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Some regularity estimates for Diffusion semigroups with Dirichlet boundary conditions I

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1 Introduction

Let $W_0 = \{w \in C([0,\infty);\mathbf{R}^d);\ w(0) = 0\}$, $\mathcal{G}$ be the Borel algebra over W_0 and μ be the Wiener measure on $(W_0,\mathcal{G})$. Let $B^i : [0,\infty) \times W_0 \to \mathbf{R}$, $i = 1,\dots,d$, be given by $B^i(t,w) = w^i(t)$, $(t,w) \in [0,\infty) \times W_0$. Then $\{(B^1(t),\dots,B^d(t));\ t \in [0,\infty)\}$ is a d-dimensional Brownian motion under μ. Let $B^0(t) = t$, $t \in [0,\infty)$. Let $\{\mathcal{F}_t\}_{t\geqq 0}$ be the Brownian filtration generated by $\{(B^1(t),\dots,B^d(t); t \in [0,\infty)\}$.

Let $V_0, V_1,\dots,V_d \in C_b^\infty(\mathbf{R}^N;\mathbf{R}^N)$. Here $C_b^\infty(\mathbf{R}^N;\mathbf{R}^n)$ denotes the space of $\mathbf{R}^n$-valued smooth functions defined in $\mathbf{R}^N$ whose devivatives of any order are bounded. We regard elements in $C_b^\infty(\mathbf{R}^N;\mathbf{R}^N)$ as vector fields on $\mathbf{R}^N$.

Now let $X(t,x)$, $t \in [0,\infty)$, $x \in \mathbf{R}^N$, be the solution to the Stratonovich stochastic integral equation

$$X(t,x) = x + \sum_{i=0}^d \int_0^t V_i(X(s,x)) \circ dB^i(s). \tag{1}$$

Then there is a unique solution to this equation. Moreover we may assume that $X(t,x)$ is continuous in t and smooth in x and $X(t,\cdot) : \mathbf{R}^N \to \mathbf{R}^N$, $t \in [0,\infty)$, is a diffeomorphism with probability one.

Let $A = A_d = \{v_0, v_1,\dots,v_d\}$, be an alphabet, a set of letters, and A^* be the set of words consisting of A including the empty word which is denoted by 1. For $u = u^1\cdots u^k \in A^*$, $u^j \in A$, $j = 1,\dots,k$, $k \geqq 0$, we denote by $n_i(u)$, $i = 0,\dots,d$, the cardinal of $\{j \in \{1,\dots,k\}; u^j = v_i\}$. Let $|u| = n_0(u) + \ldots + n_d(u)$, a length of u, and $\| u \| = |u| + n_0(u)$ for $u \in A^*$. Let $\mathbf{R}\langle A\rangle$ be the $\mathbf{R}$-algebra of non-commutative polynomials on A, $\mathbf{R}\langle\langle A\rangle\rangle$ be the $\mathbf{R}$-algebra of non-commutative formal power series on A.

Let $r : A^* \setminus \{1\} \to \mathcal{L}(A)$ denote the right normed bracketing operator inductively given by

$$r(v_i) = v_i, \qquad i = 0,1,\dots,d,$$

and

$$r(v_i u) = [v_i, r(u)] = v_i r(u) - r(u) v_i, \qquad i = 0,1,\dots,d,\ u \in A^* \setminus \{1\}.$$

Let $A^{**} = \{u \in A^*;\ u \neq 1, v_0\}$, $A^{**}_m = \{u \in A^{**};\ ||u|| = m\}$, and $A^{**}_{\leqq m} = \{u \in A^{**};\ \| u \| \leqq m\}$, $m \geqq 1$.

We can regard vector fields $V_0, V_1,\dots,V_d$ as first differential operators over $\mathbf{R}^N$. Let $\mathcal{DO}(\mathbf{R}^N)$ denotes the set of linear differential operators with smooth coefficients over $\mathbf{R}^N$. Then $\mathcal{DO}(\mathbf{R}^N)$ is a non-commutative algebra over $\mathbf{R}$. Let $\Phi : \mathbf{R}\langle A\rangle \to \mathcal{DO}(\mathbf{R}^N)$ be a homomorphism given by

$$\Phi(1) = Identity, \qquad \Phi(v_{i_1}\cdots v_{i_n}) = V_{i_1}\cdots V_{i_n}, \qquad n \geqq 1,\ i_1,\dots,i_n = 0,1,\dots,d.$$

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Then we see that

$$\Phi(r(v_i u)) = [V_i, \Phi(r(u))], \qquad i = 0, 1, \dots, d, \; u \in A^* \setminus \{1\}.$$

Now we introduce a condition (UFG) for a system of vector field $\{V_0, V_1, \dots, V_d\}$ as follows.
(UFG) There are an integer $\ell_0 \geqq 1$ and $\varphi_{u,u'} \in C_b^\infty(\mathbf{R}^N)$, $u \in A^{**}$, $u' \in A^{**}_{\leqq \ell_0}$, satisfying the following.

$$\Phi(r(u)) = \sum_{u' \in A^{**}_{\leqq \ell_0}} \varphi_{u,u'} \Phi(r(u')), \qquad u \in A^{**}.$$

Let P_t, $t \in [0, \infty)$ be a diffusion semigroup given by

$$P_t f(x) = E[f(X(t,x))], \qquad f \in C_b^\infty(\mathbf{R}^N).$$

Then P_t's are regarded as a linear operators in $C_b^\infty(\mathbf{R}^N)$. We also have the following.

Theorem 1 *Assume that* (UFG) *condition is satisfied. For any $n, m \geqq 0$, and $u_1, \dots, u_{n+m} \in A^{**}$, there is a $C \in (0, \infty)$ such that*

$$||\Phi(r(u_1), \cdots r(u_n)) P_t \Phi(r(u_{n+1}) \cdots r(u_{n+m})) f||_\infty \leqq C t^{-(||u_1|| + \cdots ||u_{n+m})/2} ||f||_\infty$$

for any $t \in (0,1)$, and $f \in C_b^\infty(\mathbf{R}^N)$. Here

$$||f||_\infty = \sup_{x \in \mathbf{R}^N} |f(x)|.$$

This theorem was shown by [5] under a uniform Hörmander condition and was shown by [3] in general case.

In the present paper, we assume (UFG) and the following assumptions (A1) and (A2) throughout.

(A1) $V_1^1(x) = 1$, $V_1^i(x) = 0$, $i = 2, \dots, N$, for any $x \in \mathbf{R}^N$.
(A2) $V_k^1(x) = 0$, $k = 0, 2, \dots, d$, for any $x \in \mathbf{R}^N$.

Then $X^1(t,x) = x^1 + B^1(t)$, $t \geqq 0$. Let $h \in C^\infty(\mathbf{R}^N)$ be given by $h(x) = x^1$, $x \in \mathbf{R}^N$. Then we see that $\Phi(r(v_1))h = 1$, and $\Phi(r(u))h = 0$, $u \in A^* \setminus \{1, v_1\}$. So we see that if (UFG) condition is satisfied, we see that $\varphi_{u,v_1} = 0$, for $u \in A^* \setminus \{1, v_1\}$.

Let $b_k \in C_b^\infty(\mathbf{R}^N)$, $k = 0, \dots, d$, and let

$$P_t^0 f(x) = E[\exp(\sum_{k=0}^d \int_0^t b_k(X(r,x)) \circ dB^k(r)) f(X(t,x)), \min_{r \in [0,t]} X^1(r)) > 0].$$

Then we see that

$$\frac{\partial}{\partial t} P_t^0 f(x) = L^0 P_t f(x), \qquad t > 0, \; x \in (0, \infty) \times \mathbf{R}^{N-1}$$

as generalized functions, and

$$P_t^0 f(x) = 0, \qquad t > 0, \; x \in \{0\} \times \mathbf{R}^{N-1}.$$

Here

$$L^0 = \frac{1}{2} \sum_{k=1}^d V_k^2 + V_0 + \sum_{k=1}^N b_k V_k + (b_0 + \frac{1}{2} \sum_{k=1}^d (b_k^2 + V_k b_k)).$$

Our final purpose is to show the following.

Theorem 2 *Assume that* (UFG) *condition is satisfied. Then for any* $n, m, r \geqq 0$ *and* $u_1, \ldots, u_{n+m} \in A^{**}$*, there is a* $C \in (0, \infty)$ *such that*

$$\sup_{x \in (0,\infty)\times \mathbf{R}^{N-1}} |\Phi(r(u_1)\cdots r(u_n)) adj(V_0)^r (P_t^0) \Phi(r(u_{n+1})\cdots r(u_{n+m}))f(x)|$$

$$\leqq Ct^{-(||u_1||+\cdots||u_{n+m}||/2)-r} \sup_{x \in (0,\infty)\times \mathbf{R}^{N-1}} |f(x)|$$

and

$$\int_{(0,\infty)\times \mathbf{R}^{N-1}} |\Phi(r(u_1)\cdots r(u_n)) adj(V_0)^r (P_t^0) \Phi(r(u_{n+1})\cdots r(u_{n+m}))f(x)| dx$$

$$\leqq Ct^{-(||u_1||+\cdots||u_{n+m}||/2)-r} \int_{(0,\infty)\times \mathbf{R}^{N-1}} |f(x)| dx$$

for any $t \in (0, 1]$ *and* $f \in C_b^\infty(\mathbf{R}^N)$.

Here $adj^0(V_0)(P_t^0) = P_t^0$, *and*

$$adj^{n+1}(V_0)(P_t^0) = V_0 \ adj(V_0)^n (P_t^0) - adj(V_0)^n (P_t^0) V_0, \qquad n = 0, 1, \ldots.$$

In the present paper, we prove the following theorem.

Theorem 3 *Assume that* (UFG) *condition is satisfied. Let* $A^{***} = A^{**} \setminus \{v_1\}$. *Then we have the following.*
(1) *For any* $n, m, r \geqq 0$ *and* $u_1, \ldots, u_{n+m} \in A^{***}$, *there is a* $C \in (0, \infty)$ *such that*

$$\sup_{x \in (0,\infty)\times \mathbf{R}^{N-1}} |\Phi(r(u_1)\cdots r(u_n)) adj(V_0)^r P_t^0 \Phi(r(u_{n+1})\cdots r(u_{n+m}))f(x)|$$

$$\leqq Ct^{-(||u_1||+\cdots||u_{n+m}||/2)-r} \sup_{x \in (0,\infty)\times \mathbf{R}^{N-1}} |f(x)|$$

for any $t \in (0, 1]$ *and* $f \in C_b^\infty(\mathbf{R}^N)$.
(2) *For any* $n, m, r \geqq 0$ *and* $u_1, \ldots, u_{n+m} \in A^{***}$, *there is a* $C \in (0, \infty)$ *such that*

$$\int_{(0,\infty)\times \mathbf{R}^{N-1}} |\Phi(r(u_1)\cdots r(u_n)) adj(V_0)^r (P_t^0) \Phi(r(u_{n+1})\cdots r(u_{n+m}))f(x)| dx$$

$$\leqq Ct^{-(||u_1||+\cdots||u_{n+m}||/2)-r} \int_{(0,\infty)\times \mathbf{R}^{N-1}} |f(x)| dx$$

for any $t \in (0, 1]$ *and* $f \in C_0^\infty(\mathbf{R}^N)$.

We will prove Theorem 2 in the forthcoming paper.

2 Normed spaces and Interpolation

From now on, we assume that (UFG) is satisfied. Let $(W_0, \mathcal{G}, \mu)$ be a Wiener space as in Introduction. Let H denote the associated Cameron-Martin space, $\mathcal{L}$ denote the associated Ornstein-Uhlenbeck operator, and $W^{r,p}(E)$, $r \in \mathbf{R}$, $p \in (1, \infty)$, be Watanabe-Sobolev spaces, i.e. $W^{r,p} = (I - \mathcal{L})^{-r/2}(L^p(W_0; E, d\mu))$ for any separable real Hilbert space E. Let D denote the gradient operator. Then D is a bounded linear operator from $W^{r,p}(E)$ to $W^{r-1,p}(H \otimes E)$. Let D^* denote the adjoint operator of D. (See Shigekawa [6] for details.)

Let $\tilde{A} = A^{**}_{\leqq \ell_0} \setminus \{v_1\}$. Let $V_u^{(s)} \in C_b^\infty(\mathbf{R}^N; \mathbf{R}^N)$, $u \in \tilde{A}$, $s \in (0, 1]$, be given by

$$V_u^{(s)}(x) = s^{||u||/2} \Phi(r(u))(x), \qquad x \in \mathbf{R}^N.$$

Note that $(V_u^{(u)} h)(x) = 0$, $x \in \mathbf{R}^N$, $u \in \tilde{A}$, $s \in (0, 1]$, where $h(x) = x^1$, $x = (x^1, \ldots, x^N) \in \mathbf{R}^N$.

Proposition 4 *There are* $\tilde{\varphi}_{u_1,u_2,u_3} \in C_b^\infty(\mathbf{R}^N)$, $u_1, u_2, u_3 \in \tilde{A}$, *such that*

$$[V_{u_1}^{(s)}, V_{u_2}^{(s)}] = \sum_{u_3 \in \tilde{A}} s^{0 \vee (||u_1||+||u_2||-||u_3||)/2} \tilde{\varphi}_{u_1,u_2,u_3} V_{u_3}^{(s)}, \qquad u_1, u_2 \in \tilde{A}.$$

Proof. Note that there are $c_{u_1,u_2,u_3} \in \mathbf{R}$, $u_1, u_2 \in \tilde{A}$, $u_3 \in A^{**}$ such that

$$[r(u_1), r(u_2)] = \sum_{u_3 \in A^{**}, ||u_3||=||u_1||+||u_2||} c_{u_1,u_2,u_3} r(u_3).$$

So if $||u_1|| + ||u_2|| \leqq \ell_0$, we have

$$[V_{u_1}^{(s)}, V_{u_2}^{(s)}](x) = s^{(||u_1||+||u_2||)/2} \Phi([r(u_1), r(u_2)])(x)$$

$$= \sum_{u_3 \in \tilde{A}, ||u_3||=||u_1||+||u_2||} c_{u_1,u_2,u_3} s^{||u_3||/2} \Phi(r(u_3))(x)$$

$$= \sum_{u_3 \in \tilde{A}, ||u_3||=||u_1||+||u_2||} c_{u_1,u_2,u_3} \psi_0(x^1) V_{u_3}^{(s)}(x).$$

Also, if $||u_1|| + ||u_2|| > \ell_0$, we have

$$[V_{u_1}^{(s)}, V_{u_2}^{(s)}](x) = \sum_{u_3 \in \tilde{A}, ||u_3||=||u_1||+||u_2||} c_{u_1,u_2,u_3} s^{(||u_1||+||u_2||)/2} \Phi(r(u_3))(x)$$

$$= \sum_{u_4 \in \tilde{A}, ||u_3||=||u_1||+||u_2||} c_{u_1,u_2,u_3} s^{(||u_1||+||u_2||)/2} \varphi_{u_3,u_4}(x) \Phi(r(u_4))(x)$$

$$= \sum_{u_4 \in \tilde{A}, ||u_3||=||u_1||+||u_2||} c_{u_1,u_2,u_3} s^{(||u_1||+||u_2||-||u_4||)/2} \varphi_{u_3,u_4}(x) V_{u_4}^{(s)}(x).$$

These imply our assertion. ∎

Now let $\tilde{B}^u(t), t \in [0,\infty)$, $u \in \tilde{A}$, be independent standard Brownian motions defined on a certain probability space and let $X^{(s)}(t,x)$, $t \in [0,\infty)$, $x \in \mathbf{R}^N$, $s \in (0,1]$, be a solution to the following stochastic differential equation.

$$dX^{(s)}(t,x) = \sum_{u \in \tilde{A}} V_u^{(s)}(X^{(s)}(t,x)) \circ d\tilde{B}^u(t),$$

$$X^{(s)}(0,x) = x.$$

Note that $h(X^{(s)}(t,x)) = h(x)$, $t \geqq 0$, $x \in \mathbf{R}^N$. Now let $Q_t^{(s)}$, $t \in [0,\infty)$, $s \in (0,1]$, be linear operators on $C_b^\infty(\mathbf{R}^N)$ given by

$$(Q_t^{(s)} f)(x) = E[f(X^{(s)}(t,x))], \qquad f \in C_b^\infty(\mathbf{R}^N).$$

Let

$$L^{(s)} = \frac{1}{2} \sum_{u \in \tilde{A}} s^{||u||} \Phi(r(u))^2.$$

Then we see that

$$Q_t^{(s)} f = f + \int_0^t L^{(s)} Q_r^{(s)} f dr, \qquad f \in C_b^\infty(\mathbf{R}^N).$$

By Theorem 1 in [4] we have the following.

Proposition 5 *For any $n, m \geqq 0$, and $u_1, \dots u_{n+m} \in \tilde{A}$, there eists a $C \in (0, \infty)$ such that*

$$s^{(||u_1||+\cdots||u_{n+m}||)/2}||\Phi(r(u_1))\cdots\Phi(r(u_n))Q_t^{(s)}\Phi(r(u_{n+1}))\cdots\Phi(r(u_{n+m}))f||_\infty$$
$$\leqq Ct^{-(||u_1||+\cdots||u_{n+m}||)/2}||f||_\infty$$

for any $f \in C_b^\infty(\mathbf{R}^N)$ and $s, t \in (0, 1]$.

Let $\mathcal{C}$ be the set of bounded measurable functions f defined in $\mathbf{R}^N$ such that $f(x^1, x^2, \dots, x^N)$ is smooth in $(x^2, \dots, x^N)$, and that

$$\sup_{x \in \mathbf{R}^N} |\frac{\partial^{\alpha_2+\dots+\alpha_N} f}{(\partial x^2)^{\alpha_2}\cdots(\partial x^N)^{\alpha_N}}(x)| < \infty$$

for any $\alpha_2, \dots, \alpha_N \geqq 0$.

Note that $Q^{(s)}f \in \mathcal{C}$ for any $f \in \mathcal{C}$. Then the following is an easy consequence of Proposition 5.

Corollary 6 *For any $n, m \geqq 0$, and $u_1, \dots u_{n+m} \in \tilde{A}$, there eists a $C \in (0, \infty)$ such that*

$$s^{(||u_1||+\cdots||u_{n+m}||)/2}||\Phi(r(u_1))\cdots\Phi(r(u_n))Q_t^{(s)}\Phi(r(u_{n+1}))\cdots\Phi(r(u_{n+m}))f||_\infty$$
$$\leqq Ct^{-(||u_1||+\cdots||u_{n+m}||)/2}||f||_\infty$$

for any $f \in \mathcal{C}$ and $s, t \in (0, 1]$.

Let us define normed spaces $\mathcal{D}^1_{(s)}$, $s \in (0, 1]$, and $\mathcal{H}^{-\alpha}_{(s)}$, $s \in [0, 1]$, $\alpha \in [0, 1)$, by the following. $\mathcal{D}^1_{(s)} = \mathcal{H}^{-\alpha}_{(s)} = \mathcal{C}$ as sets, and their norms are given by

$$||f||_{\mathcal{D}^1_{(s)}} = ||f||_\infty + \sum_{u \in \tilde{A}} s^{||u||/2}||\Phi(r(u))f||_\infty$$

and

$$||f||_{\mathcal{H}^{-\alpha}_{(s)}} = \sup_{t \in (0,1]} t^{\alpha/2}||Q_t^{(s)}f||_\infty$$

for $f \in \mathcal{C}$. Note that

$$||f||_{\mathcal{H}^0_{(s)}} = ||f||_\infty, \qquad f \in \mathcal{C}.$$

We have the following as an easy consequence of Corollary 6,

Proposition 7 *There is a $C_0 \in (0, \infty)$ such that*

$$||L^{(s)}Q_t^{(s)}f||_\infty \leqq C_0 t^{-1}||f||_\infty$$

and

$$||Q_t^{(s)}f||_{\mathcal{D}^1_{(s)}} \leqq C_0 t^{-1/2}||f||_\infty$$

for any $f \in \mathcal{C}$ and $s, t \in (0, 1]$.

Then we have the following.

Proposition 8 *Let $\alpha \in (0, 1)$ and $\theta \in (0, 1)$. If $\beta = (1-\theta)\alpha - \theta \geqq 0$, then there is a $C \in (0, \infty)$ such that*

$$\sup_{t \in (0,\infty)} t^{-\theta}K(t; f, \mathcal{H}^{-\alpha}_{(s)}, \mathcal{D}^1_{(s)}) \leqq C||f||_{\mathcal{H}^{-\beta}_{(s)}}$$

for $f \in \mathcal{C}$ and $s \in (0, 1]$. Here

$$K(t; f, \mathcal{H}^{-\alpha}_{(s)}, \mathcal{D}^1_{(s)}) = \inf\{||g||_{\mathcal{H}^{-\alpha}_{(s)}} + t||f - g||_{\mathcal{D}^1_{(s)}};\ g \in \mathcal{C}\}, \qquad t \in (0, \infty).$$

Remark 9 $K(t; f, \mathcal{H}^{-\alpha}_{(s)}, \mathcal{D}^1_{(s)})$ *is a real interpolation* (c.f. Berph-Löfström [1]).

Proof. Let $f \in \mathcal{C}$. Note that

$$||Q^{(s)}_t(Q^{(s)}_r f - f)||_\infty \leqq \int_0^r ||L^{(s)} Q^{(s)}_{t/2} Q^{(s)}_{(t+2z)/2} f||_\infty dz$$

$$\leqq C_0(t/2)^{-1} \int_0^r ||Q^{(s)}_{(t+2z)/2} f||_\infty dz \leqq C_0(t/2)^{-1-\beta/2} r ||f||_{\mathcal{H}^{-\beta}_{(s)}}.$$

Here C_0 is as in Corollary 6 .

On the other hand,

$$||Q^{(s)}_t(Q^{(s)}_r f - f)||_\infty \leqq 2||Q^{(s)}_t f||_\infty \leqq 2t^{-\beta/2}||f||_{\mathcal{H}^{-\beta}_{(s)}}.$$

Therefore

$$||Q^{(s)}_t(Q^{(s)}_r f - f)||_\infty \leqq (2 + 4C_0)t^{-\beta/2}(1 \wedge (rt^{-1}))||f||_{\mathcal{H}^{-\beta}_{(s)}}$$

$$\leqq (2 + 4C_0)t^{-\beta/2}(rt^{-1})^{\gamma/2}||f||_{\mathcal{H}^{-\beta}_{(s)}}.$$

Here $\gamma = \theta(1 + \alpha) = \alpha - \beta \in (0, 1)$. Therefore we see that

$$||Q^{(s)}_r f - f||_{\mathcal{H}^{-\alpha}_{(s)}} \leqq (2 + 4C_0)r^{\gamma/2}||f||_{\mathcal{H}^{-\beta}_{(s)}}.$$

Also we have

$$||Q^{(s)}_r f||_{\mathcal{D}^1_{(s)}} \leqq C_0(r/2)^{-1/2}||Q^{(s)}_{r/2} f||_\infty \leqq 4C_0 r^{-(1+\beta)/2}||f||_{\mathcal{H}^{-\beta}_{(s)}}.$$

Since we have

$$f = Q^{(s)}_r f + f - Q^{(s)}_r f, \qquad f \in \mathcal{C},$$

we see that for $t \in (0, 1]$

$$t^{-\theta}K(t; f, \mathcal{H}^{-\alpha}_{(s)}, \mathcal{D}^1_{(s)}) \leqq t^{1-\theta}||Q^{(s)}_r f||_{\mathcal{D}^1_{(s)}} + t^{-\theta}||Q^{(s)}_r f - f||_{\mathcal{H}^{-\alpha}_{(s)}}$$

$$\leqq (2 + 4C_0)(t^{1-\theta}r^{-(1+\beta)/2} + t^{-\theta}r^{\gamma/2})||f||_{\mathcal{H}^{-\beta}_{(s)}}.$$

Let $r = t^{2\theta/\gamma}$. Since $(1 - \theta)(1 + \alpha) = 1 + \beta$, we see that

$$\sup_{t\in(0,1]} t^{-\theta}K(t; f, \mathcal{H}^{-\alpha}_{(s)}, \mathcal{D}^1_{(s)}) \leqq 4(1 + 2C_0)||f||_{\mathcal{H}^{-\beta}_{(s)}}.$$

It is obvious that

$$\sup_{t\in[1,\infty)} t^{-\theta}K(t; f, \mathcal{H}^{-\alpha}_{(s)}, \mathcal{D}^1_{(s)}) \leqq ||f||_{\mathcal{H}^{-\alpha}_{(s)}} \leqq ||f||_{\mathcal{H}^{-\beta}_{(s)}}$$

Therefore we have our assertion. ∎

The following has been proved by Watanabe [7], but we give a proof.

Proposition 10 *Let* $\theta \in (0, 1)$, $p \in (1, \infty)$ *and* $r_0, r_1 \in [-1, 0]$. *If* $r_2 < (1 - \theta)r_0 + \theta r_1$, *then there is a* $C \in (0, \infty)$ *such that*

$$||F||_{W^{r_2,p}} \leqq C \sup_{t\in(0,\infty)} t^{-\theta}K(t; F, W^{r_0,p}, W^{r_1,p})$$

for any $F \in W^{\infty,\infty-} = \bigcap_{r\in\mathbf{R},p\in(1,\infty)} W^{r,p}$. *Here*

$$K(t; F, W^{r_0,p}, W^{r_1,p}) = \inf\{||G||_{W^{r_0,p}} + t||F - G||_{W^{r_1,p}};\ G \in W^\infty_{\infty-}\}.$$

Proof. Let us take an $F \in W^{\infty,\infty-}$ and fix it. Let T_t be the Ornstein-Uhlenbeck semi-group on W_0, and let

$$a = \sup_{t\in(0,\infty)} t^{-\theta} K(t; F, W^{r_0,p}, W^{r_1,p})$$

Then we see that

$$||F||_{W^{r_0\wedge r_1,p}} \leqq a.$$

So we have our assertion if $r_2 \leqq r_1 \wedge r_2$. Theerefore we may assume that $r_2 > r_1 \wedge r_2 \geqq -1$.

Note that for any $r \geqq 0$, there is a $C_r > 0$ such that

$$||(I - \mathcal{L})^r T_t g||_{W^{0,p}} \leqq C_r t^{-r} ||g||_{W^{0,p}}$$

for any $t \in (0, 1]$ and $g \in W^{\infty,\infty-}$.

For any $t \in (0, 1]$ and $\varepsilon > 0$, there is an $G_t \in W^{\infty}_{\infty-}$ such that

$$(t^{(r_1-r_0)/2})^{-\theta}||G_t||_{W^{r_0,p}} + (t^{(r_1-r_0)/2})^{1-\theta}||F - G_t||_{W^{r_1,p}} \leqq a + \varepsilon.$$

Let $\gamma = ((1 - \theta)r_0 + \theta r_1 - r_2)/2 > 0$. Then we have $r_2 - r_1 = -(1 - \theta)(r_1 - r_0) - 2\gamma$, and $r_2 - r_0 = \theta(r_1 - r_0) - 2\gamma$. So we see that

$$t^{-(\gamma+(r_2-r_0)/2}||G_t||_{W^{r_0,p}} + t^{-(\gamma+(r_2-r_0)/2)}||F - G_t||_{W^{r_1,p}} \leqq a + \varepsilon.$$

Then we have

$$\begin{aligned}
||(I - \mathcal{L})T_t F||_{W^{r_2,p}} &= ||(I - \mathcal{L})^{1+(r_2/2)}T_t F||_{W^{0,p}} \\
&\leqq ||(I - \mathcal{L})^{1+(r_2/2)}T_t G_t||_{W^{0,p}} + ||(I - \mathcal{L})^{1+(r_2/2)}T_t(F - G_t)||_{W^{0,p}} \\
&\leqq ||(I - \mathcal{L})^{1+((r_2-r_0)/2)}T_t(I - \mathcal{L})^{r_0/2}G_t||_{W^{0,p}} + ||(I - \mathcal{L})^{1+((r_2-r_1)/2)}T_t(I - \mathcal{L})^{r_1/2}(F - G_t)||_{W^{0,p}} \\
&\leqq C(t^{-(1+(r_2-r_0)/2)}||G_t||_{W^{r_0,p}} + t^{-(1+(r_2-r_1)/2)}||F - G_t||_{W^{r_1,p}}) \leqq Ct^{-1+\gamma}(a + \varepsilon)
\end{aligned}$$

for any $t \in (0, 1]$. Note that

$$F = \int_0^1 e^{-t}(I - \mathcal{L})T_t F dt + e^{-1}T_1 F.$$

Then we see that

$$||F||_{W^{r_2,p}} \leqq C(a + \varepsilon)\int_0^1 t^{-1+\gamma}dt + ae^{-1}||T_1||_{W^{r_0\wedge r_1,p}\to W^{r_2,p}}.$$

So we have the assertion. ∎

Proposition 11 *Let $p \in (1, \infty)$ and $\varepsilon \in (0, 1]$. If $p(1 - \varepsilon) < 1$, then*

$$\sup_{s\in(0,1],x^1>0} ||1_{(0,\infty)}(\min_{t\in[0,1]}(x^1 + s^{1/2}B^1(t)))||_{W^{1-\varepsilon,p}} < \infty$$

Proof. Let $Y = \min_{t\in[0,1]} B^1(r)$. Then

$$|Y(w + h) - Y(w)| \leqq \max_{t\in[0,1]} |h(t)| \leqq \int_0^1 |\frac{dh^1}{dr}(r)|dr \leqq ||h||_H$$

for any $w \in W_0$ and $h \in H$. Therefore $||DY||_H \leqq 1$ $\mu - a.s.$

Let $\varphi \in C_0^\infty(\mathbf{R})$ such that $\varphi \geqq 0$, $\varphi(z) = 0$, $|z| > 1$, and $\int_{\mathbf{R}} \varphi(z)dz = 1$. Also, let

$$\psi_r(z) = \frac{1}{r}\int_{-\infty}^{z} \varphi(r^{-1}y)dy, \qquad r \in (0,1], z \in \mathbf{R},$$

and

$$G_r(s,x^1) = \psi_r(s^{-1/2}x^1 + Y), \qquad r,s \in (0,1],\ x^1 > 0.$$

Then we see that $0 \leqq \psi_r \leqq 1$, $\psi_r(z) = 0$, $z \in (-\infty, -r]$, and $\psi_r(z) = 1$, $z \in [r,\infty)$. Also, we see that

$$DG_r(s,x^1) = \frac{1}{r}\varphi(r^{-1}(s^{-1/2}x^1 + Y))DY,$$

and so

$$E^\mu[||DG_r(s,x^1)||_H^p] \leqq r^{-p}E^\mu[\varphi(r^{-1}(s^{-1/2}x^1 + Y))^p]$$
$$\leqq r^{-p}||\varphi||_\infty^p P^\mu(|s^{-1/2}x^1 + Y| \leqq r).$$

Note that

$$\mu(|s^{-1/2}x_1 + Y| \leqq r) = \mu(Y \in [-s^{-1/2}x_1 - r, -s^{-1/2}x_1 + r])$$
$$\leqq 4(2\pi)^{-1/2}r \leqq 2r.$$

So we have

$$E^\mu[||DG_r(s,x^1)||_H^p]^{1/p} \leqq 2r^{-(1-1/p)}||\varphi||_\infty.$$

Also, note that

$$|1_{(0,\infty)}(\min_{t\in[0,1]}(x^1 + s^{1/2}B^1(t))) - G_r(s,x^1)|$$
$$= |1_{(0,\infty)}(s^{-1/2}x^1 + Y) - \psi_r(s^{-1/2}x^1 + Y)| \leqq 1_{(-r,r)}(s^{-1/2}x^1 + Y)$$

and so

$$||1_{(0,\infty)}(x^1 + s^{1/2}Y) - G_r(s,x^1)||_{L^p(d\mu)}^p \leqq 2r.$$

So we see that

$$\sup_{r\in(0,1]}(r^{-1/p}||1_{(0,\infty)}(\min_{t\in[0,1]}(x^1 + s^{1/2}B^1(t))) - G_r(s,x^1)||_{W^{0,p}} + r^{1-1/p}||G_r(s,x^1)||_{W^{1,p}})$$
$$\leqq 2 + (2 + 2||\varphi||_\infty).$$

Also, it is obvious that

$$\sup_{r\in[1,\infty)} r^{-1/p}||1_{(0,\infty)}(\max_{s\in[0,t]}(x^1 + B^1(s))||_{W^{0,p}} \leqq 1.$$

Since $1 - \varepsilon < 1/p$, we have our assertiin by Proposition 10.

∎

3 Basic Results

Let $V_{s,0}(x) = sV_0(x)$, $V_{s,i}(x) = s^{1/2}V_i(x)$, $i = 1,\ldots,d$, $s \in (0,1]$. Let us think of the following SDE with a parameter $s \in (0,1]$.

$$dX_s(t,x) = \sum_{i=0}^{d} V_{s,i}(X_s(t,x)) \circ dB^i(t),$$

$$X_s(0,x) = x \in \mathbf{R}^N.$$

Let us define a homomorphism $\Phi_s : \mathbf{R}\langle A\rangle \to \mathcal{DO}(\mathbf{R}^N)$, $s \in (0,1]$, by

$$\Phi_s(1) = Identity, \qquad \Phi_s(v_{i_1}\cdots v_{i_n}) = V_{s,i_1}\cdots V_{s,i_n}, \qquad n \geqq 1,\ i_1,\ldots,i_n = 0,1,\ldots,d.$$

Then we see the following.

$$\Phi_s(r(u))(x) = \sum_{u' \in A^{**}_{\leqq \ell_0}} s^{(||u||-||u'||)/2}\varphi_{u,u'}(x)\Phi_s(r(u'))(x), \qquad s \in (0,1],\ x \in \mathbf{R}^N$$

for any $u \in A^{**} \setminus A^{**}_{\leqq \ell_0}$. Here $\varphi_{v_k u,u'}$'s are as in the assumption (UFG).

From now on, we follow results in [4] basically . For any C_b^∞ vector field W on $\mathbf{R}^N$, we define $(X_s(t)_* W)(X(t,x)) = \sum_{i,j=1}^N \frac{\partial}{\partial x^j} X_s^i(t,x) W^j(x) \frac{\partial}{\partial x^i}$. Then $X_s(t)_*$ is a push-forward operator with respect to the diffeomorphism $X_s(t,\cdot) : \mathbf{R}^N \to \mathbf{R}^N$ for any $s \in (0,1]$. Also we see that

$$d(X_s(t)_*^{-1}\Phi_s(r(u)))(x)$$

$$= \sum_{i=0}^{d} (X_s(t)_*^{-1}\Phi_s(r(v_i u)))(x) \circ dB^i(t)$$

for any $u \in A^* \setminus \{1\}$.

Let $c_k^{(s)}(\cdot,u,u') \in C_b^\infty(\mathbf{R}^N,\mathbf{R})$, $k = 0,1,\ldots,d$, $u,u' \in A^{**}_{\leqq \ell_0}$, be given by

$$c_k^{(s)}(x;u,u') = \begin{cases} 1, & \text{if } ||v_k u|| \leqq \ell_0 \text{ and } u' = v_k u, \\ s^{(||v_k u||-||u'||)/2}\varphi_{v_k u,u'}(x), & \text{if } ||v_k u|| > \ell_0 \text{ and } ||u'|| \leqq \ell_0, \\ 0, & \text{otherwise.} \end{cases}$$

Then we have

$$d(X_s(t)_*^{-1}\Phi_s(r(u)))(x)$$

$$= \sum_{k=0}^{d} \sum_{u' \in A^{**}_{\leqq \ell_0}} c_k^{(s)}(X(t,x);u,u')(X_s(t)_*^{-1}\Phi_s(r(u')))(x) \circ dB^k(t), \quad u \in A^{**}_{\leqq \ell_0}.$$

There exists a unique solution $a_s(t,x;u,u')$, $u,u' \in A^{**}_{\leqq \ell_0}$, $s \in (0,1]$, to the following SDE

$$da_s(t,x;u,u') = \sum_{k=0}^{d} \sum_{u'' \in A^{**}_{\leqq \ell_0}} (c_k^{(s)}(X_s(t,x);u,u'')a_s(t,x;u'',u')) \circ dB^k(t) \tag{2}$$

$$a_s(0,x;u,u') = \delta_{u,u'}.$$

Then the uniqueness of SDE implies

$$(X_s(t)_*^{-1}\Phi_s(r(u)))(x) = \sum_{u' \in A^{**}_{\leqq \ell_0}} a_s(t,x;u,u')\Phi_s(r(u'))(x),\ u \in A^{**}_{\leqq \ell_0},\ s \in (0,1]. \tag{3}$$

Similarly we see that there exists a unique solution $b_s(t,x;u,u')$, $u,u' \in A^{**}_{\leqq \ell_0}$, to the SDE

$$b_s(t,x;u,u') = \delta_{u,u'} - \sum_{k=0}^{d} \sum_{u'' \in A^{**}_{\leqq \ell_0}} \int_0^t (b_s(r,x;u,u'') c_k^{(s)}(X_s(r,x);u'',u')) \circ dB^k(r). \qquad (4)$$

Then we see that

$$\sum_{u'' \in A^{**}_{\leqq \ell_0}} a_s(t,x,u,u'') b_s(t,x,u'',u') = \delta_{u,u'}, \qquad u,u' \in A^{**}_{\leqq \ell_0},$$

and that

$$\Phi_s(r(u))(x) = \sum_{u' \in A^{**}_{\leqq \ell_0}} b_s(t,x;u,u')(X_s(t)_*^{-1} \Phi_s(r(u')))(x), \qquad u \in A^{**}_{\leqq \ell_0}. \qquad (5)$$

Furthermore we see by Proposition 4 (1) that

$$a_s(t,x,u,v_1) = b_s(t,x,u,v_1) = 0, \ a.s. \qquad u \in \tilde{A}.$$

Also, we see that

$$(X_s(t)_*^{-1} \Phi_s(v_0))(x)$$
$$= \Phi_s(v_0) + \sum_{k=1}^{d} \int_0^t (X_s(r)_*^{-1} \Phi_s(r(v_k v_0))) \circ dB^k(r).$$

So we have

$$(X_s(t)_*^{-1} \Phi_s(v_0))(x) = \Phi_s(v_0)(x) + \sum_{u \in \tilde{A}} \hat{a}_s(t,x;u) \Phi_s(r(u))(x), \qquad (6)$$

and

$$\Phi_s(v_0)(x) = (X_s(t)_*^{-1} \Phi_s(v_0))(x) + \sum_{u \in \tilde{A}} \hat{b}_s(t,x;u)(X_s(t)_*^{-1} \Phi_s(r(u))(x), \qquad (7)$$

where

$$\hat{a}_s(t,x;u) = \sum_{k=1}^{d} \int_0^t a_s(r,x;v_k v_0,u) \circ dB^k(r)$$

and

$$\hat{b}_s(t,x;u) = -\sum_{u' \in \tilde{A}} b_s(t,x;u,u') \hat{a}(t,x;u').$$

Note that

$$\Phi_s(r(u))(f(X_s(t,x)) = \langle X_s(t)^* df, \Phi_s(r(u)) \rangle_x.$$

So we have

$$\Phi_s(r(u))(f(X_s(t,x))) = \sum_{u' \in A^{**}_{\leqq \ell_0}} b_s(t,x;u,u')(\Phi_s(r(u'))f)(X_s(t,x)), \qquad u \in A^{**}_{\leqq \ell_0}, \qquad (8)$$

$$(\Phi_s(r(u))f)(X_s(t,x)) = \sum_{u' \in A^{**}_{\leqq \ell_0}} a_s(t,x;u,u') \Phi_s(r(u'))(f(X_s(t,x))), \qquad u \in A^{**}_{\leqq \ell_0}, \qquad (9)$$

and

$$\Phi_s(v_0)(f(X_s(t,x)) - (\Phi_s(v_0)f)(X_s(t,x))$$

$$= \sum_{u' \in \mathcal{A}^{**}_{\leqq \ell_0}} \hat{b}_s(t,x;u')(\Phi_s(r(u'))f)(X_s(t,x)). \tag{10}$$

Let us define $k_s : [0,\infty) \times \mathbf{R}^N \times A^{**}_{\leqq \ell_0} \times W_0 \to H$ by

$$k_s(t,x;u) = (\int_0^{t\wedge \cdot} a_s(r,x;v_k,u)dr)_{k=1,\dots d}.$$

Let $M_s(t,x) = \{M_s(t,x;u,u')\}_{u,u' \in A^{**}_{\leqq \ell_0}}$ be a matrix-valued random variable given by

$$M_s(t,x;u,u') = t^{-(||u||+||u'||)/2}(k_s(t,x;u), k_s(t,x;u'))_H.$$

Then we have

$$\sup_{s\in(0,1]} \sup_{t\in(0,T]} \sup_{x\in\mathbf{R}^N} E^\mu[|\det M_s(t,x)|^{-p}] < \infty \text{ for any } p \in (1,\infty) \text{ and } T > 0.$$

Let $M_s^{-1}(t,x) = \{M_s^{-1}(t,x;u,u')\}_{u,u' \in A^{**}_{\leqq \ell_0}}$ be the inverse matrix of $M_s(t,x)$.

For any separable real Hilbert space E, let $\hat{\mathcal{K}}_0(E)$ be the set of $\{F_s\}_{s\in(0,1]}$ such that
(1) $F_s : (0,\infty) \times \mathbf{R}^N \times W_0 \to E$ is measurable map for all $s \in (0,1]$,
(2) $F_s(t,\cdot,w) : \mathbf{R}^N \to E$ is smooth for any $s \in (0,1]$, $t \in (0,\infty)$ and $w \in W_0$,
(3) $(\partial^\alpha F_s/\partial x^\alpha)(\cdot,*,w) : (0,\infty)\times\mathbf{R}^N \to E$ is continuous for any $s \in (0,1]$, $w \in W_0$ and $\alpha \in \mathbf{Z}^N_{\geqq 0}$,
(4) $(\partial^\alpha F_s/\partial x^\alpha)(t,x,\cdot) \in W^{r,p}$ for any $s \in (0,1]$, $r,p \in (1,\infty)$, $\alpha \in \mathbf{Z}^N_{\geqq 0}$, $t \in (0,\infty)$ and $x \in \mathbf{R}^N$, and
(5) for any $r,p \in (1,\infty)$, $\alpha \in \mathbf{Z}^N_{\geqq 0}$, and $T > 0$

$$\sup_{s\in(0,1],t\in(0,T]} \sup_{x\in\mathbf{R}^N} ||\frac{\partial^\alpha}{\partial x^\alpha} F_s(t,x)||_{W^{r,p}} < \infty.$$

Then we have the following.

Proposition 12 (1) $\{t^{-(||u'||-||u||)/2}a_s(t,x;u,u')\}_{s\in(0,1]}$ *and* $\{t^{-(||u'||-||u||)/2}b_s(t,x;u,u')\}_{s\in(0,1]}$ *belong to* $\hat{\mathcal{K}}_0(\mathbf{R})$ *for any* $u,u' \in A^{**}_{\leqq \ell_0}$.
(2) $\{t^{-||u||/2}k_s(t,x;u)\}_{s\in(0,1]}$ *belongs to* $\hat{\mathcal{K}}_0(H)$ *for any* $u \in A^{**}_{\leqq \ell_0}$.
(3) $\{M_s(t,x;u,u')\}_{s\in(0,1]}$, *and* $\{M_s^{-1}(t,x;u,u')\}_{s\in(0,1]}$ *belong to* $\hat{\mathcal{K}}_0(\mathbf{R})$ *for any* $u,u' \in A^{**}_{\leqq \ell_0}$.
(4) $\{\hat{a}_s(t,x;u)\}_{s\in(0,1]}$ *and* $\{\hat{b}_s(t,x;u)\}_{s\in(0,1]}$ *belong to* $\hat{\mathcal{K}}_0(\mathbf{R})$ *for any* $u \in \tilde{A}$.

Finally we have the following basic equation.

$$t^{||u||/2}(\Phi_s(u)f)(X_s(t,x))$$
$$= \sum_{u_1,u_2 \in A^{**}_{\leqq \ell_0}} a_s(t,x;u,u_1)M_s^{-1}(t,x;u_1,u_2)(D(f(X_s(t,x))), t^{-||u_2||/2}k_s(t,x;u_2))_H \tag{11}$$

for any $f \in \mathcal{C}$ and $u \in \tilde{A}$.

By Proposition 12 and Equation (11), we easily see the following.

Proposition 13 *For any* $p \in (1,\infty)$, *there is a constant* $C \in (0,\infty)$ *such that*

$$||(\Phi_s(u)f)(X_s(t,x))||_{W^{0,p}} \leqq C||f||_{\mathcal{D}^1_{(s)}},$$

and

$$||t^{||u||/2}(\Phi_s(u)f)(X_s(t,x))||_{W^{-1,p}} \leqq C||f||_\infty,$$

for any $u \in \tilde{A}$, $f \in \mathcal{C}$ *and* $s,t \in (0,1]$.

Proposition 14 *For any $\alpha \in [0,1)$ and $p \in (1,\infty)$, there is a constant $C \in (0,\infty)$ such that*

$$||f(X_s(t,x))||_{W^{-1,p}} \leqq Ct^{-\ell_0/2}||f||_{\mathcal{H}^{-\alpha}_{(s)}}$$

for any $f \in \mathcal{C}$, $s,t \in (0,1]$ and $x \in \mathbf{R}^N$.

Proof. Note that

$$f = Q^{(s)}_1 f - \int_0^1 L^{(s)} Q^{(s)}_r f dr = Q^{(s)}_1 f - \frac{1}{2}\sum_{u \in \tilde{A}} \Phi_s(r(u)) f_u,$$

where

$$f_u = \int_0^1 \Phi_s(r(u)) Q^{(s)}_t f dt.$$

By definition, we have

$$||Q^{(s)}_1 f||_\infty \leqq ||f||_{\mathcal{H}^{-\alpha}_{(s)}},$$

and

$$||f_u||_\infty \leqq \int_0^1 ||\Phi_s(r(u)) Q^{(s)}_{t/2} Q^{(s)}_{t/2} f||_\infty dt$$

$$\leqq C_0 \int_0^1 (\frac{t}{2})^{-1/2} ||Q^{(s)}_{t/2} f||_\infty dt \leqq C_0 (\int_0^1 (\frac{t}{2})^{-(1+\alpha)/2} dt) ||f||_{\mathcal{H}^{-\alpha}_{(s)}}.$$

Since

$$f(X_s(t,x)) = (Q^{(s)}_1 f)(X_s(t,x)) - \frac{1}{2}\sum_{u \in \tilde{A}} (\Phi_s(r(u)) f_u)(X_s(t,x)).$$

we have our assertion from Proposition 13. ∎

4 Main Lemma

For any $K = \{K_s\}_{s \in (0,1]} \in \hat{\mathcal{K}}_0(\mathbf{R})$, let $P^{s,K}_{(t)}$, $t > 0$, be linear operators defined in $\mathcal{C}$ given by

$$(P^{s,K}_{(t)} f)(x) = E[K_s(t,x) f_s(X_s(t,x)), \min_{r \in [0,t]} X^1_s(t,x) > 0], \qquad f \in \mathcal{C}.$$

Since $\min_{r \in [0,t]}(X^1_s(t,x)) = \min_{r \in [0,t]}(s^{1/2} B^1(t) + x^1)$ and it does not depend on $x^2, \ldots, x^N$, we see that $P^{s,K}_{(t)} f \in \mathcal{C}$ for any $f \in \mathcal{C}$ and $t \geqq 0$.

In this section, we prove the following.

Lemma 15 *For any $K_1, K_2 \in \hat{\mathcal{K}}_0(\mathbf{R})$, there is a $C \in (0,\infty)$ such that*

$$||P^{s,K_1}_{(t)} P^{s,K_2}_{(t)} \Phi_s(r(u)) f||_\infty \leqq Ct^{-\ell_0/2}||f||_\infty$$

for any $s,t \in (0,1]$, $f \in \mathcal{C}$ and $u \in \tilde{A}$.

We need some preparations to prove this lemma.

Proposition 16 *For any $K \in \hat{\mathcal{K}}_0(\mathbf{R})$, $\varepsilon \in (0,1)$ and $p \in (1/\varepsilon,\infty)$, there is a $C \in (0,\infty)$ such that*

$$||P^{s,K}_{(t)} f||_\infty \leqq C||f(X_s(t,x))||_{W^{-1+\varepsilon,p}}, \qquad s,t \in (0,1],\ f \in \mathcal{C}.$$

Proof. There is a $q \in (1,(1-\varepsilon)^{-1})$ and $r \in (1,\infty)$ such that $q^{-1}+r^{-1}+p^{-1}=1$. Then there is a $C_1 \in (0,\infty)$ such that

$$|P^{s,K}_{(t)} f(x)|$$

$$\leqq C_1 ||1_{(0,\infty)}(\min_{r\in[0,t]}(s^{-1/2}x^1 + B^1(t)))||_{W^{1-\varepsilon,q}} ||K_s(t,x)||_{W^{1,r}} ||f(X_s(t,x))||_{W^{-1+\varepsilon,p}}$$

for any $s,t \in (0,1]$, $x \in \mathbf{R}^N$ and $f \in \mathcal{C}$. So we have our assertion from Proposition 11.

∎

Proposition 17 *Let $K \in \hat{\mathcal{K}}_0(\mathbf{R})$. Then for any $\alpha \in (0,1)$ there is a $C \in (0,\infty)$ such that*

$$||P^{s,K}_{(t)} f||_\infty \leqq C t^{-\ell_0/2} ||f||_{\mathcal{H}^{-\alpha}_{(s)}}$$

for any $s,t \in (0,1]$ and $f \in \mathcal{C}$.

Proof. Let $\alpha \in (0,1)$. Then if we take a sufficiently small $\theta \in (0,1)$, there is a $\beta \in (0,1)$ such that $\alpha = (1-\theta)\beta - \theta$. Take an $\varepsilon \in (0,\theta)$. Then $-1+\varepsilon < -(1-\theta)$. Let us take a $p \in (1/\varepsilon,\infty)$.

First note that

$$||f(X_s(t,x))||_{W^{0,p}} \leqq ||f||_\infty \leqq ||f||_{\mathcal{D}^1_{(s)}}.$$

for any $s \in (0,1]$, $p \in (1,\infty)$ and $f \in \mathcal{C}$.

Also, by Proposition 14 there is a constant $C_1 \in (0,\infty)$ such that

$$||f(X_s(t,x))||_{W^{-1,p}} \leqq C_1 t^{-\ell_0/2} ||f||_{\mathcal{H}^{-\beta}_{(s)}}$$

for any $f \in \mathcal{C}$, $s,t \in (0,1]$ and $x \in \mathbf{R}^N$. Then by Propositions 7, 10, 12, and 13, we see that there are constants $C_2, C_3 \in (0,\infty)$ such that

$$||f(X_s(t,x))||_{W^{-1+\varepsilon,p}} \leqq C_2 \sup_{r\in(0,\infty)} r^{-\theta} K(r; f(X_s(t,x)); W^{-1,p}, W^{-0,p})$$

$$\leqq C_1 C_2 t^{-\ell_0/2} \sup_{r\in(0,\infty)} r^{-\theta} K(r; f; \mathcal{H}^{-\beta}_{(s)}, \mathcal{D}^1_{(s)}) \leqq C_3 t^{-\ell_0/2} ||f||_{\mathcal{H}^{-\alpha}_{(s)}}$$

for any $f \in \mathcal{C}$, $s,t \in (0,1]$ and $x \in \mathbf{R}^N$. Then by Proposition 16 we have our assetion.

∎

Now by Equations (8),(9) we have

$$(\Phi_s(r(u))P^{s,K}_{(t)} f)(x) = (P^{s,K_{00}(u)}_{(t)} f)(x) + \sum_{u'\in\tilde{A}} (P^{s,K_0(u;u')}_{(t)} \Phi_s(r(u'))f)(x) \tag{12}$$

and

$$(P^{s,K}_{(t)} \Phi_s(r(u))f)(x) = (P^{s,K_{10}(u)}_{(t)} f)(x) + \sum_{u'\in\tilde{A}} (\Phi_s(r(u')P^{s,K_1(u;u')}_{(t)} f)(x), \tag{13}$$

for any $u \in \tilde{A}$, $f \in \mathcal{C}$, $s,t \in (0,1]$ and $x \in \mathbf{R}^N$. Here

$$K_{00}(u)_s(t,x) = (\Phi_s(r(u))K_s(t,\cdot))|_{\cdot=x},$$

$$K_0(u;u')_s(t,x) = b_s(t,x;u,u')K_s(t,x), \qquad u' \in \tilde{A}.$$

$$K_{10}(u)_s(t,x) = -\sum_{u'\in\tilde{A}} (\Phi_s(r(u))(a_s(t,\cdot;u,u')K(t,\cdot))|_{\cdot=x},$$

and

$$K_1(u;u')_s(t,x) = a_s(t,x;u,u')K(t,x), \qquad u' \in \tilde{A}.$$

Also, note that by Equation (10)

$$(adj(\Phi_s(v_0))(P^{s,K}_{(t)})f)(x) = (\Phi_s(v_0)P^{s,K}_{(t)}f)(x) - (P^{s,K}_{(t)}\Phi_s(v_0)f)(x)$$

$$= (P^{s,\hat{K}_0}_{(t)}f)(x) + \sum_{u\in\tilde{A}}(P^{s,\hat{K}(u)}_{(t)}\Phi_s(r(u))f)(x) \tag{14}$$

for any $f \in \mathcal{C}$, $s,t \in (0,1]$ and $x \in \mathbf{R}^N$. Here

$$\hat{K}_{0s}(t,x) = (\Phi_s(v_0)K_s(t,\cdot))|_{\cdot=x},$$

$$\hat{K}(u)_s(t,x) = \hat{b}_s(t,x;u)K_s(t,x), \qquad u' \in \tilde{A}.$$

By Proposition 12, we see that $K_{00}(u)$, $K_0(u;u')$, $K_{10}(u)$, $K_1(u;u')$, $\hat{K}_0$, $\hat{K}(u) \in \hat{\mathcal{K}}_0(\mathbf{R})$ for any $u,u' \in \tilde{A}$.

Now let us prove Lemma 15.

Let $K_1, K_2 \in \hat{\mathcal{K}}_0(\mathbf{R})$. By Propositions 13, 17, and Equation (13) we see that for any $p \in (1,\infty)$ and $\alpha \in [0,1)$, there is a constant $C_1 \in (0,\infty)$ such that

$$||(P^{s,K_2}_{(t)}\Phi_s(r(u))f)(X_s(t,x))||_{W^{-1,p}} \leqq C_1 t^{-\ell_0/2}||f||_{\mathcal{H}^{-1/2}_{(s)}},$$

for any $u \in \tilde{A}$, $f \in \mathcal{C}$ and $s \in (0,1]$. It is obvious that for any $p \in (1,\infty)$, there is a constant $C > 0$ such that

$$||(P^{s,K_2}_{(t)}\Phi_s(r(u))f)(X_s(t,x))||_{W^{0,p}} \leqq ||f||_{\mathcal{D}^1_{(s)}},$$

for any $u \in \tilde{A}$, $f \in \mathcal{C}$, and $s \in (0,1]$.

Take an $\varepsilon \in (0,1/3)$. Then $-1+\varepsilon < -(1-1/3)$. Let us take a $p \in (1/\varepsilon,\infty)$. By Propositions 8 and 10, we see that there are constants $C_2, C_3 \in (0,\infty)$ such that

$$||(P^{s,K_2}_{(t)}\Phi_s(r(u))f)(X_s(t,x))||_{W^{-1+\varepsilon,p}}$$

$$\leqq C_2 t^{-\ell_0/2} \sup_{r\in(0,\infty)} r^{-1/3}K(r;(P^{s,K_2}_{(t)}\Phi_s(r(u))f)(X_s(t,x));W^{-1,p},W^{0,p})$$

$$\leqq C_2 t^{-\ell_0} \sup_{r\in(0,\infty)} r^{-1/3}K(r;f;\mathcal{H}^{-1/2}_{(s)},\mathcal{D}^1_{(s)}) \leqq C_3 t^{-\ell_0}||f||_\infty$$

for any $f \in \mathcal{C}$, $s,t \in (0,1]$ and $x \in \mathbf{R}^N$. Then by Proposition 16 we have Lemma 15.

This completes the proof of Lemma 15.

5 Proof of Theorem 3(1)

The following is an easy consequence of Lemma 15, Equations (12) and (13).

Corollary 18 *Let $K_1, K_2 \in \hat{\mathcal{K}}_0(\mathbf{R})$. Then for any $n \geqq 0$ there is a $C > 0$ such that*

$$\sum_{k=0}^{n+1}\sum_{u_1,\ldots,u_k\in\tilde{A}}||\Phi_s(r(u_1))\ldots\Phi_s(r(u_k))P^{s,K_1}_{(t)}P^{s,K_2}_{(t)}f||_\infty$$

$$\leqq Ct^{-\ell_0}\sum_{k=0}^{n}\sum_{u_1,\ldots,u_k\in\tilde{A}}||\Phi_s(r(u_1))\ldots\Phi_s(r(u_k))f)||_\infty$$

for any $s,t \in (0,1]$ and $f \in \mathcal{C}$.

For linear operators A and B in $\mathcal{C}$ we define $adj(A)^n(B)$, $n = 0, 1, \ldots,$ inductively by $adj(A)^0(B) = B$, and

$$adj(A)^n(B) = A(adj(A)^{n-1}(B)) - (adj(A)^{n-1}(B))A.$$

Then we see that for linear operators A, B, C in $\mathcal{C}$

$$adj(A)^n(BC) = \sum_{k=0}^{n} \binom{n}{k} adj(A)^k(B) adj(A)^{n-k}(C).$$

So by using Equations (12), (13) and (14) we have the following.

Lemma 19 *Let $n \geqq 0$ and $K_1, \ldots, K_{6n} \in \hat{\mathcal{K}}_0(\mathbf{R})$. Then there is a $C \in (0, \infty)$ such that*

$$\sum_{k,j,\ell=0}^{n} \sum_{u_1,\ldots,u_k \in \tilde{A}} \sum_{u'_1,\ldots,u'_\ell \in \tilde{A}} ||\Phi_s(r(u_1)\ldots r(u_k)) adj(\Phi_s(v_0))^j (P_{(t)}^{s,K_1} \cdots P_{(t)}^{s,K_{6n}}) \Phi_s(r(u'_1)\ldots r(u'_\ell)) f||_\infty$$

$$\leqq Ct^{-3n\ell_0} ||f||_\infty$$

for any $s, t \in (0, 1]$ and $f \in \mathcal{C}$.

Now we introduce the following notion.

Definition 20 *We say that $\{K_s\}_{s\in(0,1]} \in \hat{\mathcal{K}}_0(\mathbf{R})$ is multiplicative, if for any $m \geqq 1$ there are $n \geqq 1$ and $\{K_s^{ij}\} \in \hat{\mathcal{K}}_0(\mathbf{R})$, $i = 1, \ldots, n$, $j = 1, \ldots, m$, such that*

$$K_s(t_m, x, w)$$

$$= \sum_{i=1}^{n} K_s^{i,1}(t_1, x, w) K_s^{i2}(t_2 - t_1, X_s(t_1, x), \theta_{t_1} w) \cdots K_s^{i,m}(t_m - t_{m-1}, X_s(t_{m-1}, x), \theta_{t_{m-1}} w)$$

for any $s \in (0, 1]$ $0 < t_1 < \ldots < t_m$ and $x \in \mathbf{R}^N$.

Here $\theta_r : W_0 \to W_0$, $r \in [0, \infty)$, is given by $(\theta_r w)(t) = w(t + r) - w(r)$, $w \in W_0$, $t \in [0, \infty)$.

Proposition 21 *Let $\{K_s\}_{s\in(0,1]}, \{L_s\}_{s\in(0,1]} \in \hat{\mathcal{K}}_0(\mathbf{R})$ be multiplicative. Then $\{K_s + L_s\}_{s\in(0,1]}$ and $\{K_s L_s\}_{s\in(0,1]}$ are multiplicative.*

Proof. Let $m \geqq 2$. Since K_s and L_s are multiplicative, there are $n_1, n_2 \geqq 1$, $\{K_s^{ij}\} \in \hat{\mathcal{K}}_0(\mathbf{R})$, $i = 1, \ldots n_1$, $j = 1, \ldots, m$, and $\{L_s^{ij}\} \in \hat{\mathcal{K}}_0(\mathbf{R})$, $i = 1, \ldots n_2$, $j = 1, \ldots, m$, such that

$$K_s(t_n, x, w)$$

$$= \sum_{i=1}^{n_1} K_s^{i,1}(t_1, x, w) K_s^{i2}(t_2 - t_1, X_s(t_1, x), \theta_{t_1} w) \cdots K_s^{i,m}(t_m - t_{m-1}, X_s(t_{m-1}, x), \theta_{t_{m-1}} w),$$

and

$$L_s(t_n, x, w)$$

$$= \sum_{i=1}^{n_2} L_s^{i,1}(t_1, x, w) K_s^{i2}(t_2 - t_1, X_s(t_1, x), \theta_{t_1} w) \cdots L_s^{i,m}(t_m - t_{m-1}, X_s(t_{m-1}, x), \theta_{t_{m-1}} w),$$

for any $s \in (0, 1]$ $0 < t_1 < \ldots < t_m$ and $x \in \mathbf{R}^N$.

Then we have

$$K_s(t_n, x, w) + L_s(t_n, x, w)$$

$$= \sum_{i=1}^{n_1} K_s^{i,1}(t_1,x,w) K_s^{i2}(t_2-t_1, X_s(t_1,x), \theta_{t_1}w) \cdots K_s^{i,m}(t_m-t_{m-1}, X_s(t_{m-1},x), \theta_{t_{m-1}}w),$$

$$+ \sum_{i=1}^{n_2} L_s^{i,1}(t_1,x,w) K_s^{i2}(t_2-t_1, X_s(t_1,x), \theta_{t_1}w) \cdots L_s^{i,m}(t_m-t_{m-1}, X_s(t_{m-1},x), \theta_{t_{m-1}}w),$$

and

$$K_s(t_n,x,w) L_s(t_n,x,w)$$

$$= \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} (K_s^{i,1}(t_1,x,w) L_s^{i,1}(t_1,x,w))(K_s^{i,2}(t_2-t_1, X_s(t_1,x), \theta_{t_1}w) L_s^{j,2}(t_2-t_1, X_s(t_1,x), \theta_{t_1}w)))$$

$$\cdots (K_s^{i,m}(t_m-t_{m-1}, X_s(t_{m-1},x), \theta_{t_{m-1}}w) L_s^{j,m}(t_m-t_{m-1}, X_s(t_{m-1},x), \theta_{t_{m-1}}w))).$$

So we have our assertion.

∎

Proposition 22 *Let* $M \geqq 1$ *and* $d_s^{ijk} \in C_b^\infty(\mathbf{R}^N)$, $i,j = 1,\ldots,M$, $k = 0,1,\ldots,d$, $s \in (0,1]$. *and assume that*

$$\sup_{s\in(0,1]} \sup_{x\in\mathbf{R}^N} |\frac{\partial^{|\alpha|}}{\partial x^\alpha} d_s^{ijk}(x)| < \infty$$

for any $\alpha \in \mathcal{Z}_{\geqq 0}^N$.

Let $y^i \in \mathbf{R}$, *and* $Y_s^i(t,x)$, $i = 1,\ldots,M$, $s \in (0,1]$, $t \geqq 0$, $x \in \mathbf{R}^N$, *be the solution to the following SDE.*

$$Y_s^i(t,x) = y^i + \sum_{k=0}^d \sum_{\ell=1}^M \int_0^t d_s^{i\ell k}(X_s(r,x)) Y_s^\ell(r,x) \circ dB^k(r), \qquad i = 1,\ldots,M.$$

Then we see that $\{Y_s^i\}_{s\in(0,1]}$ *belongs to* $\hat{\mathcal{K}}_0$, *and is multiplicative for* $i,j = 1,\ldots,M$.

Also, $\{\int_0^t Y_s^i(r,x)dr\}$ *belongs to* $\hat{\mathcal{K}}_0$, *and is multiplicative.*

Proof. Let $E_s^{i,j}(t,x)$, $i,j = 1,\ldots,M$, $s \in (0,1]$, $t \geqq 0$, $x \in \mathbf{R}^N$, be the solution to the following SDE.

$$E_s^{i,j}(t,x) = \delta_{ij} + \sum_{k=0}^d \sum_{\ell=1}^M \int_0^t d_s^{i\ell k}(X_s(r,x)) E_s^{\ell,j}(r,x) \circ dB^k(r) \qquad i,j = 1,\ldots,M.$$

Then it is easy to see that $\{E_s^{i,j}\}_{s\in(0,1]} \in \hat{\mathcal{K}}_0$, and

$$Y_s^i(t,x) = \sum_{j=1}^M E_s^{i,j}(t,x) y_j.$$

Note that for $t_2 > t_1 \geqq 0$,

$$E_s^{ij}(t_2,x,w) = \sum_{\ell=1}^M E_s^{i\ell}(t_2-t_1, X(t_1,x,w), \theta_{t_1}w) E_s^{\ell j}(t_1,x,w), \quad i,j = 1,\ldots,M.$$

So we see that $\{E_s^{ij}\}_{s\in(0,1]}$, $i,j = 1,\ldots,M$, are multiplicative.

Also, we see that

$$\int_0^{t_2} E_s^{ij}(r,x,w)dr$$

$$= \int_0^{t_1} E_s^{ij}(r,x,w)dr + \sum_{\ell=1}^{M}(\int_0^{t_2-t_1} E_s^{i\ell}(r, X(t_2-t_1,x,w), \theta_{t_1}w)dr)E_s^{\ell j}(t_1,x,w), \ i,j=1,\ldots,M.$$

So we see that $\{\int_0^t E_s^{ij}(r,x)dr\}_{s\in(0,1]}$, $i,j=1,\ldots,M$, are multiplicative. These imply our assertion.

∎

Proposition 23 *Let* $\{K_s\}_{s\in(0,1]} \in \hat{\mathcal{K}}_0(\mathbf{R})$. *be multiplicative. Then* $\{\frac{\partial}{\partial x^i}K_s\}_{s\in(0,1]}$ *is multiplicative for any* $i=1,2,\ldots,N$.

Proof. Let $m \geqq 1$, and $0 < t_1 < \ldots < t_m$ and $x \in \mathbf{R}^N$. Then from the assumption there are $n \geqq 1$ and $\{K_s^{ij}\} \in \hat{\mathcal{K}}_0(\mathbf{R})$, $i=1,\ldots m$, $j=1,\ldots,n$, such that

$$K_s(t_m,x,w)$$

$$= \sum_{i=1}^{n} K_s^{i,1}(t_1,x,w)K_s^{i2}(t_2-t_1, X_s(t_1,x), \theta_{t_1}w)\cdots K_s^{i,m}(t_m-t_{m-1}, X_s(t_{m-1},x), \theta_{t_{m-1}}w).$$

Note that

$$X(t_{k+1},x) = X(t_{k+1}-t_k, X(t_k,x), \theta_{t_k}w), \qquad k=0,1,\ldots,m-1.$$

Here $t_0 = 0$. Then we have

$$\frac{\partial}{\partial x^j}X^i(t_{k+1},x)$$

$$= \sum_{\ell=1}^{N} \frac{\partial X^i}{\partial x^\ell}(t_{k+1}-t_k, X(t_k,x), \theta_{t_k}w)\frac{\partial X^\ell}{\partial x^j}(t_{k+1},x).$$

This implies that

$$\frac{\partial}{\partial x^j}X^i(t_{k+1},x)$$

$$= \sum_{\ell_k,\ell_{k-1},\ldots\ell_1=1}^{N} \frac{\partial X^{\ell_1}}{\partial x^j}(t_1,x)(\prod_{r=1}^{k-1}\frac{\partial X^{\ell_r}}{\partial x^{\ell_{r-1}}}(t_{r+1}-t_r, X(t_r,x), \theta_{t_r}w))\frac{\partial X^i}{\partial x^{\ell_k}}(t_{k+1}-t_k, X(t_k,x), \theta_{t_k}w).$$

Also, we see that

$$\frac{\partial}{\partial x^j}K_s(t_n,x,w)$$

$$= \sum_{k=1}^{n}\sum_{\ell=1}^{N}\sum_{i=1}^{m_1} K_s^{i,1}(t_1,x,w)K_s^{i2}(t_2-t_1, X_s(t_1,x), \theta_{t_1}w)\cdots K_s^{i,k-1}(t_n-t_{n-1}, X_s(t_{n-1},x), \theta_{t_{n-1}}w)$$

$$\times\frac{\partial K_s^{i,k}}{\partial x^\ell}(t_k-t_{k-1}, X_s(t_{k-1},x), \theta_{t_1}w)\frac{\partial X_s^\ell}{\partial x^j}(t_k-t_{k-1},x)$$

$$\times K_s^{i2}(t_2-t_1, X_s(t_1,x), \theta_{t_1}w)\cdots K_s^{i,n}(t_n-t_{n-1}, X_s(t_{n-1},x), \theta_{t_{n-1}}w).$$

These observation imply our assertion .

∎

We see that if $\{K_s\}_{s\in(0,1]} \in \hat{\mathcal{K}}_0(\mathbf{R})$ is multiplicative, then

$$P^{s,K}_{(nt)} = \sum_{i=1}^m P^{s,K_s^{i,1}}_{(t)} P^{s,K_s^{i,2}}_{(t)} \cdots P^{s,K_s^{i,n}}_{(t)},$$

where $\{K^{ij}_s\} \in \mathcal{K}_0(\mathbf{R})$, $i = 1, \dots m$, $j = 1, \dots, n$, are as in Definition 20.

So by Lemma 19 we have the following.

Theorem 24 *Suppose that $\{K_s\}_{s\in(0,1]} \in \hat{\mathcal{K}}_0(\mathbf{R})$ is multiplicative. Then for any $n, m, r \geqq 0$, and $u_1, \dots, u_{n+m} \in \tilde{A}$, there is a $C \in (0,\infty)$ such that*

$$||\Phi_s(u_1)\dots\Phi_s(u_n)(adj(\Phi_s(v_0))^r(P^{s,K}_{(t)})\Phi_s(u_{n+1})\dots\Phi_s(u_{n+m})f)||_\infty \leqq Ct^{-(n+m+r)\ell_0}||f||_\infty$$

for any $s, t \in (0,1]$ and $f \in \mathcal{C}$.

Now let us prove Theorem 3(1). Let $\rho_s(t,x)$ be the solution to the following SDE.

$$\rho_s(t,x)$$

$$= \exp(s^{1/2}\sum_{k=1}^d \int_0^t b_k(X_s(r,x))dB^k(r)) + s\int_0^t b_0(X_s(r,x))dB^0(r)), \quad x \in \mathbf{R}^N,\ t \geqq 0.$$

Then we see that

$$\rho_s(t,x) = 1 + s^{1/2}\sum_{k=1}^d \int_0^t b_k(X_s(r,x))\rho_s(r,x)\circ dB^k(r))$$

$$+s\int_0^t (b_0(X_s(r,x)) + \frac{1}{2}\sum_{k=1}^d b_k(X_s(r,x))^2)\rho_s(r,x)dB^0(r)).$$

So we see that $\{\rho_s\}_{s\in(0,1]} \in \hat{\mathcal{K}}_0$ and is multiplicative. Moreover, by using scale invariance of Wiener process, we can easily see that

$$P^0_s f(x) = E[\rho_s(1,x)f(X_s(1,x)), \min_{r\in[0,1]} X^1_s(r,x)) > 0] = (P^{s,\rho}_{(1)}f)(x)$$

for any $s \in (0,1]$, and $f \in C^\infty_b(\mathbf{R}^N)$.

This observation and Theorem 24 imply that for any $n, m, r \geqq 0$, $u_1, \dots, u_{n+m} \in \tilde{A}$ there is a $C \in (0,\infty)$ such that

$$s^{(||u_1||+\cdot+||u_{n+m}||)/2+r}||\Phi(r(u_1))\cdots\Phi(r(u_n))\ adj(V_0)^r(P^0_s)\Phi(r(u_{n+1}))\cdots\Phi(r(u_{n+m}))f||_\infty$$

$$\leqq C||f||_\infty$$

for any $s \in (0,1]$ and $f \in C^\infty_b$.

This proves Theorem 3 (1).

6 Dual Operators

Let $T \in (0,1]$, and $\hat{B}^k(w)(t) = B^k(T-t)$, $t \in [0,T]$, $k = 0,1,\ldots,d$. Also, let $\hat{X} : [0,T] \times \mathbf{R}^N \times W^d \to \mathbf{R}^N$ be the solution of the following SDE.

$$\hat{X}(t,x) = x - \sum_{k=0}^{d} \int_0^t V_k(\hat{X}(t,x)) \circ d\hat{B}^k(t), \qquad t \in [0,T],\ x \in \mathbf{R}^N.$$

We may assume that $\hat{X}(\cdot,*,w) : [0,T] \times \mathbf{R}^N \to \mathbf{R}^N$ is continuous for $\mu - a.s.$ Then we see that with probability one

$$X(t,x) = \hat{X}(T-t, X(T,x)), \qquad t \in [0,T],\ x \in \mathbf{R}^N$$

(c.f. Kunita [2]). So we see that for any $f, g \in C_0^\infty(\mathbf{R}^N)$

$$\int_{(0,\infty)\times\mathbf{R}^{N-1}} g(x)(P_T^0 f)(x)dx$$

$$= E^\mu[\int_{(0,\infty)\times\mathbf{R}^{N-1}} dx\ g(x) \exp(\sum_{k=0}^{d} \int_0^T b_k(X(r,x)) \circ dB^k(r))$$

$$\times f(X(T,x)) 1_{(0,\infty)}(\min_{r\in[0,T]} (y^1 + \hat{B}(r)))].$$

$$= E^\mu[\int_{(0,\infty)\times\mathbf{R}^{N-1}} dy\ g(\hat{X}(T,y)) \exp(-\sum_{k=0}^{d} \int_0^T b_k(\hat{X}(r,y)) \circ d\hat{B}^k(r)) f(y)$$

$$\times \det(\{\frac{\partial \hat{X}^i}{\partial y^j}(T,y)\}_{i,j=1,\ldots,N} 1_{(0,\infty)}(\min_{r\in[0,T]} (y^1 + \hat{B}(r)))].$$

Let $\bar{X} : [0,\infty) \times \mathbf{R}^N \times W^d \to \mathbf{R}^N$ be the solution of the following SDE.

$$\bar{X}(t,x) = x - \sum_{k=0}^{d} \int_0^t V_k(\bar{X}(t,x)) \circ dB^k(t), \qquad t \in [0,\infty),\ x \in \mathbf{R}^N.$$

Then we have

$$\int_{(0,\infty)\times\mathbf{R}^{N-1}} g(x)(P_T^0 f)(x)dx$$

$$= \int_{(0,\infty)\times\mathbf{R}^{N-1}} f(x) E^\mu[\exp(-\sum_{k=0}^{d} \int_0^T b_k(\bar{X}(r,x)) \circ dB^k(r)) \det(\bar{J}(T,x)) g(\bar{X}(T,x)),$$

$$\min_{r\in[0,T]} (x^1 + s^{1/2} B^1(r)) > 0].$$

Here

$$\bar{J}(t,x) = \{\bar{J}_j^i(t,x)\}_{i,j=1,\ldots,N}) = \{\frac{\partial \bar{X}_k^i}{\partial x^j}(t,x)\}_{i,j=1,\ldots,N}.$$

Since we have

$$d\bar{J}_i^j(t,x) = -\sum_{\ell=1}^{N} \sum_{k=0}^{d} \frac{\partial V_k^i}{\partial x^\ell}(\bar{X}(t,x)) \bar{J}_j^\ell(t,x) \circ dB^k(t),$$

we see that

$$d \det \bar{J}(t,x) = -\sum_{k=0}^{d} (div\ V_k)(\bar{X}(t,x)) \det \bar{J}(t,x) \circ dB^k(t),$$

where

$$div\ V_k(x) = \sum_{i=1}^{N} \frac{\partial V_k^i}{\partial x^i}(x), \qquad x \in \mathbf{R}^N.$$

So we have

$$\det \bar{J}(t,x) = \exp(-\sum_{k=0}^{d} \int_0^t (div\ V_k)(\bar{X}(r, \bar{X}(t,x)) \circ dB^k(r)).$$

Let $\bar{b}_k \in C_b^\infty(\mathbf{R}^N)$, $k = 0, 1, \ldots, d$, be given by

$$\bar{b}_k(x) = -b_k(x) - div\ V_k(x),$$

and let $\bar{P}_t^0$, $t \in [0, \infty)$ be a linear operator given by

$$(\bar{P}_t^0 f)(x)$$

$$= E^\mu[\exp(\sum_{k=0}^{d} \int_0^t \bar{b}_k(\bar{X}(r,x)) \circ dB^k(r)) f(\bar{X}(t,x)), \min_{r \in [0,t]} (x^1 - B^1(r)) > 0], \quad f \in C_b^\infty(\mathbf{R}^N).$$

Then we have

$$\int_{(0,\infty) \times \mathbf{R}^{N-1}} g(x)(P_t^0 f)(x) dx = \int_{(0,\infty) \times \mathbf{R}^{N-1}} f(x)(\bar{P}_t^0 g)(x) dx, \qquad t > 0, f, g \in C_0^\infty(\mathbf{R}^N).$$

Now let $\hat{X} : [0, \infty) \times \mathbf{R}^N \times W^d \to \mathbf{R}^N$ be the solution of the following SDE.

$$\hat{X}(t,x) = x + \sum_{k=1}^{d} \int_0^t V_k(\hat{X}(t,x)) \circ dB^k(t) - \int_0^t V_0(\bar{X}(t,x)) \circ dB^0(t) \qquad t \in [0,T],\ x \in \mathbf{R}^N.$$

Also, let $\tilde{b}_k \in C_b^\infty(\mathbf{R}^N)$, $k = 0, 1, \ldots, d$, be given by $\hat{b}_0 = \bar{b}_0$, and $\hat{b}_k = -\bar{b}_k$, $k = 1, \ldots, d$. Then we see that

$$(\bar{P}_t^0 f)(x)$$

$$= E^\mu[\exp(\sum_{k=0}^{d} \int_0^t \hat{b}_k(\hat{X}(r,x)) \circ dB^k(r)) f(\hat{X}(t,x)), \min_{r \in [0,t]} (x^1 + B^1(r)) > 0], \quad f \in C_b^\infty(\mathbf{R}^N).$$

Since a system of $\{-V_0, V_1, \ldots, V_d\}$ satisfies the assumptions (UFG), (A1) and (A2), we see by Theorem 24, that for any $n, m, r \geqq 0$, $u_1, \ldots, u_{n+m} \in \tilde{A}$, there is a $C \in (0, \infty)$ such that

$$t^{(||u_1|| + \cdot + ||u_{n+m}||)/2 + r} \sup_{x \in (0,\infty) \times \mathbf{R}^{N-1}} |(\Phi(r(u_1)) \cdots \Phi(r(u_n))\ adj(V_0)^r (\bar{P}_t^0)$$

$$\Phi(r(u_{n+1})) \cdots \Phi(r(u_{n+m})) f)(x)|$$

$$\leqq C \sup_{x \in (0,\infty) \times \mathbf{R}^{N-1}} |f(x)| \qquad t \in (0,1],\ f \in C_b^\infty(\mathbf{R}^N).$$

for any $t \in (0,1]$ and $f \in C_b^\infty$.

Let us denote by $\mathcal{D}_n$, $n \geqq 0$, the space of linear differential operators A in $\mathbf{R}^N$ such that there are $c_0 \in C_b^\infty(\mathbf{R}^N)$, $a_{u_1,\ldots,u_k} \in C_b^\infty(\mathbf{R}^N)$, $k \leqq n$, $u_1,\ldots,u_k \in A^{***}$, with $||u_1|| + \cdots + ||u_k|| \leqq n$, such that

$$(Af)(x) = c_0(x)f(x) + \sum_{k=1}^{n} \sum_{u_1,\ldots,u_k \in A^{***}, ||u_1||+\cdots+||u_k|| \leqq n} a_{u_1,\ldots,u_k}(x)(\Phi(r(u_1)\cdots r(u_k))f)(x),$$

for $x \in \mathbf{R}^N$ and $f \in C_b^\infty(\mathbf{R}^N)$.

It is easy to see the following.

Proposition 25 (1) *If $A \in \mathcal{D}_n$, and $B \in \mathcal{D}_m$, $n, m \geqq 0$, then $AB \in \mathcal{D}_{n+m}$.*
(2) *If $A \in \mathcal{D}_n$, $n \geqq 0$, then $[V_1, A] \in \mathcal{D}_{n+1}$, and $[V_0, A] \in \mathcal{D}_{n+2}$.*
(2) *If $A \in \mathcal{D}_n$, $n \geqq 0$, then a formal dual operator $A^* \in \mathcal{D}_n$.*

Also, we have the following by Proposition 24.

Proposition 26 *Let $n_i \geqq 0$, $i = 1, 2$, $m \geqq 0$, and $A_i \in \mathcal{D}_{n_i}$, $i = 1, 2$. Then there is a $C \in (0, \infty)$ such that*

$$\sup_{x \in (0,\infty)\times\mathbf{R}^{N-1}} |(A_1\, adj^m(V_0)(\bar{P}_t^0)A_2 f)(x)|$$
$$\leqq Ct^{-m-(n_1+n_2)/2)} \sup_{x \in (0,\infty)\times\mathbf{R}^{N-1}} |f(x)|.$$

for any $t \in (0, 1]$ and $f \in C_0^\infty(\mathbf{R}^N)$.

Note that if $W \in C_b^\infty(\mathbf{R}^N; \mathbf{R}^N)$ and if we regard W as a vector field over $\mathbf{R}^N$, then the formal adjoint operator W^* is given by

$$W^* = -W - \sum_{i=1}^{N} \frac{\partial W^i}{\partial x^i}.$$

Let $h \in C^\infty(\mathbf{R}^N)$ be given by $h(x) = x^1$, $x \in \mathbf{R}^N$. Note that if $Wh = 0$, we see that

$$\int_{(0,\infty)\times\mathbf{R}^{N-1}} g(x)(Wf)(x)dx = \int_{(0,\infty)\times\mathbf{R}^{N-1}} (W^*g)(x)f(x)dx$$

for any $f, g \in C_0^\infty(\mathbf{R}^N)$.

Then we have the following.

Proposition 27 *Let $m \geqq 0$. Then there are for any linear operator B in $\mathcal{C}$, there are $n_{m,k,i}, n'_{m,k,i} \geqq 0$, $k = 0,\ldots,m-1$, $i = 1,\ldots,5^m$, and $A_{m,k,i} \in \mathcal{D}_{n_{m,k,i}}$, $A'_{m,k,i} \in \mathcal{D}_{n'_{m,k,i}}$, $i = 1,\ldots,5^m$, such that $n_{m,k,i} + n'_{m,k,i} + 2k \leqq 2m$, $k = 0,\ldots,m-1$, $i = 1,\ldots,5^m$, and that*

$$adj(V_0^*)^m(B)$$
$$= (-1)^m adj(V_0)^m(B) + \sum_{k=0}^{m-1}\sum_{i=1}^{5^m} A_{m,k,i}\, adj(V_0)^k(B)A'_{m,k,i}.$$

Proof. It is obvious that our assertion is valid for $m = 0$. Note that

$$adj(V_0^*)^{m+1}(B)$$
$$= -adj(V_0)(adj(V_0^*)^m(B)) - (div\ V_0)(adj(V_0^*)^m(B)) + adj(V_0^*)^m(B)(div\ V_0)$$

So if our assertion is valid for m, we have

$$adj(V_0)(adj(V_0^*)^m(B))$$

$$= (-1)^m adj(V_0)^{m+1}(B) + \sum_{k=0}^{m-1}\sum_{i=1}^{5^m}(adj(V_0)(A_{m,k,i})adj(V_0)^k(B)A'_{m,k,i}$$

$$+\sum_{k=0}^{m-1}\sum_{i=1}^{5^m}A_{m,k,i}adj(V_0)^{k+1}(B)A'_{m,k,i} + \sum_{k=0}^{m-1}\sum_{i=1}^{5^m}A_{m,k,i}adj(V_0)^k(B)adj(V_0)(A'_{m,k,i}).$$

So we see that our assertion is valid for $m+1$. This completes the proof. ∎

Now let us prove Theorem 3 (2).

Let $n_i \geqq 0$, $i = 1, 2$, and $B_i \in \mathcal{D}_{n_i}, i = 1, 2$. Then we see that for $f, g \in C_0(\mathbf{R}^N)$

$$\int_{(0,\infty)\times\mathbf{R}^{N-1}} g(x)(B_1\ adj(V_0)^m(P_t^0)B_2 f)(x)dx$$

$$= \sum_{k=0}^{m}(-1)^k\binom{n}{k}\int_{(0,\infty)\times\mathbf{R}^{N-1}} g(x)(B_1V_0^kP_t^0V_0^{m-k}B_2 f)(x)dx$$

$$= \sum_{k=0}^{m}(-1)^k\binom{n}{k}\int_{(0,\infty)\times\mathbf{R}^{N-1}} ((V_0^*)^kB_1^*g)(x)(P_t^0V_0^{m-k}B_2 f)(x)dx$$

$$= \sum_{k=0}^{m}(-1)^k\binom{n}{k}\int_{(0,\infty)\times\mathbf{R}^{N-1}} (B_2^*(V_0^*)^{n-k}\hat{P}_t^0(V_0^*)^{m-k}B_1^*g)(x)f(x)dx$$

$$= (-1)^m\int_{(0,\infty)\times\mathbf{R}^{N-1}} (B_2^* adj(V_0^*)\hat{P}_t^0)B_1^*g)(x)f(x)dx.$$

So by Propositions **??**, **??**, we see that there is a $C \in (0, \infty)$ such that

$$|\int_{(0,\infty)\times\mathbf{R}^{N-1}} g(x)(B_1\ adj(V_0)^m(P_t^0)B_2 f)(x)dx|$$

$$\leqq Ct^{-m-(n_1+n_2)/2}(\sup_{x\in(0,\infty)\times\mathbf{R}^{N-1}}|g(x)|)(\int_{(0,\infty)\times\mathbf{R}^{N-1}}|f(x)|dx)$$

for any $t \in (0, 1)$, and $f, g \in C_0^\infty(\mathbf{R}^N)$. This implies that

$$\int_{(0,\infty)\times\mathbf{R}^{N-1}} |(B_1\ adj(V_0)^m(P_t^0)B_2 f)(x)|dx$$

$$\leqq Ct^{-m-(n_1+n_2)/2}(\int_{(0,\infty)\times\mathbf{R}^{N-1}}|f(x)|dx)$$

for any $t \in (0, 1)$, and $f \in C_0^\infty(\mathbf{R}^N)$.

This proves Theorem 3 (2).

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