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by

Tomoya KEMMOCHI

UNIVERSITY OF TOKYO
GRADUATE SCHOOL OF MATHEMATICAL SCIENCES
KOMABA, TOKYO, JAPAN

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Tomoya Kemmochi†

Abstract

Maximal regularity is a fundamental concept in the theory of nonlinear partial differential equations, for example, quasilinear parabolic equations, and the Navier-Stokes equations. It is thus natural to ask whether the discrete analogue of this notion holds when the equation is discretized for numerical computation. In this paper, we introduce the notion of discrete maximal regularity for the finite difference method (θ-method), and show that discrete maximal regularity is roughly equivalent to (continuous) maximal regularity for bounded operators. In addition, we show that this characterization is also true for unbounded operators in the case of the backward Euler method.

1. Introduction

We consider the following abstract Cauchy problem in a Banach space X:

$$\begin{cases} u'(t) = Au(t) + f(t), & t > 0, \\ u(0) = 0, \end{cases}$$

where u is an unknown X-valued function, f is a given one, and A is a linear operator on X. The operator A is said to have (continuous) *maximal regularity* if and only if, for some $p \in (1, \infty)$, and for every $f \in L^p(0, \infty; X)$, the above problem yields a unique solution u (the precise meaning of "solution" is described in Definition 3.4), satisfying

$$\|u'\|_{L^p(0,\infty;X)} + \|Au\|_{L^p(0,\infty;X)} \leq C\|f\|_{L^p(0,\infty;X)},$$

uniformly for f. For example, it is known that the Laplace operator and the Stokes operator have maximal regularity under suitable conditions, and that this property can

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†Graduate School of Mathematical Sciences, The University of Tokyo, Komaba 3-8-1, Meguro-ku, Tokyo, 153-8914, Japan. E-mail: kemmochi@ms.u-tokyo.ac.jp

be applied to quasilinear parabolic equations and the Navier-Stokes equations, respectively (cf. [3] and [20]). We are concerned about the numerical computation of the Cauchy problem above. As is well-known, the analytic semigroup theory and its discrete counterparts play important roles in construction and study of numerical schemes for parabolic equations (cf. [11], [12], [13], [18] and [19]). Hence, it is natural to ask whether a discrete analogue of maximal regularity holds when the above problem is discretized for numerical computations. Moreover, if this is the case, then it is expected that the discrete version of maximal regularity can be applied to the numerical analysis of nonlinear evolution equations, for example, the stability analysis and the error estimate of the finite element approximation for the equations as given above. Indeed, Geissert considered the continuous maximal regularity for the discrete Laplacian, and applied it to the semi-discrete problem of the linear and semilinear heat equation [14]. However, since Geissert dealt with only the semi-discrete problem, the results cannot be applied to the analysis of practical computations. Thus, we need to consider the time-discretized problem and the discrete version of maximal regularity.

In the present paper, we shall concentrate our attention to the discretization of the time variable, and postpone that of the space variables for further studies (cf. [15]). That is, we discretize the Cauchy problem by the finite difference method:

$$\begin{cases} \dfrac{u^{n+1}-u^n}{\tau} = Au^{n+\theta} + f^{n+\theta}, \quad n \in \mathbb{N}, \\ u^0 = 0, \end{cases}$$

where $\tau > 0$ is the time step, $\theta \in [0,1]$ is a fixed parameter, $u = (u^n)$ is an unknown X-valued sequence, and $f = (f^n)$ is a given one. We now summarize two previous works. For more details on these, one can refer to [1]. The first is Blunck's result [6]. Here the forward Euler method ($\theta = 0$) was considered, and the notion of discrete maximal regularity was introduced. The main result is to present the discrete version of the operator-valued Fourier multiplier theorem (cf. [21]) and to characterize discrete maximal regularity. While these results seem powerful, Blunck only considered the case where the time step is unity. Therefore, the result, especially the characterization of discrete maximal regularity, cannot be applied to numerical analysis straightforward. The second study we mention is that of Ashyralyev and Sobolevskiĭ [4]. They considered the backward Euler method ($\theta = 1$), with an arbitrary time step $\tau > 0$. However, they neither provided reasonable sufficient conditions for discrete maximal regularity, nor considered applications to numerical analysis. In contrast to these results, we introduce the notion of discrete maximal regularity for the general θ-method defined above (Definition 3.9), as we intend to apply discrete maximal regularity to the numerical analysis of nonlinear evolution equations.

The aim of this paper is to establish a reasonable condition for discrete maximal regularity. Our main theorem (Theorem 4.2) says that continuous maximal regularity implies the discrete version in a UMD space, under suitable conditions. We reduce the problem of discrete maximal regularity to the boundedness of the Fourier multiplier. The boundedness is deduced from the R-boundedness (Definition 2.1) of certain sets of operators, via the discrete version of Mikhlin multiplier theorem (Theorem 3.12). Many operators exist that have continuous maximal regularity. Furthermore, many approaches to continuous maximal regularity have been already studied. Therefore, even if we do not know whether a given operator has maximal regularity, this can be investigated. As a result, our sufficient condition is quite reasonable from both analytical and numerical viewpoints.

We also mention the dependence of constants appearing in estimates. From our main theorem, we can obtain an a priori estimate for the solution of the difference equation. However, the constant in this estimate may depend on the UMD space X. This is a delicate problem since X is expected to be a finite dimensional space with discretization parameters, for example, a finite element space. It may occur that the constant in the a priori estimate depends on the discretization parameter of the space variables, when we consider the finite element method. Therefore, we need to estimate the constants carefully, and we establish an applicable version of the main theorem (Corollary 4.4).

As mentioned above, we restrict our consideration within the time-discrete Cauchy problem. We will apply the results of this paper to the stability and convergence analysis of the finite element method to linear and semilinear heat equations in a forthcoming paper [15].

The plan for the rest of this paper is as follows. Section 2 and Section 3 are preliminary sections. In Section 2, we introduce the notion of R-boundedness. This plays an important role in operator-valued Fourier multiplier theorems, in both the continuous and discrete cases. Although we list some lemmas on R-boundedness, we omit most of the proofs. Section 3 is devoted to continuous and discrete maximal regularity. In this section, we start by giving the definition of a UMD space, which is a "good" Banach space. Blunck's result and our result are described for UMD spaces. Subsequently, we deal with continuous maximal regularity in subsection 3.1, and the discrete version in subsection 3.2. Our main result is described in Section 4. We also demonstrate that the opposite assertion of the main theorem holds. That is, discrete maximal regularity implies continuous maximal regularity. We conclude this paper by dealing with some additional topics in Section 5. In this section, we focus on the backward Euler method. In this case, we can show more analogous properties than in previous sections. This section consists of two parts. First, we consider the characterization of discrete maximal regularity for unbounded operators (subsection 5.1). This result corresponds to the one given by Blunck, which deals with bounded operators. Next, we obtain an a priori estimate for non-zero initial values (subsection 5.2). In the theory of nonlinear evolution equations, the choice of initial values is important. We need to obtain an a priori estimate with general initial values. Our result in this subsection is applicable to the numerical analysis of nonlinear equations.

We will use the following notation. The set of natural numbers is denoted by $\mathbb{N} = \{0, 1, 2, \dots\}$. That of positive (resp. negative) numbers is $\mathbb{R}^+ = (0, \infty)$ (resp. $\mathbb{R}^- = (-\infty, 0)$). The symbol $\mathbb{T}$ denotes the one-dimensional torus $\{z \in \mathbb{C} \mid |z| = 1\}$. Let us denote an open disk in $\mathbb{C}$ by $\mathbb{D}(a; r) = \{z \in \mathbb{C} \mid |z - a| < r\}$ for $a \in \mathbb{C}$ and $r > 0$. We write $\mathbb{D} = \mathbb{D}(0; 1)$ as an abbreviation. We set $\Sigma_\delta = \{z \in \mathbb{C} \setminus \{0\} \mid |\arg z| < \delta\}$ for $\delta \in (0, \pi]$, to express an open sector domain in $\mathbb{C}$. In particular, we write $\mathbb{H} = \Sigma_{\pi/2}$, which corresponds to the right half plain $\{z \in \mathbb{C} \mid \operatorname{Re} z > 0\}$.

For a Banach space X and an index set Λ, $X^\Lambda = \{(x^i)_{i\in\Lambda} \mid x^i \in X,\ \forall i \in \Lambda\}$ is the space of sequences valued in X. We write $c_{00}(\mathbb{Z}; X) \subset X^{\mathbb{Z}}$ to express the space of sequences with compact support, which is dense in $l^p(\mathbb{Z}; X)$ for each $p \in [1, \infty)$. For two Banach spaces X and Y, we denote the space of bounded operators from X to Y by $\mathcal{L}(X, Y)$, and we write $\mathcal{L}(X) = \mathcal{L}(X, Y)$.

2. R-boundedness

R-boundedness is a fundamental concept in this paper since it plays a crucial role in Weis's operator-valued Fourier multiplier theorem on UMD spaces [21, Theorem 3.4], as well as in its discrete version [6, Theorem 1.3]. In this section, we provide its definition (Definition 2.1), and recall some properties for later use. The lemmas stated below are given in [21], [7], [6], and references therein.

Let X and Y be Banach spaces with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively. We abbreviate the norms as $\|\cdot\|$ when no confusion can arise. Let $\{r_j\}_{j\in\mathbb{N}}$ be a sequence of independent and symmetric $\{\pm1\}$-valued random variables on $[0, 1]$. For example, the Rademacher functions $r_j(t) = \operatorname{sign}[\sin(2^{j+1}\pi t)]$.

Definition 2.1 (R-boundedness)**.** A set $\mathcal{T} \subset \mathcal{L}(X, Y)$ is said to be **R-bounded** if there exists a constant $C > 0$, such that

$$\int_0^1 \left\| \sum_{j=0}^n r_j(t) T_j x_j \right\|_Y dt \le C \int_0^1 \left\| \sum_{j=0}^n r_j(t) x_j \right\|_X dt \tag{2.1}$$

for all $n \in \mathbb{N}$, $x_0, \ldots, x_n \in X$, and $T_0, \ldots, T_n \in \mathcal{T}$. The infimum of the constant $C > 0$ satisfying (2.1) is called the **R-bound** of $\mathcal{T}$ and is denoted by $R(\mathcal{T})$.

Remark 2.2. (i) By the independence of $\{r_j\}$, the condition (2.1) is equivalent to the following one. There exists a constant $C > 0$, such that

$$\sum_{s\in\{\pm1\}^{n+1}} \left\| \sum_{j=0}^n s_j T_j x_j \right\|_Y \le C \sum_{s\in\{\pm1\}^{n+1}} \left\| \sum_{j=0}^n s_j x_j \right\|_X$$

for all $n \in \mathbb{N}$, $x_0, \ldots, x_n \in X$, and $T_0, \ldots, T_n \in \mathcal{T}$, where $s \in \{\pm1\}^{n+1}$ means that $s = (s_0, \ldots, s_n)$ is an $(n + 1)$-dimensional vector whose components are 1 or -1.

(ii) By Kahane's inequality, the condition (2.1) is equivalent to the following one. There exists $p \in [1, \infty)$ and $C > 0$, such that

$$\int_0^1 \left\| \sum_{j=0}^n r_j(t) T_j x_j \right\|_Y^p dt \le C^p \int_0^1 \left\| \sum_{j=0}^n r_j(t) x_j \right\|_X^p dt \tag{2.2}$$

for all $n \in \mathbb{N}$, $x_0, \ldots, x_n \in X$, and $T_0, \ldots, T_n \in \mathcal{T}$. If we denote the infimum of the constant $C > 0$ satisfying (2.2) by $R_p(\mathcal{T})$, then there exists $C_p > 0$, depending only on p, such that

$$C_p^{-1} R(\mathcal{T}) \le R_p(\mathcal{T}) \le C_p R(\mathcal{T}). \tag{2.3}$$

Here, the assertion of Kahane's inequality is that for every $p, q \in [1, \infty)$, there exists $C_{p,q} > 0$ such that

$$\left(\int_0^1 \left\| \sum_{j=0}^{n} r_j(t) x_j \right\|_X^p dt \right)^{1/p} \leq C_{p,q} \left(\int_0^1 \left\| \sum_{j=0}^{n} r_j(t) x_j \right\|_X^q dt \right)^{1/q} \tag{2.4}$$

for all Banach spaces X, $n \in \mathbb{N}$, and $x_0, \ldots, x_n \in X$. For the proof of (2.4), see, for example, [8, Theorem 11.1].

Lemma 2.3. *Let $\mathcal{S}, \mathcal{T} \subset \mathcal{L}(X, Y)$ be two R-bounded subsets. Then, the following statements hold.*

(i) *If $\mathcal{S} \subset \mathcal{T}$, then $R(\mathcal{S}) \leq R(\mathcal{T})$.*

(ii) *The closure $\overline{\mathcal{T}}$ in $\mathcal{L}(X, Y)$ is also R-bounded, and we have*

$$R(\overline{\mathcal{T}}) = R(\mathcal{T}).$$

(iii) *The union $\mathcal{S} \cup \mathcal{T}$ and the sum $\mathcal{S} + \mathcal{T}$ are also R-bounded, with bounds*

$$R(\mathcal{S} \cup \mathcal{T}) \leq R(\mathcal{S}) + R(\mathcal{T}), \quad R(\mathcal{S} + \mathcal{T}) \leq R(\mathcal{S}) + R(\mathcal{T}).$$

(iv) *Assume that $Y = X$, and set $\mathcal{S}\mathcal{T} = \{ST \mid S \in \mathcal{S},\ T \in \mathcal{T}\}$. Then, $\mathcal{S}\mathcal{T}$ is R-bounded, and we have*

$$R(\mathcal{S}\mathcal{T}) \leq R(\mathcal{S}) R(\mathcal{T}).$$

In particular, $\mathcal{T}^n = \{T^n \mid T \in \mathcal{T}\}$ is R-bounded, with the bound

$$R(\mathcal{T}^n) \leq R(\mathcal{T})^n$$

for $n \in \mathbb{N}$.

(v) *Let $p \in [1, \infty)$ and let (Ω, μ) be a σ-finite measure space. For $T \in \mathcal{L}(X, Y)$, we define $\tilde{T} \in \mathcal{L}(L^p(\Omega; X), L^p(\Omega; Y))$ as*

$$(\tilde{T} f)(x) = T(f(x)), \quad f \in L^p(\Omega; X), \quad x \in \Omega$$

and we set $\tilde{\mathcal{T}} = \{\tilde{T} \mid T \in \mathcal{T}\}$. Then, $\tilde{\mathcal{T}} \subset \mathcal{L}(L^p(\Omega; X), L^p(\Omega; Y))$ is R-bounded with the bound

$$R(\tilde{\mathcal{T}}) \leq C_p^2 R(\mathcal{T}),$$

where $C_p > 0$ is the constant in (2.3).

(vi) *We denote the convex hull of $\mathcal{T}$ by $\mathrm{CH}(\mathcal{T})$. Then, $\mathrm{CH}(\mathcal{T})$ is R-bounded, with the bound*

$$R(\mathrm{CH}(\mathcal{T})) \leq R(\mathcal{T}).$$

(vii) *We denote the real and complex absolute convex hulls of* $\mathcal{T}$ *by* $\mathrm{ACH}_{\mathbb{R}}(\mathcal{T})$ *and* $\mathrm{ACH}_{\mathbb{C}}(\mathcal{T})$*, respectively. Then,* $\mathrm{ACH}_{\mathbb{R}}(\mathcal{T})$ *and* $\mathrm{ACH}_{\mathbb{C}}(\mathcal{T})$ *are R-bounded, with the bounds*

$$R(\mathrm{ACH}_{\mathbb{R}}(\mathcal{T})) \leq R(\mathcal{T}), \quad R(\mathrm{ACH}_{\mathbb{C}}(\mathcal{T})) \leq 2R(\mathcal{T}).$$

Lemma 2.3 provides us with some basic and important examples.

Example 2.4. (i) Let $T \in \mathcal{L}(X, Y)$. Then, the set $\{T\}$ is R-bounded and

$$R(\{T\}) = \|T\|,$$

where the right hand side is the operator norm of T.

(ii) Assume that $Y = X$, and let $\Lambda \subset \mathbb{R}$ with $\Lambda \subset [-M, M]$ for some $M > 0$. If we set $T_\Lambda = \{\lambda I \mid \lambda \in \Lambda\}$, then T_Λ is R-bounded with the bound

$$R(T_\Lambda) \leq M,$$

where I is the identity operator on X.

(iii) Assume that $Y = X$, and let $\Lambda \subset \mathbb{C}$ with $\Lambda \subset \{z \in \mathbb{C} \mid |z| \leq M\}$ for some $M > 0$. If we set T_Λ as above, then T_Λ is R-bounded with the bound

$$R(T_\Lambda) \leq 2M.$$

The examples given above imply the following lemma.

Lemma 2.5 ([6, Remark 3.3])**.** *Let* (Ω, μ) *be a* σ*-finite measure space,* E *be a Lebesgue space* $L^p(\Omega; X)$ *for* $p \in [1, \infty)$*, and* $\mathcal{T} \subset \mathcal{L}(E)$ *be an R-bounded set. For* $f \in L^\infty(\Omega)$*, we denote the multiplication operator on* E *by* S_f*. Then, the set*

$$\mathcal{S} = \{S_g T S_h \mid T \in \mathcal{T},\ g, h \in L^\infty(\Omega),\ \|g\|_{L^\infty(\Omega)} \leq 1,\ \|h\|_{L^\infty(\Omega)} \leq 1\}$$

is R-bounded, with the bound

$$R(\mathcal{S}) \leq C R(\mathcal{T}),$$

where C *depends only on* p*.*

The next lemma is the result of a simple application of Lemma 2.3. Recall that $\mathbb{D}$ is the open unit disk in $\mathbb{C}$, and $\mathbb{H}$ is the right half plain $\{z \in \mathbb{C} \mid \operatorname{Re} z > 0\}$.

Lemma 2.6. (i) *Let* $N \colon \overline{\mathbb{D}} \to \mathcal{L}(X, Y)$ *be a function that is bounded in* $\overline{\mathbb{D}}$ *and analytic in* $\mathbb{D}$*. Assume that* $N(\partial\mathbb{D})$ *is R-bounded. Then, the set* $N(\overline{\mathbb{D}})$ *is also R-bounded, and*

$$R(N(\overline{\mathbb{D}})) = R(N(\partial\mathbb{D})).$$

(ii) *Let $N\colon \overline{\mathbb{H}} \to \mathcal{L}(X,Y)$ be a function that is bounded in $\overline{\mathbb{H}}$ and analytic in $\mathbb{H}$. Assume that $N(\partial\mathbb{H}) = N(i\mathbb{R})$ is R-bounded. Then, the set $N(\overline{\mathbb{H}})$ is also R-bounded, and*

$$R(N(\overline{\mathbb{H}})) = R(N(i\mathbb{R})).$$

We conclude this section by introducing some lemmas concerning series of operators.

Lemma 2.7 ([21, Lemma 2.4])**.** *Let S be a set and $T_n\colon S \to \mathcal{L}(X,Y)$ be a map for each $n \in \mathbb{N}$. Assume that $T_n(S)$ is R-bounded for every n, and that*

$$\sum_{n=0}^{\infty} R(T_n(S)) < \infty,$$

so that the series of operators

$$T(s) = \sum_{n=0}^{\infty} T_n(s)$$

is well-defined in $\mathcal{L}(X,Y)$ for all $s \in S$. Then, $T(S)$ is R-bounded with the bound

$$R(T(S)) \le \sum_{n=0}^{\infty} R(T_n(S)).$$

Lemma 2.8 ([6, Lemma 3.4])**.** *Let $\mathcal{T} \subset \mathcal{L}(X,Y)$ be an R-bounded set with $R(\mathcal{T}) > 0$. For $K > 0$ and $\alpha \in (0,1)$, we set*

$$\mathcal{A} = \left\{ a = (a_n) \in l^{\infty}(\mathbb{N};\mathbb{C}) \;\middle|\; |a_n| \le K \left(\frac{\alpha}{R(\mathcal{T})} \right)^n \right\},$$
$$\mathcal{M} = \left\{ \sum_{n=0}^{\infty} a_n T^n \;\middle|\; (a_n) \in \mathcal{A},\ T \in \mathcal{T} \right\}.$$

Then, $\mathcal{M}$ is well-defined in $\mathcal{L}(X,Y)$, and R-bounded with the bound

$$R(\mathcal{M}) \le \frac{2K}{1-\alpha}.$$

Lemma 2.8 yields an important property of sectorial operators. Recall that Σ_δ is an open sector domain $\{z \in \mathbb{C} \setminus \{0\} \mid |\arg z| < \delta\}$ for $\delta \in (0,\pi)$. We prove the following property here since the assertion is slightly different from the original [6, Corollary 3.5].

Corollary 2.9. *Let A be a closed and densely defined linear operator on X. Assume that there exists $\delta \in (0,\pi/2)$ such that $\Sigma_{\pi/2+\delta} \subset \rho(A)$, and set $\mathcal{T}_\theta = \{\lambda R(\lambda;A) \mid \lambda \in \Sigma_\theta\}$ for $\theta \in (0,\pi/2+\delta)$. Now, if $\mathcal{T}_{\pi/2}$ is R-bounded, then $\mathcal{T}_{\pi/2+\delta_0}$ is also R-bounded for each δ_0 satisfying*

$$0 < \delta_0 < \min\left\{ \delta,\ \arctan \frac{1}{R(\mathcal{T}_{\pi/2})} \right\}. \tag{2.5}$$

Moreover, we have

$$R(\mathcal{T}_{\pi/2+\delta_0}) \le P_1(R(\mathcal{T}_{\pi/2})), \tag{2.6}$$

where
$$P_1(X) = \frac{2}{1-\alpha}\left(1 + \frac{X}{\alpha}\right) + X \tag{2.7}$$
is a polynomial of degree one, with $\alpha = R(\mathcal{T}_{\pi/2})\tan\delta_0$.

Proof. By assumption, the set
$$\mathcal{T} = \{\lambda R(\lambda; A) \mid \lambda \in \overline{\Sigma_{\pi/2}} \setminus \{0\}\}$$
is well-defined ($\overline{\Sigma_{\pi/2}}$ is the closure in $\mathbb{C}$) and R-bounded with $R(\mathcal{T}) = R(\mathcal{T}_{\pi/2})$ since $\mathcal{T}_{\pi/2} \subset \mathcal{T} \subset \overline{\mathcal{T}_{\pi/2}}$. Take δ_0 as in (2.5) and set $\alpha = R(\mathcal{T})\tan\delta_0$ and $K = \sqrt{1 + (R(\mathcal{T})/\alpha)^2}$. Note that $\alpha \in (0, 1)$. For $\mathcal{T}$, α, and K, we define $\mathcal{A}$ and $\mathcal{M}$ as in Lemma 2.8. We will prove later that
$$s, t \in \mathbb{R}, \quad t \neq 0, \quad \frac{|s|}{|t|} \leq \frac{\alpha}{R(\mathcal{T})} \implies (s + it)R(s + it; A) \in \mathcal{M}. \tag{2.8}$$
Once (2.8) is obtained, our assertion can be established. Indeed, since we have
$$\frac{\alpha}{R(\mathcal{T})} = \tan\delta_0,$$
it follows that the set
$$\tilde{\mathcal{T}} = \left\{\lambda R(\lambda; A) \;\middle|\; \frac{\pi}{2} - \delta_0 < |\arg\lambda| < \frac{\pi}{2} + \delta_0\right\}$$
is contained in $\mathcal{M}$ by (2.8) and $\tilde{\mathcal{T}}$ is R-bounded by Lemma 2.8, with the bound
$$R(\tilde{\mathcal{T}}) \leq \frac{2K}{1-\alpha} \leq \frac{2}{1-\alpha}\left(1 + \frac{R(\mathcal{T}_{\pi/2})}{\alpha}\right).$$
Then, owing to the fact that $\mathcal{T}_{\pi/2+\delta_0} = \mathcal{T}_{\pi/2} \cup \tilde{\mathcal{T}}$, we can obtain (2.6).

Now, we prove (2.8). We remark that by the R-boundedness of $\mathcal{T}_{\pi/2}$, A satisfies the estimate
$$\|R(\lambda; A)\| \leq \frac{R(\mathcal{T}_{\pi/2})}{|\lambda|} \quad \text{for} \quad \lambda \in \overline{\Sigma_{\pi/2}} \setminus \{0\}.$$
We fix $t \in \mathbb{R}$, and assume that $|s| \leq |t|\alpha/R(\mathcal{T})$. Then, $F(s) = R(s+it; A)$ is well-defined. Since $F^{(n)}(s) = (-1)^n n! R(s + it; A)^{n+1}$, and
$$\|sR(it; A)\| \leq |s|\frac{R(\mathcal{T}_{\pi/2})}{|t|} \leq \alpha < 1,$$
the function F can be expanded in a Taylor series as
$$F(s) = \sum_{n=0}^{\infty}(-s)^n R(it; A)^{n+1},$$

so that we have

$$(s+it)R(s+it;A) = \sum_{n=0}^{\infty} \frac{(s+it)(-s)^n}{(it)^n} [itR(it;A)]^{n+1}. \tag{2.9}$$

Now, we set $a_n = (s+it)(-s)^n/(it)^n$. Then, through an elementary calculation, we can obtain

$$|a_n| \leq K \left(\frac{\alpha}{R(\mathcal{T})} \right)^n \tag{2.10}$$

for all $n \in \mathbb{N}$. Noting that $itR(it;A) \in \mathcal{T}$, we can deduce (2.8) from (2.9) and (2.10), which results in the desired conclusion. ■

3. Maximal regularity and discrete maximal regularity

In this section, we introduce main notions and some properties relating to continuous maximal regularity and discrete maximal regularity. We shall prove some of these where necessary. For the proofs omitted in the following, and for more details, see [9], [21], and [6].

Throughout this section, X denotes a Banach space with a norm $\|\cdot\|$. We begin by giving the definition of UMD spaces, which is essential for Mikhlin-type multiplier theorems.

Definition 3.1 (UMD space)**.** A Banach space X is called a **UMD space** if the Hilbert transform H, defined on $\mathcal{S}(\mathbb{R};X)$, can be extended to a bounded operator on $L^p(\mathbb{R};X)$ for some $p \in (1,\infty)$. Here, $\mathcal{S}(\mathbb{R};X)$ is the space of rapidly-decreasing X-valued functions, and H is defined as

$$Hu(t) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{u(s)}{t-s} ds, \quad t \in \mathbb{R}, \quad u \in \mathcal{S}(\mathbb{R};X),$$

where the integral is the principal value.

The name "UMD" is derived from *unconditionally martingale differences*. We now collect some important properties and examples of UMD spaces. For the proofs, see [2, section III.4] and references therein.

Proposition 3.2. (i) *If X is a UMD space, then the Hilbert transform on X is bounded on $L^p(\mathbb{R};X)$ for any $p \in (1,\infty)$.*

(ii) *Let X be a UMD space and Y be a Banach space. If Y is isomorphic to X, then Y is also a UMD space.*

(iii) *Every Hilbert space is a UMD space.*

(iv) *Every finite-dimensional Banach space is a UMD space.*

(v) *If two Banach spaces X and Y are UMD spaces, then the product space $X \times Y$ is a UMD space.*

(vi) *If X is a UMD space, then the dual space X^* is one as well.*

(vii) *If X is a UMD space and (Ω, μ) is a σ-finite measure space, then $L^p(\Omega; X)$ is a UMD space for $p \in (1, \infty)$.*

(viii) *Let X be a UMD space and $M \subset X$ be a closed subspace. Then, M itself, and the quotient space X/M, are UMD spaces.*

Example 3.3. Let (Ω, μ) be a σ-finite measure space.

(i) The one-dimensional space $\mathbb{C}$ is a UMD space.

(ii) The usual Lebesgue space $L^q(\Omega) = L^q(\Omega; \mathbb{C})$ is a UMD space for $q \in (1, \infty)$.

(iii) The Lebesgue space $L^p(\mathbb{R}^+; L^q(\Omega))$ is a UMD space for $p, q \in (1, \infty)$.

(iv) Suppose that Ω is an open subset of $\mathbb{R}^d$. Then, the Sobolev space $W^{m,q}(\Omega)$ is a UMD space for $m \in \mathbb{N}$ and $q \in (1, \infty)$, since it is isomorphic to a closed subspace of m-products of $L^q(\Omega)$.

3.1. Maximal regularity

We consider the abstract Cauchy problem in X:

$$\begin{cases} u'(t) = Au(t) + f(t), \quad t > 0, \\ u(0) = 0, \end{cases} \tag{3.1}$$

where $f\colon \mathbb{R}^+ \to X$ is a given function, $u\colon \mathbb{R}^+ \to X$ is the unknown, and A is a linear operator on X with a domain $D(A)$.

Definition 3.4 (maximal L^p-regularity)**.** Suppose that $p \in (1, \infty)$. A linear operator A has **maximal L^p-regularity** if, for every $f \in L^p(\mathbb{R}^+; X)$, (3.1) has a unique solution u, which fulfills the following properties:

(i) $u(t) \in D(A)$ for almost every $t > 0$,

(ii) u is strongly differentiable in X for almost every $t > 0$,

(iii) there exists a constant $C > 0$, which is independent of $f \in L^p(\mathbb{R}^+; X)$, such that

$$\|u'\|_{L^p(\mathbb{R}^+;X)} + \|Au\|_{L^p(\mathbb{R}^+;X)} \le C\|f\|_{L^p(\mathbb{R}^+;X)}. \tag{3.2}$$

In Definition 3.4, we do not require that $u \in L^p(\mathbb{R}^+; X)$. However, in the case where $0 \in \rho(A)$, maximal L^p-regularity implies that $u \in L^p(\mathbb{R}^+; X)$, since $\|A \cdot\|$ is an equivalent norm to $\| \cdot \|$ in X.

Proposition 3.5. *If a linear operator A has maximal L^{p_0}-regularity for some $p_0 \in (1, \infty)$, then A has maximal L^p-regularity for every $p \in (1, \infty)$.*

For the proof, see [9, Theorem 4.2]. By virtue of Proposition 3.5, we say that A has **maximal regularity** if A has maximal L^p-regularity for some $p \in (1, \infty)$. In order to distinguish this with the discrete case below, we occasionally call this term as continuous maximal regularity.

The next proposition presents a well-known necessary condition for maximal regularity [9, Theorem 2.1].

Proposition 3.6. *If a linear operator A has maximal regularity, then A generates a bounded and analytic semigroup on X.*

As a sufficient condition for maximal regularity, the result of Dore and Venni [10] is well-known. On the other hand, in [21], Weis *characterized* maximal regularity by the R-boundedness of some sets of operators.

Theorem 3.7 ([21, Corollary 4.4])**.** *Let X be a UMD space and $T(t)$ be a bounded and analytic semigroup on X, with the generator A. Then, the following conditions are equivalent:*

(a) *The operator A has maximal regularity.*

(b) *The set $\{\lambda R(\lambda; A) \mid \lambda \in i\mathbb{R} \setminus \{0\}$ is R-bounded.*

(c) *The sets $\{T(t) \mid t > 0\}$ and $\{tAT(t) \mid t > 0\}$ are R-bounded.*

3.2. Discrete maximal regularity

We next discretize the notion of maximal regularity. First, we need consider the discrete problem for (3.1). In this paper, we use the finite difference method to discretize the time variable. That is, we consider the discrete Cauchy problem in X:

$$\begin{cases} \dfrac{u^{n+1} - u^n}{\tau} = Au^{n+\theta} + f^{n+\theta}, \quad n \in \mathbb{N}, \\ u^0 = 0, \end{cases} \tag{3.3}$$

where $\tau > 0$ is the time step, $\theta \in [0, 1]$ is a fixed parameter, $f = (f^n)_n \in X^{\mathbb{N}}$ is a given sequence, $u = (u^n)_n \in X^{\mathbb{N}}$ is an unknown sequence, and

$$v^{n+\theta} = (1 - \theta)v^n + \theta v^{n+1}$$

for $v = (v^n) \in X^{\mathbb{N}}$. In general, the discretization (3.3) is called the **$\boldsymbol{\theta}$-method**. It is known as the **forward Euler method** when $\theta = 0$, and the **backward Euler method** when $\theta = 1$. Note that the solvability of (3.3) is equivalent to the invertibility of $I - \theta\tau A$, since (3.3) can be rewritten as

$$(I - \theta\tau A)u^{n+1} = (I + (1 - \theta)\tau A)u^n + \tau f^{n+\theta}. \tag{3.4}$$

In particular, if (3.3) is solvable, then the solution must be unique.

For the space $X^{\mathbb{N}}$, we introduce some notations.

Definition 3.8. Let $p \in (1, \infty)$.

(i) We define the discrete L^p-norm $\| \cdot \|_{l^p_\tau(\mathbb{N};X)}$ as

$$\|v\|_{l^p_\tau(\mathbb{N};X)} = \left(\sum_{n=0}^{\infty} \|v^n\|_X^p \tau \right)^{1/p}.$$

for $v = (v^n) \in l^p(\mathbb{N}; X)$.

(ii) For $v = (v^n) \in X^{\mathbb{N}}$, $\tau > 0$, and $\theta \in [0,1]$, we define the sequences $D_\tau v$, Av, and v_θ as

$$(D_\tau v)^n = \frac{v^{n+1} - v^n}{\tau}, \quad (Av)^n = Av^n, \quad (v_\theta)^n = v^{n+\theta}.$$

Now, we can define the discrete version of maximal L^p-regularity.

Definition 3.9 (maximal l^p-regularity)**.** Suppose that $p \in (1, \infty)$ and $\theta \in [0,1]$. A linear operator A has **maximal l^p-regularity** if, for every $\tau > 0$ small enough and $f \in l^p(\mathbb{N}; X)$, (3.3) has a unique solution $u = (u^n) \in X^{\mathbb{N}}$, satisfying

$$\|D_\tau u\|_{l^p_\tau(\mathbb{N};X)} + \|Au_\theta\|_{l^p_\tau(\mathbb{N};X)} \leq C\|f_\theta\|_{l^p_\tau(\mathbb{N};X)}, \tag{3.5}$$

where $C > 0$ is independent of $\tau > 0$ and f. We say that A has **discrete maximal regularity** if A has maximal l^p-regularity for some $p \in (1, \infty)$.

We characterize maximal l^p-regularity by the boundedness of the Fourier multiplier. Hereafter, we assume that $A \in \mathcal{L}(X)$, and that A is an infinitesimal generator of a bounded and analytic semigroup on X, so that (3.3) is solvable. When $\mathbb{T} \setminus \{1\} \subset \rho(T_\tau)$, we set

$$M_\tau(z) = (I - \theta\tau A)^{-1}(z-1)R(z; T_\tau), \quad z \in \mathbb{T}, \tag{3.6}$$

and set

$$(T_{M_\tau} f)^n = \left[\mathcal{F}^{-1}(M_\tau \mathcal{F} f)\right]^n, \quad f \in c_{00}(\mathbb{Z}; X), \quad n \in \mathbb{Z}, \tag{3.7}$$

where

$$T_\tau = (I - \theta\tau A)^{-1}(I + (1-\theta)\tau A). \tag{3.8}$$

In the present subsection, $\mathcal{F}$ and $\mathcal{F}^{-1}$ are the Fourier transforms on $\mathbb{Z}$ and on $\mathbb{T}$, respectively.

Lemma 3.10. *Let $p \in (1, \infty)$, and let $A \in \mathcal{L}(X)$ be an infinitesimal generator of a bounded and analytic semigroup on X. Suppose that $\mathbb{T} \setminus \{1\} \subset \rho(T_\tau)$. Then, the following assertions are equivalent:*

(a) *The operator A has maximal l^p-regularity.*

(b) *The multiplier operator T_{M_τ} can be extended to a bounded operator on $l^p(\mathbb{Z}; X)$, and its operator norm is bounded by a constant independent of $\tau > 0$.*

Proof. From (3.4), $D_\tau u$ is written as

$$(D_\tau u)^n = (I - \theta\tau A)^{-1}\left[(T_\tau - I)\sum_{j=0}^{n-1} T_\tau^{n-j-1} f^{j+\theta} + f^{n+\theta}\right]$$

for $n \in \mathbb{N}$. Therefore, by a basic computation, we can obtain

$$(D_\tau u)^n = (T_{M_\tau}\tilde{f}_\theta)^n, \quad n \in \mathbb{N},$$

where $\tilde{f}_\theta$ is the zero-extension of f_θ to $\mathbb{Z}$. Hence, we obtain the equivalence of the assertions. ■

Now, we consider under what circumstances the condition $\mathbb{T}\setminus\{1\} \subset \rho(T_\tau)$ is fulfilled. We define a fractional function $g_{\theta,\tau}$, as

$$g_{\theta,\tau}(\zeta) = \frac{\zeta - 1}{\theta\tau\zeta + (1-\theta)\tau}, \quad \zeta \in \mathbb{C}. \tag{3.9}$$

Assume that $g_{\theta,\tau}(\mathbb{T}\setminus\{1\}) \subset \rho(A)$, and let $\lambda \in \mathbb{T}\setminus\{1\}$. Then, noting that $-(1-\theta)/\theta \notin \mathbb{T}\setminus\{1\}$, we have

$$\lambda I - T_\tau = [\theta\tau\lambda + (1-\theta)\tau](g_{\theta,\tau}(\lambda)I - A)(I - \theta\tau A)^{-1}, \tag{3.10}$$

which implies that $\lambda \in \rho(T_\tau)$. What is left is to determine the set $g_{\theta,\tau}(\mathbb{T}\setminus\{1\})$. By a simple calculation, we have

$$g_{\theta,\tau}(\mathbb{T}\setminus\{1\}) = \begin{cases} C\left(\dfrac{-1}{(1-2\theta)\tau}; \dfrac{1}{(1-2\theta)\tau}\right)\setminus\{0\}, & 0 \le \theta < \dfrac{1}{2}, \\ i\mathbb{R}\setminus\{0\}, & \theta = \dfrac{1}{2}, \\ C\left(\dfrac{1}{(2\theta-1)\tau}; \dfrac{1}{(2\theta-1)\tau}\right)\setminus\{0\}, & \dfrac{1}{2} < \theta \le 1, \end{cases} \tag{3.11}$$

where $C(a;r) = \{z \in \mathbb{C} \mid |z-a| = r\}$, for $a \in \mathbb{C}$ and $r > 0$. Since A generates a bounded and analytic semigroup, $g_{\theta,\tau}(\mathbb{T}\setminus\{1\}) \subset \overline{\mathbb{H}}\setminus\{0\} \subset \rho(A)$ is satisfied when $1/2 \le \theta \le 1$ ($\overline{\mathbb{H}}$ is the closure in $\mathbb{C}$). Therefore, we need no condition for $\mathbb{T}\setminus\{1\} \subset \rho(T_\tau)$ in this case. However, an additional condition is needed when $0 \le \theta < 1/2$. We then give the following condition (S) (cf. Figure 1).

(S) The operator A satisfies the following:

$$\sigma(A) \subset \mathbb{D}\left(\frac{-1}{(1-2\theta)\tau}; \frac{1}{(1-2\theta)\tau}\right) \cup \{0\}.$$

Here, $\mathbb{D}(a;r) = \{z \in \mathbb{C} \mid |z-a| < r\}$ is an open disk for $a \in \mathbb{C}$ and $r > 0$. Note that the condition (S) is satisfied if τ is sufficiently small, since the spectrum is a bounded set in the case of $A \in \mathcal{L}(X)$. Now, we have a sufficient condition for $\mathbb{T}\setminus\{1\} \subset \rho(T_\tau)$.

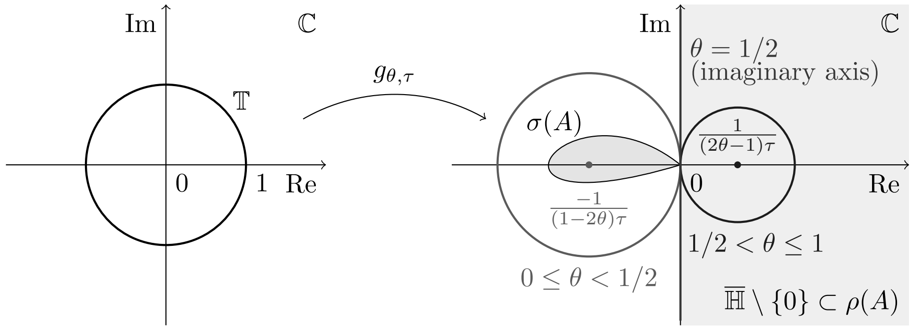

Figure 1: The set $g_{\theta,\tau}(\mathbb{T} \setminus \{1\})$ and the condition (S).

Lemma 3.11. *Let $\theta \in [0,1]$. Assume that $A \in \mathcal{L}(X)$ is an infinitesimal generator of a bounded and analytic semigroup on X. We suppose that the condition* (S) *is fulfilled when $0 \le \theta < 1/2$. Then, we have $\mathbb{T} \setminus \{1\} \subset \rho(T_\tau)$. In particular, the two assertions in the previous lemma are equivalent.*

From the viewpoint of Lemma 3.10, we need to examine the boundedness of the multiplier operator. For this purpose, Blunck proved the following multiplier theorem. This is the discrete version of Weis's operator-valued Fourier multiplier theorem [21, Theorem 3.4].

Theorem 3.12 ([6, Theorem 1.3])**.** *Let X be a UMD space, $J = (-\pi, \pi) \setminus \{0\}$, and $M\colon J \to \mathcal{L}(X)$. Set*

$$T_M f = \mathcal{F}^{-1}[\tilde{M}\mathcal{F}f], \quad f \in c_{00}(\mathbb{Z}; X), \tag{3.12}$$

where $\tilde{M}(z) = M(\arg z)$ for $z \in \mathbb{T} \setminus \{1\}$. Assume that M is differentiable, and that the set

$$\mathcal{T} = \{M(t) \mid t \in J\} \cup \{(e^{it} - 1)(e^{it} + 1)M'(t) \mid t \in J\} \tag{3.13}$$

is R-bounded. Then, T_M can be extended to a bounded operator on $l^p(\mathbb{Z}; X)$, for all $p \in (1, \infty)$. Moreover, its operator norm is bounded as

$$\|T_M\|_{\mathcal{L}(l^p(\mathbb{Z};X))} \le C_{X,p} R(\mathcal{T}), \tag{3.14}$$

where $C_{X,p} > 0$ depends only on p and X.

For the proof, see [6] or Appendix A. In view of numerical analysis, it is troublesome that the constant $C_{X,p}$ in (3.14) depends on the Banach space X, since X is supposed to be the finite element space, which depends on the discretization parameter. Tracing the constants in the proof, the dependence on X comes from Lemmas 3.14 and 3.15, given below. In the course of stating them, we use the following notation.

Definition 3.13. (i) We decompose the interval $(0, \pi)$ into a family of intervals $(J_j)_{j \in \mathbb{Z}}$, where

$$J_j = [a_j, b_j), \quad a_j = \begin{cases} \pi - 2^{-j-1}\pi, & j \ge 0, \\ 2^{j-1}\pi, & j < 0, \end{cases} \quad b_j = \begin{cases} \pi - 2^{-j-2}\pi, & j \ge 0, \\ 2^j\pi, & j < 0. \end{cases}$$

(ii) We decompose $\mathbb{T} \setminus \{\pm 1\}$ into a family of arcs $(\Delta_j)_{j\in\mathbb{Z}}$, where

$$\Delta_j = \{e^{it} \mid t \in (-J_j) \cup J_j\}.$$

(iii) For $m \in BV(\Delta_j;\mathbb{C})$, we denote the variation of m on Δ_j by $\operatorname{var}_{\Delta_j} m$, i.e.,

$$\operatorname{var}_{\Delta_j} m = \sup \left\{ \sum_{l=1}^{N} |m(e^{\sigma i t_l}) - m(e^{\sigma i t_{l-1}})| \;\middle|\; \begin{array}{l} N \in \mathbb{N},\ \sigma \in \{\pm 1\}, \\ a_j = t_0 < \cdots < t_N = b_j \end{array} \right\}.$$

(iv) For $m \in L^\infty(\mathbb{T};\mathbb{C}) \cap BV(\Delta_j;\mathbb{C})$, we set

$$\operatorname{Var}_{\Delta_j} m = \max\left\{\operatorname{var}_{\Delta_j} m,\ \|m\|_{L^\infty(\mathbb{T};\mathbb{C})}\right\}.$$

Lemma 3.14 ([5, Theorem 3.6])**.** *Let $p \in (1,\infty)$ and X be a UMD space. We define S_j as*

$$S_j f = \mathcal{F}^{-1}[\chi_{\Delta_j}\mathcal{F} f], \quad f \in c_{00}(\mathbb{Z};X).$$

Then, there exists $C_1 > 0$ depending only on p and X, satisfying

$$C_1^{-1}\|f\|_{l^p(\mathbb{Z};X)} \le \int_0^1 \left\| \sum_{j\in\mathbb{Z}} r_j(t) S_j f \right\|_{l^p(\mathbb{Z};X)} dt \le C_1 \|f\|_{l^p(\mathbb{Z};X)}, \tag{3.15}$$

for all $f \in c_{00}(\mathbb{Z};X)$, where $(r_j)_{j\in\mathbb{Z}}$ a sequence of independent and symmetric $\{\pm1\}$-valued random variables on $[0,1]$.

Lemma 3.15 ([5, Theorem 4.5])**.** *Let $p \in (1,\infty)$ and X be a UMD space. For $m \in L^\infty(\mathbb{T};\mathbb{C})$, we define the operator T_m on $c_{00}(\mathbb{Z};X)$ as*

$$T_m f = \mathcal{F}^{-1}[m\mathcal{F} f], \quad f \in c_{00}(\mathbb{Z};X).$$

Assume that m satisfies $\operatorname{var}_{\Delta_j} m < \infty$ for all $j \in \mathbb{Z}$. Then, T_m can be extended to a bounded operator on $l^p(\mathbb{Z};X)$, and its operator norm is bounded as

$$\|T_m\|_{l^p(\mathbb{Z};X)} \le C_2 \sup_{j\in\mathbb{Z}} \operatorname{Var}_{\Delta_j} m, \tag{3.16}$$

where $C_2 > 0$ depends only on p and X.

From the proof of Theorem 3.12 in Appendix A, we have the following.

Proposition 3.16. *Let $p \in (1,\infty)$, and let X be a UMD space. Then, the constant $C_{X,p}$ in (3.14) can be expressed as*

$$C_{X,p} = c_p C_1^2 C_2,$$

where $c_p > 0$ is a constant depending only on p, and C_1 and C_2 are the constants in (3.15) and (3.16), respectively.

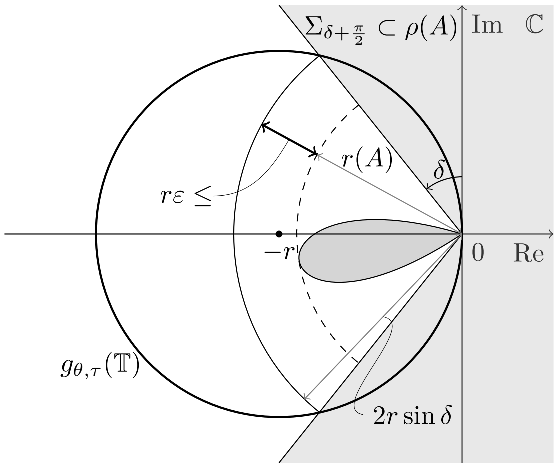

Figure 2: The condition $(\mathrm{NR})_{\delta,\varepsilon}$. In this figure, we set $r = 1/[(1-2\theta)\tau]$.

Now, let us denote the above constants by $C_1(p, X)$ and $C_2(p, X)$, respectively. Since Lemmas 3.14 and 3.15 deal with multiplier operators of scalar-valued functions, we can easily obtain

$$C_1(p, X_0) = C_1(p, X), \quad C_2(p, X_0) \leq C_2(p, X),$$

for a closed subspace $X_0 \subset X$. Therefore, we can state the following assertion.

Corollary 3.17. *Let X be a UMD space, and $X_0 \subset X$ be a closed subspace. Furthermore, let $J = (-\pi, \pi) \setminus \{0\}$, and $M\colon J \to \mathcal{L}(X_0)$. Set T_M as* (3.12) *for $f \in c_{00}(\mathbb{Z}; X_0)$, and $\mathcal{T}$ as* (3.13)*. Assume that M is differentiable, and that the set $\mathcal{T}$ is R-bounded. Then, T_M can be extended to a bounded operator on $l^p(\mathbb{Z}; X_0)$ for all $p \in (1, \infty)$. Moreover, its operator norm is bounded as*

$$\|T_M\|_{\mathcal{L}(l^p(\mathbb{Z};X_0))} \leq CR(\mathcal{T}),$$

where $C > 0$ depends only on p and X, but is independent of X_0.

4. Main result

Our main result is based on the characterization given in Lemma 3.10, such that the condition (S) is assumed when $\theta \in [0, 1/2)$. However, in order to obtain a uniform estimate for τ, the condition (S) is not sufficient. Therefore, we consider the stronger condition given below (cf. Figure 2).

$(\mathrm{NR})_{\delta,\varepsilon}$ The following two conditions are fulfilled:

- **(NR1)** $S(A) \subset \mathbb{C} \setminus \Sigma_{\delta+\pi/2}$,
- **(NR2)** the operator A satisfies

$$(1-2\theta)\tau r(A) + \varepsilon \leq 2\sin\delta.$$

Here, $\Sigma_\varphi = \{z \in \mathbb{C} \setminus \{0\} \mid |\arg z| < \varphi\}$ is an open sector,

$$S(A) = \{\langle x^*, Ax\rangle \mid x \in X,\ x^* \in X^*,\ \|x\| = \|x^*\| = \langle x^*, x\rangle = 1\}$$

is the numerical range of A, and $r(A) = \sup_{z\in S(A)} |z|$ (not the spectral radius of A).

Remark 4.1. (i) One can see that the condition $(\mathrm{NR})_{\delta,\varepsilon}$ is stronger than the condition (S), since $\sigma(A) \subset \overline{S(A)}$, where the overbar is the closure in $\mathbb{C}$. See, for example, [17, Theorem 3.9 in Chapter 1].

(ii) If a linear operator A is an infinitesimal generator of a bounded and analytic semigroup on X, then the condition (NR1) is fulfilled for some $\delta \in (0, \pi/2)$. Therefore, if one wants to achieve the condition $(\mathrm{NR})_{\delta,\varepsilon}$, it suffices to fix ε small enough, and to consider τ satisfying

$$\tau \le \frac{2\sin\delta - \varepsilon}{(1-2\theta)r(A)}.$$

Theorem 4.2 (Discrete maximal regularity for the θ-method)**.** *Let X be a UMD space, and let $p \in (1,\infty)$ and $\theta \in [0,1]$. Assume that $A \in \mathcal{L}(X)$ has maximal L^p-regularity. We suppose that A satisfies the condition $(\mathrm{NR})_{\delta,\varepsilon}$ for some $\delta \in (0,\pi/2)$ and $\varepsilon > 0$, when $\theta \in [0,1/2)$. Then, A has maximal l^p-regularity.*

Proof. Let M_τ, T_{M_τ}, and T_τ be the operators defined by (3.6), (3.7), and (3.8), respectively. In view of Lemma 3.10, it suffices to show that the operator T_{M_τ} is bounded in $l^p(\mathbb{Z}; X)$. Let $J = (-\pi,\pi) \setminus \{0\}$, and define $\tilde{M}_\tau \colon J \to \mathcal{L}(X)$ as

$$\tilde{M}_\tau(t) = M_\tau(e^{it}) = (I - \theta\tau A)^{-1}(e^{it} - 1)R(e^{it}; T_\tau), \quad t \in J.$$

By virtue of Theorem 3.12, we only need to show that the set

$$\mathcal{T}_\tau = \{\tilde{M}_\tau(t) \mid t \in J\} \cup \{(e^{it}-1)(e^{it}+1)\tilde{M}_\tau'(t) \mid t \in J\}$$

is R-bounded uniformly for $\tau > 0$. We set

$$\mathcal{T}_1 = \{\tilde{M}_\tau(t) \mid t \in J\}, \quad \mathcal{T}_2 = \{(e^{it}-1)(e^{it}+1)\tilde{M}_\tau'(t) \mid t \in J\}$$

and we calculate $\mathcal{T}_1$ and $\mathcal{T}_2$.

Let $\lambda = e^{it}$ for $t \in J$. Then, from (3.10), we have

$$R(\lambda; T_\tau) = \frac{1}{\theta\tau\lambda + (1-\theta)\tau}(I - \theta\tau A)R(g_{\theta,\tau}(\lambda); A),$$

where $g_{\theta,\tau}$ is defined by (3.9). Therefore, setting $\mu = g_{\theta,\tau}(\lambda)$, we have

$$\tilde{M}_\tau(t) = \mu R(\mu; A), \tag{4.1}$$

which implies that

$$\mathcal{T}_1 = \{\mu R(\mu; A) \mid \mu \in g_{\theta,\tau}(\mathbb{T} \setminus \{1\})\}. \tag{4.2}$$

Let us calculate $\mathcal{T}_2$. Since one can obtain

$$\tilde{M}_\tau'(t) = (I - \theta\tau A)^{-1} i e^{it} R(e^{it}; T_\tau)[I - (e^{it} - 1)R(e^{it}; T_\tau)],$$

we have

$$(e^{it} + 1)(e^{it} - 1)\tilde{M}_\tau'(t) = ie^{it}(e^{it} + 1)\tilde{M}_\tau(t)[I - (I - \theta\tau A)\tilde{M}_\tau(t)].$$

Moreover, by (4.1), we can calculate

$$I - (I - \theta\tau A)\tilde{M}_\tau(t) = I - (I - \theta\tau A)\mu R(\mu; A) = (1 - \theta\tau\mu)[I - \mu R(\mu; A)],$$

where $\mu = g_{\theta,\tau}(e^{it})$. Therefore, we have

$$(e^{it} + 1)(e^{it} - 1)\tilde{M}_\tau'(t) = ie^{it}(e^{it} + 1)(1 - \theta\tau\mu)\tilde{M}_\tau(t)[I - \tilde{M}_\tau(t)].$$

Noting that

$$(z + 1)(1 - \theta\tau g_{\theta,\tau}(z)) \in C(1; 1)$$

for $z \in \mathbb{T}$, regardless of θ and τ, we can obtain

$$R(\mathcal{T}_2) \leq 4R(\mathcal{T}_1)(1 + R(\mathcal{T}_1)), \tag{4.3}$$

provided that $\mathcal{T}_1$ is R-bounded. Hence, it suffices to prove the R-boundedness of $\mathcal{T}_1$. In the following, we set

$$\mathcal{T}_0 = \{isR(is; A) \mid s \in \mathbb{R} \setminus \{0\}\}, \tag{4.4}$$

which is R-bounded by the maximal L^p-regularity of A.

Case 1. We assume that $1/2 \leq \theta \leq 1$. In this case, from (4.2), (3.11), Theorem 3.7 (b), and Lemma 2.6, $\mathcal{T}_1$ is R-bounded with the bound

$$R(\mathcal{T}_1) \leq R(\mathcal{T}_0), \tag{4.5}$$

which is a uniform estimate for τ.

Case 2. We then assume that $0 \leq \theta < 1/2$. This case is not as simple as Case 1. We set

$$\gamma = \gamma_{\theta,\tau} = g_{\theta,\tau}(\mathbb{T}) = C\left(\frac{-1}{(1 - 2\theta)\tau}; \frac{1}{(1 - 2\theta)\tau}\right),$$

and $\dot{\gamma} = \gamma \setminus \{0\}$. Then, $\mathcal{T}_1$ is written as $\mathcal{T}_1 = \{\mu R(\mu; A) \mid \mu \in \dot{\gamma}\}$. Take $\delta_0 \in (0, \delta)$, satisfying

$$0 < \delta_0 < \arctan \frac{1}{R(\mathcal{T}_0)}.$$

We decompose $\dot{\gamma}$ into two parts, Γ_1 and Γ_2 (cf. Figure 3), as

$$\Gamma_1 = \left\{\mu \in \dot{\gamma} \;\middle|\; |\arg \mu| < \delta_0 + \frac{\pi}{2}\right\}, \quad \Gamma_2 = \left\{\mu \in \dot{\gamma} \;\middle|\; |\arg \mu| \geq \delta_0 + \frac{\pi}{2}\right\},$$

and we set $\mathcal{S}_j = \{\mu R(\mu; A) \mid \mu \in \Gamma_j\}$ for $j = 1, 2$. By Corollary 2.9, $\mathcal{S}_1$ is R-bounded with the bound

$$R(\mathcal{S}_1) \leq P_1(R(\mathcal{T}_0)), \tag{4.6}$$

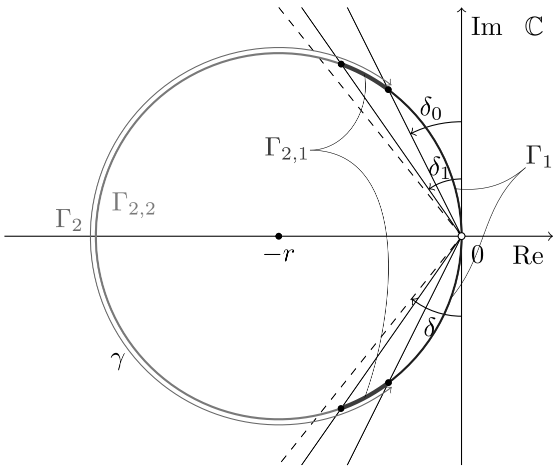

Figure 3: The arcs in the proof of the main theorem.

where P_1 is a polynomial of degree one, defined by (2.7). Note that the set $\mathcal{T}_{\pi/2}$ in Corollary 2.9 is R-bounded, and its R-bound is equal to that of $\mathcal{T}_0$ here, by Lemma 2.6. What we left is to show that $\mathcal{S}_2$ is R-bounded.

We first prove that there exists $\eta > 0$ independent of τ, such that

$$\|R(\mu; A)\| \le (1-2\theta)\tau\eta \tag{4.7}$$

for $\mu \in \Gamma_2$. Take $\delta_1 \in (\delta_0, \delta)$ sufficiently close to δ, so that

$$2(\sin\delta - \sin\delta_1) = \frac{\varepsilon}{2} \tag{4.8}$$

is satisfied. We additionally decompose Γ_2 into two parts, $\Gamma_{2,1}$ and $\Gamma_{2,2}$ (cf. Figure 3), as

$$\Gamma_{2,1} = \left\{\mu \in \Gamma_2 \;\middle|\; |\arg\mu| < \delta_1 + \frac{\pi}{2}\right\}, \quad \Gamma_{2,2} = \left\{\mu \in \Gamma_2 \;\middle|\; |\arg\mu| \ge \delta_1 + \frac{\pi}{2}\right\}.$$

It is well-known that

$$\|R(\mu; A)\| \le \frac{1}{\operatorname{dist}(\mu; S(A))}, \quad \mu \in \mathbb{C} \setminus \overline{S(A)},$$

where $\overline{S(A)}$ is the closure in $\mathbb{C}$ (cf. [17, Theorem 3.9 in Chapter 1]). Thus, we compute the distance $\operatorname{dist}(\mu; S(A))$. We set $r = 1/[(1-2\theta)\tau]$, which is the radius of the circle γ. Assume that $\mu \in \Gamma_{2,1}$. Then, since $\mu \in \Sigma_{\delta+\pi/2}$ and $S(A) \subset \mathbb{C} \setminus \Sigma_{\delta+\pi/2}$, by the condition $(\mathrm{NR})_{\delta,\varepsilon}$, we have

$$\operatorname{dist}(\mu; S(A)) \ge \operatorname{dist}(\mu; \partial\Sigma_{\delta+\pi/2}) = |\mu| \sin\left(\delta + \frac{\pi}{2} - |\arg\mu|\right).$$

Noting that $|\mu| = 2r\sin(|\arg\mu| - \pi/2)$, we have

$$|\mu| \sin\left(\delta + \frac{\pi}{2} - |\arg\mu|\right) = 2r\sin\left(|\arg\mu| - \frac{\pi}{2}\right)\sin\left(\delta + \frac{\pi}{2} - |\arg\mu|\right)$$

$$\geq 2r \sin\delta_0 \sin(\delta - \delta_1).$$

Therefore, we obtain

$$\|R(\mu; A)\| \leq \frac{1}{2r \sin\delta_0 \sin(\delta - \delta_1)} = \frac{(1-2\theta)\tau}{2\sin\delta_0 \sin(\delta - \delta_1)}, \quad \mu \in \Gamma_{2,1}. \tag{4.9}$$

Next, we assume that $\mu \in \Gamma_{2,2}$. In this case, we have

$$\operatorname{dist}(\mu; S(A)) \geq |\mu| - r(A) \geq 2r \sin\delta_1 - r(A).$$

By the condition $(\mathrm{NR})_{\delta,\varepsilon}$ and (4.8), we obtain

$$2r \sin\delta_1 - r(A) = [2r \sin\delta - r(A)] - 2r(\sin\delta - \sin\delta_1) \geq \varepsilon r - \frac{\varepsilon r}{2} = \frac{\varepsilon r}{2},$$

which implies

$$\|R(\mu; A)\| \leq \frac{(1-2\theta)\tau}{\varepsilon/2}, \quad \mu \in \Gamma_{2,1}. \tag{4.10}$$

From (4.9) and (4.10), we can obtain (4.7), with

$$\eta = \max\left\{ \frac{1}{2\sin\delta_0 \sin(\delta - \delta_1)}, \ \frac{2}{\varepsilon} \right\}.$$

We are now ready to demonstrate the R-boundedness of $\mathcal{S}_2$. Fix $\mu_0 \in \Gamma_2$ arbitrarily. Then, $R(\mu_0; A)$ can be expanded in a Taylor series as

$$R(\mu; A) = \sum_{n=0}^{\infty} (\mu_0 - \mu)^n R(\mu_0; A)^{n+1},$$

provided that $\mu \in \rho(A)$ and $|\mu - \mu_0| < \|R(\mu_0; A)\|^{-1}$. Set

$$r_0 = \frac{1}{(1-2\theta)\tau\eta} = \frac{r}{\eta}, \quad \mathcal{S}(\mu) = \left\{ R(\zeta; A) \;\middle|\; \zeta \in \Gamma_2, \ |\zeta - \mu| < \frac{1}{4} r_0 \right\},$$

for $\mu \in \Gamma_2$. Then, noting that $r_0 \leq \|R(\mu_0; A)\|^{-1}$ by (4.7), we have

$$\begin{aligned}
R(\mathcal{S}(\mu_0)) &= R\left(\left\{ \sum_{n=0}^{\infty} (\mu_0 - \mu)^n R(\mu_0; A)^{n+1} \;\middle|\; \mu \in \Gamma_2, \ |\mu - \mu_0| < \frac{1}{4} r_0 \right\} \right) \\
&\leq \sum_{n=0}^{\infty} R\left(\left\{ (\mu_0 - \mu)^n R(\mu_0; A)^{n+1} \;\middle|\; \mu \in \Gamma_2, \ |\mu - \mu_0| < \frac{1}{4} r_0 \right\} \right) \\
&\leq \sum_{n=0}^{\infty} \left(\frac{1}{2} r_0 \right)^n \|R(\mu_0; A)^{n+1}\| \\
&\leq \sum_{n=0}^{\infty} \left(\frac{1}{2} \right)^n \|R(\mu_0; A)\| \leq \frac{2}{r_0}.
\end{aligned}$$

Here, we applied Lemma 2.7 in the second step. That is, $\mathcal{S}(\mu)$ is R-bounded, and

$$R(\mathcal{S}(\mu)) \leq 2(1-2\theta)\tau\eta \tag{4.11}$$

for every $\mu \in \Gamma_2$. Now, we set $B(\mu) = \{\zeta \in \gamma \mid |\zeta - \mu| < r_0/4\}$ for $\mu \in \gamma$, so that

$$\gamma = \bigcup_{\mu \in \gamma} B(\mu).$$

Since γ is compact, there exist $N_0 \in \mathbb{N}$ and $\mu_0, \ldots, \mu_{N_0} \in \gamma$ satisfying

$$\gamma = \bigcup_{j=0}^{N_0} B(\mu_j).$$

Moreover, since the ratio of the radii of γ and $B(\mu)$ is independent of τ, we can take N_0 independently of τ. Thus, we have

$$\{R(\mu; A) \mid \mu \in \Gamma_2\} = \bigcup_{\substack{0 \leq j \leq N_0, \\ \mu_j \in \Gamma_2}} \mathcal{S}(\mu_j),$$

which implies that the set $\{R(\mu; A) \mid \mu \in \Gamma_2\}$ is R-bounded, and that

$$R(\{R(\mu; A) \mid \mu \in \Gamma_2\}) \leq \sum_{\substack{0 \leq j \leq N_0, \\ \mu_j \in \Gamma_2}} R(\mathcal{S}(\mu_j)) \leq 2(N_0+1)(1-2\theta)\tau\eta$$

by (4.11). Noting that $|\mu| \leq 2/[(1-2\theta)\tau]$ for $\mu \in \gamma$, we can obtain the R-boundedness of $\mathcal{S}_2$ with the uniform bound

$$R(\mathcal{S}_2) \leq 8(N_0+1)\eta, \tag{4.12}$$

which is the desired assertion. ∎

Now, we examine what the constant C from Theorem 4.2 depends on. We use the notation as in the proof above. We denote the constant from Theorems 4.2 and 3.12 by C_{DMR} and C_{mul}, respectively.

Corollary 4.3. *The constant C_{DMR} is bounded as*

$$C_{\mathrm{DMR}} \leq C_{\mathrm{mul}} C_0,$$

where C_0 is a constant depending only on δ, ε, and the R-bound of $\mathcal{T}_0$ defined by (4.4). Moreover, C_0 is independent of τ, θ, X, and the individual operator A.

Proof. By virtue of Theorem 3.12, (4.2) and (4.3), we have

$$C_{\mathrm{DMR}} \leq C_{\mathrm{mul}} \cdot R(\mathcal{T}_1)(4R(\mathcal{T}_1)+5).$$

Thus, we shall estimate the R-bound of $\mathcal{T}_1$. When $1/2 \le \theta \le 1$, we can obtain

$$C_{\mathrm{DMR}} \le C_{\mathrm{mul}} \cdot R(\mathcal{T}_0)(4R(\mathcal{T}_0) + 5)$$

from (4.5). In the case where $0 \le \theta < 1/2$, we have

$$R(\mathcal{T}_1) \le P_1(R(\mathcal{T}_0)) + 8(N_0 + 1)\eta \tag{4.13}$$

from (4.6) and (4.12). By the definition of N_0, it can be seen that

$$N_0 + 1 \le \frac{2\pi}{1/(4\eta)}. \tag{4.14}$$

Thus, we only need to determine the upper bound of η. By performing simple computations, one can obtain

$$\delta - \delta_1 \ge \sin\delta - \sin\delta_1 = \frac{\varepsilon}{4}$$

and

$$\sin(\delta - \delta_1) \ge \sin\frac{\varepsilon}{4} \ge \frac{\varepsilon}{2\pi}.$$

Therefore, we have

$$\frac{1}{\eta} \le \frac{\pi}{\varepsilon}\max\left\{\frac{2}{\pi},\ \frac{1}{\sin\delta_0}\right\} = \frac{\pi}{\varepsilon}\frac{1}{\sin\delta_0}. \tag{4.15}$$

Now, taking

$$\delta_0 = \min\left\{\frac{\delta}{2},\ \arctan\frac{1}{2R(\mathcal{T}_0)}\right\},$$

which implies

$$\alpha = \min\left\{R(\mathcal{T}_0)\tan\frac{\delta}{2},\ \frac{1}{2}\right\},$$

we can obtain

$$R(\mathcal{T}_1) \le P_2(R(\mathcal{T}_0))$$

by (4.13), (4.14), and (4.15), where P_2 is a polynomial of degree two, and depends only on δ and ε. Hence, we can complete the proof. ■

From Corollaries 3.17 and 4.3, we deduce the following assertion. This is an applicable version of our main theorem to the finite element method.

Corollary 4.4. *Let X be a UMD space, and $X_0 \subset X$ be a closed subspace. Suppose that $p \in (1,\infty)$ and $\theta \in [0,1]$. Assume that $A \in \mathcal{L}(X_0)$ has maximal L^p-regularity and satisfies the condition $(\mathrm{NR})_{\delta,\varepsilon}$ for some $\delta \in (0,\pi/2)$ and $\varepsilon > 0$. Then, A has maximal l^p-regularity uniformly for X_0. That is, (3.3) in X_0 is uniquely solvable, and we have the inequality*

$$\|D_\tau u\|_{l^p_\tau(\mathbb{N};X_0)} + \|Au_\theta\|_{l^p_\tau(\mathbb{N};X_0)} \le C\|f_\theta\|_{l^p_\tau(\mathbb{N};X_0)},$$

where $C > 0$ depends only on p, δ, ε, X, and the R-bound of $\mathcal{T}_0$ defined by (4.4). Moreover, this constant is independent of τ, θ, f, X_0, and the individual operator A.

In view of this corollary, we can assume that X is a Lebesgue space and that X_0 is a finite element space.

Roughly speaking, Theorem 4.2 says that continuous maximal regularity implies discrete maximal regularity. This is also true in the opposite direction. In order to show this assertion, we apply Blunck's result. We refer the reader to [6, Proposition 1.4] for the proof. Although Blunck did not mention the dependence of C below, one can obtain it by tracing the proof carefully.

Proposition 4.5. *Let X be a Banach space, and let $M \in L^\infty(\mathbb{T}; \mathcal{L}(X))$. Suppose that the operator T_M defined by* (3.7) *can be extended to a bounded operator on $l^p(\mathbb{Z}; X)$ for some $p \in (1, \infty)$. Then, the set*

$$\mathcal{T}_M = \{M(z) \mid z \text{ is a Lebesgue point of } M.\}$$

is R-bounded with the bound

$$R(\mathcal{T}_M) \leq C \|T_M\|_{\mathcal{L}(l^p(\mathbb{Z};X))},$$

where $C > 0$ depends only on p. Here, we denote the extended operator of T_M by the same symbol.

Theorem 4.6. *Let X be a UMD space, and $A \in \mathcal{L}(X)$ be an infinitesimal generator of a bounded and analytic semigroup on X. Furthermore, let $p \in (1, \infty)$ and $\theta \in [0, 1]$. Assume that the condition* (S) *is fulfilled when $0 \leq \theta < 1/2$, and that A has maximal l^p-regularity. Then, A has maximal L^p-regularity.*

Proof. Since A has maximal l^p-regularity, the operator T_{M_τ} defined by (3.7) is bounded in $l^p(\mathbb{Z}; X)$ uniformly for $\tau > 0$. Combining this fact with Proposition 4.5, we can obtain

$$R(\mathcal{T}_{M_\tau}) \leq C_0, \quad \forall \tau > 0$$

for some $C_0 > 0$ independent of $\tau > 0$, where

$$\mathcal{T}_{M_\tau} = \{M_\tau(\lambda) \mid \lambda \in \mathbb{T} \setminus \{1\}\}$$

and M_τ is defined by (3.6). Now, we show that the set $\mathcal{T}_0 = \{\mu R(\mu; A) \mid \mu \in i\mathbb{R} \setminus \{0\}\}$ is R-bounded. Recall that (4.2) and (3.11) hold. Therefore, no further argument is needed when $\theta = 1/2$. We assume that $1/2 < \theta \leq 1$. Set $h_{\theta,\tau}(\zeta) = (1 - e^{(2\theta-1)\tau z})/[(2\theta - 1)\tau]$ for $\zeta \in \mathbb{C}$. Then, it is easy to see that $h_{\theta,\tau}(\mu) \in g_{\theta,\tau}(\mathbb{T} \setminus \{1\})$ for $\mu \in i\mathbb{R} \setminus \{0\}$ and $h_{\theta,\tau}(\mu) \to \mu$ as $\tau \downarrow 0$. Thus, for $n \in \mathbb{N}$, $x_0, \ldots, x_n \in X$, and $\mu_0, \ldots, \mu_n \in i\mathbb{R} \setminus \{0\}$, we obtain

$$\begin{aligned} \int_0^1 \left\| \sum_{j=0}^n r_j(t) \mu_j R(\mu_j; A) x_j \right\| dt &= \lim_{\tau \downarrow 0} \int_0^1 \left\| \sum_{j=0}^n r_j(t) h_{\theta,\tau}(\mu_j) R(h_{\theta,\tau}(\mu_j); A) x_j \right\| dt \\ &\leq C_0 \int_0^1 \left\| \sum_{j=0}^n r_j(t) x_j \right\| dt, \end{aligned}$$

which implies that $\mathcal{T}_0$ is R-bounded. Here, we applied Lebesgue's convergence theorem in the first equality. The proof is almost the same in the case where $0 \leq \theta < 1/2$. ■

It is known that maximal L^p-regularity is independent of $p \in (1,\infty)$. That is, if the operator A has maximal L^{p_0}-regularity for some $p_0 \in (1,\infty)$, then A has maximal L^p-regularity for all $p \in (1,\infty)$. Combining this fact with Theorems 4.2 and 4.6, we have p-independence for maximal l^p-regularity.

Corollary 4.7. *Let X be a UMD space, let $A \in \mathcal{L}(X)$ be an infinitesimal generator of a bounded and analytic semigroup on X, and let $\theta \in [0,1]$. Assume that the condition $(\mathrm{NR})_{\delta,\varepsilon}$ is fulfilled when $0 \le \theta < 1/2$, and that A has maximal l^{p_0}-regularity for some $p_0 \in (1,\infty)$. Then, A has maximal l^p-regularity for all $p \in (1,\infty)$.*

5. The case of the backward Euler method

In this section, we focus on the backward Euler method. In this case, we can establish several more analogous properties than in the general cases investigated in the previous section. This section consists of two independent topics. We first consider characterizing discrete maximal regularity for unbounded operators. The result is, in a sense, an extension of Blunck's characterization of discrete maximal regularity for power-bounded operators. The next topic is the derivation of an a priori estimate for non-zero initial values. In the continuous case, it is well-known that an a priori estimate (3.2) is valid for non-zero initial values with some modifications. With this in mind, we discretize the proof and establish the estimate similar to (3.5) with appropriate initial values.

5.1. Characterization of discrete maximal regularity

In [6], Blunck considered discrete maximal regularity for the forward Euler method, and characterized it as continuous maximal regularity. However, his proof is valid only in the case where the operator A is bounded, as long as the forward Euler method is considered. Our aim is to characterize discrete maximal regularity for unbounded operators. Therefore, we consider the backward Euler method:

$$\begin{cases} \dfrac{u^{n+1}-u^n}{\tau} = Au^{n+1} + f^{n+1}, & n \in \mathbb{N}, \\ u^0 = 0, \end{cases}$$

so that the iteration operator T_τ in (3.8) is merely a resolvent of A, i.e.,

$$T_\tau = (I-\tau A)^{-1} = \tau^{-1} R(\tau^{-1}; A)$$

if the operator $I-\tau A$ is invertible. Then, we can characterize discrete maximal regularity in a way similar to Blunck. The following theorem corresponds to Blunck's characterization [6, Theorem 1.3]. Note that the operator $I-\tau A$ is invertible for each $\tau > 0$, if the operator A generates a bounded semigroup.

Theorem 5.1. *Let X be a UMD space, and let A be a linear operator on X that generates a bounded and analytic semigroup $T(t)$ on X. Set $T_\tau = (I-\tau A)^{-1}$ for $\tau > 0$. Then, the following statements are equivalent:*

(a) *The operator A has discrete maximal regularity for $\theta = 1$.*

(b) *The set $\{(\lambda - 1)T_\tau R(\lambda; T_\tau) \mid \lambda \in \mathbb{T} \setminus \{1\}\}$ is R-bounded uniformly for $\tau > 0$.*

(c) *The set $\{T_\tau^n,\ n(T_\tau - I)T_\tau^n \mid n \in \mathbb{N}\}$ is R-bounded uniformly for $\tau > 0$.*

(d) *The operator A has continuous maximal regularity.*

(e) *The set $\{\mu R(\mu; A) \mid \mu \in i\mathbb{R} \setminus \{0\}\}$ is R-bounded.*

(f) *The set $\{T(t),\ tAT(t) \mid t > 0\}$ is R-bounded.*

Proof. The equivalence (d) $\iff$ (e) $\iff$ (f) is a result given by Weis (Theorem 3.7). The implication (e) $\implies$ (b) $\implies$ (a) is our main result, and the opposite, (a) $\implies$ (b) $\implies$ (e), is shown in Theorem 4.6. Hence, it suffices to show that (1) (f) $\implies$ (c), and (2) (c) $\implies$ (f).

(1) (f) $\implies$ (c). It suffices to show that the set $\{T_\tau^n,\ n(T_\tau - I)T_\tau^n \mid n \in \mathbb{N},\ n \geq 1\}$ is R-bounded uniformly for $\tau > 0$. It is well-known that the powers of the resolvent $R(\lambda; A)^n$ can be expressed as

$$R(\lambda; A)^n = \frac{1}{(n-1)!} \int_0^\infty t^{n-1} e^{-\lambda t} T(t) dt \tag{5.1}$$

in $\mathcal{L}(X)$, for $\lambda \in \rho(A)$ with $\operatorname{Re} \lambda > 0$, and $n \in \mathbb{N}$ with $n \geq 1$. Therefore, we have

$$T_\tau^n = \frac{1}{\tau^n (n-1)!} \int_0^\infty t^{n-1} e^{-t/\tau} T(t) dt$$

and, noting that $T_\tau - I = \tau A T_\tau$,

$$n(T_\tau - I)T_\tau^n = \frac{1}{\tau^n (n-1)!} \int_0^\infty t^{n-1} e^{-t/\tau} tAT(t) dt$$

for $\tau > 0$, and $n \in \mathbb{N}$ with $n \geq 1$. These equations imply (c), by the formula $\int_0^\infty t^n e^{-\alpha t} dt = n! \alpha^{-n-1}$, for $\alpha > 0$ and $n \in \mathbb{N}$, and by Lemma 2.3.

(2) (c) $\implies$ (f). Set $\mathcal{S}_\tau = \{T_\tau^n,\ n(T_\tau - I)T_\tau^n \mid n \in \mathbb{N}\}$, and assume that there exists $C > 0$ independent of $\tau > 0$, satisfying $\mathcal{R}(\mathcal{S}_\tau) \leq C$ for each τ. Define $A_\tau = (T_\tau - I)/\tau$, which is the Yosida approximation of A. Then, as is well-known, we have

$$\lim_{\tau \downarrow 0} e^{tA_\tau} x = T(t)x \quad \text{in } X \tag{5.2}$$

for every $t > 0$ and every $x \in X$. Moreover, the convergence is uniform on each bounded interval. Here, for $B \in \mathcal{L}(X)$, e^B is the usual exponential of B, i.e.,

$$e^B = \sum_{n=0}^\infty \frac{B^n}{n!}.$$

Now, we show that the set $\{T(t)\}_t$ is R-bounded. Since

$$e^{tA_\tau} = e^{-t/\tau} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{t}{\tau}\right)^n T_\tau^n \in \overline{\mathrm{CH}(\mathcal{S}_\tau)},$$

the set $\{e^{tA_\tau} \mid t > 0\}$ is R-bounded, and its R-bound does not exceed C. Thus, for $N \in \mathbb{N}$, $s_j > 0$, and $x_j \in X$ $(j = 0, \dots, N)$, we have

$$\begin{aligned} \int_0^1 \left\| \sum_{j=1}^N r_j(t) T(s_j) x_j \right\| dt &= \lim_{\tau \downarrow 0} \int_0^1 \left\| \sum_{j=1}^N r_j(t) e^{s_j A_\tau} x_j \right\| dt \\ &\leq C \int_0^1 \left\| \sum_{j=1}^N r_j(t) x_j \right\| dt, \end{aligned}$$

which implies that $\{T(t)\}_t$ is R-bounded. We next establish the R-boundedness of $\{tAT(t)\}_t$. Note that for every $t > 0$ and $x \in X$, one can obtain

$$\lim_{\tau \downarrow 0} tA_\tau e^{tA_\tau} x = tAT(t)x \quad \text{in } X$$

in a way similar to the proof of (5.2), and the convergence is uniform on each bounded interval. We claim that the set $\mathcal{S}'_\tau = \{(n+1)(T_\tau - I)T_\tau^n \mid n \in \mathbb{N}\}$ is R-bounded. Indeed, since

$$\mathcal{S}'_\tau = \{T_\tau - I\} \cup \{(1 + n^{-1}) n T_\tau^n \mid n \in \mathbb{N},\ n \geq 1\},$$

the set $\mathcal{S}'_\tau$ is R-bounded with the bound

$$R(\mathcal{S}'_\tau) \leq \|T_\tau\| + 1 + 2R(\mathcal{S}_\tau) \leq 1 + 3C.$$

Therefore, since

$$\begin{aligned} tA_\tau e^{tA_\tau} &= e^{-t/\tau} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{t}{\tau}\right)^{n+1} (T_\tau - I) T_\tau^n \\ &= e^{-t/\tau} \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \left(\frac{t}{\tau}\right)^{n+1} (n+1)(T_\tau - I) T_\tau^n \in \overline{\mathrm{CH}(\mathcal{S}'_\tau)}, \end{aligned}$$

the set $\{tA_\tau e^{tA_\tau}\}_t$ is R-bounded, and its R-bound does not exceed $1 + 3C$. This implies the R-boundedness of $\{tAT(t)\}_t$ in the same way as above. Hence, we can complete the proof. ∎

5.2. A priori estimate with non-zero initial values

Let X be a Banach space, and let A be a linear operator on X. In the theory of nonlinear evolution equations, the choice of initial value is important. Therefore, we need to obtain an a priori estimate of maximal regularity (3.2) with non-zero initial values. It is known that the desired estimate is valid for $u(0) \in (X, D(A))_{1-1/p,p}$, which is the real interpolation space, provided that $0 \in \rho(A)$. The estimate is as follows:

$$\|u'\|_{L^p(\mathbb{R}^+;X)} + \|Au\|_{L^p(\mathbb{R}^+;X)} \leq C(\|f\|_{L^p(\mathbb{R}^+;X)} + \|u_0\|_{1-1/p,p}), \tag{5.3}$$

where u is the solution of

$$\begin{cases} u'(t) = Au(t) + f(t), \quad t > 0, \\ u(0) = u_0, \end{cases}$$

with $f \in L^p(\mathbb{R}^+; X)$ and $u_0 \in (X, D(A))_{1-1/p,p}$. Here, $\|\cdot\|_{1-1/p,p}$ is the usual norm of $(X, D(A))_{1-1/p,p}$. We equip the norm $\|A \cdot\|_X$ with $D(A)$, which is equivalent to the graph norm of A.

The aim of this subsection is to establish the discrete version of (5.3) in the case of the backward Euler method. This problem in a bounded interval has already been considered by Ashyralyev and Sobolevskiĭ in [4]. We obtain the same estimate in the case where the interval is unbounded. We consider the following problem:

$$\begin{cases} \dfrac{u^{n+1} - u^n}{\tau} = Au^{n+1} + f^{n+1}, \quad n \in \mathbb{N}, \\ u^0 = u_0 \end{cases} \tag{5.4}$$

for $f = (f^n) \in l^p(\mathbb{N}; X)$ and $u_0 \in (X, D(A))_{1-1/p,p}$, with $p \in (1, \infty)$. Recall that $v_1 = (v^{n+1})_n$ for $v = (v^n) \in X^{\mathbb{N}}$.

Theorem 5.2. *Let X be a Banach space, and let A be a linear operator on X. Suppose that $p \in (1, \infty)$ and $\tau > 0$. Assume that $0 \in \rho(A)$, and that A has discrete maximal regularity for $\theta = 1$. Then, for each $f = (f^n) \in l^p(\mathbb{N}; X)$ and $u_0 \in (X, D(A))_{1-1/p,p}$, there exists a unique solution of* (5.4) *$u = (u^n)$, satisfying*

$$\|D_\tau u\|_{l^p_\tau(\mathbb{N};X)} + \|Au_1\|_{l^p_\tau(\mathbb{N};X)} \leq C(\|f_1\|_{l^p_\tau(\mathbb{N};X)} + \|u_0\|_{1-1/p,p}), \tag{5.5}$$

where $C > 0$ is independent of τ, f, and u_0.

For the proof of this theorem, we demonstrate the following embedding result. The proof is essentially the same as in [4, Theorem 3.1 in Chapter 2]. Recall that $T_\tau = (I - \tau A)^{-1}$.

Lemma 5.3. *Let X be a Banach space, and let A be a linear operator on X. Suppose that $p \in (1, \infty)$ and $\tau > 0$. Assume that $0 \in \rho(A)$, and that A generates a bounded and analytic semigroup on X. Define a Banach space E^p_τ as*

$$E^p_\tau = \left\{ x \in X \;\middle|\; \sum_{n=0}^{\infty} \|AT^{n+1}_\tau x\|^p_X < \infty \right\}$$

with a norm

$$\|x\|_{E^p_\tau} = \|x\|_X + \left(\sum_{n=0}^{\infty} \|AT^{n+1}_\tau x\|^p_X \right)^{1/p}.$$

Then, the embedding

$$(X, D(A))_{1-1/p,p} \hookrightarrow E_\tau^p$$

holds uniformly for $\tau > 0$.

Proof. Let $T(t)$ be the semigroup generated by A, and assume that $T(t)$ satisfies

$$\|T(t)\| \le C_1, \quad \|tAT(t)\| \le C_2, \qquad \forall t > 0.$$

Then, as is well-known in the theory of interpolation spaces, the norm $\|\cdot\|_{1-1/p,p}$ satisfies

$$\|x\|_X + \|AT(\cdot)x\|_{L^p(\mathbb{R}^+;X)} \le \max\{C_1, C_2, (p-1)^{1/p}\}\|x\|_{1-1/p,p}. \tag{5.6}$$

For the proof see, for example, [16, Proposition 6.2]. By (5.1), we have

$$AT_\tau^{n+1}x = \frac{1}{\tau^{n+1}n!}\int_0^\infty t^n e^{-t/\tau} AT(t)x dt$$

for $x \in (X, D(A))_{1-1/p,p}$ and $n \in \mathbb{N}$, which implies

$$\|AT_\tau^{n+1}x\|_X^p \le \frac{1}{\tau^{n+1}n!}\int_0^\infty t^n e^{-t/\tau} \|AT(t)x\|_X^p dt$$

by Hölder's inequality. Thus, we have

$$\begin{aligned}\sum_{n=0}^\infty \|AT_\tau^{n+1}x\|_X^p &\le \frac{1}{\tau}\int_0^\infty \sum_{n=0}^\infty \frac{1}{n!}\left(\frac{t}{\tau}\right)^n e^{-t/\tau}\|AT(t)x\|_X^p dt \\ &= \frac{1}{\tau}\int_0^\infty \|AT(t)x\|_X^p dt,\end{aligned}$$

which implies

$$\|x\|_{E_\tau^p} \le \|x\|_X + \|AT(\cdot)x\|_{L^p(\mathbb{R}^+;X)}. \tag{5.7}$$

Hence, we can complete the proof, due to (5.6) and (5.7). ■

Proof of Theorem 5.2. Let $v = (v^n)$ be the solution of

$$\begin{cases} (D_\tau v)^n = Av^{n+1} + f^{n+1}, & n \in \mathbb{N}, \\ v^0 = 0, \end{cases}$$

and let $w = (w^n)$ be that of

$$\begin{cases} (D_\tau w)^n = Aw^{n+1}, & n \in \mathbb{N}, \\ w^0 = u_0. \end{cases}$$

It is obvious that $u = v + w$, and thus it suffices to estimate v and w individually. By the discrete maximal regularity of A, v is estimated as

$$\|D_\tau v\|_{l_\tau^p(\mathbb{N};X)} + \|Av_1\|_{l_\tau^p(\mathbb{N};X)} \le C_{\mathrm{DMR}}\|f_1\|_{l_\tau^p(\mathbb{N};X)}, \tag{5.8}$$

where $C_{\mathrm{DMR}} > 0$ is independent of τ. On the other hand, since

$$(D_\tau w)^n = Aw^{n+1} = AT_\tau^{n+1}u_0,$$

we have

$$\|D_\tau w\|_{l^p_\tau(\mathbb{N};X)} + \|Aw_1\|_{l^p_\tau(\mathbb{N};X)} \le 2\|u_0\|_{E^p_\tau}.$$

Therefore, Lemma 5.3 implies

$$\|D_\tau w\|_{l^p_\tau(\mathbb{N};X)} + \|Aw_1\|_{l^p_\tau(\mathbb{N};X)} \le C_{\mathrm{emb}}\|u_0\|_{1-1/p,p}, \tag{5.9}$$

where $C_{\mathrm{emb}} > 0$ is independent of τ. As a consequence of (5.8) and (5.9), we can establish (5.5). ■

Noting that C_{emb} in the above proof is independent of X, we can obtain an applicable version of Theorem 5.2, in the same way as Corollary 4.4.

Corollary 5.4. *Let X be a Banach space, $X_0 \subset X$ be a closed subspace, and A be a linear operator on X_0. Suppose that $p \in (1,\infty)$ and $\tau > 0$. Assume that $0 \in \rho(A)$, and that A has discrete maximal regularity for $\theta = 1$. Then, for each $f = (f^n) \in l^p(\mathbb{N}; X_0)$ and $u_0 \in (X_0, D(A))_{1-1/p,p}$, there exists a unique solution of (5.4) $u = (u^n)$, satisfying*

$$\|D_\tau u\|_{l^p_\tau(\mathbb{N};X)} + \|Au_1\|_{l^p_\tau(\mathbb{N};X)} \le C(\|f_1\|_{l^p_\tau(\mathbb{N};X)} + \|u_0\|_{1-1/p,p}),$$

where $C > 0$ is independent of τ, f, u_0, and the Banach space X_0. Here, the norm of u_0 is that of $(X, D(A))_{1-1/p,p}$.

Appendix

A. Proof of Blunck's multiplier theorem

In this section, we prove Theorem 3.12. Although the proof is the same as that in [6], we estimate the constants carefully. We use the notation from Definition 3.13, and Lemmas 3.14 and 3.15. We introduce a basic lemma, which can be shown with an elementary computation.

Lemma A.1. *Let $j \in \mathbb{Z}$ and $t \in I_j \cup (-I_j)$. Then, we have*

$$|1 - e^{it}||1 + e^{it}| \le 2^{|j|}.$$

We first demonstrate the Marcinkiewicz-type multiplier theorem.

Theorem A.2. *Let X be a UMD space, and $p \in (1,\infty)$. Assume that $M\colon \mathbb{T} \to \mathcal{L}(X)$ is a map expressed as*

$$M = \sum_{j\in\mathbb{Z}} \chi_{\Delta_j} m M_j,$$

where

- *$m \in L^\infty(\mathbb{T};\mathbb{C})$, and m satisfies $\sup_{j\in\mathbb{Z}} \operatorname{var}_{\Delta_j} m < \infty$,*
- *$M_j \in \mathcal{L}(X)$ for each $j \in \mathbb{Z}$, and the set $\mathcal{M} = \{M_j \mid j \in \mathbb{Z}\}$ is R-bounded.*

Then, the operator T_M, defined by (3.12), can be extended to a bounded linear operator in $l^p(\mathbb{Z};X)$, which is denoted by the same symbol T_M. Moreover, there exists a constant $C_{\mathrm{Mar}} > 0$ such that

$$\|T_M\|_{\mathcal{L}(l^p(\mathbb{Z};X))} \le C_{\mathrm{Mar}} R(\mathcal{M}) \sup_{j\in\mathbb{Z}} \operatorname{Var}_{\Delta_j} m,$$

where C_{Mar} depends only on X and p.

Proof. Define $\tilde{M}_j \in \mathcal{L}(l^p(\mathbb{Z};X))$ as

$$(\tilde{M}_j f)^n = M_j(f^n), \quad f = (f^n) \in l^p(\mathbb{Z};X), \quad n \in \mathbb{Z}$$

for each $j \in \mathbb{Z}$, and set $\tilde{\mathcal{M}} = \{\tilde{M}_j \mid j \in \mathbb{Z}\}$. Then, by Lemma 2.3 (v), $\tilde{\mathcal{M}} \subset \mathcal{L}(l^p(\mathbb{Z};X))$ is R-bounded, and there exists $c_p > 0$ such that

$$R(\tilde{\mathcal{M}}) \le c_p R(\mathcal{M}),$$

where c_p depends only on p. Noting that

$$S_j(T_M f) = \tilde{M}_j(S_j(T_m f))$$

for $j \in \mathbb{Z}$ and $f \in c_{00}(\mathbb{Z};X)$, and using Lemmas 3.14 and 3.15, we have

$$\begin{aligned}
\|T_M f\|_{l^p(\mathbb{Z};X)} &\le C_1 \int_0^1 \left\| \sum_{j\in\mathbb{Z}} r_j(t) S_j(T_M f) \right\|_{l^p(\mathbb{Z};X)} dt \\
&= C_1 \int_0^1 \left\| \sum_{j\in\mathbb{Z}} r_j(t) \tilde{M}_j(S_j(T_m f)) \right\|_{l^p(\mathbb{Z};X)} dt \\
&\le C_1 c_p R(\mathcal{M}) \int_0^1 \left\| \sum_{j\in\mathbb{Z}} r_j(t) S_j(T_m f) \right\|_{l^p(\mathbb{Z};X)} dt \\
&\le C_1^2 c_p R(\mathcal{M}) \|T_m f\|_{l^p(\mathbb{Z};X)} \\
&\le C_1^2 c_p R(\mathcal{M}) C_2 \left(\sup_{j\in\mathbb{Z}} \operatorname{Var}_{\Delta_j} m \right) \|f\|_{l^p(\mathbb{Z};X)}
\end{aligned}$$

for all $f \in c_{00}(\mathbb{Z};X)$. This implies the desired assertion, with $C_{\mathrm{Mar}} = c_p C_1^2 C_2$. ■

Now, we are ready to demonstrate Blunck's multiplier theorem.

Proof of Theorem 3.12. We set

$$\delta_{j,k} = 2^{-k}(b_j - a_j) = \begin{cases} 2^{-k-|j|-2}\pi, & j \geq 0, \\ 2^{-k-|j|-1}\pi, & j < 0, \end{cases},$$
$$b_{j,k,l} = a_j + (l-1)\delta_{j,k}$$

for $j \in \mathbb{Z}$ and $k, l \geq 1$. Furthermore, we set

$$M_k(e^{it}) = \sum_{j\in\mathbb{Z},\ \sigma\in\{\pm1\}} \left[M(\sigma a_j) + \sigma \sum_{l=1}^{2^k} \chi_{\sigma[b_{j,k,l}, b_j)}(t)\delta_{j,k} M'(\sigma b_{j,k,l}) \right].$$

Step 1. We show

$$\lim_{k\to\infty} M_k(e^{it}) = M(t)$$

for all $t \in J$ in $\mathcal{L}(X)$. We fix $t \in J$, and we can assume that $t \in I_j$ for some $j \in \mathbb{Z}$, without loss of generality. Then, $M_k(e^{it})$ is expressed as

$$M_k(e^{it}) = M(a_j) + \sum_{l=1}^{2^k} \chi_{[b_{j,k,l}, b_j)}(t)\delta_{j,k} M'(b_{j,k,l}). \tag{A.1}$$

Suppose that $l_0 \in \mathbb{N}$ satisfies

$$b_{j,k,l_0} \leq t < b_{j,k,l_0+1}.$$

Then, the second term of the right hand side in (A.1) can be rewritten as

$$\sum_{l=1}^{2^k} \chi_{[b_{j,k,l}, b_j)}(t)\delta_{j,k} M'(b_{j,k,l}) = \sum_{l=1}^{l_0} (b_{j,k,l+1} - b_{j,k,l}) M'(b_{j,k,l}),$$

which is an approximation of

$$\int_{a_j}^{t} M'(t)dt.$$

Thus, we have

$$M_k(e^{it}) \to M(a_j) + \int_{a_j}^{t} M'(t)dt = M(t)$$

in $\mathcal{L}(X)$, as $k \to \infty$.

Step 2. We prove

$$\|M_k(e^{it})\|_{\mathcal{L}(X)} \leq (1+\pi)R(\mathcal{T})$$

for all $k \geq 1$ and $t \in J$. Fix $t \in \sigma I_j$, for $j \in \mathbb{Z}$ and $\sigma \in \{\pm1\}$. From the R-boundedness of $\mathcal{T}$ and Lemma A.1, we have

$$\|M(t)\|_{\mathcal{L}(X)} \leq R(\mathcal{T})$$

and

$$\|M'(t)\|_{\mathcal{L}(X)} = |e^{it}+1|^{-1}|e^{it}-1|^{-1}\|(e^{it}+1)(e^{it}-1)M'(t)\|_{\mathcal{L}(X)} \leq 2^{|j|}R(\mathcal{T})$$

Then, noting that $\delta_{j,k} \leq 2^{-k-|j|}\pi$, we obtain

$$\|M_k(e^{it})\|_{\mathcal{L}(X)} \leq \|M(\sigma a_j)\|_{\mathcal{L}(X)} + \sum_{l=1}^{2^k} \delta_{j,k}\|M'(b_{j,k,l})\|_{\mathcal{L}(X)} \leq (1+\pi)R(\mathcal{T}).$$

Step 3. Set

$$m^{(k,l,\sigma)}(e^{it}) = \begin{cases} 1, & l=0, \\ \sum_{\nu\in\mathbb{Z}} \chi_{\sigma[b_{\nu,k,l},b_\nu)}(t), & l \geq 1, \end{cases}$$
$$M_j^{(k,l,\sigma)} = \begin{cases} M(\sigma a_j), & l=0, \\ (b_j - a_j)M'(\sigma b_{j,k,l}), & l \geq 1, \end{cases}$$

for $j \in \mathbb{Z}$, $k \geq 1$, $l = 0, 1, \ldots, 2^k$, $\sigma \in \{\pm 1\}$, and $t \in J$. Then, M_k is divided as

$$M_k = \sum_{\sigma\in\{\pm1\}} \left(M_{k,0,\sigma} + 2^{-k}\sigma \sum_{l=1}^{2^k} M_{k,l,\sigma} \right),$$

where

$$M_{k,l,\sigma} = \sum_{j\in\mathbb{Z}} \chi_{\Delta_j} m^{(k,l,\sigma)} M_j^{(k,l,\sigma)}. \tag{A.2}$$

Step 4. We show that $\{M_j^{(k,l,\sigma)} \mid j \in \mathbb{Z}\}$ is R-bounded with the bound

$$R(\{M_j^{(k,l,\sigma)}\}_j) \leq \pi R(\mathcal{T}),$$

for each $k \geq 1$ and $l \in \mathbb{N}$. In the case where $l = 0$, this is obvious, since

$$\{M_j^{(k,0,\sigma)}\}_j = \{M(\sigma a_j)\}_j \subset \mathcal{T}$$

regardless of k. Therefore we suppose that $l \geq 1$. It follows from Lemma A.1 that

$$b_j - a_j = 2^{-s_j}\pi(1-e^{it})^{-1}(1+e^{it})^{-1}2^{|j|} \cdot (1-e^{it})(1+e^{it}),$$

and

$$|2^{-s_j}\pi(1-e^{it})^{-1}(1+e^{it})^{-1}2^{|j|}| \leq \frac{\pi}{2},$$

for $t \in I_j \cap (-I_j)$ and $j \in \mathbb{Z}$, where

$$s_j = \begin{cases} 2, & j \geq 0, \\ 1, & j < 0. \end{cases}$$

Therefore, we have

$$\begin{aligned}
R(\{M_j^{(k,l,\sigma)}\}_j) &= R(\{(b_j - a_j)M'(\sigma b_{j,k,l}) \mid j \in \mathbb{Z}\}) \\
&\le R(\{(b_j - a_j)M'(t) \mid j \in \mathbb{Z},\ t \in I_j \cup (-I_j)\}) \\
&\le \pi R(\{(1 - e^{it})(1 + e^{it})M'(t) \mid j \in \mathbb{Z},\ t \in I_j \cup (-I_j)\}) \\
&\le \pi R(\mathcal{T}).
\end{aligned}$$

Step 5. We show that there exists $C_{X,p} > 0$, depending only on X and p, satisfying

$$\|T_{M_k} f\|_{l^p(\mathbb{Z};X)} \le C_{X,p} R(\mathcal{T}) \|f\|_{l^p(\mathbb{Z};X)},$$

for all $k \ge 0$ and $f \in c_{00}(\mathbb{Z}; X)$. The set $\{M_j^{(k,l,\sigma)}\}_j$ is R-bounded by step 4, and by the definition of $m^{(k,l,\sigma)}$,

$$\mathrm{Var}_{\Delta_j} m^{(k,l,\sigma)} = 1$$

for each $k \ge 1$, $l \in \mathbb{N}$, and $\sigma \in \{\pm 1\}$. From the expression (A.2), we can apply Theorem A.2, obtaining

$$\|T_{M_{k,l,\sigma}}\|_{\mathcal{L}(l^p(\mathbb{Z};X))} \le C_{\mathrm{Mar}} \pi R(\mathcal{T}).$$

Hence, we have

$$\begin{aligned}
\|T_{M_k} f\|_{l^p(\mathbb{Z};X)} &\le \sum_{\sigma \in \{\pm 1\}} \left(\|T_{M_{k,0,\sigma}}\| + 2^{-k} \sum_{l=1}^{2^k} \|T_{M_{k,l,\sigma}}\| \right) \|f\|_{l^p(\mathbb{Z};X)} \\
&\le 4 C_{\mathrm{Mar}} \pi R(\mathcal{T}) \|f\|_{l^p(\mathbb{Z};X)},
\end{aligned}$$

for all $f \in c_{00}(\mathbb{Z}; X)$.

Step 6. Now, we are ready to conclude. Fix $f \in c_{00}(\mathbb{Z}; X)$ arbitrarily. Then, by steps 1 and 2, and Lebesgue's convergence theorem, we have

$$\lim_{k \to \infty} (T_{M_k} f)^n = (T_M f)^n$$

in X, for each $n \in \mathbb{Z}$. Therefore, from step 5 and Fatou's lemma, we obtain

$$\begin{aligned}
\|T_M f\|_{l^p(\mathbb{Z};X)} &= \left(\sum_{n \in \mathbb{Z}} \lim_{k \to \infty} \|(T_{M_k} f)^n\|_X^p \right)^{1/p} \\
&\le \liminf_{k \to \infty} \left(\sum_{n \in \mathbb{Z}} \|(T_{M_k} f)^n\|_X^p \right)^{1/p} \\
&= \liminf_{k \to \infty} \|T_{M_k} f\|_{l^p(\mathbb{Z};X)} \\
&\le C_{X,p} R(\mathcal{T}) \|f\|_{l^p(\mathbb{Z};X)},
\end{aligned}$$

which is the desired assertion, with

$$C_{X,p} = 4\pi C_{\mathrm{Mar}} = 4\pi c_p C_1^2 C_2$$

by Theorem A.2. ■

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ADDRESS:
Graduate School of Mathematical Sciences, The University of Tokyo
3–8–1 Komaba Meguro-ku, Tokyo 153, JAPAN

TEL +81-3-5465-7001 FAX +81-3-5465-7012