UTMS 2015–4 March 02, 2015

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Analysis of the dipole simulation method for two-dimensional Dirichlet problems in Jordan regions with analytic boundaries*

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February 26, 2015

In this paper, we show well-posedness and error estimate of the dipole simulation method applied to Dirichlet problems in Jordan regions.

Key words: Dipole simulation method, Method of fundamental solutions, Charge simulation method, Dirichlet problem

2010 Mathematics Subject Classification: 65N80, 65N35, 65N12, 35J05, 35J08

1. Introduction and the main result

Let Ω be a Jordan region in the two-dimensional Euclidean space $\mathbb{R}^2$. We consider the following Dirichlet problem for the Laplace equation:

$$\begin{cases} \triangle u = 0 & \text{in } \Omega, \\ u = f & \text{on } \Gamma, \end{cases} \tag{1.1}$$

where Γ denotes the boundary $\partial\Omega$ of Ω and $\triangle$ the Laplace operator. Throughout this paper, we identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$.

As is well-known, the Dirichlet problem (1.1) appears in many fields in mathematical physics and engineering so that its rapid solver is highly required. The method of fundamental solutions (MFS) is one of the popular rapid solvers for (1.1). In MFS, we first take the charge points

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$\{y_k\}_{k=1}^N \subset \mathbb{C} \setminus \overline{\Omega}$, the collocation points $\{x_j\}_{j=1}^N \subset \Gamma$ and then find an approximate solution of the form

$$u^{(N)}(x) = \sum_{k=1}^{N} Q_k E(x - y_k), \tag{1.2}$$

where $E(x)$ denotes a fundamental solution of $\triangle$ with the singularity at the origin and $\{Q_k\}_{k=1}^N$ the coefficient to be determined by the boundary condition;

$$u^{(N)}(x_j) = f(x_j) \quad (j = 1, 2, \ldots, N). \tag{1.3}$$

The most typical choice of E is

$$E(x) = -\frac{1}{2\pi} \log|x|,$$

and under this selection MFS is also called as the charge simulation method (CSM). Whereas the method itself is quite simple and the implementation is easy, it is rather difficult to establish the well-posedness, stability and convergence. As a matter of fact, the first mathematical analysis of CSM was done by Katsurada and Okamoto [5]. They considered (1.1) in the case where Ω is a disk D_ρ with radius ρ having the origin as its center, and showed the well-posedness (cf. [5, Theorems 1]) and the exponential convergence (cf. [5, Theorem 2]) with at most one exceptional N under the choice of the charge points $\{y_k\}_{k=1}^N$ and the collocation points $\{x_j\}_{j=1}^N$ as

$$y_j = R\omega^{j-1}, \quad x_j = \rho\omega^{j-1} \quad (j = 1, 2, \ldots, N), \tag{1.4}$$

where $\omega = \exp(2\pi \mathrm{i}/N)$ and $R > \rho$. Unlike the finite difference method or the finite element method, the well-posedness is not so obvious. In fact, when we take the charge points $\{y_k\}_{k=1}^N$ as $y_k = R\omega^{k-1/2}$ $(k = 1, 2, \ldots, N)$, there cannot exist an approximate solution of the form (1.2) satisfying (1.3) when N is even (cf. Katsurada [6, Theorem 8.2]). After this pioneering work, the well-posedness and exponential convergence of CSM are well established for a Jordan region with the analytic boundary, an annular region and an elliptic region. Furthermore CSM is applied to compute numerical conformal mappings in various regions, and offers a high-precision and simple numerical scheme (cf. Amano et.al [1] and references therein).

Not to mention, another choice of E is possible. In [6], Katsurada concentrated his attention to the case disk $\Omega = D_\rho$ and proposed the dipole simulation method (DSM), in which E is given as follows:

$$E(x, y) = -\frac{1}{2\pi} \frac{(n_y \mid x - y)}{\|x - y\|^2}, \tag{1.5}$$

where $n_y = y/|y|$ and $(\cdot \mid \cdot)$ denotes the Euclidean inner product on $\mathbb{R}^2$. It has been shown that DSM composed of (1.2) and (1.3), where E is defined by (1.5), is well-posed (cf. [6, Theorem 5.1]) and that the exponential convergence (cf. Comments before [6, Theorem 5.2]) is guaranteed under the choice of the dipole points $\{y_k\}_{k=1}^N$ and the collocation points $\{x_j\}_{j=1}^N$ is the same as (1.4). Recently, Ogata [8] generalized Katsurada's DSM and examined its effectiveness through numerical experiments. Thus, he treated (1.1) in the case where Ω is a Jordan region in $\mathbb{R}^2$ and considered the approximate solution of the form

$$u^{(N)}(x) = \sum_{k=1}^{N} Q_k D(x, y_k; n_k), \tag{1.6}$$

where $\{y_k\}_{k=1}^N \subset \mathbb{R}^2 \setminus \overline{\Omega}$ are the dipole points, $\{n_k\}_{k=1}^N$ are the unit vectors, which are called the dipole moments, that n_k represents the direction of the dipole located at y_k and D is defined as

$$D(x, y_k; n_k) = -\frac{1}{2\pi} \frac{(n_k \mid x - y_k)}{\|x - y_k\|^2}.$$

In fact, DSM's approximate solution can be represented as the real part of a holomorphic function;

$$u^{(N)}(x) = u^{(N)}(z) = \mathrm{Re}\left[\sum_{k=1}^{N} Q_k \frac{n_k}{z - \zeta_k}\right],$$

where $z = x + \mathrm{i}y$, $\zeta_k = \xi_k + \mathrm{i}\eta_k$ and $n_k = n_k^{(1)} + \mathrm{i}n_k^{(2)}$ in which $x = (x, y)^{\mathrm{T}}$, $y_k = (\xi_k, \eta_k)^{\mathrm{T}}$ and $n_k = (n_k^{(1)}, n_k^{(2)})^{\mathrm{T}}$. Inspired by the above expression, the complex dipole simulation method which is an approximation technique for holomorphic functions is proposed in our previous paper [10].

In [8], moreover, he applied DSM to compute numerical conformal mappings, which makes us to be able to remove the difficulty of computing arguments, therefore, his method offers much easier and simpler scheme for numerical conformal mappings. Indeed, his method can be extended to compute bidirectional numerical conformal mappings (cf. S. and Ogata [11]). However, there is no mathematical result in [8].

The purpose of the present paper is to give ways of arranging the dipole points $\{y_k\}_{k=1}^N$ and the collocation points $\{x_j\}_{j=1}^N$ and defining the dipole moments $\{n_k\}_{k=1}^N$ that guarantee the well-posedness and the exponential convergence of DSM composed of (1.6) and (1.3). As a preliminary step to this end, we first consider the case where Γ is a circle $\gamma_\rho = \{z \in \mathbb{C} \mid |z| = \rho\}$ with $\rho > 0$. Introducing the dipole and the collocation points as (1.4) and the dipole moments $\{n_k\}_{k=1}^N$ as $n_k = y_k/\|y_k\|$ we establish the well-posedness (cf. Theorem 3.2) and the exponential convergence (cf. Theorem 3.3). In order to extend the results to more general regions, we follow Katsurada [7] and introduce the notion of the *peripheral conformal mapping*. Actually, the following definition is a generalization of Katsurada's one. Set $\mathscr{A}_{\rho_2,\rho_1} = \{z \in \mathbb{C} \mid \rho_2 < |z| < \rho_1\}$ with $\rho_1 > \rho_2 > 0$.

Definition 1.1. For a Jordan curve Γ in $\mathbb{C}$ and a constant $r > 0$, the mapping Ψ from a neighborhood of γ_r to $\mathbb{C}$ is called a peripheral conformal mapping of Γ with the reference radius r if the following two conditions are satisfied:

1. Ψ maps γ_r onto Γ;
2. $\Psi\colon \mathscr{A}_{\kappa^{-1}r,\kappa r} \to \mathbb{C}$ is a schlicht function with some $\kappa > 1$.

For any analytic Jordan curve, there exists a peripheral conformal mapping; Actually, by using its Fourier expansion based on an analytic parameterization, we can construct Ψ concretely (cf. [7, Remark 3.1]).

In what follows, we assume that there exists a peripheral conformal mapping Ψ of Γ with the reference radius ρ, and that Γ is regular. Then, letting $R \in]\rho, \kappa\rho[$, we propose an arrangement of the dipole and the collocation points and a definition of the dipole moments as

$$y_j = \Psi(R\omega^{j-1}), \quad x_j = \Psi(\rho\omega^{j-1}), \quad n_j = \frac{\omega^{j-1}\Psi'(R\omega^{j-1})}{|\Psi'(R\omega^{j-1})|} \quad (j = 1, 2, \ldots, N). \tag{1.7}$$

In order to describe our result, we need a function space $\mathscr{X}_{\varepsilon,s}$ and its norm $\|\cdot\|_{\varepsilon,s}$ which were originally introduced by Arnold [2]. Let $\mathscr{T}$ be the set of all finite Fourier series on $S^1 := \mathbb{R}/\mathbb{Z}$; $\mathscr{T}$ denotes the set of all functions of the form

$$f(\tau) = \sum_{n\in\mathbb{Z}} \hat{f}(n) \mathrm{e}^{2\pi \mathrm{i} n\tau} \quad (\tau \in S^1),$$

where $\hat{f}(n)$ are complex numbers and all but finite number of them are zeros. For each $(\varepsilon, s) \in]0,+\infty[\times\mathbb{R}$, we introduce

$$(f,g)_{\varepsilon,s} = \sum_{n\in\mathbb{Z}} \hat{f}(n)\overline{\hat{g}(n)}\varepsilon^{2|n|}\underline{n}^{2s} \qquad (f,g \in \mathscr{T}),$$

$$\|f\|_{\varepsilon,s} = \sqrt{(f,f)_{\varepsilon,s}} = \sqrt{\sum_{n\in\mathbb{Z}} |\hat{f}(n)|^2\varepsilon^{2|n|}\underline{n}^{2s}} \quad (f \in \mathscr{T}), \tag{1.8}$$

where $\underline{n} := \max\{2\pi|n|, 1\}$. They are an inner product and a norm of $\mathscr{T}$, respectively. Then, $\mathscr{X}_{\varepsilon,s}$ is defined as the completion of $\mathscr{T}$ with $\|\cdot\|_{\varepsilon,s}$ and it forms a Hilbert space. Moreover, $H^s(\Gamma)$ denotes the standard Sobolev space.

We are now in a position to state the main result of this paper, where the well-posedness and the exponential convergence of DSM under the arrangement and the definition (1.7) are established by applying the results for a circle.

Theorem 1.2. *Assume that there exists a peripheral conformal mapping Ψ of Γ with the reference radius ρ. Let $R \in]\rho, \kappa\rho[$ and suppose*

$$1 \le \delta \le \kappa; \quad \delta = 1 \Longrightarrow t > 1/2; \quad \delta = \kappa \Longrightarrow t < -1/2. \tag{1.9}$$

Suppose that the dipole and the collocation points and the dipole moments are defined as (1.7) *and that the boundary data satisfies $f_\rho \in \mathscr{X}_{\delta,t}$, where $f_\rho(\tau) = f(\Psi(\rho \mathrm{e}^{2\pi\mathrm{i}\tau}))$ for $\tau \in S^1$.*
(i) *For a sufficiently large $N \in \mathbb{N}$, there exists a unique $\{Q_k\}_{k=1}^N$ satisfying* (1.6) *and* (1.3). *Thus, an approximate solution of DSM is actually exists uniquely.*
(ii) *Assume further that*

$$R \le \sqrt{\kappa}\rho, \quad \delta > 1; \quad R = \sqrt{\kappa}\rho \Longrightarrow s > 1/2. \tag{1.10}$$

Then there exist constants $\mu = \mu(\delta) \in]0,1[$ and C such that the error estimate

$$\|u - u^{(N)}\|_{H^s(\Gamma)} \le C\mu^N \|f_\rho\|_{\delta,t}$$

holds true for a sufficiently large $N \in \mathbb{N}$, where C is independent of N.

This theorem is a readily obtainable corollary of Theorem 4.4 below. Therefore, hereafter we aim to prove Theorem 4.4 instead of Theorem 1.2 itself.

The contents of this paper are as follows. In Section 2, we collect several notions and results which will be used in analysis below. Section 3 is devoted to the case where Γ is a circle and we prove the well-posedness (cf. Theorem 3.2) and the exponential convergence (cf. Theorem 3.3). The general case is studied in Section 4 and the proof of Theorem 4.4 is described there. We conclude this paper by summarizing the results and giving some concluding remarks in Section 5.

Let us end this section with some notation to be used in this paper. We denote the lexicographic order on $]0,+\infty[\times\mathbb{R}$ by $\geq$, that is, for $(\varepsilon_\mu, s_\mu) \in]0,+\infty[\times\mathbb{R}$ $(\mu = 1,2)$, $(\varepsilon_1, s_1) \geq (\varepsilon_2, s_2)$ is defined as $\varepsilon_1 > \varepsilon_2 \vee (\varepsilon_1 = \varepsilon_2 \wedge s_1 \geq s_2)$. Furthermore we also use the relation $(\varepsilon_1, s_1) > (\varepsilon_2, s_2)$ defined as $(\varepsilon_1, s_1) \geq (\varepsilon_2, s_2) \wedge (\varepsilon_1, s_1) \neq (\varepsilon_2, s_2)$. For each $N \in \mathbb{N}$, we set $\Delta_N := \{j/N \in S^1 \mid j = 0,1,\ldots,N-1\}$. For all $m,n \in \mathbb{Z}$, $m \equiv n$ always means $m \equiv n \pmod N$.

2. Preliminaries

2.1. Function spaces $\mathscr{X}_{\varepsilon,s}$

As to the Hilbert space $\mathscr{X}_{\varepsilon,s}$, we use the following elementary result which seems not to be new for specialists.

Proposition 2.1. (i) *For all $n \in \mathbb{Z}$, the nth Fourier coefficient mapping*

$$\mathscr{T} \ni f \longmapsto \hat{f}(n) = \int_0^1 f(\theta) \mathrm{e}^{-2\pi i n\theta}\,\mathrm{d}\theta$$

has a unique bounded linear extension to $\mathscr{X}_{\varepsilon,s}$. Therefore, using these extended Fourier coefficients, we can define the norm $\|f\|_{\varepsilon,s}$ of $f \in \mathscr{X}_{\varepsilon,s}$ by (1.8).
(ii) *If $(\varepsilon_1, s_1) > (\varepsilon_2, s_2)$, then a natural inclusion $\mathscr{X}_{\varepsilon_1,s_1} \hookrightarrow \mathscr{X}_{\varepsilon_2,s_2}$ exists and is compact. Especially, we can define the union of all $\mathscr{X}_{\varepsilon,s}$; $\mathscr{X} = \bigcup_{\varepsilon,s} \mathscr{X}_{\varepsilon,s}$.*

See for details [7, Lemma 4.1]. The spaces $H^s := \mathscr{X}_{1,s}$ are periodic Sobolev spaces whose elements are distributions with period 1. The space $H^s(\Gamma)$ is defined as the set of functions whose composition with a parameterization of Γ on S^1 belongs to H^s, and its norm is given by the H^s norm of this composition. H^0 is identical with L^2 which is a space of measurable functions with period 1 which are square integrable over a period. For all $\varepsilon > 1$, the elements of $\mathscr{X}_{\varepsilon,s}$ are infinitely differentiable. For all $s > 1/2$, the elements of $\mathscr{X}_{1,s}$ are continuous functions. Finally we note that the dual space of $\mathscr{X}_{\varepsilon,s}$ is isomorphic to $\mathscr{X}_{\varepsilon^{-1},-s}$, therefore we identify them: $(\mathscr{X}_{\varepsilon,s})' = \mathscr{X}_{\varepsilon^{-1},-s}$.

2.2. Integral operators

Fix $R \in]\rho, \kappa\rho[$ and suppose that there exists some function Q defined on $\Gamma_R = \Psi(\gamma_R)$ such that the boundary data f of (1.1) can be written as a double-layer potential:

$$f(x) = \int_{\Gamma_R} \frac{-1}{2\pi} \frac{(n_y \mid x-y)}{\|x-y\|^2} Q(y)\,\mathrm{d}s_y \quad (x \in \Gamma), \tag{2.1}$$

where n_y denotes the outward unit normal vector of Γ_R at $y \in \Gamma_R$ and $\mathrm{d}s_y$ the line element of Γ_R. Then the exact solution u of (1.1) is as follows:

$$u(x) = \int_{\Gamma_R} \frac{-1}{2\pi} \frac{(n_y \mid x-y)}{\|x-y\|^2} Q(y)\,\mathrm{d}s_y \quad (x \in \Omega).$$

At this moment, our problem is reduced to find an approximation of Q. Of course, the function Q does not necessarily exist, therefore, we have to consider the above problem in the Hilbert space $\mathscr{X}_{\varepsilon,s}$ prescribed before.

In order to introduce an integral operator, we give S^1-parameterizations of Γ, Γ_R, f and Q as follows:

$$\begin{aligned}
&\Gamma\colon S^1 \ni \tau \longmapsto \Psi(\rho e^{2\pi i\tau}) \in \mathbb{C},\\
&\Gamma_R\colon S^1 \ni \tau \longmapsto \Psi(R e^{2\pi i\tau}) \in \mathbb{C},\\
&F(\tau) := f(\Psi(\rho e^{2\pi i\tau})) \quad (\tau \in S^1),\\
&q(\tau) := Q(\Psi(R e^{2\pi i\tau})) \quad (\tau \in S^1).
\end{aligned}$$

Then we can represent (2.1) as

$$F(\tau) = \int_0^1 \mathrm{Re}\left\{ \frac{-Re^{2\pi i\theta}\Psi'(Re^{2\pi i\theta})}{\Psi(\rho e^{2\pi i\tau}) - \Psi(Re^{2\pi i\theta})} \right\} q(\theta)\, d\theta \quad (\tau \in S^1).$$

Thus, if we define an integral operator A as

$$\begin{aligned}
Aq(\tau) &= \int_0^1 a(\tau,\theta) q(\theta)\, d\theta && (\tau \in S^1), \\
a(\tau,\theta) &= \mathrm{Re}\left\{ \frac{-Re^{2\pi i\theta}\Psi'(Re^{2\pi i\theta})}{\Psi(\rho e^{2\pi i\tau}) - \Psi(Re^{2\pi i\theta})} \right\} && (\tau,\theta \in S^1),
\end{aligned} \tag{2.2}$$

then the boundary condition in (1.1) is equivalent to $F = Aq$. Eventually, our problem is reduced to find an approximation of the above q.

2.3. Approximate function space

We introduce an approximate function space defined on S^1 for q as follows:

$$\mathscr{D}^{(N)} = \left\{ \sum_{k=1}^{N} Q_k \delta\left(\cdot - \frac{k-1}{N}\right) \,\middle|\, (Q_k) \in \mathbb{C}^N \right\},$$

where δ is the Dirac delta function on S^1. Concerning $\mathscr{D}^{(N)}$, the following proposition holds true. The following result, which is described in [7, Lemma 4.3] and Ogata and Katsurada [9, Lemma 2] for example, is well-known.

Proposition 2.2. (i) *For all $v \in \mathscr{D}^{(N)}$, the sequence $\{\hat{v}(n)\}_{n\in\mathbb{Z}}$ is periodic with respect to n with period N, that is, $\hat{v}(n) = \hat{v}(m)$ $(n \equiv m)$.*
(ii) *If $(\varepsilon, s) < (1, -1/2)$, then $\mathscr{D}^{(N)} \subset \mathscr{X}_{\varepsilon,s}$.*

2.4. Discrete Fourier transform

The following proposition, which will be used in order to show the well-posedness of DSM when Γ is a circle, states that the discrete Fourier transform is an isomorphism.

Proposition 2.3. *Suppose that* $(\delta,t) > (1,1/2)$ *and* $f \in \mathscr{X}_{\delta,t}$. *Then,*

$$\sum_{n\equiv p} \hat{f}(n) = 0 \quad (\forall p \in \Lambda_N) \iff f = 0 \quad \text{on } \Delta_N.$$

See for details Arnold and Wendland [3, Lemma 2.1]. Note that the above condition on f implies that f is at least continuous.

2.5. Potential theory

The following proposition is used to show that $\mathscr{A}$, which is some extension of A and will be defined later, is injective.

Proposition 2.4. *Suppose that* Γ *is a* C^2*-regular Jordan curve,* Ω *the interior simply-connected region of* Γ *and* Q *a continuous function on* Γ. *If*

$$\int_\Gamma \frac{-1}{2\pi} \frac{(n_y \mid x-y)}{\|x-y\|^2} Q(y)\,\mathrm{d}s_y = 0 \quad (x \in \Omega)$$

holds, then $Q \equiv 0$.

This proposition can be proved by using a standard potential theory, so that we omit its proof.

3. DSM in a disk

When Γ is a circle, the well-posedness and the exponential convergence of DSM is studied in [6]. However, the settings in this paper is different from that of [6], and the complete proof seems not to be given in [6]. Therefore, we state results and proofs for DSM in the case where Γ is a circle in this section.

Let Ω be a disk with radius ρ having the origin as its center: $\Omega = D_\rho$. In this case, we can take the peripheral conformal mapping Ψ as the identity mapping, and the integral operator A is reduced to an integral operator L defined as

$$Lq(\tau) = \int_0^1 \mathrm{Re}\left\{\frac{-R\mathrm{e}^{2\pi\mathrm{i}\theta}}{\rho\mathrm{e}^{2\pi\mathrm{i}\tau} - R\mathrm{e}^{2\pi\mathrm{i}\theta}}\right\} q(\theta)\,\mathrm{d}\theta \quad (\tau \in S^1)$$

for $q \in C(S^1)$. If we define a function G as

$$G(\tau) := \mathrm{Re}\left\{\frac{-R}{\rho\mathrm{e}^{2\pi\mathrm{i}\tau} - R}\right\} \quad (\tau \in S^1),$$

then Lq can be represented as the convolution of L and q;

$$Lq = G * q. \tag{3.1}$$

By direct calculation, the Fourier series expansion of G is

$$G(\tau) = 1 + \frac{1}{2} \sum_{n\in\mathbb{Z}\setminus\{0\}} \left(\frac{\rho}{R}\right)^{|n|} \mathrm{e}^{2\pi\mathrm{i}n\tau} \quad (\tau \in S^1). \tag{3.2}$$

Then the nth Fourier coefficient of Lq can be calculated as

$$(Lq)^\wedge(n) = \hat{G}(n)\hat{q}(n) \quad (n \in \mathbb{Z}), \quad \hat{G}(n) = \begin{cases} 1 & (n=0), \\ \dfrac{1}{2}\left(\dfrac{\rho}{R}\right)^{|n|} & (n \neq 0) \end{cases}$$

owing to (3.1) and (3.2).

In order to deal with the considered problem on the Hilbert space $\mathscr{X}_{\varepsilon,s}$, we have to extend L to $\mathscr{X}_{\varepsilon,s}$.

Lemma 3.1. *For each $(\varepsilon,s) \in]0,+\infty[\times\mathbb{R}$, we define an operator $\mathscr{L}\colon \mathscr{X}_{\varepsilon,s} \to \mathscr{X}_{\varepsilon R/\rho,s}$ as $\mathscr{L}q = G * q$. Then, $\mathscr{L}$ is a bounded linear extension of L and an isomorphism.*

Proof. For all $q \in \mathscr{X}_{\varepsilon,s}$, we have

$$\|\mathscr{L}q\|^2_{\varepsilon R/\rho,s} = \sum_{n\in\mathbb{Z}} |(\mathscr{L}q)^\wedge(n)|^2 \left(\frac{\varepsilon R}{\rho}\right)^{2|n|} \underline{n}^{2s} = |\hat{q}(0)|^2 + \frac{1}{4}\sum_{n\in\mathbb{Z}\setminus\{0\}} |\hat{q}(n)|^2\varepsilon^{2|n|}\underline{n}^{2s}.$$

Therefore we obtain

$$\frac{1}{4}\|q\|^2_{\varepsilon,s} \leq \|\mathscr{L}q\|^2_{\varepsilon,s} \leq \|q\|^2_{\varepsilon,s}. \tag{3.3}$$

The linearity of $\mathscr{L}$ is clear, and its boundedness follows from the right inequality of (3.3). The bijectivity can be shown easily, so we omit the detail of it. The continuity of $\mathscr{L}^{-1}$ follows from the left inequality of (3.3). □

We can here rewrite the boundary condition (1.3) by virtue of the extended operator $\mathscr{L}$. We take $q^{(N)} \in \mathscr{D}^{(N)}$ arbitrarily and write it as

$$q^{(N)} = \sum_{k=1}^{N} Q_k\delta\left(\cdot - \frac{k-1}{N}\right).$$

Then we have

$$\mathscr{L}q^{(N)}(\tau) = \sum_{k=1}^{N} 2\pi R Q_k \operatorname{Re}\left\{\frac{-1}{2\pi}\frac{\omega^{k-1}}{\rho e^{2\pi i\tau} - y_k}\right\}.$$

Therefore the unique solvability of (1.3) is equivalent to that of

$$\mathscr{L}q^{(N)} = F \quad \text{on } \Delta_N. \tag{3.4}$$

As to the unique solvability of (3.4), the following theorem holds, which assures the well-posedness of DSM when Ω is a disk.

Theorem 3.2. *Let $0 < \rho < R$ and $(\delta,t) > (1,1/2)$. Then, for all $F \in \mathscr{X}_{\delta,t}$, there uniquely exists $q^{(N)} \in \mathscr{D}^{(N)}$ which satisfies* (3.4)*, and its Fourier coefficients are given by*

$$\hat{q}^{(N)}(p) = \left(\sum_{m\equiv p}\hat{F}(m)\right)\Big/\varphi_p^{(N)}(\rho) \quad (p \in \Lambda_N), \tag{3.5}$$

where

$$\varphi_p^{(N)}(\rho) = \sum_{m\equiv p}\hat{G}(m).$$

Proof. By Proposition 2.3, (3.4) is equivalent to

$$\sum_{m\equiv p} \hat{G}(m)\hat{q}^{(N)}(m) = \sum_{m\equiv p} \hat{F}(m) \quad (\forall p \in \Lambda_N).$$

Since $\hat{q}^{(N)}(m)$ is periodic with respect to m with period N because of Proposition 2.2, (3.4) is equivalent to

$$\varphi_p^{(N)}(\rho)\hat{q}^{(N)}(p) = \sum_{m\equiv p} \hat{F}(m) \quad (\forall p \in \Lambda_N). \tag{3.6}$$

Since $\varphi_p^{(N)}(\rho) \neq 0$ for all $p \in \mathbb{Z}$, (3.6) is uniquely solvable and its Fourier coefficients are given by (3.5). □

We next give the error estimate of DSM, which asserts the exponential convergence of DSM.

Theorem 3.3. *Let* $0 < \rho < R$, $(\delta,t) > (1,1/2)$ *and* (ε,s) *satisfies the following conditions:*

$$\max\left\{\delta\left(\frac{\rho}{R}\right)^2, \frac{1}{\delta}\right\} \leq \varepsilon \leq \min\left\{\frac{1}{\delta}\left(\frac{R}{\rho}\right)^2, \delta\right\}; \quad \varepsilon = \delta \Longrightarrow s \leq t; \quad \varepsilon = \frac{R}{\rho} \Longrightarrow s < -\frac{1}{2}. \tag{3.7}$$

Then, there exist some positive constant $C = C(\varepsilon,s,\delta,t,\rho,R,\|\mathscr{L}\|,\|\mathscr{L}^{-1}\|)$ *and real constant* $P = P(\varepsilon,s,\delta,t)$ *such that for all* $F \in \mathscr{X}_{\delta,t}$*, all* $N \in \mathbb{N}$ *and the unique solution* $q^{(N)} \in \mathscr{D}^{(N)}$ *of* $\mathscr{L}q^{(N)} = F$ *on* Δ_N*, of which the existence is assured by Theorem 3.2, the following error estimate holds:*

$$\|F - \mathscr{L}q^{(N)}\|_{\varepsilon,s} \leq CN^P \left(\frac{\varepsilon}{\delta}\right)^{N/2} \|F\|_{\delta,t}.$$

Remark 3.4. The first inequalities in (3.7) on δ and ε are seemed to be rather complicated. We subsidiary use a graph in order to understand the condition graphically. We set

$$\begin{aligned}
H_1 &= \{(\delta,\varepsilon) \mid 1 \leq \delta \leq R/\rho, \varepsilon = \delta^{-1}\},\\
H_2 &= \{(\delta,\varepsilon) \mid R/\rho \leq \delta \leq (R/\rho)^2, \varepsilon = (R/\rho)^2\delta^{-1}\},\\
L_1 &= \{(\delta,\varepsilon) \mid R/\rho \leq \delta \leq (R/\rho)^2, \varepsilon = \delta(\rho/R)^2\},\\
L_2 &= \{(\delta,\varepsilon) \mid 1 \leq \delta \leq R/\rho, \varepsilon = \delta\},\\
C_1 &= (R/\rho, \rho/R), \quad C_2 = ((R/\rho)^2, 1),
\end{aligned}$$

and $\mathscr{I}$ as a closed region surrounded by $H_1 \cup L_1 \cup H_2 \cup L_2$. Then δ and ε satisfy the first inequalities in (3.7) if and only if $(\delta,\varepsilon) \in \mathscr{I}$ (see Figure 1).

Remark 3.5. The exponent P in Theorem 3.3 can be taken as follows:

$$P = P(\varepsilon,s,\delta,t) = \begin{cases}
\max\{s-t, 0, -t\} & ((\delta,\varepsilon) = C_1),\\
\max\{s-t, 0, s\} & ((\delta,\varepsilon) = C_2),\\
\max\{s-t, -t\} & ((\delta,\varepsilon) \in H_1 \setminus \{C_1\}),\\
\max\{s-t, s\} & ((\delta,\varepsilon) \in H_2 \setminus \{C_2\}),\\
\max\{s-t, 0\} & ((\delta,\varepsilon) \in L_1 \setminus \{C_1, C_2\}),\\
s-t & (\text{otherwise}).
\end{cases}$$

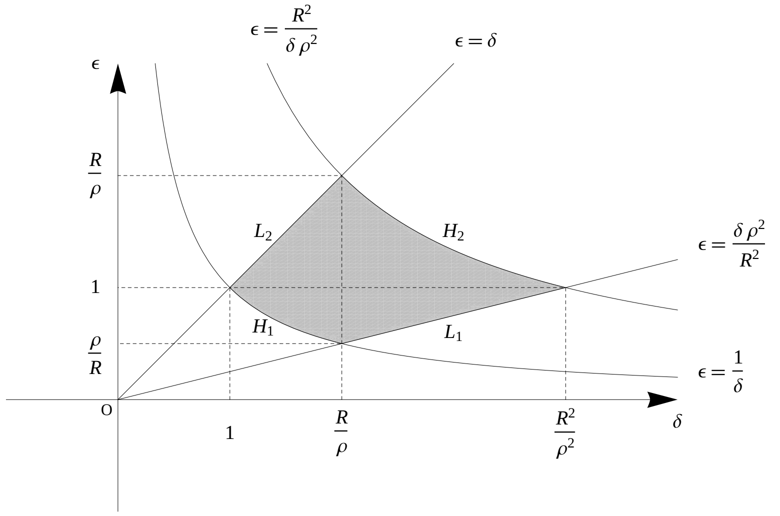

Figure 1: The region $\mathscr{I}$ of (δ,ε)

In order to prove Theorem 3.3, we need the following lemma.

Lemma 3.6. *Suppose that* $0<\rho<R$, $(\delta,t)>(1,1/2)$ *and* (ε,s) *satisfies* (3.7)*. Then there exists some positive constant* $C=C(\varepsilon,s,\delta,t,\rho,R)$ *such that for all* $F\in\mathscr{X}_{\delta,t}$, $q\in\mathscr{X}_{\delta\rho/R,t}$ *with* $\mathscr{L}q=F$ *and* $q^{(N)}\in\mathscr{D}^{(N)}$ *with* $\mathscr{L}q^{(N)}=F$ *on* Δ_N*, of which the unique existence is assured by Theorem 3.2, the following estimate holds:*

$$\|q-q^{(N)}\|_{\varepsilon\rho/R,s}\le CN^{P(\varepsilon,s,\delta,t)}\left(\frac{\varepsilon}{\delta}\right)^{N/2}\|q\|_{\delta\rho/R,t}.$$

We postpone the proof of Lemma 3.6 to the appendix, and give the proof of Theorem 3.3 by virtue of Lemma 3.6.

Proof of Theorem 3.3. Since $\mathscr{L}$ is an isomorphism owing to Lemma 3.6, we have

$$\begin{aligned}\|F-\mathscr{L}q^{(N)}\|_{\varepsilon,s}&=\|\mathscr{L}q-\mathscr{L}q^{(N)}\|_{\varepsilon,s}\le C\|q-q^{(N)}\|_{\varepsilon\rho/R,s}\\&\le CN^{P(\varepsilon,s,\delta,t)}\left(\frac{\varepsilon}{\delta}\right)^{N/2}\|q\|_{\delta\rho/R,t}\le CN^{P(\varepsilon,s,\delta,t)}\left(\frac{\varepsilon}{\delta}\right)^{N/2}\|\mathscr{L}q\|_{\delta,t}\\&=CN^{P(\varepsilon,s,\delta,t)}\left(\frac{\varepsilon}{\delta}\right)^{N/2}\|F\|_{\delta,t},\end{aligned}$$

where C denotes some positive constant independent of N and may mean another one with respect to each expression here and hereafter. □

4. DSM in a Jordan region

At first, we shall extend an integral operator A defined by (2.2) to $\mathscr{X}_{\varepsilon,s}$. To this end, we define a perturbation operator K as

$$K = A - L. \tag{4.1}$$

If q is a continuous function on S^1, then we have

$$Kq(\tau) = \int_0^1 k(\tau,\theta) q(\theta)\,\mathrm{d}\theta,$$

where

$$k(\tau,\theta) = \mathrm{Re}\left\{ \frac{-R\mathrm{e}^{2\pi\mathrm{i}\theta}\Psi'(R\mathrm{e}^{2\pi\mathrm{i}\theta})}{\Psi(\rho\mathrm{e}^{2\pi\mathrm{i}\tau}) - \Psi(R\mathrm{e}^{2\pi\mathrm{i}\theta})} + \frac{R\mathrm{e}^{2\pi\mathrm{i}\theta}}{\rho\mathrm{e}^{2\pi\mathrm{i}\tau} - R\mathrm{e}^{2\pi\mathrm{i}\theta}} \right\} \quad (\tau,\theta \in S^1).$$

Thus the lth Fourier coefficient of Kq can be calculated as

$$(Kq)^\wedge(l) = \sum_{m\in\mathbb{Z}} \hat{k}(l,m)\hat{q}(-m),$$

where $\hat{k}(l,m)$ is the double Fourier coefficient defined as

$$\hat{k}(l,m) = \int_0^1 \int_0^1 k(\tau,\theta)\mathrm{e}^{-2\pi\mathrm{i}(l\tau+m\theta)}\,\mathrm{d}\tau\,\mathrm{d}\theta.$$

We require estimates on $\hat{k}(l,m)$ to extend K to $\mathscr{X}_{\varepsilon,s}$.

Lemma 4.1. *There exists some positive constant C independent of N such that*

$$|\hat{k}(l,m)| \le C\kappa^{-|l|}\left(\frac{R}{\kappa\rho}\right)^{|m|}$$

holds for all $l,m \in \mathbb{Z}$.

By using above estimates, we can extend K as follows:

Lemma 4.2. *Suppose that $(\varepsilon,s) > (R/(\kappa\rho), 1/2)$ and $(\delta,t) < (\kappa,-1/2)$. If we define $\mathscr{K} : \mathscr{X}_{\varepsilon,s} \to \mathscr{X}_{\delta,t}$ as*

$$(\mathscr{K}q)^\wedge(l) = \sum_{m\in\mathbb{Z}} \hat{k}(l,m)\hat{q}(-m), \quad l \in \mathbb{Z},$$

then $\mathscr{K}$ is a bounded linear extension of K and compact.

Proof. For all $q \in \mathscr{X}_{\varepsilon,s}$ we have

$$\begin{aligned}
\|\mathscr{K}q\|_{\delta,t}^2 &= \sum_{l\in\mathbb{Z}} |(\mathscr{K}q)^\wedge(l)|^2\delta^{2|l|}\underline{l}^{2t} = \sum_{l\in\mathbb{Z}} \left| \sum_{m\in\mathbb{Z}} \hat{k}(l,m)\hat{q}(-m) \right|^2 \delta^{2|l|}\underline{l}^{2t} \\
&\le \sum_{l\in\mathbb{Z}} \left(\sum_{m\in\mathbb{Z}} |\hat{k}(l,m)|^2 \varepsilon^{-2|m|}\underline{m}^{-2s} \right) \left(\sum_{m\in\mathbb{Z}} |\hat{q}(-m)|^2 \varepsilon^{2|m|}\underline{m}^{2s} \right) \delta^{2|l|}\underline{l}^{2t} \\
&\le C \sum_{l\in\mathbb{Z}} \left(\frac{\delta}{\kappa}\right)^{2|l|} \underline{l}^{2t} \sum_{m\in\mathbb{Z}} \left(\frac{1}{\varepsilon}\frac{R}{\kappa\rho}\right)^{2|m|} \underline{m}^{-2s} \|q\|_{\varepsilon,s}^2 \le C\|q\|_{\varepsilon,s}^2.
\end{aligned}$$

This implies that $\mathscr{K}$ is a bounded linear operator.

In order to see the compactness of $\mathscr{K}$, we take $(\delta',t') \in]0,+\infty[\times\mathbb{R}$ to satisfy $(\delta,t) < (\delta',t') < (\kappa,-1/2)$, and decompose it as follows:

$$\begin{array}{ccccc} \mathscr{K}: & \mathscr{X}_{\varepsilon,s} & \longrightarrow & & \mathscr{X}_{\delta,t} \\ & & \tilde{\mathscr{K}} \searrow & & \nearrow i \\ & & & \mathscr{X}_{\delta',t'} & \end{array}$$

Here $\tilde{\mathscr{K}} : \mathscr{X}_{\varepsilon,s} \to \mathscr{X}_{\delta',t'}$ is a bounded linear operator defined as well as $\mathscr{K}$ and i a natural inclusion, which is a compact operator, assured its existence by Proposition 2.1. Since $\mathscr{K} = i \circ \tilde{\mathscr{K}}$, $\mathscr{K}$ is compact. □

The following corollary immediately follows from the above lemma.

Corollary 4.3. *If* (ε,s) *satisfies*

$$\left(\frac{R}{\kappa\rho},\frac{1}{2}\right) < (\varepsilon,s) < \left(\frac{\kappa\rho}{R},-\frac{1}{2}\right), \tag{4.2}$$

then the operator $\mathscr{K} : \mathscr{X}_{\varepsilon,s} \to \mathscr{X}_{\varepsilon R/\rho,s}$ *is compact.*

When (ε,s) satisfies the condition (4.2), we define $\mathscr{A} : \mathscr{X}_{\varepsilon,s} \to \mathscr{X}_{\varepsilon R/\rho,s}$ as $\mathscr{A} = \mathscr{K} + \mathscr{L}$. Then $\mathscr{A}$ is an extension of A. We can now state the most general version of Theorem 1.2.

Theorem 4.4. *Suppose that* $R \in]\rho,\kappa\rho[$ *and* $(1,1/2) < (\delta,t) < (\kappa,-1/2)$. *Then the following hold true:*
(i) *For sufficiently large* $N \in \mathbb{N}$ *and all* $F \in \mathscr{X}_{\delta,t}$, *there exists a unique* $q^{(N)} \in \mathscr{D}^{(N)}$ *such that*

$$\mathscr{A} q^{(N)}(x) = F(x), \quad x \in \Delta_N.$$

(ii) *Suppose further that* (ε,s) *satisfies the following conditions:*

$$\max\left\{\delta\left(\frac{\rho}{R}\right)^2,\frac{1}{\delta}\right\} \le \varepsilon \le \min\left\{\frac{1}{\delta}\left(\frac{R}{\rho}\right)^2,\delta\right\}; \quad \varepsilon = \delta \Longrightarrow s < t; \quad \varepsilon = \frac{R}{\rho} \Longrightarrow s < -\frac{1}{2}$$

and $(\varepsilon,s) > ((R/\rho)^2\kappa^{-1},1/2)$. *Then there exists some positive constant C which depends on* ε, *s,* δ, *t,* ρ, *R,* $\|\mathscr{L}\|$, $\|\mathscr{L}^{-1}\|$, $\|\mathscr{A}\|$, $\|\mathscr{A}^{-1}\|$ *such that*

$$\|F - \mathscr{A} q^{(N)}\|_{\varepsilon,s} \le C N^{P(\varepsilon,s,\delta,t)} \left(\frac{\varepsilon}{\delta}\right)^{N/2} \|F\|_{\delta,t}.$$

In order to prove the above theorem, we need the following two lemmas.

Lemma 4.5. *Suppose that* $R \in]\rho,\kappa\rho[$ *and that* (ε,s) *satisfies* (4.2). *Then* $\mathscr{A}$ *is bounded and isomorphic.*

Proof. The boundedness of $\mathscr{A}$ is clear. Concerning that $\mathscr{A}$ is isomorphic, we only have to show that $\mathscr{A}$ is injective since $\mathscr{A}$ is a Fredholm operator with index 0. We take $q \in \operatorname{Ker}\mathscr{A}$ arbitrarily. Since $\mathscr{L}$ is an isomorphism, $\mathscr{A}q = 0$ is equivalent to $q = -\mathscr{L}^{-1}\mathscr{K}q$. Then we have $\mathscr{K}q \in \mathscr{X}_{\kappa,t}$ for all $t < -1/2$ since $\mathscr{K}\colon \mathscr{X}_{\varepsilon,s} \to \mathscr{X}_{\delta,t}$ defines a bounded linear operator when $(\varepsilon,s) > (R/(\kappa\rho),1/2)$ and $(\delta,t) < (\kappa,-1/2)$ are satisfied due to Lemma 4.2. Therefore $q = -\mathscr{L}^{-1}\mathscr{K}q \in \mathscr{X}_{\kappa\rho/R,t}$. Note that $\kappa\rho/R > 1$. Defining a function Q on Γ_R as

$$Q(\Psi(R\mathrm{e}^{2\pi\mathrm{i}\tau})) = q(\tau) \quad (\tau \in S^1),$$

$Q\colon \Gamma_R \to \mathbb{C}$ is continuous. Then we have

$$\begin{aligned}
\mathscr{A}q = 0 &\iff \int_0^1 \operatorname{Re}\left\{\frac{-R\mathrm{e}^{2\pi\mathrm{i}\theta}\Psi'(R\mathrm{e}^{2\pi\mathrm{i}\theta})}{\Psi(\rho\mathrm{e}^{2\pi\mathrm{i}\tau}) - \Psi(R\mathrm{e}^{2\pi\mathrm{i}\theta})}\right\} q(\theta)\,\mathrm{d}\theta = 0 \quad (\forall\tau \in S^1)\\
&\iff \underbrace{\int_{\Gamma_R} \frac{-1}{2\pi}\frac{(n_y \mid x-y)}{\|x-y\|^2} Q(y)\,\mathrm{d}s_y}_{=:u(x)} = 0 \quad (\forall x \in \Gamma).
\end{aligned}$$

The function u is harmonic in the interior simply-connected region Ω_R of Γ_R, and especially continuous on $\overline{\Omega}$. Thus we have $u = 0$ in Ω thanks to the maximum principle for harmonic functions. Furthermore we have $u = 0$ in Ω_R because of the identity theorem for real analytic functions (see for instance Axler, Bourdon and Ramey [4]). Hence $Q \equiv 0$ follows from Proposition 2.4, and this yields $q \equiv 0$. □

Lemma 4.6. *Suppose that the following conditions are satisfied: $R \in]\rho,\kappa\rho[$; $(1,1/2) < (\delta,t) < (\kappa,-1/2)$; (3.7) and $(\varepsilon,s) > ((R/\rho)^2\kappa^{-1},1/2)$. Then there exists some positive constant C which depends on ε, s, δ, t, ρ, R, $\|\mathscr{L}\|$, $\|\mathscr{L}^{-1}\|$, $\|\mathscr{A}\|$, $\|\mathscr{A}^{-1}\|$ such that for all $N \in \mathbb{N}$, all $q \in \mathscr{X}_{\delta\rho/R,t}$ and all $q^{(N)} \in \mathscr{D}^{(N)}$ satisfying $\mathscr{A}q = \mathscr{A}q^{(N)}$ on Δ_N, the following estimate holds:*

$$\|q - q^{(N)}\|_{\varepsilon\rho/R,s} \le CN^{P(\varepsilon,s,\delta,t)}\left(\frac{\varepsilon}{\delta}\right)^{N/2}\left(\|q\|_{\delta\rho/R,t} + \|q - q^{(N)}\|_{\varepsilon\rho/R,s}\right),$$

where C is independent of N.

Proof. Since both $\mathscr{L}$ and $\mathscr{A}$ are isomorphic, the following estimate holds:

$$\|q - q^{(N)}\|_{\varepsilon\rho/R,s} \le C\|\mathscr{A}(q^{(N)} - q)\|_{\varepsilon,s} \le C\|\mathscr{L}^{-1}\mathscr{A}(q^{(N)} - q)\|_{\varepsilon\rho/R,s}.$$

Here we put

$$w_N = q^{(N)}, \quad w = q^{(N)} - \mathscr{L}^{-1}\mathscr{A}(q^{(N)} - q).$$

Then we have

$$\begin{aligned}
&w_N \in \mathscr{D}^{(N)}, \quad \mathscr{L}^{-1}\mathscr{A}(q^{(N)} - q) = w_N - w, \quad \mathscr{L}w = \mathscr{L}w_N \quad \text{on } \Delta_N,\\
&w = q + \mathscr{L}^{-1}(\mathscr{L} - \mathscr{A})(q^{(N)} - q).
\end{aligned}$$

Therefore, by Theorem 3.3 which gives the error estimate of DSM when Ω is a disk, we have

$$\begin{aligned} \|q^{(N)} - q\|_{\varepsilon\rho/R,s} &\le C\|\mathscr{L} w_N - \mathscr{L} w\|_{\varepsilon,s} \le C N^{P(\varepsilon,s,\delta,t)} \left(\frac{\varepsilon}{\delta}\right)^{N/2} \|\mathscr{L} w\|_{\delta,t} \\ &\le C N^{P(\varepsilon,s,\delta,t)} \left(\frac{\varepsilon}{\delta}\right)^{N/2} \|w\|_{\delta\rho/R,t}. \end{aligned} \tag{4.3}$$

Moreover we have

$$\begin{aligned} \|w\|_{\delta\rho/R,t} &= \|q + \mathscr{L}^{-1}(\mathscr{L} - \mathscr{A})(q^{(N)} - q)\|_{\delta\rho/R,t} \le \|q\|_{\delta\rho/R,t} + C\|(\mathscr{L} - \mathscr{A})(q^{(N)} - q)\|_{\delta,t} \\ &\le \|q\|_{\delta\rho/R,t} + C\|q^{(N)} - q\|_{\varepsilon\rho/R,s} \le C\left(\|q\|_{\delta\rho/R,t} + \|q^{(N)} - q\|_{\varepsilon\rho/R,s}\right), \end{aligned} \tag{4.4}$$

where we use the boundedness of $\mathscr{L} - \mathscr{A} = -\mathscr{K} : \mathscr{X}_{\varepsilon\rho/R,s} \to \mathscr{X}_{\delta,t}$. Combining (4.3) with (4.4), we obtain the desired estimate. □

Proof of Theorem 4.4. At first, we remark that

$$N^{P(\varepsilon,s,\delta,t)} \left(\frac{\varepsilon}{\delta}\right)^{N/2} = o(1) \quad \text{as } N \to \infty$$

holds since $(\varepsilon, s) < (\delta, t)$. Therefore, by Lemma 4.6, for a sufficiently large $N \in \mathbb{N}$ and all $q^{(N)} \in \mathscr{D}^{(N)}$ with $\mathscr{A} q^{(N)} = \mathscr{A} q$ on Δ_N, we have

$$\|q - q^{(N)}\|_{\varepsilon\rho/R,s} \le C N^{P(\varepsilon,s,\delta,t)} \left(\frac{\varepsilon}{\delta}\right)^{N/2} \|q\|_{\delta\rho/R,t}. \tag{4.5}$$

Therefore $\mathscr{A} q^{(N)} = 0$ on Δ_N yields $q^{(N)} = 0$. Since $\mathscr{A} q^{(N)} = F$ on Δ_N is equivalent to the system of finite linear equations, this shows the unique solvability of the considered functional equation.

Finally we prove the second statement. Since $\mathscr{A}$ is an isomorphism of $\mathscr{X}_{\delta\rho/R,t}$ onto $\mathscr{X}_{\delta,t}$, for $F \in \mathscr{X}_{\delta,t}$, there uniquely exists $q \in \mathscr{X}_{\delta\rho/R,t}$ which satisfies $\mathscr{A} q = F$. Then we have

$$\|F - \mathscr{A} q^{(N)}\|_{\varepsilon,s} = \|\mathscr{A} q - \mathscr{A} q^{(N)}\|_{\varepsilon,s} \le C\|q - q^{(N)}\|_{\varepsilon\rho/R,s} \quad \text{and} \quad \|q\|_{\delta\rho/R,t} \le C\|F\|_{\delta,t}.$$

Hence we obtain the desired error estimate by the above two inequalities and (4.5). □

5. Concluding remarks

In the present paper, we introduced the concept of peripheral conformal mapping following Katsurada [7], and used it to arrange the dipole and collocation points and to define the dipole moments. Under this situation, we proved the well-posedness and the exponential convergence of DSM.

One of researches to be continued is to extend this result to multiply-connected region's case. However, it may be considered that the original DSM cannot be applied to potential problem in multiply-connected region, therefore we may need some modification.

A. Proof of Lemma 3.6

We here prove Lemma 3.6. The basic idea is the same as [7].

Firstly note that $\hat{q}^{(N)}(p)$ can be represented as

$$\hat{q}^{(N)}(p) = \left(\sum_{m \equiv p} \hat{G}(m)\hat{q}(m) \right) / \varphi_p^{(N)}(\rho).$$

We decompose and estimate $\|q - q^{(N)}\|_{\varepsilon\rho/R,s}$ as follows:

$$\begin{aligned}\|q - q^{(N)}\|^2_{\varepsilon\rho/R,s} &= |\hat{q}(0) - \hat{q}^{(N)}(0)|^2 + \sum_{n \in \mathbb{Z}\setminus\{0\}} |\hat{q}(n) - \hat{q}^{(N)}(n)|^2 \left(\frac{\varepsilon\rho}{R}\right)^2 \underline{n}^{2s} \\ &\le T_1 + (2\pi)^{2s}\{T_2 + 2T_3 + 2T_4\},\end{aligned}$$

where $\Lambda_N := \{p \in \mathbb{Z} \mid -N/2 < p \le N/2\}$ and

$$T_1 = |\hat{q}(0) - \hat{q}^{(N)}(0)|^2, \quad T_2 = \sum_{n \in \Lambda_N \setminus\{0\}} |\hat{q}(n) - \hat{q}^{(N)}(n)|^2 \left(\frac{\varepsilon\rho}{R}\right)^{2|n|} |n|^{2s},$$

$$T_3 = \sum_{n \in \mathbb{Z}\setminus\Lambda_N} |\hat{q}(n)|^2 \left(\frac{\varepsilon\rho}{R}\right)^{2|n|} |n|^{2s}, \quad T_4 = \sum_{n \in \mathbb{Z}\setminus\Lambda_N} |\hat{q}^{(N)}(n)|^2 \left(\frac{\varepsilon\rho}{R}\right)^{2|n|} |n|^{2s}.$$

We frequently use the following proposition without proof for proving Lemma 3.6.

Proposition A.1. (i) *For arbitrary $N \in \mathbb{N}$ we have $|\varphi_0^{(N)}(\rho)| \ge 1$.*
(ii) *For each $n \in \Lambda_N \setminus \{0\}$ we have $|\varphi_n^{(N)}(\rho)| \ge 2^{-1}(\rho/R)^{|n|}$.*
(iii) *For all $\varepsilon \in]0,1[$ and all $t \in \mathbb{R}$ there exists some positive constant $C_{\varepsilon,t}$ such that*

$$\max_{p \in \Lambda_N\setminus\{0\}} \left\{ \left(\frac{N}{|p|}\right)^t \varepsilon^{N-2|p|} \right\} \le C_{\varepsilon,t}$$

holds for all $N \in \mathbb{N}$.
(iv) *For all $(\varepsilon, s) < (1, -1)$ there exists some positive constant $C_{\varepsilon,s}$ such that*

$$\sum_{m \in I(p)} |m|^s \varepsilon^{|m|} \le C_{\varepsilon,s} N^s \varepsilon^{N-|p|}$$

holds for all $N \in \mathbb{N}$ and all $p \in \Lambda_N$, where $I(p) = \{p + lN \mid l \in \mathbb{Z}\setminus\{0\}\}$.

In the remainder of this section we estimate each T_j ($j = 1,2,3,4$). Since

$$\hat{q}(0) - \hat{q}^{(N)}(0) = \left[\sum_{m \in I(0)} \hat{G}(m)\hat{q}(0) - \sum_{m \in I(0)} \hat{G}(m)\hat{q}(m) \right] / \varphi_0^{(N)}(\rho)$$

we have

$$T_1 = |\hat{q}(0) - \hat{q}^{(N)}(0)|^2 \le T_{11} + T_{12},$$

where $I(p) := \{p + lN \mid l \in \mathbb{Z}\setminus\{0\}\}$ and

$$T_{11} = \frac{2}{|\varphi_0^{(N)}(\rho)|^2} \left| \sum_{m \in I(0)} \hat{G}(m)\hat{q}(0) \right|^2, \quad T_{12} = \frac{2}{|\varphi_0^{(N)}(\rho)|^2} \left| \sum_{m \in I(0)} \hat{G}(m)\hat{q}(m) \right|^2.$$

From Proposition A.1 (i) and the assumption $\delta(\rho/R)^2 \le \varepsilon$ we obtain

$$T_{11} \le 2\left(\sum_{l=1}^{\infty}\left(\frac{\rho}{R}\right)^{lN}\right)^2 |\hat{q}(0)|^2 \le C_{11}\left(\frac{\rho}{R}\right)^{2N}\|q\|^2_{\delta\rho/R,t}$$
$$\le \begin{cases} C_{11}\left(\frac{\varepsilon}{\delta}\right)^N \|q\|^2_{\delta\rho/R,t} & \text{if } \varepsilon = \delta(\rho/R)^2 \wedge s-t \le 0, \\ C_{11}N^{2(s-t)}\left(\frac{\varepsilon}{\delta}\right)^N \|q\|^2_{\delta\rho/R,t} & \text{otherwise.} \end{cases}$$

Here and hereafter $C^{\text{superscript}}_{\text{subscript}}$ denotes some constant independent of N and it may represent another constant in each symbol. By Proposition A.1 (i) we have

$$T_{12} \le 2\left(\sum_{m\in I(0)}\frac{1}{2}\left(\frac{\rho}{R}\right)^{|m|}|\hat{q}(m)|\right)^2 \le \frac{1}{2}\left(\sum_{m\in I(0)}|\hat{q}(m)|^2\left(\frac{\delta\rho}{R}\right)^{2|m|}\underline{m}^{2t}\right)\underline{\left(\sum_{m\in I(0)}\frac{1}{\delta^{2|m|}}\frac{1}{\underline{m}^{2t}}\right)}$$
$$\le C_{12}\delta^{-2N}N^{-2t}\|q\|^2_{\delta\rho/R,t} \le \begin{cases} C_{12}N^{-2t}\left(\frac{\varepsilon}{\delta}\right)^N \|q\|^2_{\delta\rho/R,t} & \text{if } \delta^{-1} = \varepsilon \wedge s \le 0, \\ C_{12}N^{2(s-t)}\left(\frac{\varepsilon}{\delta}\right)^N \|q\|^2_{\delta\rho/R,t} & \text{otherwise.} \end{cases}$$

Here note that the underlined infinite sum is estimated by using Proposition A.1 (iv) and we use the assumption $\delta^{-1} \le \varepsilon$.

Next we estimate T_2. For $n \in \Lambda_N \setminus \{0\}$ we have

$$\hat{q}(n) - \hat{q}^{(N)}(n) = \left[\sum_{m\in I(n)}\hat{G}(m)\hat{q}(n) - \sum_{m\in I(n)}\hat{G}(m)\hat{q}(m)\right]/\varphi_n^{(N)}(\rho),$$

therefore

$$|\hat{q}(n) - \hat{q}^{(N)}(n)|^2 \le \frac{2}{|\varphi_n^{(N)}(\rho)|^2}\left[\left|\sum_{m\in I(n)}\hat{G}(m)\hat{q}(n)\right|^2 + \left|\sum_{m\in I(n)}\hat{G}(m)\hat{q}(m)\right|^2\right]$$

holds. Thus we obtain $T_2 \le T_{21} + T_{22}$, where

$$T_{21} = \sum_{n\in\Lambda_N\setminus\{0\}}\frac{2}{|\varphi_n^{(N)}(\rho)|^2}\left|\sum_{m\in I(n)}\hat{G}(m)\hat{q}(n)\right|^2\left(\frac{\varepsilon\rho}{R}\right)^{2|n|}|n|^{2s},$$
$$T_{22} = \sum_{n\in\Lambda_N\setminus\{0\}}\frac{2}{|\varphi_n^{(N)}(\rho)|^2}\left|\sum_{m\in I(n)}\hat{G}(m)\hat{q}(m)\right|^2\left(\frac{\varepsilon\rho}{R}\right)^{2|n|}|n|^{2s}.$$

As to T_{21}, from Proposition A.1 (ii), we have

$$\frac{1}{|\varphi_n^{(N)}(\rho)|^2}\left|\sum_{m\in I(n)}\hat{G}(m)\right|^2 \le 4\left(\frac{R}{\rho}\right)^{2|n|}\left|\sum_{m\in I(n)}\frac{1}{2}\left(\frac{\rho}{R}\right)^{|m|}\right|^2 \le C_{21}\left(\frac{\rho}{R}\right)^{2(N-2|n|)}$$

for $n \in \Lambda_N \setminus \{0\}$, therefore

$$T_{21} \le C_{21}\sum_{n\in\Lambda_N\setminus\{0\}}|\hat{q}(n)|^2\left(\frac{\rho}{R}\right)^{2(N-2|n|)}\left(\frac{\varepsilon\rho}{R}\right)^{2|n|}|n|^{2s}$$
$$\le C_{21}N^{2\max\{s-t,0\}}\left(\frac{\varepsilon}{\delta}\right)^N\|q\|^2_{\delta\rho/R,t}\underbrace{\sup_{n\in\Lambda_N\setminus\{0\}}\left\{|n|^{2(s-t)}N^{-2\max\{s-t,0\}}\left\{\frac{\delta}{\varepsilon}\left(\frac{\rho}{R}\right)^2\right\}^{N-2|n|}\right\}}_{=:A_{21}}.$$

Taking care of $\varepsilon \geq \delta(\rho/R)^2$ the above supremum A_{21} is bounded as follows due to Proposition A.1 (iii):

$$A_{21} \leq \begin{cases} 1 & \text{if } \varepsilon = \delta(\rho/R)^2 \wedge s \leq t, \\ 4^{-(s-t)} & \text{if } \varepsilon = \delta(\rho/R)^2 \wedge s > t, \\ C_{-2(s-t),\delta\varepsilon^{-1}(\rho/R)^2} N^{2(s-t)} & \text{if } \varepsilon > \delta(\rho/R)^2 \wedge s \leq t, \\ C_{-2(s-t),\delta\varepsilon^{-1}(\rho/R)^2} & \text{if } \varepsilon > \delta(\rho/R)^2 \wedge s > t. \end{cases}$$

Therefore we obtain

$$T_{21} \leq \begin{cases} C_{21} \left(\frac{\varepsilon}{\delta}\right)^N \|q\|^2_{\delta\rho/R,t} & \text{if } \varepsilon = \delta(\rho/R)^2 \wedge s \leq t, \\ C_{21} N^{2(s-t)} \left(\frac{\varepsilon}{\delta}\right)^N \|q\|^2_{\delta\rho/R,t} & \text{otherwise.} \end{cases}$$

For $n \in \Lambda_N \setminus \{0\}$ we have

$$\begin{aligned} \frac{1}{|\varphi_n^{(N)}(\rho)|^2} \left| \sum_{m\in I(n)} \hat{G}(m)\hat{q}(m) \right|^2 &\leq \left(\frac{R}{\rho}\right)^{2|n|} \sum_{m\in I(n)} |\hat{q}(m)|^2 \left(\frac{\delta\rho}{R}\right)^{2|m|} \underline{m}^{2t} \sum_{m\in I(n)} \frac{1}{\delta^{2|m|}} \frac{1}{\underline{m}^{2t}} \\ &\leq C_{22} \left(\frac{R}{\rho}\right)^{2|n|} \frac{1}{\delta^{2(N-|n|)}} N^{-2t} \sum_{m\in I(n)} |\hat{q}(m)|^2 \left(\frac{\delta\rho}{R}\right)^{2|m|} \underline{m}^{2t} \end{aligned}$$

by Proposition A.1 (ii) and (iv). Thus we have

$$\begin{aligned} T_{22} &\leq C_{22} \sum_{n\in\Lambda_N\setminus\{0\}} \left(\frac{R}{\rho}\right)^{2|n|} \frac{1}{\delta^{2(N-|n|)}} N^{-2t} \sum_{m\in I(n)} |\hat{q}(m)|^2 \left(\frac{\delta\rho}{R}\right)^{2|m|} \underline{m}^{2t} \left(\frac{\varepsilon\rho}{R}\right)^{2|n|} |n|^{2s} \\ &\leq C_{22} \|q\|^2_{\delta\rho/R,t} N^{-2t+2\max\{t,0\}} \left(\frac{\varepsilon}{\delta}\right)^N \underbrace{\sup_{n\in\Lambda_N\setminus\{0\}} \left\{ \left(\frac{1}{\varepsilon\delta}\right)^{N-2|n|} |n|^{2s} N^{-2\max\{s,0\}} \right\}}_{=:A_{22}}. \end{aligned}$$

Since $\varepsilon \geq \delta^{-1}$ we have the following estimate on A_{22} thanks to Proposition A.1 (iii):

$$A_{22} \leq \begin{cases} 1 & \text{if } \varepsilon = \delta^{-1} \wedge s \leq 0, \\ 4^{-s} & \text{if } \varepsilon = \delta^{-1} \wedge s > 0, \\ C_{-2s,(\varepsilon\delta)^{-1}} N^{2s} & \text{if } \varepsilon > \delta^{-1} \wedge s \leq 0, \\ C_{-2s,(\varepsilon\delta)^{-1}} & \text{if } \varepsilon > \delta^{-1} \wedge s > 0. \end{cases}$$

Therefore we obtain

$$T_{22} \leq \begin{cases} C_{22} N^{-2s} \left(\frac{\varepsilon}{\delta}\right)^N \|q\|^2_{\delta\rho/R,t} & \text{if } \varepsilon = \delta^{-1} \wedge s \leq 0, \\ C_{22} N^{2(s-t)} \left(\frac{\varepsilon}{\delta}\right)^N \|q\|^2_{\delta\rho/R,t} & \text{otherwise.} \end{cases}$$

Concerning T_3 we have

$$\begin{aligned} T_3 &\leq \sum_{n\in\mathbb{Z}\setminus\Lambda_N} |\hat{q}(n)|^2 \left(\frac{\delta\rho}{R}\right)^{2|n|} |n|^{2t} \sup_{n\in\mathbb{Z}\setminus\Lambda_N} \left\{ \left(\frac{\varepsilon}{\delta}\right)^{2|n|} |n|^{2(s-t)} \right\} \\ &\leq \|q\|^2_{\delta\rho/R,t} N^{2(s-t)} \left(\frac{\varepsilon}{\delta}\right)^N \underbrace{\sup_{n\in\mathbb{Z}\setminus\Lambda_N} \left\{ \left(\frac{\varepsilon}{\delta}\right)^{2|n|-N} \left(\frac{|n|}{N}\right)^{2(s-t)} \right\}}_{=:A_3}. \end{aligned}$$

Remarking that $(\varepsilon,s)\le(\delta,t)$ we have

$$A_3 \le \begin{cases} 4^{-(s-t)} & \text{if } \varepsilon=\delta \vee (\varepsilon<\delta \wedge s\le t), \\ C & \text{if } \varepsilon<\delta \text{ and } s>t. \end{cases}$$

Here C is some positive constant. Thus we obtain

$$T_3 \le C_3 N^{2(s-t)} \left(\frac{\varepsilon}{\delta}\right)^N \|q\|^2_{\delta\rho/R,t}.$$

Finally as to T_4 we have

$$T_4 = \sum_{p\in\Lambda_N}\sum_{l\in\mathbb{Z}\setminus\{0\}} |\hat{q}^{(N)}(p+lN)|^2 \left(\frac{\varepsilon\rho}{R}\right)^{2|p+lN|} |p+lN|^{2s} = T_{41}+T_{42},$$

where

$$T_{41} = \sum_{l\in\mathbb{Z}\setminus\{0\}} |lN|^{2s}\left(\frac{\varepsilon\rho}{R}\right)^{2|lN|} |\hat{q}^{(N)}(0)|^2, \quad T_{42} = \sum_{p\in\Lambda_N\setminus\{0\}} \left\{ \sum_{l\in\setminus\{0\}} |p+lN|^{2s}\left(\frac{\varepsilon\rho}{R}\right)^{2|p+lN|} |\hat{q}^{(N)}(p)|^2 \right\}.$$

Here note that the infinite series

$$\sum_{l\in\mathbb{Z}\setminus\{0\}} |p+lN|^{2s}\left(\frac{\varepsilon\rho}{R}\right)^{2|p+lN|} \qquad (\forall p\in\Lambda_N)$$

is absolutely convergent because of the assumption $(\varepsilon,s)<(R/\rho,-1/2)$. Making use of Proposition A.1 (i) we have $|\hat{q}^{(N)}(0)|^2 \le C\|q\|^2_{\delta\rho/R,t}$ and from Proposition A.1 (iv) this yields an estimate

$$\begin{aligned} T_{41} &\le C_{2s,\varepsilon\rho/R} N^{2s}\left(\frac{\varepsilon\rho}{R}\right)^{2N} \cdot C\|q\|^2_{\delta\rho/R,t} \\ &\le \begin{cases} C_{41}N^{2s}\left(\frac{\varepsilon}{\delta}\right)^N \|q\|^2_{\delta\rho/R,t} & \text{if } \varepsilon=\delta^{-1}(R/\rho)^2 \wedge t\ge 0, \\ C_{41}N^{2(s-t)}\left(\frac{\varepsilon}{\delta}\right)^N \|q\|^2_{\delta\rho/R,t} & \text{otherwise.} \end{cases} \end{aligned}$$

Concerning T_{42} we first have an estimate of $\hat{q}^{(N)}(p)$ for $p\in\Lambda_N\setminus\{0\}$

$$|\hat{q}^{(N)}(p)|^2 \le \frac{1}{(2\pi)^{2t}} \sum_{m\equiv p} |\hat{q}(m)|^2 \left(\frac{\delta\rho}{R}\right)^{2|m|} \underline{m}^{2t} \left(\frac{R}{\rho}\right)^{2|p|} \left[\frac{1}{\delta^{2|p|}}\frac{1}{|p|^{2t}} + C_{-2t,\delta^{-2}} N^{-2t}\left(\frac{1}{\delta}\right)^{2(N-|p|)}\right].$$

Then we have

$$\begin{aligned} T_{42} &\le \sum_{p\in\Lambda_N\setminus\{0\}} \left(\sum_{l\in\mathbb{Z}\setminus\{0\}} |p+lN|^{2s}\left(\frac{\varepsilon\rho}{R}\right)^{2|p+lN|}\right) \frac{1}{(2\pi)^{2t}} \sum_{m\equiv p} |\hat{q}(m)|^2\left(\frac{\delta\rho}{R}\right)^{2|m|} \underline{m}^{2t}\left(\frac{R}{\rho}\right)^{2|p|} \\ &\qquad \times \left[\frac{1}{\delta^{2|p|}}\frac{1}{|p|^{2t}} + C_{-2t,\delta^{-2}}N^{-2t}\left(\frac{1}{\delta}\right)^{2(N-|p|)}\right] \\ &\le C_{2s,\varepsilon\rho/R} N^{2[s+\max\{-t,0\}]}\left(\frac{\varepsilon}{\delta}\right)^N \|q\|^2_{\delta\rho/R,t} \\ &\quad \times \underbrace{\sup_{p\in\Lambda_N\setminus\{0\}} \left[\frac{N^{-2\max\{-t,0\}}}{|p|^{2t}}\left\{\varepsilon\delta\left(\frac{\rho}{R}\right)^2\right\}^{N-2|p|} + C_{-2t,\delta^{-2}} N^{-2[t+\max\{-t,0\}]}\left\{\frac{\varepsilon}{\delta}\left(\frac{\rho}{R}\right)^2\right\}^{N-2|p|}\right]}_{=:A_{42}} \end{aligned}$$

Noting that $\varepsilon \le \delta^{-1}(R/\rho)^2$ we have

$$A_{42} \le \begin{cases} C' & \text{if } \varepsilon = \delta^{-1}(R/\rho)^2, \\ C'N^{-2t} & \text{otherwise} \end{cases}$$

due to Proposition A.1 (iii). Therefore we obtain

$$T_{42} \le \begin{cases} C_{42}N^{2s}\left(\dfrac{\varepsilon}{\delta}\right)^N \|q\|^2_{\delta\rho/R,t} & \text{if } \varepsilon = \delta^{-1}(R/\rho)^2 \wedge t \ge 0, \\ C_{42}N^{2(s-t)}\left(\dfrac{\varepsilon}{\delta}\right)^N \|q\|^2_{\delta\rho/R,t} & \text{otherwise.} \end{cases}$$

Combining the above estimates we obtain

$$\|q - q^{(N)}\|^2_{\varepsilon\rho/R,s} \le CN^{2P(\varepsilon,s,\delta,t)}\left(\frac{\varepsilon}{\delta}\right)^N \|q\|^2_{\delta\rho/R,t}$$

as desired.

References

[1] K. Amano, D. Okano, H. Ogata and M. Sugihara, *Numerical conformal mappings onto the linear slit domain*, Jpn. J. Ind. Appl. Math. **29** (2012), no. 2, 165–186.

[2] D. N. Arnold, *A Spline-Trigonometric Galerkin Method and an Exponentially Convergent Boundary Integral Method*, Math. Comp. **41** (1983), no. 164, 383–397.

[3] D. N. Arnold and W. L. Wendland, *The convergence of spline collocation for strongly elliptic equations on curves*, Numer. Math. **47** (1985), 317–341.

[4] S. Axler, P. Bourdon and W. Ramey, *Harmonic function theory*, Second edition, Springer-Verlag, New York, 2001.

[5] M. Katsurada and H. Okamoto, *A mathematical study of the charge simulation method I*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **35** (1988), no. 3, 507–518.

[6] M. Katsurada, *A mathematical study of the charge simulation method II*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **36** (1989), no. 1, 135–162.

[7] M. Katsurada, *A mathematical study of the charge simulation method by use of peripheral conformal mappings*, Mem. Inst. Sci. Tech. Meiji Univ. **37** (1998), no. 8, 195–212.

[8] H. Ogata, *Dipole simulation method for two-dimensional potential problems*, NOLTA, IEICE **5** (2014), no. 1, 2–14.

[9] H. Ogata and M. Katsurada, *Convergence of the invariant scheme of the method of fundamental solutions for two-dimensional potential problems in a Jordan region*, Jpn. J. Indust. Appl. Math. **31** (2014), no. 1, 231–262.

[10] K. Sakakibara and M. Katsurada, *A mathematical analysis of the complex dipole simulation method*, submitted.

[11] K. Sakakibara and H. Ogata, *A bidirectional method for numerical conformal mappings based on the dipole simulation method*, in preparation.

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