde{b}(t) - b(t)\|_{Y^2(\mathbb{R}^2_h \times \mathbb{T}^1_v)}$ can be estimated small enough uniformly in t since the difference of b_0 and $\tilde{b}_0$ is sufficiently small.

We then have to show the existence of a weak solution to the perturbed Navier-Stokes equations:

$$\begin{cases} \partial_t v - \Delta v + v \cdot \nabla v + \tilde{b} \cdot \nabla v + v \cdot \nabla \tilde{b} + \nabla q = 0, \quad \text{in} \quad \mathbb{R}^2_{\mathrm{h}} \times \mathbb{T}^1_{\mathrm{v}} \times (0,\infty), \\ \operatorname{div} v = 0, \quad \text{in} \quad \mathbb{R}^2_{\mathrm{h}} \times \mathbb{T}^1_{\mathrm{v}} \times (0,\infty), \\ v(0) = \tilde{v}_0, \quad \text{on} \quad \mathbb{R}^2_{\mathrm{h}} \times \mathbb{T}^1_{\mathrm{v}}. \end{cases} \tag{1.4}$$

For the vertor field v that satisfies above eqations, we find that $v + \tilde{b}$ satisfies (1.1) with initial data $\tilde{v}_0 + \tilde{b}_0$. Since the fifth term of the left-hand side of the above equation $v \cdot \nabla \tilde{b}$ has singularity at $t = 0$, it is difficult to get the energy inequality by integration on $\mathbb{R}^2_h \times \mathbb{T}^1_v \times (0, t)$ and show the existece of a weak solution to (1.4) directly. To avoid this, we construct a unique local-in-time mild solution v to (1.4) on $(0,T]$ for some $T>0$ with initial data $\tilde{v}_0$ in a subspace of $L^2(\mathbb{R}_h^2\times\mathbb{T}_v^1)$, after that, we show the existence of global-in-time weak solution with initial data $v(T)$. The local-in-time mild solution is constructed as in [18] for two-dimensional case. we follow his approach. To show the existence of a weak solution with initial data $v(T)$, we first construct a unique solution to approximated equations to (1.4) with energy inequality that is independent of approximation parameter. Next, taking limit to the approximated solution, we obtain a weak solution to (1.4).

Finally, we prove the decay of $\|v(t)\|_{L^2(\mathbb{R}_h^2\times\mathbb{T}_v^1)}$ as $t\to\infty$. To prove this, since the domain is vertically periodic, we can apply the Fourier expansion to v with respect to x_v variable:

$$\begin{aligned}v(x_h,x_v,t) &= v^0(x_h,t)+\sum_{j\neq 0} v^j(x_h,t)\ \mathrm{e}^{2\pi ij}\\ &=: v^0+v_{os}.\end{aligned}$$

Using orthogonality of the Fourier series, we can derive the equation that v^0 satisfies. Since the averaged term v^0 is independent of x_v, we can apply two-dimensional argument as in [18] to get the decay of $\|v_0(t)\|_{L^2(\mathbb{R}_h^2\times\mathbb{T}_v^1)}$ as $t\to\infty$. Unfortunately, because of the non-linearity of (1.4) and dependence of v_{os} on x_v variable, it is difficult to show the decay to the oscillating term by using same way as the averaged term. However, we can avoid this difficulty using Poincáre-type inequality and get the decay of $\|v_{os}\|_{L^2(\mathbb{R}_h^2\times\mathbb{T}_v^1)}$. It is worth to mention that there was no result on asymptotic stability to the three-dimensional Oseen vortex under three-dimensional perturbation, even if basic flows or initial perturbation are small, and domain has no boundary. Our result is somewhat restrictive in terms of domain. We hope to get similar result on $\mathbb{R}^3$ under large L^2-initial perturbation in future work.

This paper is organized as follows. In first section, we define notations and notions and state our main theorem. In section 2 the solutions to NS that contain the three dimensional Oseen vortex are constructed by using the Fujita-Kato iteration method. We state Maekawa's decomposition to the Oseen type flows in section 3. The existence of the solutions to the perturbed Navier-Stokes equations with logarithmic energy estimate is proved in section 4. In section 5 we establish energy estimate for the low-frequency part to the zero Fourier mode. In this section some lemmas that leads the energy decay to the oscillating part are shown. The final section we establish the energy decay which implies the the asymptotic stability for the solution that constructed in second section.

2 Notations and Main results

In this section, we firstly define some notations and notions to state our two main theorem. Secondly, we mention them.

Notations

- The norm in a Banach space B is denoted by $\|\cdot\|_B$.
- $C_0^\infty(M)$ denotes the set of all smooth and compactly supported functions in a manifold M.
- $\mathcal{S}$ denotes the space of all rapidly decreasing functions in the sense of Schwartz. $\mathcal{S}'$ denotes its topological dual, i.e. the space of tempered distribution.
- $\mathcal{F}f$ and $\hat{f}$ denote the Fourier transform

$$\mathcal{F}f(\xi) = \hat{f}(\xi) := \frac{1}{(2\pi)^{\frac{n}{2}}} \int\limits_{\mathbb{R}^n} \mathrm{e}^{-x\cdot\xi} f(x)dx.$$

- $L^p(\mathbb{R}^n)$ denotes the Lebesgue spaces for $1 \le p \le \infty$ with the standard norm.
- $L^{p,q}(\mathbb{R}^n)$ denotes the Lorentz spaces for $1 < p < \infty$ and $1 \le q \le \infty$ with the quasi norm

$$\|f\|_{L^{p,q}} = p^{\frac{1}{q}} \Big(\int\limits_0^\infty t^q |\{x \in \mathbb{R}^n : |f(x)| > t\}|^{\frac{q}{p}} \Big) \frac{dt}{t} \Big)^{\frac{1}{q}}$$
$$\|f\|_{L^{p,\infty}} = \sup t |\{x \in \mathbb{R}^n : |f(x)| > t\}|^{\frac{1}{p}}.$$

- For $s \in \mathbb{R}$, $H^s(\mathbb{R}^n)$ denotes the Bessel potential spaces $H^s(\mathbb{R}^n) := \{f \in \mathcal{S}' : \|f\|_{H^s} := \|(1+|\xi|)^s \hat{f}\|_{L^2} < \infty\}$ and the Riesz potential space $\dot{H}^s := \{f \in \mathcal{S}' : \|f\|_{\dot{H}^s} := \||\xi|^s \hat{f}\|_{L^2} < \infty\}$.

We define vertically anisotropic function spaces to define the mild solutions to (2.3) that include the three dimensional Oseen vortex.

Definition 2.1. Let $\Omega = \mathbb{R}^3$ *or* $\mathbb{R}_h^2 \times \mathbb{T}_v^1$. The vertically anisotropic space $X^p(\Omega)$, $X_p(\Omega)$ $(1 \le p \le \infty)$, $Y^q((\Omega))$ and $Y_q(\Omega)$ $(1 < q < \infty)$ are the space of functions that are locally L^1 and satisfy

$$\|f\|_{X^p} := \sup_{x_v \in \mathbb{R}} (\int_{\mathbb{R}^2_h} |f(x_h, x_v)|^p dx_h)^{\frac{1}{p}} < \infty,$$

$$\|f\|_{X_q} := (\int_{\mathbb{R}^2} (\sup_{x_v \in \mathbb{R}} |f(x_h, x_v)|)^p dx_h)^{\frac{1}{p}} < \infty,$$

$$\|f\|_{Y^q} := \sup_{x_v \in \mathbb{R}} \sup_{\lambda > 0} \lambda(|\{x_h \in \mathbb{R}^2 : |f(x_h, x_v)| > \lambda\}|)^{\frac{1}{q}} < \infty,$$

$$\|f\|_{Y_q} := \sup_{\lambda > 0} \lambda(|\{x_h \in \mathbb{R}^2 : \sup_{x_h \in \mathbb{R}} |f(x_h, x_v)| > \lambda\}|)^{\frac{1}{q}} < \infty$$

respectively, where $|S|$ denotes the Lebesgue measure of S.

Remark 2.2. Y^q is larger than Y_q. Indeed, for $x_v \in \mathbb{R}$ and $\lambda > 0$, we find

$$\{x_h \in \mathbb{R}^2 : \sup_{x_v \in \mathbb{R}} |f(x_h, x_v)| > \lambda\} \supset \{x_h \in \mathbb{R}^2 : |f(x_h, x_v)| > \lambda\}.$$

This implies

$$\begin{aligned} \|f\|_{Y^q} &= \sup_{x_v \in \mathbb{R}} \sup_{\lambda > 0} \lambda(|\{x_h \in \mathbb{R}^2 : |f(x_h, x_v)| > \lambda\}|)^{\frac{1}{q}} \\ &\le \sup_{x_v \in \mathbb{R}} \sup_{\lambda > 0} \lambda(|\{x_h \in \mathbb{R}^2 : \sup_{x \in \mathbb{R}} |f(x_h, x_v)| > \lambda\}|)^{\frac{1}{q}} \\ &= \|f\|_{Y_q}. \end{aligned}$$

Definition 2.3. Let $T > 0$. Let $v_0 \in L^2_x(\mathbb{R}^2 \times \mathbb{T}^1)$ and $b \in L^\infty_t Y^2_x((0,T) \times (\mathbb{R}^2 \times \mathbb{T}^1))$ be a solution to (1.1) with initial data $b_0 \in Y^2$ satisfying following estimates

$$\sup_{0 \le \tau \le T} \|b(\tau)\|_{Y^2(\mathbb{R}^2 \times \mathbb{T}^1)} \le C\|b_0\|_{Y^2(\mathbb{R}^2 \times \mathbb{T}^1)} \tag{2.1}$$

$$\sup_{0 \le \tau \le T} \tau^{\frac{1}{4}} \|b(\tau)\|_{X^4(\mathbb{R}^2 \times \mathbb{T}^1)} \le C\|b_0\|_{Y^2(\mathbb{R}^2 \times \mathbb{T}^1)}. \tag{2.2}$$

A functions $v \in L^\infty_t L^2_x \cap L^2_t H^1_{x,}((0,T) \times (\mathbb{R}^2 \times \mathbb{T}^1))$ is called a weak solution to the perturbed Navier-Stokes equations by $\tilde{b}$ with initial data $v_0 \in L^2_x(\mathbb{R}^2 \times \mathbb{T}^1)$ if

$$\begin{cases} \partial_t v - \Delta v + \mathrm{div}(v \otimes v + v \otimes b + b \otimes v) + \nabla q = 0 \\ \mathrm{div} v = 0 \end{cases} \tag{2.3}$$

in $(0,T) \times (\mathbb{R}^2 \times \mathbb{T}^1)$ in the sense of distribution with $q \in L^1_t L^1_{x,loc}((0,T) \times (\mathbb{R}^2 \times \mathbb{T}^1))$;
for all $\phi \in C^\infty_0(\mathbb{R}^2 \times \mathbb{T}^1)$

$$t \mapsto \langle v(t), \phi \rangle \tag{2.4}$$

is continuous at any $t \in [0,T)$;

$$\|v(t) - v_0\|_{L^2(\mathbb{R}^2\times\mathbb{T}^1)} \to 0 \tag{2.5}$$

as $t \to +0$;

$$\|v(t)\|^2_{L^2(\mathbb{R}^2\times\mathbb{T}^1)} + 2\int_1^t \|\nabla v(\tau)\|^2_{L^2(\mathbb{R}^2\times\mathbb{T}^1)} d\tau \le C_1\|v_0\|^2_{L^2(\mathbb{R}^2\times\mathbb{T}^1)}(1+t)^{C_2\|b_0\|^4_{Y^2(\mathbb{R}^2\times\mathbb{T}^1)}} \tag{2.6}$$

for all $t > 1$, where $C_1, C_2 > 0$ is independent of t.

Now, we state the main results in this paper : the existence of Oseen type solutions and asymptotic stability for this solutions.

Theorem 2.4. *Let $\Omega = \mathbb{R}^3$ or $\mathbb{R}^2 \times \mathbb{T}^1$. Let $u_0 \in Y^2(\Omega)$. Then there exists a positive number δ such that, if $\|u_0\|_{Y^2(\Omega)} \le \delta$, there exists a unique mild solutions $u \in C_tY^2_x((\Omega))_x \times (0,\infty)_t)$ of (1.1) satisfying*

$$u(x,t) = \mathrm{e}^{t\Delta}u_0 - \int_0^t \mathrm{e}^{(t-\tau)\Delta}P\mathrm{div}u(\tau)\otimes u(\tau))d\tau \quad \text{in} \quad Y^2(\Omega)$$

for all $t \in (0,T)$, where $\mathrm{e}^{t\Delta}$ and P are the heat kernel and the Helmholtz projection respectively, and

$$\sup_{0<t<T} \|u(t)\|_{Y^2(\Omega)} \le C\|u_0\|_{Y^2(\Omega)}, \tag{2.7}$$

$$\sup_{0<t<T} t^{\frac{1}{4}}\|u(t)\|_{X^4(\Omega)} \le C\|u_0\|_{Y^2(\Omega)}, \tag{2.8}$$

$$u(t) \to u_0 \quad \textit{weakly} * \quad \text{in} \quad Y^2(\Omega) + X^p(\Omega) \quad \textit{as } t \to 0 \tag{2.9}$$

where $\frac{1}{p} = \frac{1}{r} + \frac{1}{4}$ for all $\frac{1}{2} < \frac{1}{r} < \frac{3}{4}$.

Remark 2.5. For $u_0 \in Y_2(\Omega)$, unique existence of the unique mild solution to (2.3) can be proved as in Theorem 2.4

The following corollary is the direct consequence of Theorem2.4.

Corollary 2.6. *Let $u_0 \in Y^2(\mathbb{R}^3)$ satisfying $\lambda u_0(\lambda x) = u_0(x)$ for all $\lambda > 0$. Then there exists $\delta > 0$ such that, if $\|u_0\|_{Y^2(\mathbb{R}^3)} < \delta$, there exists a unique self-similar mild solution $u \in L^\infty_t Y^2_x(\mathbb{R}^3 \times (0,\infty))$ to (1.1) satisfying (2.9) and $u(x,t) = \lambda u(\lambda x, \lambda^2 t)$.*

Theorem 2.7. *There exists a constant $\delta > 0$ such that for any $u_0 \in Y^2(\mathbb{R}^2_h \times \mathbb{T}^1_v) + L^\infty_v \overline{C_0^\infty}^{L^{2,\infty}_h}(\mathbb{R}^2_h \times \mathbb{T}^1_v)$ of the form*

$$u_0 = b_0 + v_0, \quad \|b_0\|_{Y^2(\mathbb{R}^2_h \times \mathbb{T}^1_v)} < \delta, \quad v_0 \in L^\infty_v \overline{C_0^\infty}^{L^{2,\infty}_h}(\mathbb{R}^2_h \times \mathbb{T}^1_v) \tag{2.10}$$

there exists a solutions $u = \tilde{b} + \tilde{v} \in L^\infty_t Y^2_x((\mathbb{R}^2_h \times \mathbb{T}^1_v)_x \times (0,\infty)_t) + L^\infty_t L^2_x((\mathbb{R}^2_h \times \mathbb{T}^1_v)_x \times (0,\infty)_t)$ to (1.1) in the sense of distribution with initial data u_0 which satisfies

$$\lim_{t\to\infty} \|u(t) - b(t) - \mathrm{e}^{t\Delta} v_0\|_{L^2(\mathbb{R}^2_h \times \mathbb{T}^1_v)} = 0, \tag{2.11}$$

where, for $\tilde{b}_0 \in Y^2(\mathbb{R}^2_h \times \mathbb{T}^1_v)$ with $\|\tilde{b}_0\|_{Y^2(\mathbb{R}^2_h \times \mathbb{T}^1_v)} < \delta$ and $\tilde{v}_0 \in L^\infty_v C^\infty_{0,v}(\mathbb{R}^2_h \times \mathbb{T}^1_v)$ satisfying $u_0 = \tilde{b}_0 + \tilde{v}_0$, $\tilde{b} \in Y^2(\mathbb{R}^2_h \times \mathbb{T}^1_v)$ is the solutions to (1.1) with initial data $\tilde{b}_0$ which is constructed in Theorem2.4 *and $\tilde{v}$ is the weak solution to the perturbed Navier-Stokes equations defined in* Definition 2.3 *with $\tilde{b}$ and $\tilde{v}_0$*

3 Construction of Oseen type solutions

In this section, we prove 2.4 by constructing an Oseen type solution to the Navier-Stokes equations.

The next estimates for the heat semigroup on our anisotropic spaces play a key role in this paper.

Lemma 3.1. 1. *Let $1 \le q \le r \le \infty$ and $\alpha = (\alpha_1, \alpha_2)$ be a multi-index. Then*

$$\|\partial_h^{\alpha_1} \partial_v^{\alpha_2} \mathrm{e}^{t\Delta} f\|_{X^r} \le C t^{-\frac{n-1}{2}(\frac{1}{r} - \frac{1}{q}) - \frac{|\alpha|}{2}} \|f\|_{X^q} \tag{3.1}$$

for all $t > 0$ and $f \in X^q$, where the constant $C > 0$ depends only on n and α.

2. *Let $1 < q < r < \infty$ and $\alpha = (\alpha_1, \alpha_2)$. Then*

$$\|\partial_h^{\alpha_1} \partial_v^{\alpha_2} \mathrm{e}^{t\Delta} f\|_{X^r} \le C t^{-\frac{n-1}{2}(\frac{1}{r} - \frac{1}{q}) - \frac{|\alpha|}{2}} \|f\|_{X^q} \tag{3.2}$$

for all $t > 0$ and $f \in Y^q$, where the constant $C > 0$ depends only on n and α. e

3. *Let $1 \le q \le r \le \infty$.Then*

$$\|(\mathrm{e}^{t\Delta} - \mathrm{e}^{s\Delta}) f\|_{X^r} \le C(t-s)^\theta t^{-\theta - \frac{n-1}{2}(\frac{1}{q} - \frac{1}{r})} \|f\|_{X^q} \tag{3.3}$$

for all $0 < s < t$ and $f \in X^q$, where the constant $C > 0$ depends only on n.

4. *Let* $1<q\le r<\infty$. *Then the composite operator* $\mathrm{e}^{t\Delta}P\mathrm{div}$ *extends to a bounded operator from* X^q *to* X^r *with*

$$\|\mathrm{e}^{t\Delta}P\mathrm{div}F\|_{X^r}\le Ct^{\frac{n-1}{2}(\frac{1}{r}-\frac{1}{q})-\frac{1}{2}}\|F\|_{X^q} \tag{3.4}$$

for all $t>0$ *and* $F\in X^q$, *where the constant* $C>0$ *depends only on* n.

5. *Let* $1<q\le r<\infty$ *and* $0<\theta<1$. *Then*

$$\|(\mathrm{e}^{s\Delta}-\mathrm{id})\mathrm{e}^{t\Delta}P\mathrm{div}F\|_{X^r}\le Ct^{-\frac{n-1}{2}(\frac{1}{q}-\frac{1}{r})-\theta-\frac{1}{2}}s^{\theta}\|F\|_{X^q}. \tag{3.5}$$

for all $s,t>0$ *and* $F\in X^r$, *where the constant* $C>0$ *depends only on* n.

Proof. Since

$$\mathrm{e}^{t\Delta}f=G_t^n*f,\quad G_t^n(x):=(4\pi)^{-\frac{n}{2}}\exp(-\frac{|x|^2}{4t})$$

and

$$|G_t^1*(G_t^{n-1}*f)|\le(G_t^1*(G_t^{n-1}*|f|)^r)^{\frac{1}{r}}.$$

Put $x=(x_1,\cdots,x_{n-1},x_n)=(x',x_n)$. From the Young inequality

$$\|\mathrm{e}^{t\Delta}f(\cdot,x_v)\|^r_{L^r_{x'}}=\int\limits_{\mathrm{R}^{n-1}}\int\limits_{\mathrm{R}}G_t^1(x_n-\xi_n)(\int\limits_{\mathrm{R}^{n-1}}G_t^{n-1}(x'-\xi')|f(\xi',\xi_n)|d\xi')^r d\xi_n dx'$$
$$\le\int\limits_{\mathrm{R}}G_t^1(x_n-\xi_n)t^{-\frac{r(n-1)}{2}(\frac{1}{q}-\frac{1}{r})}\|f(\cdot,\xi_n)\|^r_{L^q_y}d\zeta$$
$$=t^{-\frac{r(n-1)}{2}(\frac{1}{q}-\frac{1}{r})}G_t^1*\|f(\cdot,\xi_n)\|^r_{L^q_y}.$$

This implies (3.1).(3.2) follows from interpolation. Let us prove (3.3). Since

$$\begin{aligned}(\mathrm{e}^{t\Delta}-\mathrm{e}^{s\Delta})f&=\int\limits_s^t\frac{d}{d\tau}\mathrm{e}^{\tau\Delta}fd\tau\\&=\int\limits_s^t\Delta\mathrm{e}^{\tau\Delta}fd\tau,\end{aligned}$$

then we find from (3.1) that

$$
\begin{aligned}
\|(\mathrm{e}^{t\Delta}-\mathrm{e}^{s\Delta})f\|_{X^r} &\le C\int_s^t \|\Delta \mathrm{e}^{\tau\Delta} f\|_{X^r}d\tau\\
&\le C\|f\|_{X^q}\int_s^t \tau^{-1-\frac{n-1}{2}(\frac{1}{q}-\frac{1}{r})}d\tau\\
&\le Cs^{-\theta-\frac{n-1}{2}(\frac{1}{q}-\frac{1}{r})}\|f\|_{X^q}\int_s^t \tau^{-1+\theta}d\tau\\
&\le Cs^{-\theta-\frac{n-1}{2}(\frac{1}{q}-\frac{1}{r})}(t-s)^{\theta}.
\end{aligned}
$$

We write the composite operator as convolution form

$$
(\mathrm{e}^{t\Delta}P\mathrm{div}F)_j=\sum_{1\le k,l\le 3}K_{j,k,l,t}*F_{k,l}
$$

where

$$
K_{j,k,l,t}(x)=\partial_l G_t^n(x)\delta_{j,k}+\int_t^\infty \partial^3_{jkl}G^n_\tau(x)d\tau.
$$

Let $\alpha=(\alpha_1,\alpha_2)$ be a multi-index with length three. Then we find from (3.1) that

$$
\begin{aligned}
&\|\int_t^\infty \partial^3_{jkl}G^n_\tau(x)d\tau * F_{k,l}\|_{L^r_{x'}}\\
&\le \int_t^\infty \|\partial^3_{jkl}G^n_\tau * F_{kl}\|_{L^r_{x'}}d\tau\\
&=\int_t^\infty \|\partial^{\alpha_1}_{x'}\partial^{\alpha_2}_{x_n}G^n_\tau * F_{kl}\|_{L^r_{x'}}d\tau\\
&\le C\int_t^\infty \tau^{-\frac{|\alpha_2|}{2}-\frac{n-1}{2}(\frac{1}{q}-\frac{1}{r})}(\partial^{\alpha_1}_{x'}G^1_\tau(x_n)*\|F_{k,l}(\cdot,x_n)\|^r_{L^q_{x'}})^{\frac{1}{r}}d\tau.
\end{aligned}
$$

Thus it follows that

$$\begin{aligned}
&\|\int_t^\infty \partial^3_{jkl} G^n_\tau(x) d\tau * F_{k,l}\|_{X^r} \\
&\le C\|F_{k,l}\|_{X^q} \int_t^\infty \tau^{-\frac{3}{2}-\frac{n-1}{2}(\frac{1}{q}-\frac{1}{r})} d\tau \\
&\le C t^{-\frac{1}{2}-\frac{n-1}{2}(\frac{1}{q}-\frac{1}{r})} \|F_{j,k}\|_{X^q}.
\end{aligned}$$

This implies (3.4). We find from (3.4) that

$$\begin{aligned}
&(\mathrm{e}^{s\Delta}-\mathrm{id})\mathrm{e}^{t\Delta} P\mathrm{div} F = \int_t^{s+t} \frac{d}{d\tau} \mathrm{e}^{\tau\Delta} P\mathrm{div} F d\tau \\
&= \int_t^{s+t} \Delta \mathrm{e}^{\frac{\tau}{2}\Delta} \mathrm{e}^{\frac{\tau}{2}\Delta} P\mathrm{div} F d\tau.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\|\int_t^{s+t} \Delta \mathrm{e}^{\frac{\tau}{2}\Delta} \mathrm{e}^{\frac{\tau}{2}\Delta} P\mathrm{div} F d\tau\|_{X^r} \\
&\le C \int_t^{s+t} \tau^{-1} \|\mathrm{e}^{\frac{\tau}{2}\Delta} P\mathrm{div} F\|_{X^r} d\tau \\
&\le C\|F\|_{X^q} \int_t^{s+t} \tau^{-\frac{3}{2}-\frac{n-1}{2}(\frac{1}{q}-\frac{1}{r})} d\tau \\
&\le C\|F\|_{X^q} t^{-\theta-\frac{1}{2}-\frac{n-1}{2}(\frac{1}{q}-\frac{1}{r})} \int_t^{s+t} \tau^{\theta-1} d\tau \\
&\le C\|F\|_{X^q} t^{-\theta-\frac{1}{2}-\frac{n-1}{2}(\frac{1}{q}-\frac{1}{r})} s^\theta.
\end{aligned}$$

This implies (3.5) □

Let $T > 0$ and $u_0 \in Y^2$. We inductively define the function u_j as follows.

$$u_1 = \mathrm{e}^{t\Delta} u_0 \tag{3.6}$$

$$u_{j+1} = \mathrm{e}^{t\Delta} u_1 - \int_0^t \mathrm{e}^{(t-\tau)\Delta} P\mathrm{div}(u_j(\tau)\otimes u_j(\tau)) d\tau \tag{3.7}$$

for all $t \in (0, T)$ and positive integer j. First, we have to show uniform boundedness of $t^{\frac{1}{4}} \|u_j(t)\|_{X^4}$ and $\|u_j(t)\|_{Y^2}$ on j to prove Theorem2.4,.

Lemma 3.2. *There exists a positive constant C and C_0 such that, for any positive integer j,*

$$\sup_{t>0} t^{\frac{1}{4}} \|u_1(t)\|_{X^4} \leq C\|u_0\|_{Y^2} \tag{3.8}$$

$$\sup_{t>0} t^{\frac{1}{4}} \|u_{j+1}(t)\|_{X^4} \leq \sup_{t>0} t^{\frac{1}{4}} \|u_1(t)\|_{X^4} + C_0(\sup_{t>0} t^{\frac{1}{4}} \|u_j(t)\|_{X^4})^2. \tag{3.9}$$

Proof. (3.8) is the direct consequence of (3.2). By definition of u_{j+1}, we find

$$\begin{aligned}
&\|u_{j+1}(t)\|_{X^4} \\
&\leq \|u_1(t)\|_{X^4} + \int_0^t \|e^{(t-\tau)\Delta} P\mathrm{div}(u_j(\tau) \otimes u_j(\tau))\|_{X^4} d\tau,
\end{aligned}$$

using (3.4), we get

$$\begin{aligned}
&\leq \|u_1(t)\|_{X^4} + C\int_0^t (t-\tau)^{-\frac{1}{2}-\frac{1}{4}} \|u_j(\tau) \otimes u_j(\tau)\|_{X^2} d\tau \\
&\leq \|u_1(t)\|_{X^4} + C\int_0^t (t-\tau)^{-\frac{1}{2}-\frac{1}{4}} \|u_j(\tau)\|_{X^4}^2 d\tau \\
&\leq \|u_1\|_{X^4} + C(\sup \tau^{\frac{1}{4}} \|u_j(\tau)\|_{X^4})^2 \int_0^t (t-\tau)^{-\frac{3}{4}} \tau^{-\frac{1}{2}} d\tau \\
&\leq \|u_1\|_{X^4} + C(\sup \tau^{\frac{1}{4}} \|u_j(\tau)\|_{X^4})^2 t^{-\frac{1}{4}}
\end{aligned}$$

for all $t \in (0, \infty)$. This prove the lemma. □

Lemma 3.3. *There exists a positive constant C_1 such that, for any positive integer j, then*

$$\sup_{t>0} \|u_1(t)\|_{Y^2} \leq \|u_0\|_{Y^2} \tag{3.10}$$

$$\sup_{t>0} \|u_{j+1}(t)\|_{Y^2} \leq \sup_{t>0} \|u_1(t)\|_{Y^2} + C_1(\sup_{t>0} t^{\frac{1}{4}} \|u_j(t)\|_{X^4})^2. \tag{3.11}$$

Proof. (3.10) is the direct consequence of (3.2). Let us show (3.11). We use duality argument. Let $\phi \in C^\infty_{0,\infty}$, then we find

$$\begin{aligned}
&|\langle u_{j+1}(t), \phi\rangle| \\
&\le |\langle u_1(t), \phi\rangle| + \int_0^t |\langle e^{(t-\tau)\Delta} P\mathrm{div}(u_j(\tau)\otimes u_j(\tau)), \phi\rangle| d\tau \\
&\le |\langle u_1(t), \phi\rangle| + \int_0^t |\langle e^{(t-\tau)\Delta} P\mathrm{div} u_j(\tau)\otimes u_j(\tau), \phi\rangle| d\tau \\
&\le |\langle u_1(t), \phi\rangle| + \|\phi\|_{L^1_v L^{2,1}_h} \int_0^t \|e^{(t-\tau)\Delta} P\mathrm{div} u_j(\tau)\otimes u_j(\tau)\|_{X^4} d\tau \qquad (3.12)
\end{aligned}$$

using (3.4), then we get

$$\begin{aligned}
&\le \|u_1(t)\|_{Y^2}\|\phi\|_{L^1_v L^{2,1}_h} + C(\sup_{t>0} \tau^{\frac14}\|u_j(\tau)\|_{X^4})^2 \|\phi\|_{L^1_v L^2_h} \int_0^t (t-\tau)^{-\frac12}\tau^{-\frac12} d\tau \\
&\le \|u_1(t)\|_{Y^2}\|\phi\|_{L^1_v L^{2,1}_h} + C(\sup_{t>0} \tau^{\frac14}\|u_j(\tau)\|_{X^4})^2 \|\phi\|_{L^1_v L^2_h} \qquad (3.13)
\end{aligned}$$

for all $t > 0$. Since $C^\infty_{0,\sigma}$ is dense in $L^1_v L^{2,1}_h$, the above estimate leads (3.11). ☐

Next, we show the uniform bound of $\sup_{t>0} t^{\frac14}\|u_{j+1}(t) - u_j(t)\|_{X^4}$ and $\sup_{t>0}\|u_{j+1}(t) - u_j(t)\|_{Y^2}$ for all $j \ge 1$.

Lemma 3.4. *There exists a positive constant C_2, such that, for all positive integer j,*

$$\sup_{t>0} t^{\frac14}\|u_2(t) - u_1(t)\|_{X^4} \le C_2(\sup_{t>0} t^{\frac14}\|u_1(t)\|_{X^4})^2, \qquad (3.14)$$

$$\begin{aligned}
&\sup_{t>0} t^{\frac14}\|u_{j+2}(t) - u_{j+1}(t)\|_{X^4} \\
&\le C_2(\sup_{t>0} t^{\frac14}\|u_j(t)\|_{X^4} + \sup_{t>0} t^{\frac14}\|u_{j+1}(t)\|_{X^4}) \sup_{t>0} t^{\frac14}\|u_{j+1}(t) - u_j(t)\|_{X^4}
\end{aligned} \qquad (3.15)$$

Proof. By definition of u_2, we find

$$\begin{aligned}
&\|u_2(t) - u_1(t)\|_{X^4} \\
&\le \int_0^t \|e^{(t-\tau)\Delta} P\mathrm{div}(u_1(\tau)\otimes u_1(\tau))\|_{X^4} d\tau,
\end{aligned}$$

using (3.1),

$$
\begin{aligned}
&\le \int_0^t (t-\tau)^{-\frac{3}{4}} \|u_1(\tau)\|_{X^4}^2 d\tau \\
&\le C(\sup_{\tau>0} \tau^{\frac{1}{4}} \|u_1(\tau)\|_{X^4})^2 \int_0^t (t-\tau)^{-\frac{3}{4}} \tau^{-\frac{1}{2}} d\tau \\
&\le C(\sup_{\tau>0} \tau^{\frac{1}{4}} \|u_1(\tau)\|_{X^4})^2 t^{-\frac{1}{4}}.
\end{aligned}
$$

This lead (3.14). Similarly, we see

$$
\begin{aligned}
&\|u_{j+2}(t) - u_{j+1}(t)\|_{X^4} \\
&\le \int_0^t \|\mathrm{e}^{(t-\tau)\Delta} P\mathrm{div}(u_{j+1}(\tau) \otimes u_{j+1}(\tau) - u_j(\tau) \otimes u_j(\tau))\|_{X^4} d\tau.
\end{aligned} \tag{3.16}
$$

Using (3.4), we get

$$
\begin{aligned}
\mathrm{RHS}(3.16) &\le C \int_0^t \|\mathrm{e}^{(t-\tau)\Delta} P\mathrm{div}((u_{j+1}(\tau) - u_j(\tau)) \otimes u_{j+1}(\tau) - u_j(\tau) \\
&\quad \otimes (u_{j+1}(\tau) - u_j(\tau))\|_{X^4} d\tau \\
&\le C \int_0^t (t-\tau)^{-\frac{3}{4}} (\|u_{j+1}(\tau)\|_{X^4} + \|u_j(\tau)\|_{X^4}) \|u_{j+1}(\tau) - u_j(\tau)\|_{X^4} d\tau \\
&\le C(\sup_{\tau>0} \tau^{\frac{1}{4}} \|u_j\|_{X^4}) \sup_{\tau>0} (\tau^{\frac{1}{4}} \|u_{j+1}(\tau) - u_j(\tau)\|_{X^4}) \int_0^t (t-\tau)^{-\frac{3}{4}} \tau^{-\frac{1}{2}} d\tau. \\
&\le C t^{-\frac{1}{4}} (\sup_{\tau>0} \tau^{\frac{1}{4}} \|u_j\|_{X^4}) \sup_{\tau>0} (\tau^{\frac{1}{4}} \|u_{j+1}(\tau) - u_j(\tau)\|_{X^4}).
\end{aligned}
$$

This estimate implies (3.15). □

Lemma 3.5. *There exists a positive constant C_3 such that, for any positive integer j, then*

$$\sup_{t>0}\|u_2(t)-u_1(t)\|_{Y^2} \le C_3(\sup_{t>0} t^{\frac14}\|u_1(t)\|_{X^4})(\sup_{t>0} t^{\frac14}\|u_1(t)\|_{X^4})^2, \tag{3.17}$$

$$\begin{aligned}&\sup_{t>0}\|u_{j+2}(t)-u_{j+1}(t)\|_{Y^2}\\ &\le C_3(\sup_{t>0} t^{\frac14}\|u_j(t)\|_{X^4}+\sup_{t>0} t^{\frac14}\|u_{j+1}(t)\|_{X^4})\sup_{t>0} t^{\frac14}\|u_{j+1}(t)-u_j(t)\|_{X^4}.\end{aligned} \tag{3.18}$$

Proof. We use duality argument. Let $\phi \in C^\infty_{0,\sigma}$. Then (3.4) implies

$$\begin{aligned}&|\langle u_2(t)-u_1(t),\phi\rangle|\\ &\le |\int_0^t \langle \mathrm{e}^{(t-\tau)\Delta}P\mathrm{div}(u_1(\tau)\otimes u_1(\tau)),\phi\rangle| d\tau\\ &\le C\int_0^t (t-\tau)^{-\frac12}\|u_1(\tau)\|^2_{X^4}\|\phi\|_{L^1_vL^2_h}d\tau\\ &\le C\sup_{\tau>0}\tau^{\frac14}(\|u_1(\tau)\|_{X^4})^2\|\phi\|_{L^1_vL^{2,1}_h}\int_0^t (t-\tau)^{-\frac12}\tau^{-\frac12}d\tau.\end{aligned}$$

This implies (3.17). Using (3.4) again, we get

$$\begin{aligned}&|\langle u_{j+2}(t)-u_{j+1}(t),\phi\rangle|\\ &\le \int_0^t |\langle \mathrm{e}^{(t-\tau)\Delta}P\mathrm{div}(u_{j+1}(\tau)\otimes u_{j+1}(\tau)-u_j(\tau)\otimes u_j(\tau)),\phi\rangle| d\tau\\ &\le \int_0^t (t-\tau)^{-\frac12}(\|u_{j+1}(\tau)\|_{X^4}+\|u_j(\tau)\|_{X^4})\|u_{j+1}(\tau)-u_j(\tau)\|_{X^2}\|\phi\|_{L^1_vL^2_h}d\tau\\ &\le CA_\infty\sup_{t>0}(\tau^{\frac14}\|u_{j+1}(\tau)-u_j(\tau)\|_{X^4})\|\phi\|_{L^1_vL^{2,1}_h}\int_0^t (t-\tau)^{-\frac12}\tau^{-\frac12}d\tau\\ &\le CA_\infty\|\phi\|_{L^1_vL^{2,1}_h}\sup_{t>0}(\tau^{\frac14}\|u_{j+1}(\tau)-u_j(\tau)\|_{X^4})\end{aligned}$$

for all $t>0$. Since C_0^∞ is dense in $L^1_vL^{2,1}_h$, we have (3.18). □

From Lemma 3.2, there exists some $0 < \alpha < \frac{1}{4C_0}$, if $\sup_{t>0} t^{\frac{1}{4}} \|u_1(t)\|_{X^4} < \alpha$, then

$$\sup_{t>0} t^{\frac{1}{4}} \|u_j(t)\|_{X^4} \leq A_\infty := \frac{1-\sqrt{1-4C_0\alpha}}{2C_0}. \tag{3.19}$$

Take $\|b_0\|_{Y^2}$ so small that $2C_2A_\infty < 1$ and $2C_3A_\infty < 1$, then we find from Lemma3.4 and Lemma 3.5 that

$$\sum_{j\geq 0} \sup_{t>0} t^{\frac{1}{4}} \|u_{j+1}(t) - u_j(t)\|_{X^4} < \infty,$$
$$\sum_{j\geq 0} \sup_{t>0} \|u_{j+1}(t) - u_j(t)\|_{Y^2} < \infty.$$

Then $u_j = u_0 + \sum_{j=0}^{j-1}(u_{j+1} - u_j)$ converge in $\mathcal{A}$ and $L_t^\infty Y_x^2$. where $\mathcal{A}$ is a vector valued measureable functions of $f(x,t)$ in $\mathbb{R}^3 \times (0,\infty)$ such that $\|f\|_{\mathcal{A}} = \sup_{t>0} t^{\frac{1}{4}} \|f(t)\|_{X^4} < \infty$. We denote $\lim_{j\to\infty} u_j$ as u.

Let us show continuity of $\|u(t)\|_{X^4}$ and $\|u(t)\|_{Y^2}$.

Lemma 3.6. *Let u be a mild solutions to (1.1) satisfying*

$$\sup_{t>0} \|u(t)\|_{Y^2} + \sup_{t>0} t^{\frac{1}{4}} \|u(t)\|_{X^4} < \infty.$$

Then $\|u(t)\|_{X^4}$ is continuous on $(0,\infty)$

Proof. It suffices to show $\lim_{s\to t-0} \|u(t) - u(s)\|_{X^4} = 0$. Let $0 < s < t < \infty$. Then we find

$$\begin{aligned}
&\|u(t) - u(s)\|_{X^4} \\
&\leq \|e^{t\Delta} u_0 - e^{s\Delta} u_0\|_{X^4} \\
&\quad + \int_s^t \|e^{(t-\tau)\Delta} P\mathrm{div}(u(\tau)\otimes u(\tau))\|_{X^4} d\tau \\
&\quad + \int_0^s \|e^{(t-\tau)\Delta} P\mathrm{div}(u(\tau)\otimes u(\tau)) - e^{(s-\tau)\Delta} P\mathrm{div}(u(\tau)\otimes u(\tau))\|_{X^4} d\tau \\
&=: I_1 + I_2 + I_3.
\end{aligned}$$

First, using (3.3), we find

$$\begin{aligned}
I_1 &= \|(e^{(t-s)\Delta} - \mathrm{id}) e^{s\Delta} u_0\|_{X^4} \\
&\leq C(t-s)^\theta s^{-\theta} \|u_0\|_{X^4}.
\end{aligned}$$

Second, we see from (3.4) that

$$
\begin{aligned}
I_2 &\le C\int_s^t (t-\tau)^{\frac14}\|u(\tau)\|_{X^4}^2\,d\tau \\
&\le C(\sup_{\tau>0}\tau^{\frac14}\|u(\tau)\|_{X^4})^2\int_s^t (t-\tau)^{-\frac34}\tau^{-\frac12}\,d\tau \\
&\le C(\sup_{\tau>0}\tau^{\frac14}\|u(\tau)\|_{X^4})^2 s^{-\frac12}\int_s^t (t-\tau)^{-\frac34}\,d\tau \\
&\le C(\sup_{\tau>0}(\tau^{\frac14}\|u(\tau)\|_{X^4}))^2 s^{-\frac12}(t-s)^{\frac14}
\end{aligned}
$$

.

Finally, using (3.5), we obtain

$$
\begin{aligned}
I_3 = C &\le \int_0^s \|(\mathrm{e}^{(t-s)\Delta}-\mathrm{id})\mathrm{e}^{(s-\tau)\Delta}P\mathrm{div}(u(\tau)\otimes u(\tau))\|_{X^4}\,d\tau \\
&\le C(t-s)^{\theta}\int_0^s (s-\tau)^{-\frac{\theta}{2}-\frac34}\|u(\tau)\|_{X^4}^2\,d\tau \\
&\le C(t-s)^{\theta}(\sup_{\tau>0}\tau^{\frac14}\|u(\tau)\|_{X^4})^2\int_0^s (s-\tau)^{-\frac{\theta}{2}-\frac34}\tau^{-\frac12}\,d\tau \\
&\le C(t-\tau)^{\theta}(\sup_{\tau>0}\tau^{\frac14}\|u(\tau)\|_{X^4})^2 s^{-\frac{\theta}{2}-\frac14}.
\end{aligned}
$$

Therefore, $\|u(t)-u(s)\|_{X^4}\to 0$ as $s\to t-0$. The lemma is proved. ☐

Lemma 3.7. *Let u be a mild solution for (1.1) satisfying*

$$
\sup_{t>0}\|u(t)\|_{Y^2}+\sup_{t>0}t^{\frac14}\|u(t)\|_{X^4}<\infty.
$$

Then $\|u(t)\|_{Y^2}$ is continuous on $(0,\infty)$.

Proof. We use duality argument. It suffices to show $\lim_{s\to t-0}$. Let $\phi \in C_0^\infty$ and $0 < s < t < \infty$. Then we find that

$$\begin{aligned}
&|\langle u(t) - u(s), \phi\rangle| \\
&\leq |\langle \mathrm{e}^{t\Delta}u_0 - \mathrm{e}^{s\Delta}u_0, \phi\rangle| \\
&+ \int_s^t |\langle \mathrm{e}^{(t-\tau)\Delta} P\mathrm{div}(u(\tau)\otimes u(\tau)), \phi\rangle| d\tau \\
&+ \int_0^s |\langle (\mathrm{e}^{(t-s)\Delta} - \mathrm{id})\mathrm{e}^{(s-\tau)\Delta} P\mathrm{div}(u(\tau)\otimes u(\tau)), \phi\rangle| d\tau. \\
&=: I_1 + I_2 + I_3.
\end{aligned}$$

Decompose Gaussian kernel as $G_s(x_h, x_v) = G_s(x_v)^v G_s(x_h)^h$, then we find

$$\begin{aligned}
|I_1| &\leq |\langle G^v_{t-s} * (\mathrm{e}^{s\Delta}u_0) - \mathrm{e}^{s\Delta}u_0, G^v_{t-s} * \phi\rangle| + |\langle \mathrm{e}^{s\Delta}u_0, G^v_{t-s} * \phi - \phi\rangle| \\
&\leq C\|G^v_{t-s} * (\mathrm{e}^{s\Delta}u_0) - \mathrm{e}^{s\Delta}u_0\|_{Y^2}\|G^h_{t-s} * \phi\|_{L^1_v L^{2,1}_h} \\
&\quad + C\|\mathrm{e}^{s\Delta}u_0\|_{Y^2}\|G^h_{t-s} * \phi - \phi\|_{L^1_v L^{2,1}_h} \\
&=: I_{1,1} + I_{1,2}.
\end{aligned}$$

The Lebesgue dominated convergence theorem yields $I_{1,1} \to 0$ as $s \to t$. Using continuity of $G^h_{t-s} * \phi$ in $L^1_v L^{2,1}_h$ on t, we find $I_{2,1} \to 0$ as $s \to t$. Thus, $|I_1|$ converge to 0 as $s \to t$. It follows from (3.4) that

$$\begin{aligned}
|I_2| &\leq C\int_s^t \|\mathrm{e}^{(t-\tau)\Delta} P\mathrm{div}(u(\tau)\otimes u(\tau))\|_{X^2}\|\phi\|_{L^1_v L^{2,1}_h} d\tau \\
&\leq C\int_s^t (t-\tau)^{-\frac{1}{2}}\|u(\tau)\|^2_{X^4}\|\phi\|_{L^1_v L^{2,1}_h} d\tau \\
&\leq C(\sup_{\tau>0} \tau^{\frac{1}{4}}\|u(\tau)\|_{X^4})^2\|\phi\|_{L^1_v L^{2,1}_h}\int_s^t (t-\tau)^{-\frac{1}{2}}\tau^{-\frac{1}{2}} d\tau \\
&\leq Cs^{-\frac{1}{2}}(\sup_{\tau>0} \tau^{\frac{1}{4}}\|u(\tau)\|_{X^4})^2\|\phi\|_{L^1_v L^{2,1}_h}\int_s^t (t-\tau)^{-\frac{1}{2}} d\tau \\
&\leq Cs^{-\frac{1}{2}}(t-s)^{\frac{1}{2}}(\sup_{\tau>0} \tau^{\frac{1}{4}}\|u(\tau)\|_{X^4})^2\|\phi\|_{L^1_v L^{2,1}_h}.
\end{aligned}$$

This implies $I_2 \to 0$ as $s \to t$. Let $0 < \theta < \frac{1}{4}$. Using (3.3), we find

$$\begin{aligned}
|I_3| &\leq \int_0^s \|(\mathrm{e}^{(t-s)\Delta} - \mathrm{id})\mathrm{e}^{(s-\tau)\Delta} P\mathrm{div}(u(\tau)\otimes u(\tau))\|_{X^2}\|\phi\|_{L^1_v L^2_h} d\tau \\
&\leq C\int_0^s (s-\tau)^{-\frac{3}{4}-\theta}(t-s)^\theta \|u(\tau)\|^2_{X^4} d\tau \\
&\leq C(t-s)^\theta (\sup_{\tau>0} \tau^{\frac{1}{4}}\|u(\tau)\|_{X^4})^2 \|\phi\|_{L^1_v L^{2,1}_h} \int_0^s (s-\tau)^{-\frac{3}{4}}\tau^{-\frac{1}{2}} d\tau \\
&\leq C(t-s)^\theta s^{-\frac{1}{4}} (\sup_{\tau>0} \tau^{\frac{1}{4}}\|u(\tau)\|_{X^4})^2 \|\phi\|_{L^1_v L^{2,1}_h}.
\end{aligned}$$

This implies $I_3 \to 0$ as $s \to t$. We have required continuity on $(0,\infty)$. ☐

The following Lemma implies the continuity to the initial data.

Lemma 3.8. *Let $\frac{4}{3} < r < 2$ and $\frac{1}{p} = \frac{1}{r} + \frac{1}{4}$. Let u be a mild solution for (1.1) satisfying*

$$\sup_{t>0}\|u(t)\|_{Y^2} + \sup_{t>0} t^{\frac{1}{4}}\|u(t)\|_{X^4} < \infty.$$

Then

$$u(t) \to u_0 \quad \textit{weakly}* \quad \textit{in} \quad Y^2 + X^p.$$

Proof. We use duality argument. Let $\phi \in C_0^\infty(\mathbb{R}^3)$ and $t > 0$. Then

$$\begin{aligned}
&|\langle u(s) - u, \phi\rangle| \\
&\leq |\langle \mathrm{e}^{s\Delta}u_0 - u_0, \phi\rangle| \\
&+ \int_0^t |\langle \mathrm{e}^{(t-\tau)\Delta} P\mathrm{div}(u(\tau)\otimes u(\tau)), \phi\rangle| d\tau \\
&=: I_1 + I_2.
\end{aligned}$$

Decomposing the Gaussian kernel as $G_s(x_h, x_v) = G_s^v(x_v)G_s^h(x_h)$, then we find

$$|I_1| = |\langle G_t^v u_0 - u_0, G_h^h \phi\rangle| + |\langle u_0, G_t^h * \phi - \phi\rangle|.$$

Since

$$G_t^v * u_0 \to u_0 \quad \textit{weakly}^* \quad \text{in} \quad Y^2 \quad \textit{as} \quad t \to 0, \tag{3.20}$$

$$G_t^h * \phi \to \phi \quad \text{in} \quad L^1_v L^{2,1}_h \quad \textit{as} \quad t \to 0, \tag{3.21}$$

we have

$$|I_1| \to 0 \quad as \quad t \to 0. \tag{3.22}$$

Next, let $\frac{4}{3} < r < 2$ and $\frac{1}{p} = \frac{1}{r} + \frac{1}{4}$, then we obtain

$$\begin{aligned}
|I_2| &\le \int_0^t \langle K_\tau^v * (u(\tau) \otimes u(\tau)) : K_\tau^h * \phi \rangle d\tau \\
&\le \int_0^t \|K_\tau^v * (u(\tau) \otimes u(\tau))\|_{X^p + X^2} \|K_\tau^h * \phi\|_{L_v^1 L^{p'} \cap L_v^1 L^2} d\tau \\
&\le C \int_0^t (t-\tau)^{-\frac{1}{2}} \|u(\tau) \otimes u(\tau)\|_{X^p + X^2} \|\phi\|_{L_v^1 L^{p'} \cap L_v^1 L^2} d\tau.
\end{aligned}$$

Using the Hölder inequality, we find

$$\begin{aligned}
&\le C \int_0^t \|u(\tau)\|_{X^4 + X^r} \|u(\tau)\|_{X^4} \|\phi\|_{L_v^1 L^{p'} \cap L_v^1 L^2} d\tau \\
&\le C \int_0^t \|u(\tau)\|_{Y^2} \|u(\tau)\|_{X^4} \|\phi\|_{L_v^1 L^{p'} \cap L_v^1 L^2} d\tau \\
&\le C (\sup_{\tau > 0} \|(\tau)\|_{Y^2}) (\sup_{\tau > 0} \tau^{\frac{1}{4}} \|u(\tau)\|_{X^4}) \|\phi\|_{L_v^1 L^{p'} \cap L_v^1 L^2} \int_0^t (t-\tau)^{-\frac{1}{2}} \tau^{-\frac{1}{4}} d\tau \\
&\le C t^{\frac{1}{4}} (\sup_{\tau > 0} \|(\tau)\|_{Y^2}) (\sup_{\tau > 0} \tau^{\frac{1}{4}} \|u(\tau)\|_{X^4}) \|\phi\|_{L_v^1 L^{p'} \cap L_v^1 L^2}.
\end{aligned}$$

This implies $I_2 \to 0$ as $t \to 0$. The Lemma is proved. □

The following proposition implies the uniqueness of u.

Proposition 3.9. *Let $u_0 \in Y^2$ sufficiently small, then there exists at most one solutions u to (1.1) with initial data $u_0 \in Y^2$ satisfying*

$$\sup_{t > 0} t^{\frac{1}{4}} \|u(t)\|_{X^4} \le C \|u_0\|_{Y^2} \tag{3.23}$$

Proof. Let u_1 and u_2 be two solution to the Navier-Stokes equations satisfying

$$\sup_{t > 0} t^{\frac{1}{4}} \|u_j(t)\|_{X^4} \le C \|u_0\|_{Y^2}, \quad j = 1, 2.$$

Then we obtain

$$
\begin{aligned}
&\|u_1(t)-u_2(t)\|_{X^4} \le \int_0^t \|\mathrm{e}^{(t-\tau)\Delta}P\mathrm{div}(u_1(\tau)\otimes u_1(\tau)-u_2(\tau)\otimes u_2(\tau))\|_{X^4}d\tau \\
&\le C(\sup_{\tau>0}\tau^{\frac14}\|u_1(\tau)\|_{X^4}+\sup_{\tau>0}\tau^{\frac14}\|u_2(\tau)\|_{X^4}\sup_{\tau>0})\tau^{\frac14}\|u_1(\tau)-u_2(\tau)\|_{X^4} \\
&\quad\times\int_0^t (t-\tau)^{-\frac34}\tau^{\frac12}d\tau \\
&\le C't^{-\frac14}\|u_0\|_{Y^2}\sup_{\tau>0}\tau^{\frac14}\|u_1(\tau)-u_2(\tau)\|_{X^4}.
\end{aligned}
$$

If $\|u_0\|_{Y^2}$ is sufficiently small so that $C'\|u_0\|_{Y^2}<1$, we find

$$\sup_{\tau>0}\tau^{\frac14}\|u_1(\tau)-u_2(\tau)\|_{X^4}\equiv 0.$$

The Proposition is proved. $\square$

Assume u_j is periodic with respect to x_v. By definition, u_{j+1} is also periodic in x_v. Since $X^q(\mathbb{R}^2_h\times\mathbb{T}^1_v)$ is closed, we see that the limit function u is periodic with respect to x_v. We complete the proof of Theorem2.4.

Now, we prove (1.1) is locally-in-time well-posed for large initial data if its singularity is sufficiently small.

Theorem 3.10. *Let $2<q<\infty$. Then there exists a positive constant $\epsilon>0$ such that for every $u_0\in Y^2(\mathbb{R}^3)$ satisfying*

$$\limsup_{\lambda\to 0}\lambda|\{x\in\mathbb{R}^3 : |u_0(x)|>\lambda\}|<\epsilon \tag{3.24}$$

there exists $T>0$ and a mild solution $u\in Y^2$ to (1.1) on $\mathbb{R}^3$.

Proof. It is sufficient to show that there exists $T>0$ such that

$$\sup_{0<t<T} t^{\frac14}\|\mathrm{e}^{t\Delta}u_0\|_{X^4}\le\frac{1}{2C_1}.$$

By assumption, u_0 can be decomposed as

$$u_0=u_{0,1}+u_{0,2},\quad \text{where } u_{0,1}\in Y^2(\mathbb{R}^3),\ \|u_{0,1}\|_{Y^2}<\epsilon \text{ and } u_{0,2}\in X^4(\mathbb{R}^3).$$

(3.4) implies $\sup_{0<t<T} t^{\frac14}\|\mathrm{e}^{t\Delta}u_0\|_{X^4}\le C(\epsilon+t^{\frac14}\|u_{0,2}\|_{X^4})$. Put $\epsilon=\frac{1}{4CC_1}$. Let $T^{\frac14}<\frac{1}{4CC_1\|u_{0,2}\|_{X^4}}$. Then $\sup_{0<t<T} t^{\frac14}\|\mathrm{e}^{t\Delta}u_0\|_{X^4}<\frac{1}{2C_1}$. The Theorem is proved. $\square$

4 Maekawa's decompotion of basic flow and their estimate

In this section, we decompose the basic flow as in the Maekawa's paper [18] to show the asymptotic stability of the Oseen type Navier-Stokes flows. For $T>0$, Let denote $Q_{vper,T}$ the anisotropic space time set $(\mathbb{R}^2_h \times \mathbb{T}^1_v) \times (0,T)_t$.

Firstly, Let us recall Maekawa's decomposition of basic flows in [18].

Proposition 4.1. *(Maekawa's decomposition of basic flow and their estimate in two-dimensional case) There exists a constant $\delta>0$ such that, for any $b_0 \in L^{2,\infty}(\mathbb{R}^2)$ with $\|b_0\|_{L^{2,\infty}(\mathbb{R}^2)} \le \delta$ and $T>1$, the solution b to two dimensional Navier-Stokes equation (1.1) with initial data b_0 is decomposed as $b = b_T + b^T$, where b_T and b^T with $b_T, b^T \in C_{w^*,t}L^{2,\infty}_x(\mathbb{R}^2 \times (0,\infty))$ satisfy*

$$\sup_{t>0} \|b_T(t)\|_{L^{2,\infty}(\mathbb{R}^2)} + \sup_{t>0}(t+T)^{\frac{1}{4}} \|b_T(t)\|_{L^4(\mathbb{R}^2)} \le C\|b_0\|_{L^{2,\infty}(\mathbb{R}^2)} \tag{4.1}$$

$$\sup_{t>0} \|b_T(t)\|_{L^{2,\infty}(\mathbb{R}^2)} + \sup_{t>0} t^{\frac{1}{4}} \|b_T(t)\|_{L^4(\mathbb{R}^2)} \le C\|b_0\|_{L^{2,\infty}(\mathbb{R}^2)} \tag{4.2}$$

and b^T also satisfies the energy estimate

$$\|b^T(t)\|^2_{L^2(\mathbb{R}^2)} + \int_1^t \|\nabla b^T(\tau)\|^2_{L^2(\mathbb{R}^2)} d\tau \le C\|b_0(\mathbb{R}^2)\|^2_{L^{2,\infty}(\mathbb{R}^2)} \log(1+T) \tag{4.3}$$

for all $t>1$.

The following proposition is the Maekawa's decomposition to the three dimensional Oseen type solution.

Proposition 4.2. *(Maekawa's decomposition of the Oseen type basic flow and its estimate) There exists a constant $\delta>0$ such that, for any $b_0 \in Y^2(\mathbb{R}^2 \times \mathbb{T}^1)$ with $\|b_0\|_{Y^2(\mathbb{R}^2\times\mathbb{T}^1)} \le \delta$ and $T>1$, the solution b to (1.1) with initial data b_0 is decomposed as $b = b_T + b^T$, where b_T and b^T with $b_T, b^T \in C_{w^*,t}Y^2_x((\mathbb{R}^2 \times \mathbb{T}^1) \times (0,\infty))$ satisfy*

$$\sup_{t>0} \|b_T(t)\|_{Y^2(\mathbb{R}^2\times\mathbb{T}^1)} + \sup_{t>0}(t+T)^{\frac{1}{4}} \|b_T(t)\|_{X^4(\mathbb{R}^2\times\mathbb{T}^1)} \le C\|b_0\|_{Y^2(\mathbb{R}^2\times\mathbb{T}^1)} \tag{4.4}$$

$$\sup_{t>0} \|b_T(t)\|_{Y^2(\mathbb{R}^2\times\mathbb{T}^1)} + \sup_{t>0} t^{\frac{1}{4}} \|b_T(t)\|_{X^4(\mathbb{R}^2\times\mathbb{T}^1)} \le C\|b_0\|_{Y^2(\mathbb{R}^2\times\mathbb{T}^1)} \tag{4.5}$$

and b^T also satisfies the energy estimate

$$\|b^T(t)\|^2_{L^2(\mathbb{R}^2\times\mathbb{T}^1)} + \int_1^t \|\nabla b^T(\tau)\|^2_{L^2(\mathbb{R}^2\times\mathbb{T}^1)} d\tau \le C\|b_0\|^2_{Y^2(\mathbb{R}^2\times\mathbb{T}^1)} \log(1+T) \tag{4.6}$$

for all $t > 1$.

To show the Proposition 4.1, we have to decompose the initial data to the basic flow b.

Lemma 4.3. *Let $T > 1$ and $b \in Y^2(\mathbb{R}^2_h \times \mathbb{T}^1_v)$. Then there exists a positive constant C such that b_0 can be decomposed as $b_0 = b_{0,T} + b_0^T$ satisfying*

$$\|b_{0,T}\|_{Y^2(\mathbb{R}^2\times\mathbb{T}^1)} + T^{\frac{1}{4}}\|b_{0,T}\|_{X^4(\mathbb{R}^2\times\mathbb{T}^1)} \le C\|b_0\|_{Y^2(\mathbb{R}^2\times\mathbb{T}^1)} \tag{4.7}$$

$$\|b_0^T\|_{Y^2(\mathbb{R}^2\times\mathbb{T}^1)} + \frac{(2-q)^{\frac{1}{2}}}{T^{\frac{1}{q}-\frac{1}{2}}}\|b_0^T\|_{X^q(\mathbb{R}^2\times\mathbb{T}^1)} \le C\|b_0\|_{Y^2(\mathbb{R}^2\times\mathbb{T}^1)}, \tag{4.8}$$

for all $q \in [\frac{4}{3}, 2)$.

Proof. It follows from Lemma 3.2 in [18] that

$$\begin{aligned}
&\|b_{0,T}(\cdot,x_v)\|_{L^{2,\infty}_h} + T^{\frac{1}{4}}\|b_{0,T}(\cdot,x_v)\|_{L^4_h} \le C\|b_0(\cdot,x_v)\|_{L^{2,\infty}_h}\\
&\|b_0^T(\cdot,x_v)\|_{L^{2,\infty}} \le C\|b_0(\cdot,x_v)\|_{L^{2,\infty}_h}\\
&\|b_0^T(\cdot,x_v)\|_{L^q_h} \le C\frac{T^{\frac{1}{q}-\frac{1}{2}}}{(2-q)^{\frac{1}{2}}}\|b_0(\cdot,x_v)\|_{L^{2,\infty}}.
\end{aligned}$$

This inequalities imply the Lemma. □

proof of Proposition4.2. Let $\delta >$ be sufficient small. Then, by definition, $\|b_{0,T}\|_{Y^2}, \|b_0^T\|_{Y^2} \le \delta$. Using contraction principle as in [18], we can construct a unique mild solution to the following integral equation with initial data $b_{0,T}$

$$b_T(t) = \mathrm{e}^{t\Delta}b_{0,T} - \int_0^t \mathrm{e}^{(t-\tau)\Delta}P\mathrm{div}(b_T(\tau)\otimes b(\tau))d\tau, \tag{4.9}$$

where $\mathrm{e}^{t\Delta}$ and P are the heat semigroup and the Helmholtz projection on $\mathbb{R}^2_h\times\mathbb{T}^1_v$ respectively. Moreover, the solution b_T satisfies

$$\sup_{t>0}\|b_T(t)\|_{Y^2(\mathbb{R}^2\times\mathbb{T}^1)} + \sup_{t>0}(t+T)^{\frac{1}{4}}\|b_T(t)\|_{X^4(\mathbb{R}^2\times\mathbb{T}^1)} \le C\|b_{0,T}\|_{Y^2(\mathbb{R}^2\times\mathbb{T}^1)}.$$

in $\mathrm{Q}_{vper,T}$. Similarly, there exits a functions b^T satisfying

$$b^T(t) = \mathrm{e}^{t\Delta} b_0^T - \int_0^t \mathrm{e}^{t-\tau\Delta} P\mathrm{div}(b^T(\tau)\otimes b(\tau))d\tau,$$

and

$$\sup_{t>0} \|b_T(t)\|_{Y^2(\mathbb{R}^2\times\mathbb{T}^1)} + \sup_{t>0} t^{\frac{1}{4}} \|b^T(t)\|_{X^4(\mathbb{R}^2\times\mathbb{T}^1)} \leq C\|b_0\|_{Y^2(\mathbb{R}^2\times\mathbb{T}^1)}.$$

Note that b_T and b^T satisfies $b = b_T + b^T$. Now, we prove the energy estimate (4.6). First, we have to check $b^T(t) \in L^2(\mathbb{R}^2\times\mathbb{T}^1)$ for all $t \geq 1$. Indeed, it follows from (3.4) that

$$\begin{aligned}
&\|\mathrm{e}^{t\Delta} b_0^T\|_{L^2(\mathbb{R}^2\times\mathbb{T}^1)} \leq \|\mathrm{e}^{t\Delta} b_0^T\|_{X^2((\mathbb{R}^2\times\mathbb{T}^1))} \\
&\leq Ct^{-(\frac{1}{q}-\frac{1}{2})}\|b_0^T\|_{X^q(\mathbb{R}^2\times\mathbb{T}^1)}, \quad \text{for all } q \in [\frac{4}{3}, 2),
\end{aligned} \tag{4.10}$$

and

$$\begin{aligned}
&\| \int_0^t \mathrm{e}^{(t-\tau)\Delta} P\mathrm{div}(b^T(\tau)\otimes b(\tau))d\tau\|_{L^2(\mathbb{R}^2\times\mathbb{T}^1)} \\
&\leq C\int_0^t \|\mathrm{e}^{(t-\tau)\Delta} P\mathrm{div}(b^T(\tau)\otimes b(\tau))\|_{X^2(\mathbb{R}^2\times\mathbb{T}^1)} d\tau \\
&\leq C(\sup_{t>0} \tau^{\frac{1}{4}}\|b^T(\tau)\|_{X^4(\mathbb{R}^2\times\mathbb{T}^1)})(\sup_{\tau>0} \tau^{\frac{1}{4}}\|b(\tau)\|_{X^4(\mathbb{R}^2\times\mathbb{T}^1)}) \int_0^t (T-\tau)^{-\frac{1}{2}}\tau^{-\frac{1}{2}} d\tau \\
&\leq C\|b_0\|^2_{Y^2(\mathbb{R}^2\times\mathbb{T}^1)}.
\end{aligned} \tag{4.11}$$

Thus, we get $b^T(t) \in L^2(\mathbb{R}^2\times\mathbb{T}^1)$. Next, since b^T satisfies

$$\partial_t b^T - \Delta b^T + b\cdot\nabla b^T + \nabla q = 0, \ \mathrm{div}\ b^T = 0,$$

it follows from integration by part that

$$\|b^T(t)\|^2_{L^2(\mathbb{R}^2\times\mathbb{T}^1)} + \int_1^t \|\nabla b(\tau)\|^2_{L^2(\mathbb{R}^2\times\mathbb{T}^1)} d\tau = \|b^T(1)\|^2_{L^2(\mathbb{R}^2\times\mathbb{T}^1)} \tag{4.12}$$

for all $t \geq 1$. From (4.10) and (4.11), the right hand side of (4.12) satisfies

$$\|b(1)\|^2_{L^2(\mathbb{R}^2\times\mathbb{T}^1)} \leq C\|b_0\|_{Y^2(\mathbb{R}^2_h\times\mathbb{T}^1_v)}(\frac{T^{\frac{1}{q}-\frac{1}{2}}}{(2-q)^{\frac{1}{2}}} + \|b_0\|_{Y^2(\mathbb{R}^2_h\times\mathbb{T}^1_v)}) \tag{4.13}$$

for all $q \in [\frac{4}{3}, 2)$. Taking q so that $2 - q = \frac{1}{4\log(1+T)}$, we finally obtain

$$\|b^T(t)\|^2_{L^2(\mathrm{R}^2\times\mathrm{T}^1)} + \int_1^t \|\nabla b^T(\tau)\|^2_{L^2(\mathrm{R}^2\times\mathrm{T}^1)} d\tau \le C\|b_0\|_{Y^2}(\|b_0\|_{Y^2} + \log(1+T)).$$

□

5 Logarithmic energy estimates for perturbed equations with their construction

In this section, we construct a weak solution to the perturbed Navier-Stoks equations v defined in the second section with initial data $v_0 \in L^\infty_v C^\infty_{0,h}(\mathbb{R}^2_h \times \mathbb{T}^1_v)$. Firstly, we construct a local-in-time mild solution on $(0, T_*)$. Secondly, we establish the global-in-time weak solution with initial data $v(T_*)$.

Proposition 5.1. *Let $\delta > 0$ be sufficiently small and $v_0 \in X^{\frac{4}{3}}(\mathbb{R}^2_h \times \mathbb{T}^1_v) \cap X^4(\mathbb{R}^2_h \times \mathbb{T}^1_v)$. Let us assume that $b \in L^\infty_t Y^2_x(\mathbb{R}^2_h \times \mathbb{T}^1_v T_*)$ satisfies*

$$\sup_{t>0} \|b(t)\|_{Y^2(\mathbb{R}^2_h\times\mathbb{T}^1_v)} + \sup_{t>0} t^{\frac{1}{4}} \|b(t)\|_{X(\mathbb{R}^2_h\times\mathbb{T}^1_v)} \le \delta.$$

Then there exist $T_* > 0$ and a unique mild solution $v \in Y^{\frac{4}{3}}(\mathbb{R}^2_h \times \mathbb{T}^1_v \times (0, T_*)) \cap X^4(\mathbb{R}^2_h \times \mathbb{T}^1_v \times (0, T_*))$ to (1.1) satisfying

$$v(t) = \mathrm{e}^{t\Delta} b_0 - \int_0^t \mathrm{e}^{(t-\tau)\Delta} P\mathrm{div}(v(\tau) \otimes v(\tau) + v(\tau) \otimes b(\tau) + b(\tau) \otimes v(\tau)) d\tau \tag{5.1}$$

and

$$\sup_{0<\tau<T_*} \|v(\tau)\|_{X^{\frac{4}{3}}(\mathbb{R}^2_h\times\mathbb{T}^1_v)} \le C\|v_0\|_{X^{\frac{4}{3}}(\mathbb{R}^2_h\times\mathbb{T}^1_v)} \tag{5.2}$$

$$\sup_{0<\tau<T_*} \|v(\tau)\|_{X^4(\mathbb{R}^2_h\times\mathbb{T}^1_v)} \le C\|v_0\|_{X^4(\mathbb{R}^2_h\times\mathbb{T}^1_v)}. \tag{5.3}$$

Proof. Put

$$N(v, w, t) := \int_0^t \mathrm{e}^{(t-\tau)\Delta} P\mathrm{div}(v(\tau) \otimes w(\tau) + v(\tau) \otimes b(\tau) + b(\tau) \otimes v(\tau)) d\tau,$$

where $v, w \in L_t^\infty(X^{\frac{4}{3}} \cap X^4)_x$. It is sufficient to show that there exist constants C_1 and C_2 such that

$$\|N(v,w,t)\|_{L_t^\infty(X^{\frac{4}{3}}\cap X^4)_x} \le C_1 t^{\frac{1}{4}} \|v\|_{L_t^\infty(X^{\frac{4}{3}}\cap X^4)_x} + C_2\|b_0\|_{Y^2}\|v\|_{L_t^\infty(X^{\frac{4}{3}}\cap X^4)_x} \tag{5.4}$$

for $v, w \in L_t^\infty(X^{\frac{4}{3}} \cap X^4)_x(Q_{Y_*,per})$. Using (3.4), we find

$$\begin{aligned}
&\|N(v,w,t)\|_{X^{\frac{4}{3}}(\mathbb{R}_h^2\times\mathbb{T}_v^1)} && (5.5)\\
&\le \int_0^t \|e^{(t-\tau)\Delta} P\mathrm{div}(v(\tau)\otimes w(\tau) + v(\tau)\otimes b(\tau) + b(\tau)\otimes v(\tau))\|_{X^{\frac{4}{3}}(\mathbb{R}_h^2\times\mathbb{T}_v^1)} d\tau \\
&\le C\int_0^t (t-\tau)^{-\frac{3}{4}}(\|v(\tau)\|_{X^{\frac{4}{3}}(\mathbb{R}_h^2\times\mathbb{T}_v^1)}\|w(\tau)\|_{X^4(\mathbb{R}_h^2\times\mathbb{T}_v^1)} && (5.6)\\
&\quad + 2\|v(\tau)\|_{X^{\frac{4}{3}}(\mathbb{R}_h^2\times\mathbb{T}_v^1)}\|b(\tau)\|_{X^4(\mathbb{R}_h^2\times\mathbb{T}_v^1)})d\tau \\
&\le C_1 t^{\frac{1}{4}}(\sup_{0<\tau<t}\|v(\tau)\|_{X^{\frac{4}{3}}(\mathbb{R}_h^2\times\mathbb{T}_v^1)})(\sup_{0<\tau<t}\|w(\tau)\|_{X^4(\mathbb{R}_h^2\times\mathbb{T}_v^1)}) \\
&\quad + C_2(\sup_{0<\tau<t}\|v(\tau)\|_{X^{\frac{4}{3}}(\mathbb{R}_h^2\times\mathbb{T}_v^1)})(\sup_{\tau>0}\tau^{\frac{1}{4}}\|b(\tau)\|_{X^4(\mathbb{R}_h^2\times\mathbb{T}_v^1)}). && (5.7)
\end{aligned}$$

Similarly, we find

$$\begin{aligned}
&\|N(v,w,t)\|_{X^4(\mathbb{R}_h^2\times\mathbb{T}_v^1)} \le \int_0^t \|e^{(t-\tau)\Delta}P\mathrm{div}(v(\tau)\otimes w(\tau) + v(\tau)\otimes b(\tau) && (5.8)\\
&+ b(\tau)\otimes v(\tau))\|_{X^4(\mathbb{R}_h^2\times\mathbb{T}_v^1)}d\tau \\
&\le C\int_0^t (t-\tau)^{-\frac{3}{4}}(\|v(\tau)\|_{X^4(\mathbb{R}_h^2\times\mathbb{T}_v^1)}\|w(\tau)\|_{X^4(\mathbb{R}_h^2\times\mathbb{T}_v^1)} && (5.9)\\
&+ \|v(\tau)\|_{X^4}\|b(\tau)\|_{X^4})d\tau \\
&\le C_1 t^{\frac{1}{4}}(\sup_{0<\tau<t}\|v(\tau)\|_{X^4}(\mathbb{R}_h^2\times\mathbb{T}_v^1))(\sup_{0<\tau<t}\|w(\tau)\|_{X^4(\mathbb{R}_h^2\times\mathbb{T}_v^1)}) && (5.10)\\
&+ C_2(\sup_{0<\tau<t}\|v(\tau)\|_{X^4(\mathbb{R}_h^2\times\mathbb{T}_v^1)})(\sup_{\tau>0}\tau^{\frac{1}{4}}\|b(\tau)\|_{X^4(\mathbb{R}_h^2\times\mathbb{T}_v^1)}). && (5.11)
\end{aligned}$$

Let T_* be sufficiently small. Using contraction principle, we get the proof as in the proof of Theorem2.4. □

We construct a global-in-time weak solution to the perturbed Navier-Stokes equations on $(0,\infty)$ with initial data $v(T_*) \in L^2(\mathbb{R}_h^2\times\mathbb{T}_v^1)$. Firstly, we construct a solution to the mollified perturbed Navier-Stokes equations. Secondly, taking limit for it, we get a solution to the perturbed Navier-Stokes equations.

Let ψ be the standard mollifier and $(f)_\rho(x)$ denote $\frac{1}{\rho^3}\psi(\frac{\cdot}{\rho}) * f$. The following proposition assert that there exist a weak solutions to the mollified perturbed Navier-Stokes equations with initial data $v_0 \in L^2(\mathbb{R}^2_h \times \mathbb{T}^1_v)$.

Proposition 5.2. *Let $0 < \rho < 1$ and $T > 0$. Let $b \in L^\infty_t Y^2_x(Q_{vper,T})$ and be a mild solution to (1.1) with non-zero initial data $b_0 \in Y^2$ satisfying*

$$\sup_{t>0}\|b(t)\|_{Y^2(\mathbb{R}^2_h\times\mathbb{T}^1_v)} + \sup_{t>0}(t+1)^{\frac{1}{4}}\|b(t)\|_{X^4(\mathbb{R}^2_h\times\mathbb{T}^1_v)} \leq C\|b_0\|_{Y^2(\mathbb{R}^2_h\times\mathbb{T}^1_v)}. \tag{5.12}$$

Then there exists a unique weak solution $v^\rho \in (L^\infty_t L^2_x \cap L^2_t H^1_x \cap H^1_t L^2_x)(Q_{vper,T})$ to the mollified perturbed Navier-Stokes equation

$$\partial_t v^\rho - \Delta v^\rho + (v^\rho)_\rho \cdot \nabla v^\rho + b \cdot \nabla v^\rho + v^\rho \cdot \nabla b + \nabla q = 0, \tag{5.13}$$

$$\operatorname{div}\, v = 0 \tag{5.14}$$

with initial data $v_0 \in L^2(\mathbb{R}^2_h \times \mathbb{T}^1_v)$ satisfying

$$\int_0^t -\langle v^\rho, \partial_t\phi\rangle + \langle \nabla v^\rho : \nabla\phi\rangle - \langle v^\rho \otimes (v^\rho)_\rho + (b)_\rho \otimes v^\rho + v^\rho \otimes (b)_\rho : \nabla\phi\rangle d\tau$$
$$= \langle v_0, \phi\rangle \tag{5.15}$$

for any $\phi \in C^\infty_{0,\sigma}(Q_{vper,T})$. Moreover, v^ρ satisfies the energy estimate

$$\|v^\rho(t)\|_{L^2(\mathbb{R}^2_h\times\mathbb{T}^1_v)} + \int_0^t \|\nabla v^\rho(\tau)\|^2_{L^2(\mathbb{R}^2_h\times\mathbb{T}^1_v)} d\tau$$
$$\leq C_1(1+t)^{C_2\|b_0\|^4_{Y^2(\mathbb{R}^2_h\times\mathbb{T}^1_v)}}\|v_0\|^2_{L^2(\mathbb{R}^2_h\times\mathbb{T}^1_v)} \tag{5.16}$$

for all $t \in (0, T)$, where constants C_1 and C_2 are independent of ρ.

Proof. Let $v, w \in L^\infty_t L^2_x(Q_{vper,T})$. We define N_ρ as

$$N_\rho(v, w, t) \tag{5.17}$$
$$:= \int_0^t e^{(t-\tau)\Delta} P\operatorname{div}(v(\tau) \otimes (w)_\rho(\tau) + v(\tau) \otimes (b)_\rho(\tau) + (b)_\rho(\tau) \otimes v(\tau))d\tau. \tag{5.18}$$

First, we show that there exits a positive constant $0 < T_* < 1$ and $v^\rho \in L^\infty_t L^2_x(Q_{vper,T_*}) \cap L^2_t H^1_x(Q_{vper,T_*})$ such that

$$v^\rho(t) = \mathrm{e}^{t\Delta} v_0 - N_\rho(v^\rho, v^\rho, t), \quad in \quad (L^\infty_t L^2_x \cap L^2_t \dot{H^1}_x)(Q_{vper,T_*}). \tag{5.19}$$

It follows from integration by parts that

$$\|\mathrm{e}^{t\Delta} v_0\|_{L^\infty_t L^2_x(Q_{vper,T_*})} + \|\mathrm{e}^{t\Delta} v_0\|_{L^2_t \dot{H}^1_x(Q_{vper,T_*})} \leq \|v_0\|_{L^2(\mathbb{R}^2\times\mathbb{T}^1)}.$$

Since

$$\begin{aligned}
&(\int_0^t \|v(\tau)\otimes(w)_\rho(\tau)\|^2_{L^2(\mathbb{R}^2_h\times\mathbb{T}^1_v)} + \|(b)_\rho(\tau)\otimes v(\tau)\|^2_{L^2(\mathbb{R}^2_h\times\mathbb{T}^1_v)} \\
&\quad + \|v(\tau)\otimes(b)_\rho(\tau)\|^2_{L^2(\mathbb{R}^2_h\times\mathbb{T}^1_v)} d\tau)^{\frac{1}{2}} \\
&\leq C_1(\int_0^t \|v(\tau)\|_{L^2(\mathbb{R}^2_h\times\mathbb{T}^1_v)}\|(w)_\rho(\tau)\|_{L^\infty(\mathbb{R}^2_h\times\mathbb{T}^1_v)} \\
&\quad + \|(b)_\rho(\tau)\|_{L^\infty(\mathbb{R}^2_h\times\mathbb{T}^1_v)}\|v(\tau)\|_{L^2(\mathbb{R}^2_h\times\mathbb{T}^1_v)} d\tau)^{\frac{1}{2}} \\
&\leq C_1\rho^{-\frac{3}{2}} T_*^{\frac{1}{2}} (\|v\|_{L^\infty_t L^2_x(Q_{vper,T_*})}\|w\|_{L^\infty_t L^2_x(Q_{vper,T_*})} \\
&\quad + \|b\|_{L^\infty_t X^4_x(Q_{vper,T_*})}\|v\|_{L^\infty_t L^2_x(Q_{vper,T_*})}),
\end{aligned}$$

it follows from energy estimate that

$$\begin{aligned}
&\|N_\rho(v,w,t)\|_{L^\infty_t L^2_x(Q_{vper,T_*})} + \|N(v,w,t)\|_{L^2_t \dot{H}^1_x(Q_{vper,T_*})} \\
&\leq C_1\rho^{-\frac{3}{2}} T_*^{\frac{1}{2}} (\|v\|_{L^\infty_t L^2_x(Q_{vper,T_*})}\|w\|_{L^\infty_t L^2_x(Q_{vper,T_*})} \\
&\quad + \|b\|_{L^\infty_t X^4_x(Q_{vper,T_*})}\|v\|_{L^\infty_t L^2_x(Q_{vper,T_*})}).
\end{aligned}$$

Thus, if we take T_* so small that

$$\begin{aligned}
T_*^{\frac{1}{2}} < \min(1, \rho^{\frac{3}{2}} &\frac{\|b\|_{L^\infty_t X^4_x(Q_{vper,T_*})} + 2\|v_0\|_{L^2(\mathbb{R}^2_h\times\mathbb{T}^1_v)}}{4C_1\|b\|_{L^\infty_t X^4_x(Q_{vper,T_*})}} \\
&- \frac{\sqrt{(\|b\|_{L^\infty_t X^4_x(Q_{vper,T_*})} + 2\|v_0\|_{L^2(\mathbb{R}^2_h\times\mathbb{T}^1_v)})^2 - \|b\|_{L^\infty_t X^4_x(Q_{vper,T_*})}}}{4C_1\|b\|^2_{L^\infty_t X^4_x(Q_{vper,T_*})}}),
\end{aligned}$$

there exists a unique mild solution v to

$$v^\rho(t) = \mathrm{e}^{t\Delta} v_0 - \int_0^t \mathrm{e}^{(t-\tau)\Delta} P\mathrm{div}(v^\rho(\tau)\otimes(v^\rho)_\rho(\tau) + v^\rho(\tau)\otimes(b)_\rho(\tau) + (b)_\rho(\tau)\otimes v^\rho(\tau))d\tau$$

on $t \in (0, T_*)$.

Next, we show the a priori bound for v. This leads the existence of global-in-time weak solution to (5.16). Integration by parts to (5.13) yields

$$\begin{aligned}
&\frac{1}{2}\partial_t \|v^\rho(t)\|^2_{L^2(\mathbb{R}^2_h\times\mathbb{T}^1_v)} + \|\nabla v^\rho(t)\|^2_{L^2(\mathbb{R}^2_h\times\mathbb{T}^1_v)} \\
&\leq |\langle b(t)\otimes v^\rho(t) : \nabla v^\rho(t)\rangle| \\
&\leq C\|b(t)\otimes v^\rho(t)\|_{L^2(\mathbb{R}^2_h\times\mathbb{T}^1_v)}\|\nabla v^\rho(t)\|_{L^2(\mathbb{R}^2_h\times\mathbb{T}^1_v)}. \qquad (5.20)
\end{aligned}$$

Using interpolation inequality and the Young inequality, we get

$$\begin{aligned}
(5.20) &\leq C\|b(t)\|_{X^4(\mathbb{R}^2_h\times\mathbb{T}^1_v)}\|v^\rho(t)\|^{\frac{1}{2}}_{L^2(\mathbb{R}^2_h\times\mathbb{T}^1_v)}\|\nabla v^\rho(t)\|^{\frac{3}{2}}_{L^2(\mathbb{R}^2_h\times\mathbb{T}^1_v)} \\
&\leq C\|b(t)\|^4_{X^4(\mathbb{R}^2_h\times\mathbb{T}^1_v)}\|v^\rho(t)\|^2_{L^2(\mathbb{R}^2_h\times\mathbb{T}^1_v)} + \frac{1}{2}\|\nabla v^\rho(t)\|^2_{L^2(\mathbb{R}^2_h\times\mathbb{T}^1_v)}. \qquad (5.21)
\end{aligned}$$

Applying the Gronwall inequality to (5.20) and (5.21), we obtain

$$\begin{aligned}
&\|v^\rho(t)\|^2_{L^2} + \int_0^t \|\nabla v^\rho(\tau)\|^2_{L^2} d\tau \\
&\leq \exp(C\int_0^t \|b(\tau)\|^4_{X^4} d\tau)\|v_0\|^2_{L^2} \leq C_2(1+t)^{C_3\|b_0\|^4_{Y^2}}\|v_0\|^2_{L^2}.
\end{aligned}$$

Thus, we get a priori estimate $\|u(t)\|_{L^2} \leq C_2(1+T)^{C_3\|b_0\|^4_{Y^2}}\|v_0\|_{L^2}$. Using this estimate, we can extend the maximal existence time by

$$\begin{aligned}
&\min(1, \rho^3(\frac{\|b\|_{L^\infty_t X^4_x} + 2C_2(1+T)^{C_3\|b_0\|^4_{Y^2}}\|v_0\|_{L^2}}{4C_1\|b\|_{L^\infty_t X^4_x}} \\
&- \frac{\sqrt{(\|b\|_{L^\infty_t X^4_x} + 2C_2(1+T)^{C_3\|b_0\|^4_{Y^2}}\|v_0\|_{L^2})^2 - \|b\|^2_{L^\infty_t X^4_x}}}{4C_1\|b\|^2_{L^\infty_t X^4_x}}).
\end{aligned}$$

Since T is finite, we can use same argument until the existence time become greater than T. The proposition is proved. □

Now, let us prove the existence of the perturbed Navier-Stokes equation for L^2-initial data.

Proposition 5.3. *Let $T > 0$, $v_0 \in L^2_x(\mathbb{R}^2_h \times \mathbb{T}^1_v)$ and $b \in L^\infty_t Y^2_x((0,T)\times(\mathbb{R}^2_h \times \mathbb{T}^1_v))$ be a mild solution to (1.1) with initial data $b_0 \in Y^2$ satisfying*

$$\sup_{t>0}\|b(t)\|_{Y^2(\mathbb{R}^2_h\times\mathbb{T}^1_v)} + \sup_{t>0}(t+1)^{\frac{1}{4}}\|b(t)\|_{X^4(\mathbb{R}^2_h\times\mathbb{T}^1_v)} \leq C\|b_0\|_{Y^2(\mathbb{R}^2_h\times\mathbb{T}^1_v)}. \qquad (5.22)$$

Then there exists a weak solution v to the perturbed Navier-Stokes equation

$$\begin{cases} \partial_t v - \Delta v + \operatorname{div}(v \otimes v + v \otimes b + b \otimes v) + \nabla q = 0 \\ \operatorname{div}\ v = 0 \\ v(0) = v_0 \end{cases} \tag{5.23}$$

in $(0,T) \times (\mathbb{R}^2 \times \mathbb{T}^1)$ with $q \in L^1_t L^1_{x,loc}((0,T) \times (\mathbb{R}^2 \times \mathbb{T}^1))$ satisfying the following properties;

(i) For all $\phi \in C_0^\infty(\mathbb{R}^2 \times \mathbb{T}^1)$

$$t \mapsto \langle v(t), \phi \rangle \tag{5.24}$$

is continuous at any $t \in [0,T)$.

(ii) (Continuity at $t = 0$)

$$\|v(t) - v_0\|_{L^2(\mathbb{R}^2 \times \mathbb{T}^1)} \to 0 \tag{5.25}$$

as $t \to +0$

(iii) (Energy estimate)

$$\begin{aligned} &\|v(t)\|^2_{L^2(\mathbb{R}^2 \times \mathbb{T}^1)} + 2 \int_1^t \|\nabla v(\tau)\|^2_{L^2(\mathbb{R}^2 \times \mathbb{T}^1)} d\tau \\ &\le C_1 \|v_0\|^2_{L^2(\mathbb{R}^2 \times \mathbb{T}^1)} (1+t)^{C_2 \|b_0\|^4_{Y^2(\mathbb{R}^2 \times \mathbb{T}^1)}} \end{aligned} \tag{5.26}$$

for all $t > 1$, where $C_1, C_2 > 0$ is independent of t.

Proof. We have shown the existence of a solution to the mollified equations with energy estimate. We have to get uniform estimate in ρ to $\|\partial_t v^\rho\|_{L^2_t \dot{H}^{-s}_x}$ for $s > \frac{3}{2}$ to take limit to the mollified equations. Let $\phi \in L^2_t H^s_x(Q_{vper,T})$. Then, using the Hölder inequality and embedding $L^\infty \hookrightarrow H^s$, we find that

$$\begin{aligned} |\langle \partial_t v^\rho(t), \phi(t) \rangle| &\le |\langle \nabla v^\rho(t), \nabla \phi(t) \rangle| + |\langle v^\rho(t) \otimes (v)_\rho(t) : \nabla \phi(t) \rangle| \\ &+ |\langle (b)_\rho(t) \otimes v^\rho(t) : \nabla \phi(t) \rangle| + |\langle v^\rho(t) \otimes (b)_\rho(t) : \nabla \phi(t) \rangle| \\ &\le \|\nabla v^\rho(t)\|_{L^2(\mathbb{R}^2_h \times \mathbb{T}^1_v)} \|\nabla \phi(t)\|_{L^2(\mathbb{R}^2_h \times \mathbb{T}^1_v)} \qquad (5.27) \\ &+ \|v^\rho(t) \otimes (v^\rho)_\rho(t)\|_{L^1(\mathbb{R}^2_h \times \mathbb{T}^1_v)} \|\nabla \phi(t)\|_{L^\infty(\mathbb{R}^2_h \times \mathbb{T}^1_v)} \\ &+ \|(b)_\rho(t) \otimes v^\rho(t)\|_{L^2(\mathbb{R}^2_h \times \mathbb{T}^1_v)} \|\nabla \phi(t)\|_{L^2(\mathbb{R}^2_h \times \mathbb{T}^1_v)} \qquad (5.28) \\ &+ \|v^\rho \otimes (b)_\rho(t)\|_{L^2(\mathbb{R}^2_h \times \mathbb{T}^1_v)} \|\nabla \phi(t)\|_{L^2(\mathbb{R}^2_h \times \mathbb{T}^1_v)} \end{aligned}$$

(5.29)

$$\le \|\nabla v^\rho(t)\|_{L^2(\mathbb{R}^2_h\times\mathbb{T}^1_v)}\|\nabla\phi(t)\|_{L^\infty(\mathbb{R}^2_h\times\mathbb{T}^1_v)} \tag{5.30}$$
$$+ (\|v^\rho(t)\|^2_{L^2(\mathbb{R}^2_h\times\mathbb{T}^1_v)} + 2\|\|b(t)\|_{L^4_h(\mathbb{R}^2)}\|v^\rho(t)\|_{L^4_h(\mathbb{R}^2)}\|_{L^2_v})\|\nabla\phi(t)\|_{L^2(\mathbb{R}^2_h\times\mathbb{T}^1_v)}$$
$$\le C(\|\nabla v^\rho(t)\|_{L^2(\mathbb{R}^2_h\times\mathbb{T}^1_v)} + \|v^\rho(t)\|^2_{L^2(\mathbb{R}^2_h\times\mathbb{T}^1_v)} \tag{5.31}$$
$$+ 2\|b(t)\|_{X^4(\mathbb{R}^2_h\times\mathbb{T}^1_v)}\|v^\rho(t)\|^{\frac{1}{2}}_{L^2(\mathbb{R}^2_h\times\mathbb{T}^1_v)}\|\nabla v^\rho(t)\|^{\frac{3}{2}}_{L^2(\mathbb{R}^2_h\times\mathbb{T}^1_v)})\|\phi(t)\|_{H^3(\mathbb{R}^2_h\times\mathbb{T}^1_v)}$$
$$\le C(\|\nabla v^\rho(t)\|_{L^2(\mathbb{R}^2_h\times\mathbb{T}^1_v)} + \|v^\rho(t)\|^2_{L^2(\mathbb{R}^2_h\times\mathbb{T}^1_v)} \tag{5.32}$$
$$+ \|b(t)\|^4_{X^4(\mathbb{R}^2_h\times\mathbb{T}^1_v)}\|v^\rho(t)\|^2_{L^2(\mathbb{R}^2_h\times\mathbb{T}^1_v)} + \|\nabla v^\rho(t)\|^2_{L^2(\mathbb{R}^2_h\times\mathbb{T}^1_v)})\|\phi(t)\|_{H^3(\mathbb{R}^2_h\times\mathbb{T}^1_v)}. \tag{5.33}$$

Therefore, from (5.16), the above estimate and the Aubin-Lions theorem, we can select subsequence $\{v^{\rho_j}\}_{\rho_j} \subset \{v^\rho\}_\rho$ such that

$$v^{\rho_j} \to v \quad \textit{weakly}^* \quad \text{in} \quad L^\infty_t L^2_x(Q_{vper,T}) \tag{5.34}$$
$$\nabla v^{\rho_j} \to v \quad \textit{weakly} \quad \text{in} \quad L^2_t L^2_x(Q_{vper,T}) \tag{5.35}$$
$$v^{\rho_j} \to v \quad \text{in} \quad L^2_t L^2_{loc,x}(Q_{vper,T}). \tag{5.36}$$

Moreover, the limit functions v satisfies the energy estimate

$$\|v(t)\|_{L^2(\mathbb{R}^2_h\times\mathbb{T}^1_v)} + \int_0^t \|\nabla v(\tau)\|^2_{L^2(\mathbb{R}^2_h\times\mathbb{T}^1_v)}d\tau$$
$$\le C_1(1+t)^{C_2\|b_0\|^4_{Y^2(\mathbb{R}^2_h\times\mathbb{T}^1_v)}}\|v_0\|^2_{L^2(\mathbb{R}^2_h\times\mathbb{T}^1_v)}, \tag{5.37}$$

and the perturbed Navier-Stokes equation. From the estimates above, it follows that

$$t \to \langle v(t), \phi\rangle \tag{5.38}$$

is continuous on $[0,T)$ for all $\phi \in L^2(\mathbb{R}^2_h \times \mathbb{T}^1_v)$, and

$$\|v(t) - v_0\|_{L^2} \to 0 \quad \text{as} \quad t \to 0. \tag{5.39}$$

The proposition is proved. □

Fix $T > 0$. Then, from Proposition 5.1 and Proposition 5.3, we have a global weak solutions $v \in L^\infty_t L^2_x(Q_{T,vper})$ to the perturbed Navier-Stokes equations with initial data $v_0 \in (X^{\frac{4}{3}} \cap X^4)(\mathbb{R}^2_h \times \mathbb{T}^1_v)$. Moreover, since $v(T_1) \in (X^{\frac{4}{3}} \cap X^4)(\mathbb{R}^2_h \times \mathbb{T}^1_v)$ for all $0 < T_1 < T_*$, it follows that

$$\|v(t)\|^2_{L^2(\mathbb{R}^2\times\mathbb{T}^1)} + \int_{T_1}^t \|\nabla v(\tau)\|^2_{L^2(\mathbb{R}^2\times\mathbb{T}^1)}d\tau \le C_1(1+t)^{C_2\|b_0\|^4_{Y^2}}\|v(T_1)\|^2_{L^2(\mathbb{R}^2\times\mathbb{T}^1)} \tag{5.40}$$

for all $T_1 < t < T$, where $C > 0$ is independent of t. Hereafter, we denote T_1 as 1 for simplicity.

Let us assume that $v_0 \in L^\infty_v \overline{C^\infty_0}_h^{L^{2,\infty}}(\mathbb{R}^2_h \times \mathbb{T}^1_v)$ with $\mathrm{div} v_0 = 0$. Then for small $\epsilon > 0$ the initial perturbation v_0 can be decomposed into

$$v_0 = v_{0,\epsilon} + w_{0,\epsilon}, \quad \text{where } \|w_{0,\epsilon}\|_{Y^2(\mathbb{R}^2_h\times\mathbb{T}^1_v)} < \epsilon,\ v_{0,\epsilon} \in L^\infty_v \overline{C^\infty_0}_h^{L^2}(\mathbb{R}^2_h \times \mathbb{T}^1_v). \tag{5.41}$$

Therefore, the initial data u_0 to the Navier-Stokes equation in Theorem 2.7 can be decomposed into

$$u_0 = b_0 + v_0 = b_0 + (w_{0,\epsilon} + v_{0,\epsilon}) \tag{5.42}$$
$$= (b_0 + w_{0,\epsilon}) + v_{0,\epsilon} \tag{5.43}$$
$$=: \tilde{b}_0 + \tilde{v}_0, \tag{5.44}$$

Let ϵ sufficiently small, then there exits a unique mild solution $\tilde{b}$ with initial data $\tilde{b}_0$. Moreover, let $T > 1$, then using Proposition 5.1 and Proposition 5.3, we see that there exists a weak solution to the perturbed Navier-Stokes equations with initial data $\tilde{v}_0$. Now, we write $\tilde{b}$ and $\tilde{v}$ as b and v respectively for simplicity.

The following proposition is the logarithmic energy estimate for v.

Proposition 5.4. *Fix sufficiently small $\epsilon > 0$ and $\delta > 0$. Let $b_0 \in Y(\mathbb{R}^2_h \times \mathbb{T}^1_v)$ satisfy $\|b_0\|_{Y^2(\mathbb{R}^2_h\times\mathbb{T}^1_v)} < \delta + \epsilon$ and $b \in Y^2(\mathbb{R}^2_h \times \mathbb{T}^1_v)$ be the mild solution with initial data b_0 such that*

$$\sup_{t>0}\|b(t)\|_{Y^2(\mathbb{R}^2_h\times\mathbb{T}^1_v)} + \sup_{t>0} t^{\frac{1}{4}} \|\sup_{t>0}\|_{X^4(\mathbb{R}^2_h\times\mathbb{T}^1_v)} \leq C\|b_0\|_{Y^2(\mathbb{R}^2_h\times\mathbb{T}^1_v)}$$

for some constant C. Then a solution v to the perturbed Navier-Stokes with b obtained by Proposition 5.1 *and* Proposition 5.3 *with initial data v_0 satisfies*

$$\|v(t)\|^2_{L^2(\mathbb{R}^2\times\mathbb{T}^1)} + \int_1^t \|\nabla v(\tau)\|^2_{L^2(\mathbb{R}^2\times\mathbb{T}^1)} d\tau \leq C_\epsilon + C\delta^2 \log(1+t) \tag{5.45}$$

for $t > 1$ where C_ϵ and C are independent of t.

Proof. First, from Proposition 4.2 there exist b_T and b^T such that $b = b_T + b^T$ satisfying (4.4), (4.5) and (4.6). Put $v^T := v - b^T$, then we find that v^T satisfies

$$\partial_t v^T - \Delta v^T + \mathrm{div}(v^T \otimes v^T + v^T \otimes b_T + b_T \otimes v^T - b_T \otimes b^T) + \nabla q = 0, \tag{5.46}$$
$$\mathrm{div}\ v^T = 0. \tag{5.47}$$

It follows from integration by parts that

$$\frac{1}{2}\partial_t\|v^T\|^2_{L^2(\mathbb{R}^2_h\times\mathbb{T}^1_v)}+\|\nabla v^T\|^2_{L^2(\mathbb{R}^2_h\times\mathbb{T}^1_v)}=\langle v^T\otimes b_T-b^T\otimes b_T:\nabla v\rangle. \tag{5.48}$$

Using the Hölder inequality and the Young inequality, we find

$$\begin{aligned}|\langle v^T\otimes b_T:\nabla v^T\rangle|&\le C\|v^T\otimes b_T\|_{L^2(\mathbb{R}^2_h\times\mathbb{T}^1_v)}\|\nabla v^T\|_{L^2(\mathbb{R}^2_h\times\mathbb{T}^1_v)}\\&\le C\|b_T\|^4_{X^4(\mathbb{R}^2_h\times\mathbb{T}^1_v)}\|v^T\|^2_{L^2(\mathbb{R}^2_h\times\mathbb{T}^1_v)}+\frac{1}{4}\|\nabla v^T\|^2_{L^2(\mathbb{R}^2_h\times\mathbb{T}^1_v)}\end{aligned} \tag{5.49}$$

and

$$\begin{aligned}|\langle b^T\otimes b_T:\nabla v\rangle|&\le \|b_T\otimes b^T\|_{L^2(\mathbb{R}^2_h\times\mathbb{T}^1_v)}\|v^T\|_{L^2(\mathbb{R}^2_h\times\mathbb{T}^1_v)}\\&\le C\|b^T\|^2_{X^4(\mathbb{R}^2_h\times\mathbb{T}^1_v)}\|b_T\|^2_{X^4(\mathbb{R}^2_h\times\mathbb{T}^1_v)}+\frac{1}{4}\|v^T\|^2_{L^2(\mathbb{R}^2_h\times\mathbb{T}^1_v)},\end{aligned} \tag{5.50}$$

Using the Gronwall inequality, we find for $t\in(1,T]$ that

$$\begin{aligned}&\|v^T(t)\|^2_{L^2(\mathbb{R}^2_h\times\mathbb{T}^1_v)}+\int_1^t\|\nabla v^T(\tau)\|^2_{L^2(\mathbb{R}^2_h\times\mathbb{T}^1_v)}d\tau\\&\le C\exp\Big(\int_1^t\|b_T(\tau)\|^4_{X^4(\mathbb{R}^2_h\times\mathbb{T}^1_v)}d\tau\Big)\Big(\|v^T(1)\|^2_{L^2(\mathbb{R}^2_h\times\mathbb{T}^1_v)}\\&\qquad+\int_1^t\|b^T(\tau)\|^2_{X^4}\|b_T(\tau)\|^2_{X^4}d\tau\Big)\\&\le C\exp\Big(C_1\int_1^t(T+\tau)^{-1}d\tau\Big)\Big(\|v^T(1)\|^2_{L^2(\mathbb{R}^2_h\times\mathbb{T}^1_v)}\\&\qquad+\|b^T\|^4_{Y^2(\mathbb{R}^2_h\times\mathbb{T}^1_v)}\int_1^t\tau^{-\frac{1}{2}}(T+\tau)^{-\frac{1}{2}}d\tau\Big)\\&\le C(\|v^T(1)\|^2_{L^2(\mathbb{R}^2_h\times\mathbb{T}^1_v)}+\delta^4)\end{aligned} \tag{5.51}$$

Since $v=v^T+b^T$, it follows from energy inequality (4.6) that

$$\begin{aligned}\|v^T(1)\|^2_{L^2}&\le 2(\|v(1)\|^2_{L^2(\mathbb{R}^2_h\times\mathbb{T}^1_v)}+\|b^T(t)\|^2_{L^2(\mathbb{R}^2_h\times\mathbb{T}^1_v)})\\&\le C_\epsilon+C\|b_0\|^4_{X^4(\mathbb{R}^2_h\times\mathbb{T}^1_v)}\log(1+T).\end{aligned} \tag{5.52}$$

Then we obtain

$$\begin{aligned}
&\|v(t)\|^2_{L^2(\mathbb{R}^2_h\times\mathbb{T}^1_v)} + \int_1^t \|\nabla v(\tau)\|^2_{L^2(\mathbb{R}^2_h\times\mathbb{T}^1_v)} d\tau \\
&\leq C(\|v^T(t)\|^2_{L^2(\mathbb{R}^2_h\times\mathbb{T}^1_v)} + \int_1^t \|\nabla v^T(\tau)\|^2_{L^2(\mathbb{R}^2_h\times\mathbb{T}^1_v)} d\tau \\
&+ \|b^T(t)\|^2_{L^2(\mathbb{R}^2_h\times\mathbb{T}^1_v)} + \int_1^t \|b^T(\tau)\|^2_{L^2(\mathbb{R}^2_h\times\mathbb{T}^1_v)} d\tau) \\
&\leq C_\epsilon + C\delta^2 \log(1+T). \qquad (5.53)
\end{aligned}$$

Teke $t = T$, then we have (5.45). □

6 Estimates for vertically averaged part

In this section, we show some lemmas that enable us to get the L^2-decay for the weak solution to the perturbed Navier-Stokes equations. The decay estimate of v in this section is possible for any v that is constructed as the limit function of solutions v^ρ obtained by Proposition 5.2.

Applying the Fourier expansion to v with respect to x_v, we can decompose v into averaged part v_a and oscillating part v_{os};

$$\begin{aligned}
v(x_h, x_v, t) &= \sum_{k\in\mathbb{Z}} v_k(x_h, t)\mathrm{e}^{2\pi i x_v k} = v_0(x_h, t) + \sum_{k\neq 0} v_k(x_h, t)\mathrm{e}^{2\pi x_v k} \\
&=: v_a(x_h, t) + v_{os}(x_h, x_v, t).
\end{aligned}$$

Because of orthogonality of the Fourier series, it follows from (5.45) that

$$\|v_a(t)\|^2_{L^2(\mathbb{R}^2)} + \int_1^t \|\nabla_h v_a\|^2_{L^2(\mathbb{R}^2)} \leq C_\epsilon + C\delta^2 \log(1+t) \qquad (6.1)$$

$$\|v_{os}(t)\|^2_{L^2(\mathbb{R}^2_h\times\mathbb{T}^1_v)} + \int_1^t \|\nabla v_{os}\|^2_{L^2(\mathbb{R}^2_h\times\mathbb{T}^1_v)} \leq C_\epsilon + C\delta^2 \log(1+t). \qquad (6.2)$$

We first show the following proposition to prove the decay of averaged part.

Proposition 6.1. *Let $T > 0$. Put $w_a := (-\Delta_h)^{-\frac{1}{4}} v_a$, where $(-\Delta_h)^s f = \mathcal{F}^{-1}(|\xi_h|^s \mathcal{F} f)$ for $s \in \mathbb{R}$. Then there exist constants $C > 0$ and $M > 0$ such that*

$$
\begin{aligned}
&\|w_a(t)\|^2_{L^2(\mathbb{R}^2)} + \int_1^t \|\nabla_h w_a(t)\|^2_{L^2(\mathbb{R}^2)} d\tau \\
&\le C(1+t)^{M\delta^2}(1 + \log(1+t) + \sup_{1\le\tau\le t} \|v_{os}(\tau)\|_{L^2(\mathbb{R}^2_h \times \mathbb{T}^1_v)} \log(1+t))
\end{aligned}
\tag{6.3}
$$

for all $1 < t \le T$.

Proof. Integrate (2.3) with respect to x_v, then

$$
\partial_t v_a^1 - \Delta_h v_a^1 + \operatorname{div} \int_{\mathbb{T}^1} (v^1 v + b^1 v + v^1 b) dx_v + \partial_1 q = 0 \tag{6.4}
$$

$$
\partial_t v_a^2 - \Delta_h v_a^2 + \operatorname{div} \int_{\mathbb{T}^1} (v^2 v + b^2 v + v^2 b) dx_v + \partial_2 q = 0 \tag{6.5}
$$

$$
\partial_t v_a^3 - \Delta_h v_a^3 + \operatorname{div} \int_{\mathbb{T}^1} (v^3 v + b^3 v + v^3 b) dx_v = 0. \tag{6.6}
$$

(6.4) (6.5) are the two dimensional perturbed Navier-Stokes system and (6.6) is two dimensional heat equation respectively. It follows from integration by parts that

$$
\begin{aligned}
&\frac{1}{2}\partial_t \|w_a\|^2_{L^2(\mathbb{R}^2)} + \|\nabla w_a\|^2_{L^2(\mathbb{R}^2)} \\
&\le |\int_{\mathbb{R}^2}\int_{\mathbb{T}^1} (v\otimes v + b\otimes v + v\otimes b) dx_v : \nabla_h (-\Delta_h)^{-\frac{1}{4}} w_a dx_h| \\
&= |\int_{\mathbb{R}^2}\int_{\mathbb{T}^1} ((v_a + v_{os})\otimes(v_a + v_{os}) + b\otimes(v_a + v_{os}) \\
&+ (v_a + v_{os})\otimes b) dx_v : \nabla(-\Delta_h)^{-\frac{1}{4}} w_a dx_h| \\
&= |\int_{\mathbb{R}^2}\int_{\mathbb{T}^1} (v_a\otimes v_a + v_{os}\otimes v_{os} + b\otimes v_a + b\otimes v_{os} + v_a\otimes b \\
&+ v_{os}\otimes b) dx_v : \nabla(-\Delta_h)^{-\frac{1}{4}} w_a dx_h| \\
&=: I_1 + I_2 + I_3 + I_4 + I_5 + I_6.
\end{aligned}
\tag{6.7}
$$

Estimate for I_1 The Sobolev embedding

$$
\|v_a\|_{L^4(\mathbb{R}^2)} \le C\|(-\Delta_h)^{\frac{1}{4}} v_a\|_{L^2(\mathbb{R}^2)} \tag{6.8}
$$

and the interpolation inequality

$$\|(-\Delta_h)^{\frac{1}{4}} v_a\|_{L^2(\mathbb{R}^2)} \le C\|v_a\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}} \|\nabla_h v_a\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}} \tag{6.9}$$

yield

$$\begin{aligned}
|I_1| &\le C\|v_a\|_{L^4(\mathbb{R}^2)}^2 \|(-\Delta_h)^{\frac{1}{4}} w_a\|_{L^2(\mathbb{R}^2)} \\
&\le C\|(-\Delta_h)^{\frac{1}{4}} v_a\|_{L^2(\mathbb{R}^2)}^2 \|(-\Delta_h)^{\frac{1}{4}} w_a\|_{L^2(\mathbb{R}^2)} \\
&\le C\|v_a\|_{L^2(\mathbb{R}^2)} \|(-\Delta_h)^{\frac{1}{2}} v_a\|_{L^2(\mathbb{R}^2)} \|(-\Delta_h)^{\frac{1}{4}} w_a\|_{L^2(\mathbb{R}^2)} \\
&\le C\|\nabla_h v_a\|_{L^2(\mathbb{R}^2)} \|(-\Delta_h)^{\frac{1}{4}} w_a\|_{L^2(\mathbb{R}^2)}^2 \\
&\le C\|\nabla_h v_a\|_{L^2(\mathbb{R}^2)} \|w_a\|_{L^2(\mathbb{R}^2)} \|\nabla_h w_a\|_{L^2(\mathbb{R}^2)}.
\end{aligned}$$

Applying the Young inequality to the last inequality, we find

$$|I_1| \le C\|\nabla_h v_a\|_{L^2(\mathbb{R}^2)}^2 \|w_a\|_{L^2(\mathbb{R}^2)}^2 + \frac{1}{8}\|\nabla_h w_a\|_{L^2(\mathbb{R}^2)}^2$$

Estimate for I_2 Using the Schwarz inequality, (6.8), (6.9) and the Young inequality, we find

$$\begin{aligned}
|I_2| &\le C\Big\|\int_{\mathbb{T}^1} v_{os}\otimes v_{os}\,dx_v\Big\|_{L^2(\mathbb{R}_h^2\times\mathbb{T}_v^1)} \|(-\Delta_h)^{\frac{1}{4}} w_a\|_{L^2(\mathbb{R}^2)} \\
&\le C\int_{\mathbb{T}} \|v_{os}\|_{L_h^4(\mathbb{R}^2)} dx_v \|w_a\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}} \|\nabla_h w_a\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}}, \\
&\le C\int_{\mathbb{T}} \|(-\Delta_h)^{\frac{1}{4}} v_{os}\|_{L_h^2(\mathbb{R}^2)}^2 dx_v \|w_a\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}} \|\nabla_h w_a\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}} \\
&\le C\int_{\mathbb{T}} \|v_{os}\|_{L_h^2(\mathbb{R}^2)} dx_v \|\nabla_h v_{os}\|_{L_h^2(\mathbb{R}^2)} dx_v \|w_a\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}} \|\nabla_h w_a\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}} \\
&\le C\|v_{os}\|_{L^2(\mathbb{R}_h^2\times\mathbb{T}_v^1)} \|\nabla v_{os}\|_{L^2(\mathbb{R}_h^2\times\mathbb{T}_v^1)} \|w_a\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}} \|\nabla_h w_a\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}} \\
&\le C_1\|v_{os}\|_{L^2(\mathbb{R}_h^2\times\mathbb{T}_v^1)}^2 \|\nabla v_{os}\|_{L^2(\mathbb{R}_h^2\times\mathbb{T}_v^1)} \\
&\quad + C_2\|\nabla v_{os}\|_{L^2(\mathbb{R}_h^2\times\mathbb{T}_v^1)}^2 \|w_a\|_{L^2(\mathbb{R}^2)}^2 + \frac{1}{8}\|\nabla_h w_a\|_{L^2(\mathbb{R}^2)}^2 \\
&\le C_1\|v_{os}\|_{L^2(\mathbb{R}_h^2\times\mathbb{T}_v^1)} \|\nabla v\|_{L^2(\mathbb{R}_h^2\times\mathbb{T}_v^1)}^2 \\
&\quad + C_2\|\nabla v\|_{L^2(\mathbb{R}_h^2\times\mathbb{T}_v^1)}^2 \|w_a\|_{L^2(\mathbb{R}^2)}^2 + \frac{1}{8}\|\nabla_h w_a\|_{L^2(\mathbb{R}^2)}^2.
\end{aligned}$$

Estimate for I_3 and I_5. Using the Hölder inequality,(6.8), (6.9) and the Young inequality, we find

$$
\begin{aligned}
|I_3|+|I_5| &\le C\int\limits_{\mathbb{T}^1}\|b\|_{L^4_h(\mathbb{R}^2)}\|v_a\|_{L^4_h(\mathbb{R}^2)}dx_v\|(\Delta_h)^{\frac14}w_a\|_{L^2(\mathbb{R}^2)}\\
&\le C\|b\|_{X^4(\mathbb{R}^2_h\times\mathbb{T}^1_v)}\|v_a\|_{L^4(\mathbb{R}^2)}\|w_a\|^{\frac12}_{L^2(\mathbb{R}^2)}\|\nabla_h w_a\|^{\frac12}_{L^2(\mathbb{R}^2)},\\
&\le C\|b\|_{X^4(\mathbb{R}^2_h\times\mathbb{T}^1_v)}\|(-\Delta)^{\frac14}v_a\|_{L^2(\mathbb{R}^2)}\|w_a\|^{\frac12}_{L^2(\mathbb{R}^2)}\|\nabla_h w_a\|^{\frac12}_{L^2(\mathbb{R}^2)}\\
&\le C\|b\|_{X^4(\mathbb{R}^2_h\times\mathbb{T}^1_v)}\|w_a\|^{\frac12}_{L^2(\mathbb{R}^2)}\|\nabla_h w_a\|^{\frac32}_{L^2(\mathbb{R}^2)}\\
&\le C\|b\|^4_{X^4(\mathbb{R}^2_h\times\mathbb{T}^1_v)}\|w_a\|^2_{L^2(\mathbb{R}^2)}+\frac18\|\nabla_h w_a\|^2_{L^2(\mathbb{R}^2)}.
\end{aligned}
$$

Estimate for I_4 and I_6. Using the Hölder inequality, (6.8), (6.9) and the Pincaré inequality, we find

$$
\begin{aligned}
&|I_4|+|I_6| \le C\int\limits_{\mathbb{T}^1}\|b\|_{L^4_h(\mathbb{R}^2)}\|v_{os}\|_{L^4_h(\mathbb{R}^2)}dx_v\|(-\Delta_h)^{\frac14}w_a\|_{L^2(\mathbb{R}^2)}\\
&\le C\int\limits_{\mathbb{T}^1}\|b\|_{L^4_h(\mathbb{R}^2)}\|(-\Delta_h)^{\frac14}v_{os}\|_{L^2_h(\mathbb{R}^2)}dx_v\|(-\Delta_h)^{\frac14}w_a\|_{L^2(\mathbb{R}^2)}\\
&\le C\|b\|_{X^4(\mathbb{R}^2_h\times\mathbb{T}^1_v)}\int\limits_{\mathbb{T}^1}\|v_{os}\|^{\frac12}_{L^2_h(\mathbb{R}^2)}\|\nabla_h v_{os}\|^{\frac12}_{L^2_h(\mathbb{R}^2)}dx_v\|(-\Delta_h)^{\frac14}w_a\|_{L^2(\mathbb{R}^2)}\\
&\le C\|b\|_{X^4(\mathbb{R}^2_h\times\mathbb{T}^1_v)}\|v_{os}\|^{\frac12}_{L^2(\mathbb{R}^2_h\times\mathbb{T}^1_v)}\|\nabla v_{os}\|^{\frac12}_{L^2(\mathbb{R}^2_h\times\mathbb{T}^1_v)}\|w_a\|^{\frac12}_{L^2(\mathbb{R}^2)}\|\nabla_h w_a\|b^{\frac12}_{L^2(\mathbb{R}^2)}\\
&\le C_1\|b\|^2_{X^4(\mathbb{R}^2_h\times\mathbb{T}^1_v)}\|v_{os}\|_{L^2(\mathbb{R}^2_h\times\mathbb{T}^1_v)}\\
&\quad +C_2\|\nabla v_{os}\|^2_{L^2(\mathbb{R}^2_h\times\mathbb{T}^1_v)}\|w_a\|^2_{L^2(\mathbb{R}^2)}+\frac18\|\nabla_h w_a\|^2_{L^2(\mathbb{R}^2)}\\
&\le C_1\|b\|^2_{X^4(\mathbb{R}^2_h\times\mathbb{T}^1_v)}\|\nabla v_{os}\|_{L^2(\mathbb{R}^2_h\times\mathbb{T}^1_v)}\\
&\quad +C_2\|\nabla v_{os}\|^2_{L^2(\mathbb{R}^2_h\times\mathbb{T}^1_v)}\|w_a\|^2_{L^2(\mathbb{R}^2)}+\frac18\|\nabla_h w_a\|^2_{L^2(\mathbb{R}^2)}.
\end{aligned}
$$

Thus, from (6.7), above estimates and the Gronwall inequality, we get

$$
\|w_a(t)\|^2_{L^2_h}+\int\limits_1^t\|\nabla w_a(\tau)\|^2_{L^2_h}d\tau\le\exp(\Phi(t))\|w_a(1)\|^2_{L^2_h}+\int\limits_1^t\Psi(\tau)d\tau \tag{6.10}
$$

where

$$\Phi(t) = C_1 \int_1^t (\|\nabla v(\tau)\|_{L^2}^2 + \|b(\tau)\|_{X^4}^4) d\tau$$

$$\Psi(t) = C_2 \exp(\int_\tau^t \Phi(s)ds)(\|v_{os}(t)\|_{L^2}\|\nabla v_{os}(t)\|_{L^2}^2 + \|b(t)\|_{X^4}^2\|\nabla v_{os}(t)\|_{L^2}).$$

Using (6.2) and (2.8), we find

$$\Phi(t) \leq C_1(1 + \delta^2 \log(1+t)).$$

and

$$\int_1^t \Psi(t) d\tau$$

$$\leq C_2(1+t)^{C_1\delta^2}(\sup_{1\leq\tau\leq t}\|v_{os}(\tau)\|_{L^2}\int_1^t \|\nabla v_{os}(\tau)\|^2 d\tau + \int_1^t \|b(\tau)\|_{X^4}^2\|\nabla v_{os}(\tau)\|_{L^2} d\tau)$$

$$\leq C_2(1+t)^{C_1\delta^2}(\sup_{1\leq\tau\leq t}\|v_{os}(\tau)\|_{L^2}\int_1^t \|\nabla v_{os}(\tau)\|^2 d\tau$$

$$+ (\int_1^t \|b(\tau)\|_{X^4}^4 d\tau)^{\frac{1}{2}}(\int_1^t \|\nabla v_{os}(\tau)\|_{L^2}^2 d\tau)^{\frac{1}{2}})$$

$$\leq C_2(1+t)^{C_1\delta^2}(1 + \log(1+t) + \sup_{1\leq\tau\leq t}\|v_{os}(\tau)\|_{L^2}\log(1+t)).$$

Thus, we obtain

$$\|w_a(t)\|_{L^2(\mathbb{R}^2)}^2 + \int_1^t \|\nabla w_a(t)\|_{L^2(\mathbb{R}^2)}^2 d\tau$$

$$\leq C(1+t)^{M\delta^2}(1 + \log(1+t) + \sup_{1\leq\tau\leq t}\|v_{os}(\tau)\|_{L^2(\mathbb{R}_h^2\times\mathbb{T}_v^1)}\log(1+t)) \quad (6.11)$$

□

7 Decay estimates for perturbation

In this section, we show the decay of $\|v(t)\|_{L^2} \to 0$ as $t \to \infty$. The Poincaré inequality is useful to derive the decay to the oscillating part.

Proposition 7.1. *Let $b_0 \in Y^2(\mathbb{R}^2_h \times \mathbb{T}^1_v)$, $v_0 \in L^\infty_v C^\infty_{0,h}(\mathbb{R}^2_h \times \mathbb{T}^1_v)$ and $\delta > 0$ be sufficiently small. Let b be a mild with initial data b_0 and v is a weak solution to the perturbed Navier-Stokes equations obtained by* Proposition 5.4 *with initial data v_0. Then there exists a constant C_ϵ and C which are independent of t such that*

$$\|v(t)\|_{L^2(\mathbb{R}^2\times\mathbb{T}^1)} \leq Ct^{-\frac{1}{2}}\{C_\epsilon + C_1(1+t)^{M\delta^2}(1+\log(1+t)+\log^{\frac{3}{2}}(1+t)\}.$$

for $t \geq 1$.

Proof. Let $t \geq 1$. From (6.2) and (6.11), there exists $t_0 \in [\frac{t}{2}, t]$ such that

$$\begin{aligned}
&\|w_a(t_0)\|^2_{L^2_h} + \|v_{os}(t_0)\|^2_{L^2} + t_0(\|\nabla w_a(t_0)\|^2_{L^2_h} + \|v_{os}(t_0)\|^2_{L^2}) \\
&\leq C_\epsilon + C_1(1+t_0)^{M\delta^2}(1+\log(1+2t_0)+\log^{\frac{3}{2}}(1+2t_0)).
\end{aligned} \tag{7.1}$$

Therefore using interpolation the inequality, (7.1) and the Poincaré inequality, we have

$$\begin{aligned}
&\|v(t_0)\|^2_{L^2} \leq 2(\|v_a(t_0)\|^2_{L^2_h} + \|v_{os}(t_0)\|^2_{L^2}) \\
&\leq \|w_a(t_0)\|_{L^2_h}\|\nabla_h w_a(t_0)\|_{L^2_h} + \|v_{os}\|_{L^2}\|\nabla v_{os}\|_{L^2} \\
&\leq t_0^{-\frac{1}{2}}\{C_\epsilon + C_1(1+t_0)^{M\delta^2}(1+\log(1+2t_0)+\log^{\frac{3}{2}}(1+2t_0)\}.
\end{aligned} \tag{7.2}$$

Since v solves the perturbed Navier-Stokes equations

$$\partial_t v - \Delta v + \mathrm{div}(v\otimes v + v\otimes b + b\otimes v) + \nabla q = 0, \tag{7.3}$$

$$\mathrm{div}\ v = 0, \tag{7.4}$$

then it follows from integration by parts and the Gronwall inequality that

$$\|v(t)\|^2_{L^2} + \int_{t_0}^{t} \|\nabla v(\tau)\|^2_{L^2} d\tau \leq \exp(\int_{t_0}^{t} C\|b(s)\|^4_{X^4} ds)\|v(t_0)\|^2_{L^2}. \tag{7.5}$$

Since

$$\int_{t_0}^{t} \|b(s)\|_{X^4} ds \leq C\log\frac{t}{t_0} \leq C\log 2, \quad t_0 \in (\frac{t}{2}, t),$$

we finally obtain from (7.2) and (7.5) that

$$\begin{aligned}
&\|v(t)\|^2_{L^2} \leq C\|v_0(t_0)\|^2_{L^2} \\
&\leq Ct^{-\frac{1}{2}}\{C_\epsilon + C_1(1+t)^{M\delta^2}(1+\log(1+t)+\log^{\frac{3}{2}}(1+t)\}.
\end{aligned}$$

$\square$

Proof of theorem2.7. Fix arbitrarily small $\eta > 0$. Let $u_0 = v_0 + b_0 = \tilde{v_0} + \tilde{b_0}$. By definition $v = \tilde{v} + (\tilde{b} - b)$, where $\tilde{v}$ is a solution of the perturbed Navier-Stokes equation with initial data $\tilde{v_0}$ and $\tilde{b}$ is the solution to (1.1) with initial data $\tilde{b_0}$ as in Theorem 2.4, then we find

$$\|v(t) - \mathrm{e}^{t\Delta}v_0\|_{L^2(\mathbb{R}^2_h\times\mathbb{T}^1_v)} \le \|\tilde{v}(t)\|_{L^2(\mathbb{R}^2_h\times\mathbb{T}^1_v)} + \|\tilde{b}(t) - b(t) - \mathrm{e}^{t\Delta}v_0\|_{L^2(\mathbb{R}^2_h\times\mathbb{T}^1_v)}$$

Put $w_{0,\epsilon} = v_0 - (\tilde{b}_0 - b_0)$ and $w_\epsilon(t) = \tilde{b}(t) - b(t) - \mathrm{e}^{t\Delta}v_0$. It follows that

$$\begin{aligned}
&\|w_\epsilon(t)\|_{L^2(\mathbb{R}^2_h\times\mathbb{T}^1_v)}\\
&\le \|\mathrm{e}^{t\Delta}w_{0,\epsilon}\|_{L^2(\mathbb{R}^2_h\times\mathbb{T}^1_v)} + C\int_0^t (t-\tau)^{-\frac{1}{2}}\|b(\tau)\otimes(\tilde{b}(\tau)-b(\tau))\\
&\quad + (\tilde{b}(\tau)-b(\tau))\otimes b(\tau) + (\tilde{b}(\tau)-b(\tau))\otimes(\tilde{b}(\tau)-b(\tau))\|_{L^2(\mathbb{R}^2_h\times\mathbb{T}^1_v)}d\tau\\
&\le \|\mathrm{e}^{t\Delta}w_{0,\epsilon}\|_{L^2(\mathbb{R}^2_h\times\mathbb{T}^1_v)} + C(\sup_{\tau>0}\tau^{\frac{1}{4}}\|b(\tau)\|_{X^4(\mathbb{R}^2_h\times\mathbb{T}^1_v)} + \sup_{\tau>0}\tau^{\frac{1}{4}}\|\tilde{b}(\tau)\|_{X^4(\mathbb{R}^2_h\times\mathbb{T}^1_v)})\\
&\quad \times(\sup_{\tau>0}\tau^{\frac{1}{4}}\|\tilde{b}(\tau)-b(\tau)\|_{X^4(\mathbb{R}^2_h\times\mathbb{T}^1_v)} + (\sup_{\tau>0}\tau^{\frac{1}{4}}\|\tilde{b}(\tau)-b(\tau)\|_{X^4(\mathbb{R}^2_h\times\mathbb{T}^1_v)})^2)\\
&\quad \times\int_0^t (t-\tau)^{-\frac{1}{2}}\tau^{-\frac{1}{2}}d\tau\\
&\le \|\mathrm{e}^{t\Delta}w_{0,\epsilon}\|_{L^2(\mathbb{R}^2_h\times\mathbb{T}^1_v)} + C(\delta+\delta+\epsilon)(\epsilon+\epsilon^2)\\
&\le \|\mathrm{e}^{t\Delta}w_{0,\epsilon}\|_{L^2(\mathbb{R}^2_h\times\mathbb{T}^1_v)} + C(\delta+\epsilon)(\epsilon+\epsilon^2)
\end{aligned}$$

Choose ϵ so small that $C(\delta+\epsilon)(\epsilon+\epsilon^2) < \frac{\eta}{3}$ and $t > 0$ so large that

$$Ct^{-\frac{1}{2}}\{C_\epsilon + C_1(1+t)^{M\delta^2}(1+\log(1+t)+\log^{\frac{3}{2}}(1+t)\} < \frac{\eta}{3}$$

and

$$\|\mathrm{e}^{t\Delta}w_{0,\epsilon}\|_{L^2(\mathbb{R}^2_h\times\mathbb{T}^1_v)} < \frac{\eta}{3},$$

then we obtain

$$\|v(t) - \mathrm{e}^{t\Delta}v_0\|_{L^2(\mathbb{R}^2_h\times\mathbb{T}^1_v)} = \|u(t) - b(t) - \mathrm{e}^{t\Delta}v_0\|_{L^2(\mathbb{R}^2_h\times\mathbb{T}^1_v)} < \eta.$$

This implies Theorem 2.7. □

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