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We propose a new outflow boundary condition, a unilateral condition of Signorini's type, for the incompressible Navier-Stokes equations. The condition is a generalization of the standard free-traction condition and it's variational formulation is given by a variational inequality. We also consider a penalty approximation, which is a kind of the Robin condition, to deduce a suitable formulation for numerical computations. Under those conditions, we can obtain energy inequalities which are key features for numerical computations. The main contribution of this paper is to establish the well-posedness of the Navier-Stokes equations under those boundary conditions. In particular, we prove the unique existence of strong solutions of Ladyzhenskaya's class by the standard Galerkin's method. For the proof of the existence of pressures, however, we offer a new method of analysis.

Key words: Navier-Stokes equations, variational inequality, penalty method

2010 Mathematics Subject Classification: 35K85, 35Q30, 76D05

1 Introduction

1.1 Motivation

In numerical simulation of real-world flow problems, we often encounter some issues concerning artificial boundary conditions. A typical and important example is the blood flow problem in the large arteries, where the blood is assumed to be a viscous incompressible fluid (cf. [8], [21]). The blood vessel is modeled by a branched pipe as illustrated, for example, by Fig. 1. We are able to give a velocity profile at the *inflow boundary* S and the flow is supposed to be a perfect non-slip on the wall C. Then, the blood flow simulation is highly dependent on the choice of artificial boundary conditions posed on the *outflow boundary* Γ.

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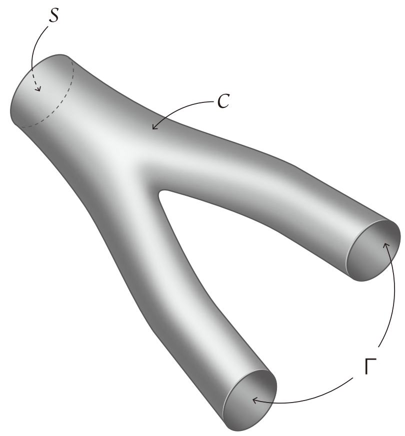

Figure 1: A branched pipe

In order to state the problem more specifically, let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, be a bounded domain and let the boundary $\partial\Omega$ be composed of three parts S, C and Γ. Those S, C and Γ are assumed to be smooth surfaces, although the whole boundary $\partial\Omega$ itself is not smooth. Then, for $T > 0$, we consider the Navier-Stokes equations

$$u_t + (u \cdot \nabla)u = \nabla \cdot \sigma(u, p) + f \qquad \text{in } \Omega \times (0, T), \tag{1a}$$

$$\nabla \cdot u = 0 \qquad \text{in } \Omega \times (0, T), \tag{1b}$$

$$u = b \qquad \text{on } S \times (0, T), \tag{1c}$$

$$u = 0 \qquad \text{on } C \times (0, T), \tag{1d}$$

$$u|_{t=0} = u_0 \qquad \text{on } \Omega \tag{1e}$$

for the velocity $u = (u_1, \ldots, u_d)$ and the pressure p with the density $\rho = 1$ and the kinematic viscosity ν of the viscous incompressible fluid under consideration. Therein, $\sigma(u, p) = (\sigma_{i,j}(u, p)) = -pI + 2\nu D(u)$ denotes the stress tensor, where $D(u) = (D_{i,j}(u)) = \left(\frac{1}{2}\left(\nabla u + \nabla u^{\mathrm{T}}\right)\right)$ the deformation-rate tensor and I the identity. The prescribed functions $f = f(x, t)$ and $u_0 = u_0(x)$ denote the external force and initial velocity, respectively. Moreover, $b = b(x, t)$ denotes the prescribed inflow velocity with $b|_{\partial S} = 0$.

A setting of the boundary condition on Γ is not a trivial task, since the flow distribution and pressure field are unknown and cannot be prescribed in many simulations. As a common outflow boundary condition, the free-traction condition or the so-called do-nothing condition

$$\tau(u, p) = 0 \quad \text{on} \quad \Gamma \tag{2}$$

is still frequently used so far (cf. [12], [10]), where

$$\tau(u, p) = \sigma(u, p)n \tag{3}$$

denotes the traction vector on $\partial\Omega$ and n the outward normal vector to $\partial\Omega$. Though this condition is enough for many problems, it sometimes causes serious numerical instability near

Γ (cf. [5, Remark 4.1], [22]). Actually, from the view-point of mathematics, the energy inequality is not guaranteed under (2) and it is a drawback of employing (2). To describe this issue, we take a *reference flow* (g, π) which is the solution of the Stokes system

$$\nabla \cdot \sigma(g, \pi) = 0, \quad \nabla \cdot g = 0 \quad \text{in } \Omega, \tag{4a}$$

$$g = b \text{ on } S, \quad g = 0 \text{ on } C, \quad g = g_0(x)\beta(t) \text{ on } \Gamma \tag{4b}$$

for all $t \in [0, T]$, where $g_0 = g_0(x) \in C_0^\infty(\Gamma)^d$ is a prescribed function satisfying

$$\int_\Gamma g_0 \cdot n \; d\Gamma = 1, \quad g_0 \cdot n \geq 0 \; \text{ on } \Gamma \tag{5}$$

and we set

$$\beta(t) = -\int_S b \cdot n \; dS.$$

(The function g is nothing but a lifting function of b.) By using this, we will find (u, p) of the form

$$u = U + g, \qquad p = P + \pi.$$

Then, the energy inequality for (1) reads as

$$\sup_{t \in [0,T]} \|U\|^2_{L^2(\Omega)^d} + 2\nu \int_0^T D_{ij}(U) D_{ij}(U) \leq C, \tag{6}$$

where C denotes a positive constant depending only on f, u_0, b and T. This inequality is of use. It plays a crucial role in the construction of a solution of the Navier-Stokes equations as is just discussed in this paper. Furthermore, it is connected with the stability of numerical schemes from the view-point of numerical computation. That is, it is preferred that the energy inequality does not spoiled after discretizations. For example, we often take some kinds of approximation to

$$\int_\Omega (u \cdot \nabla) v \cdot w \; dx$$

to ensure the energy inequality under time discretizations (cf. [23]). However, it is not certain the energy inequality (6) to hold under (2) even for the continuous case. In fact, assuming (1) admits a smooth solution $(u, p) = (U + g, P + \pi)$ in $0 \leq t \leq T$ and multiplying the both sides of (1a) by U, we have by the integration by parts

$$\frac{d}{dt}\|U\|^2_{L^2(\Omega)^d} + \underbrace{2\nu \int_\Omega D_{ij}(U) D_{ij}(U) \; dx + \frac{1}{2}\int_\Gamma u_n |U|^2 \; d\Gamma - \int_\Gamma \tau(u,p) \cdot U \; d\Gamma}_{=I}$$
$$= \int_\Omega [f - g_t - (g \cdot \nabla) g] \cdot U \; dx - \int_\Omega (U \cdot \nabla) g \cdot U \; dx. \tag{7}$$

If $I \geq 0$, we can derive (6) as will be demonstrated in Section 5; However, it is impossible to get $I \geq 0$ since we have no information about u_n on Γ under (2). (Bothe et al. [6] recently studied the well-posedness of the Navier-Stokes equations under a class of energy preserving boundary conditions; However, the common one (2) was discussed only in the case of the Stokes equations.)

With this connection, F. Boyer, F. Bruneau and P. Fabrie proposed and studied a class of nonlinear boundary conditions that ensure the energy inequality (cf. [1], [2], [3], [4]). A typical outflow condition they proposed is given as

$$\tau(u,p) = -\frac{1}{2}[u_n]_- U + 2\nu D(g)n \quad \text{on } \Gamma, \tag{8}$$

where

$$[s]_\pm = \max\{0, \pm s\}, \qquad s = [s]_+ - [s]_-.$$

Under the boundary condition (8), the identity (7) implies

$$\begin{aligned}\frac{d}{dt}\|U\|^2_{L^2(\Omega)^d} + 2\nu \int_\Omega D_{ij}(U)D_{ij}(U)\ dx + \frac{1}{2}\int_\Gamma [u_n]_+|U|^2\ d\Gamma \\ = \int_\Omega [f - g_t - (g\cdot\nabla)g]\cdot U\ dx - \int_\Omega (U\cdot\nabla)g\cdot U\ dx + \int_\Gamma 2\nu D(g)n\cdot U\ d\Gamma.\end{aligned}$$

Then, after some calculations, we obtain the energy inequality (6). Actually, they established the well-posedness of (1) with a class of boundary conditions, including (8), by Galerkin's method based on (6).

As a matter of fact, a similar boundary condition is successfully applied in actual computations, that is, in blood flow simulation for thoracic arteries. In Bazilevs et al. [5, §4], they employed the following condition. First, they introduced a regularized traction vector

$$\tilde\tau(u,p) = \tau(u,p) + [u_n]_- u$$

and considered the resistance boundary condition

$$\tilde\tau(u,p)\cdot n + R\int_\Gamma u_n\ d\Gamma + p_0 = 0, \quad \tilde\tau(u,p) - [\tilde\tau(u,p)\cdot n]n = 0 \quad \text{on } \Gamma,$$

where R and p_0 are prescribed constants that control the average of the flow rate across Γ (cf. [24], [9]). This condition is equivalently written as

$$\tau(u,p) = -[u_n]_- u - \left(R\int_\Gamma u_n\ d\Gamma + p_0\right)n. \tag{9}$$

If $b = 0$ (then we can take $g = 0$ and $\pi = 0$), we derive the energy inequality under this condition. They offered several numerical results for medical problems and did not give any mathematical considerations. On the other hand, Labeur and Wells [17] considered essentially the same condition as (9) with $R = p_0 = 0$, where they studied energy stable hybrid discontinuous finite element method but did not discuss about the well-posedness of the continuous problem.

Those previous works suggest us that it is important to control the flow-direction near the outflow boundary for stable numerical computations and that the energy inequality is a key property to check whether the flow-direction is suitable or not. Therefore, it is worth-while considering flow-direction boundary conditions, such as (8) and (9), from the view-point of numerical analysis. Furthermore, it seems that there are little works devoted to those boundary conditions from the view-point of pure analysis.

The condition (8) is useful, but it has a few difficulties. Thus, a non-trivial relationship is assumed between the traction $\tau(u,p)$ and the velocity u in (8) and we have to determine the reference velocity g before computation. On the other hand, it is not obvious that the condition (9) is suitable for the case $b \neq 0$.

In the present paper, we propose a new boundary condition. That is, in order to control the flow direction at Γ, we pose a unilateral boundary condition of Signorini's type

$$\begin{cases} u_n \geq 0, \\ \tau_n(u,p) \geq 0, \ u_n \tau_n(u,p) = 0, \ \tau_T(u) = 0 \end{cases} \text{on } \Gamma, \tag{10}$$

where

$$\tau_n(u,p) = \tau(u,p)n, \qquad \tau_T(u) = \tau(u,p) - \tau_n(u,p)n.$$

This is an analogy to Signorini's condition in the theory of elasticity (cf. [14]). Under this condition, the solution of (1) satisfies the energy inequality (cf. Theorem 4) and it is indeed a generalization of the free-traction condition (2). Namely,

$$\begin{aligned} &\text{if } u_n > 0 \text{ on } \omega \subset \Gamma, \text{ then } \tau_n(u,p) = 0 \text{ on } \omega; \\ &\text{if } u_n = 0 \text{ on } \omega \subset \Gamma, \text{ then } \tau_n(u,p) \geq 0 \text{ on } \omega. \end{aligned}$$

The condition (10) is described in terms of inequalities so that it cannot be directly applied to numerical calculations. However, we can utilize its penalty approximation

$$\tau_n(u,p) = \frac{1}{\varepsilon}[u_n]_-, \quad \tau_T(u) = 0 \quad \text{ on } \Gamma, \tag{11}$$

where $0 < \varepsilon \ll 1$ is the penalty parameter. After introducing a C^1 regularization of $[\cdot]_-$ (for example, ρ_δ in (47)), we can solve (1) with (11) by using, for example, Newton's iteration. We do not need to introduce the reference velocity g for computation. (For mathematical analysis below, we need g.) It is indeed an approximation of (10); Thus, we have (cf. the proof of Lemma 4.7)

$$(u_\varepsilon, p_\varepsilon) \to (u,p) \quad \text{as} \quad \varepsilon \to 0$$

in a certain sense or other, where (u,p) and $(u_\varepsilon, p_\varepsilon)$ denote solutions of (1) with, respectively, (10) and (11). Moreover, the condition (11) is closely related with (9) in a certain sense. Namely, although ε is originally defined as a positive constant, we set it as a function;

$$\frac{1}{\varepsilon} = [u_n]_-.$$

Then, (11) implies

$$\tau_n(u,p) = [u_n]_-^2 = -[u_n]_- u_n, \quad \tau_T(u) = 0.$$

Hence, as for the normal component, (11) and (9) are equivalent in the case $R = p_0 = 0$. This suggests that (9) is of use for the case $b \neq 0$. This is another motivation for studying (11).

Our ultimate aim is to develop the theory for the initial-boundary value problems for the Navier-Stokes equations (1) with (10) or with (11) from the standpoint both of analysis and numerical computations. The particular purpose of this paper is to establish the well-posedness of these problems. We postpone a study on time discretizations and the finite element approximation in future works; a partial result on the finite element approximation for a model (stationary) Stokes problem will be reported in Saito et al. [19].

1.2 Summary of the results

We shall give the precise statement of our main results in §2.5, after having described the variational interpretation of our target problems. However, let us summarize our results here for the reader's convenience.

First, we assume that the prescribed inflow velocity $b = b(x,t)$ satisfies $b|_{\partial S} = 0$ and

$$\beta(t) = -\int_S b \cdot n \, dS > 0, \quad t \in [0,T].$$

Consequently, we will have

$$\int_\Gamma u \cdot n \, d\Gamma = \beta(t) > 0, \quad t \in [0,T].$$

As is clearly stated in Introduction of [13], weak solutions of Leray-Hopf's class is not suitable for the purpose of application to numerical analysis. We are strongly motivated by [13] and interested in constructing of strong solutions of Ladyzhenskaya's class (cf. [15]), that is, we will find functions

$$u \in L^\infty(0,T;H^1(\Omega)^d), \quad u_t \in L^2(0,T;H^1(\Omega)^d) \cap L^\infty(0,T;L^2(\Omega)^d),$$
$$p \in L^\infty(0,T;L^2(\Omega))$$

that satisfy the Navier-Stokes equation (1) with the unilateral boundary condition (10) in the sense of distributions.

To this end, it suffices to find (U,P) satisfying the following perturbed Navier-Stokes problem.

(NS) For $t \in (0,T)$, find (U,P) such that

$$U_t + ((U+g)\cdot\nabla)U + (U\cdot\nabla)g - \nabla\cdot\sigma(U,P) = F \qquad \text{in } \Omega, \tag{12a}$$
$$\nabla \cdot U = 0 \qquad \text{in } \Omega, \tag{12b}$$
$$U = 0 \qquad \text{on } S \cup C, \tag{12c}$$
$$U_n + g_n \geq 0, \quad \tau_n(U+g, P+\pi) \geq 0 \qquad \text{on } \Gamma, \tag{12d}$$
$$(U_n + g_n)\tau_n(U+g,P+\pi) = 0, \quad \tau_T(U) = -\tau_T(g) \qquad \text{on } \Gamma, \tag{12e}$$
$$U(x,0) = U_0 \qquad \text{on } \Omega, \tag{12f}$$

where

$$F = f - g_t - (g\cdot\nabla)g,$$
$$U_0 = u_0 - g(0).$$

Actually, under some appropriate assumptions on F, U_0, and (g,π) (cf. (A1)–(A4) below), we will prove (cf. Theorem 2 in §2.5), there exists a unique solution of (NS) satisfying

$$U \in L^\infty(0,T;H^1(\Omega)^d), \quad U_t \in L^2(0,T;H^1(\Omega)^d) \cap L^\infty(0,T;L^2(\Omega)^d),$$
$$P \in L^\infty(0,T;L^2(\Omega)).$$

For the penalty problem (1) with (11), we consider the following perturbed problem.

(NS$_\varepsilon$) Let $0 < \varepsilon \ll 1$. For all $t \in (0,T)$, find $(U_\varepsilon, P_\varepsilon)$ such that

$$
\begin{aligned}
& U_{\varepsilon,t} + (U_\varepsilon + g\cdot\nabla)U_\varepsilon + (U_\varepsilon\cdot\nabla)g - \frac{1}{\rho}\nabla\cdot\sigma(U_\varepsilon,P_\varepsilon) = F && \text{in } \Omega,\\
& \nabla\cdot U_\varepsilon = 0 && \text{in } \Omega,\\
& U_\varepsilon = 0 && \text{on } S\cup C,\\
& \tau_n(U_\varepsilon+g,P_\varepsilon+\pi) = \frac{1}{\varepsilon}[U_{\varepsilon n}+g_n]_-, \quad \tau_T(U_\varepsilon) = -\tau_T(g) && \text{on } \Gamma,\\
& U_\varepsilon(x,0) = u_0 - g(0) && \text{on } \Omega,
\end{aligned}
$$

Then,

$$u_\varepsilon = U_\varepsilon + g, \qquad p_\varepsilon = P_\varepsilon + \pi$$

solve (1) with (11). Under the same assumptions on F, U_0, and (g,π), we will prove (cf. Theorem 3 in §2.5), there exists a unique solution of (NS$_\varepsilon$) satisfying

$$
\begin{gathered}
U_\varepsilon \in L^\infty(0,T;H^1(\Omega)^d), \quad U_{\varepsilon,t} \in L^2(0,T;H^1(\Omega)^d)\cap L^\infty(0,T;L^2(\Omega)^d),\\
P_\varepsilon \in L^\infty(0,T;L^2(\Omega))
\end{gathered}
$$

for a sufficiently small ε.

The plan of our analysis is as follows. We firstly give variational formulations (NS-E) and (NS$_\varepsilon$-E) of (NS) and (NS$_\varepsilon$) in §2.3 and §2.4, respectively, after having described the variational interpretations of traction vectors $\tau(u,p)$, $\tau_n(u,p)$ and $\tau_T(u)$ in §2.2. Further, (NS-E) is converted into the variational inequality problem (NS-I) and the equivalence of those two formulations is proved (cf. Theorem 1 in §2.3). We also introduce the solenoidal (divergence-free) versions (NS-I$^\sigma$) and (NS$_\varepsilon$-E$^\sigma$) of (NS-I) and (NS$_\varepsilon$-E), respectively.

Theorems 2 and 3 are divided into several propositions:

- Proposition 1 (The unique existence of U of a solution of (NS-I$^\sigma$));
- Proposition 2 (The existence of an associating pressure P with U of (NS-I));
- Proposition 3 (The uniqueness of (NS-I));
- Proposition 4 (The unique existence of U_ε of a solution of (NS$_\varepsilon$-E$^\sigma$));
- Proposition 5 (The existence of an associating pressure P_ε with U_ε of (NS$_\varepsilon$-E));
- Proposition 6 (The uniqueness of (NS$_\varepsilon$-E)).

We use a C^1 regularization ρ_δ of $[\cdot]_-$ and the standard Galerkin's method to prove Proposition 4 (cf. Section 4). Therein, several a priori estimates including

$$\|[U_{\varepsilon,n}+g_n]_-\|_{L^\infty(0,T;L^2(\Gamma))} \le C\sqrt{\varepsilon}$$

play important role. Then, we prove Proposition 1 by compactness in Section 4. Usually, we apply a version of De Rham's theorem (cf. [20, Lemma IV.1.4.3] for example) to deduce a pressure of the Navier-Stokes equations, after a velociy has been obtained. Unfortunately, it is not enough to deduce pressures P and P_ε for our problems. Actually, we have to choose suitable constants k and k_ε such that $(U, \mathring{P}+k)$ and $(U_\varepsilon, \mathring{P}_\varepsilon + k_\varepsilon)$ satisfy (NS-I) and (NS$_\varepsilon$-E),

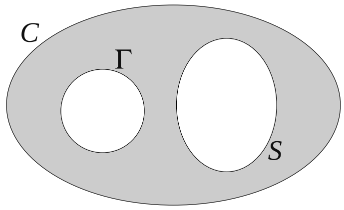

Figure 2: A smooth domain

respectively, where $\mathring{P}$ and $\mathring{P}_\varepsilon$ are associating pressures with U and U_ε, respectively. (P and P_ε are L^2 functions in Ω with zero mean values.) We discuss this issue and prove Propositions 2 and 5 in Section 3. Proofs of Propositions 3 and 6 are also mentioned in Section 3.

(NS-E) and (NS$_\varepsilon$-E) admit energy inequalities of the form (6); We derive those inequalities in Section 5.

List of problems

Name	Place	Page	Equation
(NS-E)	§2.3	11	(18)
(NS-I)	§2.3	11	(19)
(NS-I$^\sigma$)	§2.3	11	(20)
(NS$_\varepsilon$-E)	§2.4	12	(21)
(NS$_\varepsilon$-E$^\sigma$)	§2.4	12	(22)
(NS-I$^\sigma$)$^\smile$	Sec. 4	20	(45)
(NS$_\varepsilon$-E$^\sigma$)$^\smile$	Sec. 4	20	(46)
(NS$_\varepsilon$-E$^\sigma_\delta$)$^\smile$	Sec. 4	21	(49)
(NS$_\varepsilon$-E$^\sigma_{\delta m}$)$^\smile$	Sec. 4	21	(51)

2 Problems and results

2.1 Notation

We recall that $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, is a bounded domain and the boundary $\partial\Omega$ is composed of three parts S, C and Γ. Although we mostly deal with the case illustrated by Fig. 1, our discussion is also valid for the case where $\partial\Omega$ is smooth with $\overline{\Gamma} \cap \overline{C} = \emptyset$, $\overline{C} \cap \overline{S} = \emptyset$, and $\overline{S} \cap \overline{\Gamma} = \emptyset$; See, for example, Fig. 2.

We follow the standard notation, for example, of [16] and [20] as for function spaces and their norms. We employ the abbreviations:

$$\|u\| = \|u\|_\Omega = \|u\|_{0,\Omega} = \|u\|_{L^2(\Omega)^d} \text{ or } \|u\|_{L^2(\Omega)};$$
$$\|u\|_1 = \|u\|_{1,\Omega} = \|u\|_{H^1(\Omega)^d} \text{ or } \|u\|_{H^1(\Omega)};$$
$$\|u\|_\Gamma = \|u\|_{0,\Gamma} = \|u\|_{L^2(\Gamma)^d} \text{ or } \|u\|_{L^2(\Gamma)};$$
$$(u, v) = (u, v)_{L^2(\Omega)^d} \text{ or } (u, v)_{L^2(\Omega)}.$$

We frequently use the following function spaces:

$$V = \{v \in H^1(\Omega)^d \mid v = 0 \text{ on } C \cap S\}, \qquad V^\sigma = \{v \in v \mid \nabla \cdot v = 0 \text{ in } \Omega\},$$
$$V_0 = H_0^1(\Omega)^d, \qquad V_0^\sigma = \{v \in V_0 \mid \nabla \cdot v = 0 \text{ in } \Omega\},$$
$$K = \{v \in V \mid v_n + g_n \geq 0 \text{ on } \Gamma\}, \qquad K^\sigma = \{v \in K \mid \nabla \cdot v = 0 \text{ in } \Omega\},$$
$$Q = L^2(\Omega), \qquad Q_0 = L_0^2(\Omega) = \left\{ q \in Q \mid \int_\Omega q\ dx = 0 \right\},$$
$$M = \begin{cases} H^{\frac{1}{2}}(\Gamma) & \text{if } \overline{\Gamma} \cap \overline{C} = \emptyset \text{ (e.g. Fig. 2)}, \\ H_{00}^{\frac{1}{2}}(\Gamma) & \text{if } \overline{\Gamma} \cap \overline{C} \neq \emptyset \text{ (e.g. Fig. 1)}. \end{cases}$$

The spaces V and V^σ are closed subspaces of $H^1(\Omega)^d$ and are equipped with norm $\|\cdot\|_1$. The spaces V_0 and V_0^σ are also closed subspaces of $H^1(\Omega)^d$ and are equipped with norm $\|\cdot\|_1$ by virtue of the Poincaré inequality.

In general, X' denotes the dual space of a Banach space X.

Remark 2.1. The space $H_{00}^{\frac{1}{2}}(\Gamma)$ is sometimes called the *Lions-Magenes space* (cf. [16, Ch. 1, §11.5]). It is defined as the set of all $v \in L^2(\Gamma)$ satisfying $\rho^{-1/2} v \in L^2(\Gamma)$, where $\rho \in C^\infty(\overline{\Gamma})$ denotes any positive function satisfying $\rho|_{\partial\Gamma} = 0$ and, for $x_0 \in \partial\Gamma$,

$$\lim_{x \to x_0} \frac{\rho(x)}{\text{dist } (x, \partial\Gamma)} = d' > 0$$

with some $d' > 0$. As will be stated in Lemma 2.1, the set of all traces of functions in V is identical with M.

We use the following forms (the summation convection is employed):

$$a(u,v) = 2\nu \int_\Omega D_{ij}(u) D_{ij}(v)\ dx \quad (u, v \in H^1(\Omega)^d);$$
$$a_1(u,v,w) = \int_\Omega [(u \cdot \nabla) v] w\ dx \quad (u, v, w \in H^1(\Omega)^d);$$
$$b(v,p) = -\int_\Omega (\nabla \cdot v) p\ dx \quad (v \in H^1(\Omega)^d,\ p \in L^2(\Omega));$$
$$[\lambda, \eta] = \text{ the duality pairing between } \lambda \in M' \text{ and } \eta \in M;$$
$$[[\lambda, \eta]] = \text{ the duality pairing between } \lambda \in (M^d)' \text{ and } \eta \in M^d.$$

As usual, we write C to express various positive constants that depend only on Ω.

For a vector-valued function v defined on $\partial\Omega$, it's normal and tangential components are denoted, respectively, by

$$v_n = v \cdot n, \qquad v_T = v - (v_n) n.$$

2.2 The re-definition of traction vectors

For $(U, P) \in V \times Q$, we cannot define $\tau(U, P)$ as a function on Γ because of the lack of regularity. However, if (U, P) is smooth and satisfies (12a), it also satisfies, for $t \in (0, T)$

$$\int_\Gamma \tau(U,P) \cdot v\ d\Gamma = (U_t, v) + a(U, v) + a_1(U + g, U, v) \\ + a_1(U, g, v) + b(v, P) - (F, v) \qquad (v \in V),$$

where $\tau(U, p)$ is understood as a usual function on Γ.

Based on this identity, we re-define the traction vector $\tau(U, P)$ as a *functional* over M^d for a solution $(U, P) \in V \times Q$ of (NS) in the sense of distributions (More precisely, for (U, P) satisfying (18a) below). We recall the following result (cf. [11, Theorem 2.5] for $M = H_{00}^{1/2}(\Gamma)$ and [16, Theorem 8.3, Chap. 1] for $M = H^{1/2}(\Gamma)$).

Lemma 2.1. *There exists an extension operator $\mathcal{E} : M^d \to V$ such that $\mathcal{E}\eta = \eta$ on Γ and $\|\mathcal{E}\eta\|_V \le C\|\eta\|_{M^d}$ for all $\eta \in M^d$. Conversely, for any $w \in V$, we have $\eta = w|_\Gamma \in M^d$ and $\|\eta\|_{M^d} \le C\|w\|_V$.*

As a consequence, we obtain an extension operator $\mathcal{E}_n : M \to V$ such that

$$(\mathcal{E}_n\eta)_n = \eta \ , (\mathcal{E}_n\eta)_T = 0 \ \text{ on } \Gamma, \qquad \|\mathcal{E}_n\eta\|_V \le C\|\eta\|_M$$

for any $\eta \in M$. Now we propose the re-definition of $\tau(U, P)$ as follows:

$$[[\tau(U,P),\eta]] = (U_t, w_\eta) + a(U, w_\eta) + a_1(U+g, U, w_\eta) \\ + a_1(U, g, w_\eta) + b(w_\eta, P) - (F, w_\eta) \qquad (\eta \in M^d), \quad (13)$$

where $w_\eta = \mathcal{E}\eta \in V$. Actually, the right-hand side of (13) does not depend on the way of extension; Hence, this definition is well-defined. Similarly, we re-define as

$$[[\tau_T(U),\eta]] = (U_t, w_\eta) + a(U, w_\eta) + a_1(U+g, U, w_\eta) + a_1(U, g, w_\eta) \\ + b(w_\eta, P) - (F, w_\eta) \qquad (\eta \in M^d \text{ with } \eta_n = 0;\ w_\eta = \mathcal{E}\eta) \quad (14)$$

and

$$[\tau_n(U,P),\eta] = (U_t, w_\eta) + a(U, w_\eta) + a_1(U+g, U, w_\eta) \\ + a_1(U, g, w_\eta) + b(w_\eta, P) - (F, w_\eta) \qquad (\eta \in M;\ w_\eta = \mathcal{E}_n\eta). \quad (15)$$

Then,

$$[[\tau(U,P),\eta]] = [\tau_n(U,P),\eta_n] + [[\tau_T(U),\eta_T]] \qquad (\eta \in M^d). \quad (16)$$

For a solution $(U_\varepsilon, P_\varepsilon)$ of (NS$_\varepsilon$), we propose the similar re-definition. For example,

$$[\tau_n(U_\varepsilon,P_\varepsilon),\eta] = (U_{\varepsilon,t}, w_\eta) + a(U_\varepsilon, w_\eta) + a_1(U_\varepsilon+g, U_\varepsilon, w_\eta) \\ + a_1(U_\varepsilon, g, w_\eta) + b(w_\eta, P_\varepsilon) - (F, w_\eta) \qquad (\eta \in M;\ w_\eta = \mathcal{E}_n\eta). \quad (17)$$

On the other hand, we will assume that $\tau(g, \pi) \in H^1(0, T; L^2(\Gamma)^d)$ (see, (A1) below) so that we have

$$[[\tau(g,\pi),\eta]] = \int_\Gamma \tau(g,\pi)\cdot\eta \, d\Gamma \quad (\eta \in M^d).$$

2.3 Unilateral problems

Under those re-definitions presented in the previous section, we precisely interpret (NS) as follows.

(NS-E) For a.e. $t \in (0,T)$, find $(U(t),P(t)) \in V \times Q$ with $U_t(t) \in Q^d$ such that

$$\begin{aligned}
&(U_t, v) + a(U,v) \\
&\quad + a_1(U+g,U,v) + a_1(U,g,v) + b(v,P) = (F,v) && \forall v \in V_0, && \text{(18a)}\\
&b(U,q) = 0 && \forall q \in Q, && \text{(18b)}\\
&U_n + g_n \geq 0 && \text{on } \Gamma, && \text{(18c)}\\
&[\tau_n(U,P) + \tau_n(g,\pi), \eta] \geq 0 && \forall \eta \in M,\ \eta \geq 0, && \text{(18d)}\\
&[\tau_n(U,P) + \tau_n(g,\pi), U_n + g_n] = 0, && && \text{(18e)}\\
&[[\tau_T(U) + \tau_T(g), \eta]] = 0 && \forall \eta \in M, && \text{(18f)}\\
&U(x,0) = U_0 && \text{on } \Omega. && \text{(18g)}
\end{aligned}$$

Remark 2.2. If $(U,P) \in V \times Q$ satisfies (18a), then $[\tau(U,P),\eta]$ and $[[\tau_T(U),\eta]]$ are well-defined by (15) and (14).

(NS-E) can be converted into the following variational inequality problem.

(NS-I) For a.e. $t \in (0,T)$, find $(U(t),P(t)) \in K \times Q$ with $U_t(t) \in Q^d$ such that

$$\begin{aligned}
&(U_t, v-U) + a(U,v-U) + a_1(U+g,U,v-U) \\
&\quad + a_1(U,g,v-U) + b(v-U,P) \geq (F,v-U) - [[\tau(g,\pi), v-U]] && \forall v \in K, && \text{(19a)}\\
&b(U,q) = 0 && \forall q \in Q, && \text{(19b)}\\
&U(x,0) = U_0 && \text{on } \Omega. && \text{(19c)}
\end{aligned}$$

The following solenoidal version of (NS-I) will be of use later.

(NS-I$^\sigma$) For a.e. $t \in (0,T)$, find $U(t) \in K^\sigma$ with $U_t(t) \in Q^d$ such that

$$\begin{aligned}
&(U_t, v-U) + a(U,v-U) + a_1(U+g,U,v-U) \\
&\quad + a_1(U,g,v-U) \geq (F,v-U) - [[\tau(g,\pi), v-U]] && \forall v \in K^\sigma, && \text{(20a)}\\
&U(x,0) = U_0 && \text{on } \Omega. && \text{(20b)}
\end{aligned}$$

Theorem 1. *Problems* (NS-E) *and* (NS-I) *are equivalent.*

Proof. First, letting (U,P) be a solution of (NS-E), we show (U,P) satisfies (NS-I). Let $v \in K$ be arbitrary. Since $v - U \in V$, we see from (13)

$$\begin{aligned}
(U_t, v-U) + a(U,v-U) + a_1(U+g,U,v-U) + a_1(U,g,v-U)&\\
+ b(v-U,P) - [[\tau(U,P), v-U]] = (F, v-U).&
\end{aligned}$$

Thus,

$$\begin{aligned}
(U_t, v-U) + a(U,v-U) + a_1(U+g,U,v-U) + a_1(U,g,v-U)&\\
+ b(v-U,P) - [[\tau(U,P) + \tau(g,\pi), v-U]] = (F, v-U).&
\end{aligned}$$

Since $v_n + g_n \geq 0$ a.e. Γ, by using (16), (18d) and (18e)

$$\begin{aligned}
&[[\tau(U,P) + \tau(g,\pi), v-U]] \\
&\quad = [\tau_n(U,P) + \tau_n(g,\pi), v_n - U_n] + [[\tau_T(U) + \tau_T(g), v_T - U_T]] \\
&\quad = [\tau_n(U,P) + \tau_n(g,\pi), v_n + g_n] - [\tau_n(U,P) + \tau_n(g,\pi), U_n + g_n] \geq 0.
\end{aligned}$$

Hence, (U,P) solves (NS-I).

Conversely, letting (U,P) be a solution to (NS-I), we show (U,P) satisfies (NS-E).

For any $\phi \in V_0$, substituting $v = U \pm \phi \in K$ into (19a), we immediately obtain (18a).

Let $\varphi \in V$ with $\varphi_n = 0$ on Γ be arbitrary. Substituting $v = U \pm \varphi \in K$ into (19a), we have

$$(U_t, \varphi) + a(U, \varphi) + a_1(U+g, U, \varphi) + a_1(U, g, \varphi) + b(\varphi, P) = (F, \varphi) - [[\tau_T(g), \varphi_T]].$$

This, together with (14), implies (18f). Let $w \in V$ with $w_n \geq 0$ on Γ be arbitrary. Substituting $v = w + U \in K$ into (19a), we have (18d).

Finally, substituting $v = -g \in K$ and $v = 2U + g \in K$ into (19a), we deduce

$$\begin{aligned}(U_t, U+g) + a(U, U+g) + a_1(U+g, U, U+g) + a_1(U, g, U+g)&\\ + b(U+g, P) = (F, U+g) - [[\tau(g,\pi), U+g]].&\end{aligned}$$

This, together with (13), gives (18e). $\square$

2.4 Penalty problems

We state the following variational formulations of (NS$_\varepsilon$).

(NS$_\varepsilon$E) For a.e. $t \in (0,T)$, find $(U_\varepsilon(t), P_\varepsilon(t)) \in V \times Q$ with $U_{\varepsilon,t}(t) \in Q^d$ such that

$$\begin{aligned}&(U_{\varepsilon,t}, v) + a(U_\varepsilon, v) + a_1(U_\varepsilon + g, U_\varepsilon, v) + a_1(U_\varepsilon, g, v)\\ &\qquad + b(v, P_\varepsilon) - \frac{1}{\varepsilon}\int_\Gamma [U_{\varepsilon n} + g_n]_- v_n \, d\Gamma = (F, v) - [[\tau(g,\pi), v]] && \forall v \in V, && (21a)\\ &b(U_\varepsilon, q) = 0 && \forall q \in Q, && (21b)\\ &U_\varepsilon(x, 0) = U_0 && \text{on } \Omega. && (21c)\end{aligned}$$

(NS$_\varepsilon$E$^\sigma$) For a.e. $t \in (0,T)$, find $U_\varepsilon(t) \in V^\sigma$ with $U_{\varepsilon,t}(t) \in Q^d$ such that

$$\begin{aligned}&(U_{\varepsilon,t}, v) + a(U_\varepsilon, v) + a_1(U_\varepsilon + g, U_\varepsilon, v)\\ &\qquad + a_1(U_\varepsilon, g, v) - \frac{1}{\varepsilon}\int_\Gamma [U_{\varepsilon n} + g_n]_- v_n \, ds = (F, v) - [[\tau(g,\pi), v]] && \forall v \in V^\sigma, && (22a)\\ &U_\varepsilon(x, 0) = U_0, && \text{on } \Omega. && (22b)\end{aligned}$$

2.5 Main results

We are now in a position to state the main results of this paper. Recall that (g, π) is the solution of the Stokes system (4) and $g_0 \in C_0^\infty(\Gamma)^d$ is defined by (5). We make the following assumptions.

(A1) $f \in H^1(0,T; Q^d)$ and $\tau(g,\pi)|_\Gamma \in H^1(0,T; L^2(\Gamma)^d)$.

(A2) $g \in H^2(0,T; Q^d) \cap L^\infty(0,T; V^\sigma)$ and $g_t \in L^2(0,T; V^\sigma)$.

(A3) $\beta(t) \geq \beta_0 > 0$ for $t \in [0,T]$ with some $\beta_0 > 0$ and $\beta(t) \in C^2(0,T)$.

(A4) $U_0 \in V_0^\sigma \cap H^2(\Omega)^d$ and it satisfies

$$-(\nu \Delta U_0, v) = a(U_0, v) + \int_\Gamma \tau(g,\pi)|_{t=0} v \, d\Gamma \quad (v \in V^\sigma).$$

Remark 2.3. Conditions (A1) and (A2) implies $F \in H^1(0,T;Q^d)$ and $F \in L^\infty(0,T;Q^d)$.

Remark 2.4. Condition (A2) leads to $g \in L^\infty(0,T;Q^d)$.

Remark 2.5. On Γ, $\tau(g,\pi)|_{t=0}$ is well-defined by (A1).

Theorem 2. *Assume that* (A1)–(A4) *are satisfied. When $d = 2$, there exists a unique*

$$U \in L^\infty(0,T;V^\sigma), \quad U_t \in L^2(0,T;V^\sigma) \cap L^\infty(0,T;Q^d), \tag{23a}$$

$$P \in L^\infty(0,T;Q) \tag{23b}$$

satisfying (NS-I) *for any $T \in (0,\infty)$. In particular, (U,P) is the unique solution of* (1) *with* (10) *in the sense of distributions. When $d = 3$, the same conclusion holds for a smaller time interval $(0,T_*]$, where T_* denotes a positive constant depending on U_0.*

Theorem 3. *Assume that* (A1)–(A4) *are satisfied. When $d = 2$, there exists a unique*

$$U_\varepsilon \in L^\infty(0,T;V^\sigma), \quad U_{\varepsilon,t} \in L^2(0,T;V^\sigma) \cap L^\infty(0,T;Q^d), \tag{24a}$$

$$P_\varepsilon \in L^\infty(0,T;Q) \tag{24b}$$

satisfying (NS$_\varepsilon$-E) *for any $T \in (0,\infty)$ and a sufficiently small ε. More precisely, there exists $\varepsilon_0 > 0$, which depends only on F, g, U_0, Ω and T, such that* (NS$_\varepsilon$-E) *admits a unique solution $(U_\varepsilon, P_\varepsilon)$ satisfying* (24) *for any $\varepsilon \in (0,\varepsilon_0]$. When $d = 3$, the same conclusion holds for a smaller time interval $(0,T_*]$, where T_* denotes a positive constant depending on U_0.*

The proof of Theorems 2 and 3 are divided into the following propositions where (A1)–(A4) are always assumed unless otherwise stated explicitly.

Proposition 1 (Existence of U)**.** *When $d = 2$, there exists a unique U satisfying* (23a) *and* (NS-I$^\sigma$) *for any $T \in (0,\infty)$. When $d = 3$, the same conclusion holds for a smaller time interval $(0,T_*]$, where T_* denotes a positive constant depending on U_0.*

Proposition 2 (Existence of P)**.** *Let U be a solution of* (NS-I$^\sigma$) *satisfying* (23a)*, then there exists a unique $P \in L^\infty(0,T;Q)$ such that (U,P) is a solution of* (NS-I).

Proposition 3 (Uniqueness)**.** *The solution of* (NS-I) *is unique.*

Proposition 4 (Existence of U_ε)**.** *When $d = 2$, there exists a unique U_ε satisfying* (24a) *and* (NS$_\varepsilon$-E$^\sigma$) *for any $T \in (0,\infty)$ and a sufficiently small ε. When $d = 3$, the same conclusion holds for a smaller time interval $(0,T_*]$, where T_* denotes a positive constant depending on U_0.*

Proposition 5 (Existence of P_ε)**.** *Let U_ε be a solution of* (NS$_\varepsilon$-E$^\sigma$) *satisfying* (24a)*, then there exists a unique $P_\varepsilon \in L^\infty(0,T;Q)$ such that $(U_\varepsilon,P_\varepsilon)$ is a solution of* (NS$_\varepsilon$-E).

Proposition 6 (Uniqueness)**.** *The solution of* (NS$_\varepsilon$-E) *is unique.*

As is stated in §1.2, Propositions 1 and 4 are proved in Section 4. Propositions 2, 5, 3 and 6 are proved in Section 3.

2.6 Review of some inequalities

We collect here some inequalities used below.

The following one is called Korn'n inequality (cf. [14, Lemma 6.2])

$$a(v,v) \geq \alpha\|v\|_1^2 \qquad (v \in V), \tag{25}$$

where $\alpha > 0$ denotes a positive constant depending only on Ω.

Lemma 2.2. *When $d = 2$,*

$$\begin{aligned} |a_1(u,v,w)| &\leq C\|u\|_{L^4(\Omega)^d}\|v\|_1\|w\|_{L^4(\Omega)^d} \\ &\leq C\|u\|^{\frac{1}{2}}\|u\|_1^{\frac{1}{2}}\|v\|_1\|w\|^{\frac{1}{2}}\|w\|_1^{\frac{1}{2}} \qquad (u,v,w \in H^1(\Omega)^d). \end{aligned} \tag{26}$$

When $d = 3$,

$$\begin{aligned} a_1(u,v,w) &\leq C\|u\|_{L^3(\Omega)}\|v\|_1\|w\|_{L^6(\Omega)} \\ &\leq C\|u\|^{\frac{1}{2}}\|u\|_1^{\frac{1}{2}}\|v\|_1\|w\|_1 \qquad (u,v,w \in H^1(\Omega)^d). \end{aligned} \tag{27}$$

Moreover, when $d = 2, 3$, we have

$$a_1(u,v,v) = \frac{1}{2}\int_\Gamma u_n|v|^2 \, d\Gamma \leq \|u_n\|_\Gamma\|v\|_{L^4(\Gamma)}^2 \leq C\|u_n\|_\Gamma\|v\|_1^2 \qquad (u,v \in V^\sigma). \tag{28}$$

Proof. It follows from Sobolev's embedding theorem and the trace theorem (cf. [13, 15]). For example, (27) is a readily obtainable consequence of Hölder's inequality, the continuous embedding $H^{\frac{1}{2}}(\Omega)(\Omega) \subset L^3(\Omega)$, and the interpolation inequality $\|v\|_{H^{\frac{1}{2}}(\Omega)} \leq C\|v\|^{\frac{1}{2}}\|v\|_1^{\frac{1}{2}}$. $\square$

Remark 2.6. Applying Young's inequality and Lemma 2.2, we have, for any $\xi > 0$,

$$|a_1(u,v,u)| \leq C\|u\|\|u\|_1\|v\|_1 \leq \xi\|u\|_1^2 + C\xi^{-1}\|u\|_1^2\|v\|_1^2, \tag{29}$$

if $d = 2$. On the other hand,

$$|a_1(u,v,u)| \leq C\|u\|^{\frac{1}{2}}\|u\|_1^{\frac{3}{2}}\|v\|_1 \leq \xi\|u\|_1^2 + C\xi^{-3}\|u\|_1^2\|v\|_1^4, \tag{30}$$

if $d = 3$.

3 Proof of Propositions 2, 3, 5 and 6

Proof of Proposition 2. (Existence) Let $\phi \in V_0 \cap V^\sigma$ be arbitrary. Substitution $v = \phi + U \in K^\sigma$ into (20) yields

$$(U_t, \phi) + a(U, \phi) + a_1(U + g, U, \phi) + a_1(U, g, \phi) = (F, \phi).$$

Then, there exists a unique $\mathring{P} \in Q_0$ (cf. [20, Lemma IV.1.4.3]) such that, for a.e. $t \in (0, T)$,

$$(U_t, \phi) + a(U, \phi) + a_1(U + g, U, \phi) + a_1(U, g, \phi) + b(v, \mathring{P}) = (F, \phi) \quad \forall \phi \in V_0 \tag{31}$$

and

$$\|\mathring{P}\| \leq C(\|U_t\| + \|U\|_1 + \|F\| + \|(U+g)\cdot\nabla U\| + \|U\cdot\nabla g\|). \tag{32}$$

We will show that there exists $k \in L^\infty(0,T)$ such that $(U, \mathring{P}+k)$ solves (NS-E).

First, by virtue of (31), (18a) is satisfied for $P = \mathring{P}+k$ with any $k \in L^\infty(0,T)$.

Recall that (16) and (20a) give

$$\begin{aligned}[][[\tau_T(U), v_T - U_T]] &+ [\tau_n(U, \mathring{P}+k), v_n - U_n] \\ &\geq -[[\tau_T(g), v_T - U_T]] - [\tau_n(g,\pi), v_n - U_n] \quad \forall v \in K^\sigma. \end{aligned} \tag{33}$$

Let $\psi \in C_0^\infty(\Gamma)^d$ be a function such that supp $\psi \subset \Gamma$ and $\psi_n = 0$. Then, since $\int_\Gamma \psi_n \, d\Gamma = 0$, there is a function $w \in V$ such that $w|_\Gamma = \psi$, $\nabla\cdot w = 0$ and $\|w\|_V \leq C\|\psi\|_{M^d}$. Substituting $v = U \pm w \in K^\sigma$ into (33), we have

$$[[\tau_T(U), \psi_T]] = [\tau_T(g), \psi_T].$$

By the density, this implies (18f). Moreover, since (33) is valid for an arbitrary $k \in L^\infty(0,T)$, we have

$$[\tau_n(U,\mathring{P}) + \tau_n(g,\pi), v_n + g_n] \geq [\tau_n(U,\mathring{P}) + \tau_n(g,\pi), U_n + g_n] \quad \forall v \in K^\sigma. \tag{34}$$

At this stage, we set

$$\gamma = \gamma(t) = \frac{1}{\beta}[\tau_n(U+g, \mathring{P}+\pi), U_n + g_n] \tag{35}$$

and take $k = \gamma$.

Then, noting $\int_\Gamma U_n \, d\Gamma = 0$ by $\nabla\cdot U = 0$ in Ω and $U|_{S\cup C} = 0$, we can calculate as

$$\begin{aligned}[][\tau_n(U,\mathring{P}+\gamma) + \tau_n(g,\pi), U_n + g_n] &= [\tau_n(U,\mathring{P}) + \tau_n(g,\pi), U_n + g_n] - \gamma\int_\Gamma g_n \, d\Gamma \\ &= [\tau_n(U,\mathring{P}) + \tau_n(g,\pi), U_n + g_n] - \gamma\beta \\ &= 0; \end{aligned}$$

which implies (18e).

For the time being, we admit

$$\gamma = \inf_{\eta\in Y}[\tau_n(U+g, \mathring{P}+\pi), \eta], \tag{36}$$

where

$$Y = \left\{\eta \in M \mid \eta \geq 0,\ \eta \not\equiv 0,\ \int_\Gamma \eta \, d\Gamma = 1\right\}.$$

For $\xi \in M$ with $\xi \geq 0$ and $\xi \not\equiv 0$, we have, by setting $m = \int_\Gamma \xi \, d\Gamma \neq 0$,

$$\begin{aligned}[][\tau_n(U,\mathring{P}+\gamma) + \tau_(g,\pi), \xi] &= [\tau_n(U,\mathring{P}) + \tau_n(g,\pi), \xi] - \gamma m \\ &= m[\tau_n(U,\mathring{P}) + \tau_n(g,\pi), \xi/m] - \gamma m \\ &\geq m\gamma - \gamma m = 0. \end{aligned}$$

Hence, we get (18d).

It remains to verify (36). Let $\eta \in Y$ be arbitrary and set $\tilde{\eta} = \beta\eta - g_n \in M$. Since $\int_\Gamma \tilde{\eta}\, d\Gamma = 0$, there exists $\tilde{v} \in V^\sigma$ such that $\tilde{v}_n|_\Gamma = \tilde{\eta}$. Then, the function $\tilde{v}$ satisfies that $\tilde{v}_n + g_n = \beta\eta \geq 0$ on Γ. Thus, $\tilde{v} \in K^\sigma$. Consequently, we have by (34)

$$
\begin{aligned}
[\tau_n(U, \mathring{P}) + \tau_n(g, \pi), \eta] &= \left[\tau_n(U, \mathring{P}) + \tau_n(g, \pi), \frac{\tilde{\eta} + g_n}{\beta}\right] \\
&= \left[\tau_n(U, \mathring{P}) + \tau_n(g, \pi), \frac{\tilde{v}_n + g_n}{\beta}\right] \\
&\geq \frac{1}{\beta}[\tau_n(U, \mathring{P}) + \tau_n(g, \pi), U_n + g_n] = \gamma;
\end{aligned}
$$

which yields (36).

(Regularity) According to the expression (35) and the definition (15), we deduce, for a.e. $t \in (0, T)$,

$$|\gamma| \leq C_0,$$

where $C_0 = C_0(t)$ denotes a positive function in $L^\infty(0, T)$ which depends only on $\|U_t\|$, $\|U\|_1$, $\|F\|$ and $\|g\|_1$. This, together with (32), gives $P \in L^\infty(0, T; Q)$.

(Uniqueness) Suppose that there is another pressure P'. Since $\mathring{P}$ and k are unique, we have

$$P' + k' = \mathring{P}, \quad k' \equiv -\frac{1}{|\Omega|}\int_\Omega P'\, dx = k.$$

Hence, $P = P'$. $\square$

Proof of Proposition 3. From Proposition 2, we know that P is uniquely determined by U; Therefore, we only need to show the uniqueness of U.

Suppose that U_1, U_2 are two solutions to (NS-I$^\sigma$). Let $w = U_1 - U_2$. From (20a), we have

$$
\begin{aligned}
&(U_{1,t}, U_2 - U_1) + a(U_1, U_2 - U_1) + a_1(U_1 + g, U_1, U_2 - U_1) \\
&\qquad + a_1(U_1, g, U_2 - U_1) \geq (F, U_2 - U_1) - [\tau(g, \pi), U_2 - U_1],
\end{aligned}
$$

and

$$
\begin{aligned}
&(U_{2,t}, U_1 - U_2) + a(U_2, U_1 - U_2) + a_1(U_2 + g, U_2, U_1 - U_2) \\
&\qquad + a_1(U_2, g, U_1 - U_2) \geq (F, U_1 - U_2) - [\tau(g, \pi), U_1 - U_2].
\end{aligned}
$$

Therefore,

$$(w_t, w) + a(w, w) + a_1(U_2 + g, w, w) \leq -a_1(w, U_1 + g, w).$$

In view of Korn's inequality (25), Lemma 2.2, Remark 2.6 and

$$a_1(U_2 + g, w, w) = \frac{1}{2}\int_\Gamma \underbrace{(U_2 \cdot n + g_n)}_{\geq 0} |w|^2\, d\Gamma \geq 0,$$

we have for any $\xi > 0$

$$\frac{1}{2}\|w(t)\|^2 + \alpha\|w(t)\|_1^2 \leq \begin{cases} \xi\|w\|_1^2 + C\xi^{-1}\|U_1 + g\|_1^2\|w\|^2 & \text{for } d = 2, \\ \xi\|w\|_1^2 + C\xi^{-3}\|U_1 + g\|_1^4\|w\|^2 & \text{for } d = 3. \end{cases}$$

Let ξ be sufficiently small so that $\alpha - \xi > \alpha/2$. In virtue of Gronwall's inequality, we obtain, for all $t \in (0,T)$,

$$\|w(t)\|^2 + \alpha \int_0^t \|w(s)\|_1^2 \, ds \le C \exp\left[Ct\|U_1 + g\|_{L^\infty(0,t;V)}\right] \|w(0)\|^2.$$

Since $w(0) = U_1(0) - U_2(0) = 0$, we conclude that $U_1 = U_2$. $\square$

We proceed to the proof of Propositions for the penalty problem. To do this, we need the following lemma. For the time being, we write $C_0(T) = C_{0,\varepsilon}(T)$ to express various positive constants depending only on the following quantities:

$$\|U_{\varepsilon,t}\|_{L^\infty(0,T;Q^d)}, \quad \|U_\varepsilon\|_{L^\infty(0,T;V^\sigma)}, \quad \|F\|_{L^\infty(0,T;Q^d)}, \quad \|g\|_{L^\infty(0,T;V^\sigma)}.$$

Lemma 3.1. *Let U_ε be a solution of* (NS_ε-E^σ)*, then, for a.e. $t \in (0,T)$,*

$$\|[U_{\varepsilon n} + g_n]_-\|_\Gamma \le C_0\sqrt{\varepsilon}. \tag{37}$$

Proof. Substituting $v = U_\varepsilon$ into (22a), it yields

$$-\frac{1}{\varepsilon}\int_\Gamma [U_{\varepsilon n} + g_n]_- U_{\varepsilon n} \, d\Gamma = (F, U_\varepsilon) - [[\tau(g,\pi), U_\varepsilon]] - (U_\varepsilon', U_\varepsilon) \\ - a(U_\varepsilon, U_{\varepsilon n}) + a_1(U_\varepsilon + g, U_\varepsilon, U_{\varepsilon n}) + a_1(U_\varepsilon, g, U_{\varepsilon n}).$$

RHS can be estimated from above in terms of $\|U_{\varepsilon,t}\|$, $\|U_\varepsilon\|_1$, $\|F\|$ and $\|g\|_1$, and the function U_ε satisfies (24a). Thus, we have

$$-\frac{1}{\varepsilon}\int_\Gamma [U_{\varepsilon n} + g_n]_- U_{\varepsilon n} \, d\Gamma \le C_0(T).$$

On the other hand, by using $g_n \ge 0$ on Γ, we see that

$$\begin{aligned} -\frac{1}{\varepsilon}\int_\Gamma [U_{\varepsilon n} + g_n]_- U_{\varepsilon n} \, d\Gamma &= -\frac{1}{\varepsilon}\int_\Gamma [U_{\varepsilon n} + g_n]_- (U_{\varepsilon n} + g_n - g_n) \, d\Gamma \\ &= \frac{1}{\varepsilon}\int_\Gamma |[U_{\varepsilon n} + g_n]_-|^2 \, d\Gamma + \frac{1}{\varepsilon}\int_\Gamma [U_{\varepsilon n} + g_n]_- g_n \, d\Gamma \\ &\ge \frac{1}{\varepsilon}\|[U_{\varepsilon n} + g_n]_-\|_\Gamma^2. \end{aligned}$$

Combining those estimates, we obtain (37). $\square$

Proof of Proposition 5. From (22), there exists a unique $\mathring{P}_\varepsilon \in Q_0$ (cf. [20, Lemma IV.1.4.3]) such that

$$(U_{\varepsilon,t}, v) + a(U_\varepsilon, v) + a_1(U_\varepsilon + g, U_\varepsilon, v) + a_1(U_\varepsilon, g, v) + b(v, \mathring{P}_\varepsilon) = (F, v) \qquad (v \in V_0)$$

and

$$\|\mathring{P}_\varepsilon\| \le C(\|U_\varepsilon'\| + \|U_\varepsilon\|_1 + \|(U_\varepsilon + g)\cdot\nabla U_\varepsilon\| + \|U_\varepsilon \cdot \nabla g\| + \|F\|).$$

Thus, we have

$$\|\mathring{P}_\varepsilon\| \le C_0(T). \tag{38}$$

We will show that there is $k_\varepsilon \in L^\infty(0,T)$ such that $(U_\varepsilon, P_\varepsilon)$ with $P_\varepsilon = \mathring{P}_\varepsilon + k_\varepsilon$ is a solution of (NS_ε-E).

Recalling (17) and using (22a), we have

$$\begin{aligned}[\tau_n(U_\varepsilon, P_\varepsilon), v_n] &= (U_{\varepsilon,t}, v) + a(U_\varepsilon, v) + a_1(U_\varepsilon + g, U_\varepsilon, v) + a_1(U_\varepsilon, g, v) + b(v, P_\varepsilon) - (F, v) \\ &= \frac{1}{\varepsilon}\int_\Gamma [U_{\varepsilon n} + g_n]_- v_n - [[\tau_n(g,\pi), v_n]] \qquad (v \in V^\sigma,\ v_T|_\Gamma = 0).\end{aligned}$$

Hence,

$$\left[\tau_n(U_\varepsilon, P_\varepsilon) + \tau_n(g,\pi) - \varepsilon^{-1}[\tau_n(U_\varepsilon, P_\varepsilon), v_n], \eta\right] = 0 \qquad (\eta \in M^\sigma), \tag{39}$$

where

$$M^\sigma = \left\{\eta \in M \mid \int_\Gamma \eta \, d\Gamma = 0\right\}.$$

Now we introduce

$$Z = \left\{\phi \in C_0^\infty(\Gamma) \mid \int_\Gamma \phi = 1\right\}$$

and take (and fix below) $\phi \in Z$. Then, for any $v \in V$, $\hat{\eta} = v_n - \alpha\phi$ with $\alpha = \int_\Gamma v_n \, d\Gamma$ belongs to M_0. Therefore, by (39),

$$\begin{aligned}&[\tau_n(U_\varepsilon, P_\varepsilon) + \tau_n(g,\pi) - \varepsilon^{-1}[U_{\varepsilon n} + g_n]_-, v_n] \\ &\quad = [\tau_n(U_\varepsilon, P_\varepsilon) + \tau_n(g,\pi) - \varepsilon^{-1}[U_{\varepsilon n} + g_n]_-, v_n - \alpha\phi] \\ &\qquad\qquad + [\tau_n(U_\varepsilon, P_\varepsilon) + \tau_n(g,\pi) - \varepsilon^{-1}[U_{\varepsilon n} + g_n]_-, \alpha\phi] \\ &\quad = \alpha[\tau_n(U_\varepsilon, P_\varepsilon) + \tau_n(g,\pi) - \varepsilon^{-1}[U_{\varepsilon n} + g_n]_-, \phi] \qquad (v \in V).\end{aligned}$$

Now, since

$$\begin{aligned}&[\tau_n(U_\varepsilon, P_\varepsilon) + \tau_n(g,\pi) - \varepsilon^{-1}[U_{\varepsilon n} + g_n]_-, \phi] \\ &\qquad\qquad = [\tau_n(U_\varepsilon, \mathring{P}_\varepsilon) + \tau_n(g,\pi) - \varepsilon^{-1}[U_{\varepsilon n} + g_n]_-, \phi] - k_\varepsilon,\end{aligned}$$

choosing

$$k_\varepsilon = [\tau_n(U_\varepsilon, \mathring{P}_\varepsilon) + \tau_n(g,\pi) - \varepsilon^{-1}[U_{\varepsilon n} + g_n]_-, \phi] \tag{40}$$

we obtain

$$[\tau_n(U_\varepsilon, P_\varepsilon) + \tau_n(g,\pi) - \varepsilon^{-1}[U_{\varepsilon n} + g_n]_-, v_n] = 0 \quad (v \in V);$$

which, together with (17), implies (21a).

It should be checked that k_ε defined as (40) actually independent of $\phi \in Z$ and it represents a function only of t. We let $\phi, \phi' \in Z$ with $\phi \not\equiv \phi'$. Then $\eta = \phi - \phi' \in M^\sigma$. Hence, by (39),

$$[\tau_n(U_\varepsilon, P_\varepsilon) + \tau_n(g,\pi) - \varepsilon^{-1}[U_{\varepsilon n} + g_n]_-, \phi] = [\tau_n(U_\varepsilon, P_\varepsilon) + \tau_n(g,\pi) - \varepsilon^{-1}[U_{\varepsilon n} + g_n]_-, \phi'],$$

which means that k_ε does not depend on the choice of $\phi \in Z$.

Finally, in view of (17), (37) and (40), we get

$$|k_\varepsilon| \le C_0(T).$$

Combining this with (38), we conclude $P_\varepsilon \in L^\infty(0,T;Q)$. □

Proof of Proposition 6. Since P_ε is uniquely determined by U_ε from Proposition 5, it suffices to show the uniqueness of (NS$_\varepsilon$-E$^\sigma$). Suppose that $U_{\varepsilon 1}$ and $U_{\varepsilon 2}$ are two solutions of (NS$_\varepsilon$-E$^\sigma$). Set $w = U_\varepsilon^1 - U_\varepsilon$. From (21a), we have, for any $v \in V^\sigma$,

$$
\begin{aligned}
&(w_t, v) + a(w, v) + a_1(U_{\varepsilon 1} + g, U_{\varepsilon 1}, v) - a_1(U_{\varepsilon 2} + g, U_{\varepsilon 2}, v) \\
&\qquad + a_1(w, g, v) - \frac{1}{\varepsilon}\int_\Gamma ([U_{\varepsilon 1}\cdot n + g_n]_- - [U_{\varepsilon 2}\cdot n + g_n]_-) v_n \, d\Gamma = 0.
\end{aligned}
$$

Substituting $v = w$ into above,

$$
\begin{aligned}
&(w_t, w) + a(w, w) - \frac{1}{\varepsilon}\int_\Gamma ([U_{\varepsilon 1}\cdot n + g_n]_- - [U_{\varepsilon 2}\cdot n + g_n]_-) w_n \, d\Gamma \\
&\qquad + a_1(U_{\varepsilon 2} + g, w, w) = -a_1(w, U_{\varepsilon 1} + g, w). \qquad (41)
\end{aligned}
$$

We can estimate as

$$
\begin{aligned}
&-\int_\Gamma ([U_{\varepsilon 1}\cdot n + g_n]_- - [U_{\varepsilon 2}\cdot n + g_n]_-) w_n \, d\Gamma \\
&= -\int_\Gamma ([U_{\varepsilon 1}\cdot n + g_n]_- - [U_{\varepsilon 2}\cdot n + g_n]_-)(U_{\varepsilon 1}\cdot n + g_n - (U_{\varepsilon 2}\cdot n + g_n)) \, d\Gamma \\
&= \int_\Gamma |[U_{\varepsilon 1}\cdot n + g_n]_- - [U_{\varepsilon 2}\cdot n + g_n]_-|^2 \, d\Gamma \\
&\quad + \int_\Gamma ([U_{\varepsilon 1}\cdot n + g_n]_-[U_{\varepsilon 2}\cdot n + g_n]_+ + [U_{\varepsilon 1}\cdot n + g_n]_+[U_{\varepsilon 2}\cdot n + g_n]_-) \, d\Gamma \ge 0 \qquad (42)
\end{aligned}
$$

and, by using Lemma 2.2,

$$
\begin{aligned}
a(w, w) + a_1(U_{\varepsilon 2} + g, w, w) &\ge \alpha\|w\|_1^2 + \frac{1}{2}\int_\Gamma (U_{\varepsilon 2}\cdot n + g_n)|w|^2 \, d\Gamma \\
&= \alpha\|w\|_1^2 + \frac{1}{2}\int_\Gamma ([U_{\varepsilon 2}\cdot n + g_n]_+ - [U_{\varepsilon 2}\cdot n + g_n]_-)|w|^2 \, d\Gamma \\
&\ge (\alpha - C'\|[U_{\varepsilon 2}\cdot n + g_n]_-\|_\Gamma)\|w\|_1^2. \qquad (43)
\end{aligned}
$$

In view of Lemma 3.1, we have $\|[U_{\varepsilon 2}\cdot n + g_n]_-\|_\Gamma \le C_0(T)\varepsilon$.

At this stage, we suppose that ε is small so that $\alpha - C'\|[U_{\varepsilon 2}\cdot n + g_n]_-\|_\Gamma \ge \alpha/2$. Then, it follows from (41), (42), and (43) that, for arbitrary $\xi > 0$,

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|w\|^2 + \frac{\alpha}{2}\|w\|_1 &\le -a_1(w, U_{\varepsilon 1} + g, w) \\
&\le \begin{cases} \xi\|w\|_1^2 + C\xi^{-1}\|U_{\varepsilon 1} + g\|_1^2\|w\|^2, & \text{for } d = 2, \\ \xi\|w\|_1^2 + C\xi^{-3}\|U_{\varepsilon 1} + g\|_1^4\|w\|^2, & \text{for } d = 3. \end{cases} \qquad (44)
\end{aligned}
$$

Setting $\xi = \alpha/4$, from (44) and Gronwall's inequality, we obtain, for a.e. $t \in (0, T]$,

$$
\|w(t)\|^2 + \int_0^t \|w(s)\|_1^2 \, ds \le C \exp\left[Ct\|U_{\varepsilon 1} + g\|_{L^\infty(0,t;V)}\right]\|w(0)\|^2.
$$

Since $w(0) = U_{\varepsilon 1}(0) - U_{\varepsilon 2}(0) = 0$, we conclude that $U_{\varepsilon 1} = U_{\varepsilon 2}$. □

4 Proof of Propositions 1 and 4

This section is devoted to the proof of the unique existence of solutions of (NS-I$^\sigma$) and (NS$_\varepsilon$-E$^\sigma$), that is, the proof of Propositions 1 and 4.

To achieve this purpose, we introduce new variables

$$\tilde{U} = \frac{U}{\beta}, \quad \tilde{P} = \frac{P}{\beta}, \quad \tilde{\pi} = \frac{\pi}{\beta}, \quad \tilde{f} = \frac{f}{\beta} \quad \text{and} \quad \tilde{g} = \frac{g}{\beta}.$$

Problem to find $(\tilde{U}, \tilde{P})$ reads as follows. For $t \in (0,T)$, find $(\tilde{U}, \tilde{P})$ such that

$$\begin{aligned}
&\tilde{U}_t + \frac{\beta'}{\beta}\tilde{U} + \beta((\tilde{U}+\tilde{g})\cdot\nabla)\tilde{U} + \beta(\tilde{U}\cdot\nabla)\tilde{g} - \nabla\cdot\sigma(\tilde{U},\tilde{P}) = \tilde{F} && \text{in } \Omega,\\
&\nabla\cdot\tilde{U} = 0 && \text{in } \Omega,\\
&\tilde{U} = 0 && \text{on } S\cup C,\\
&\tilde{U}_n + \tilde{g}_n \geq 0, \quad \tau_n(\tilde{U}+\tilde{g}, \tilde{P}+\tilde{\pi}) \geq 0 && \text{on } \Gamma,\\
&(\tilde{U}_n + \tilde{g}_n)\tau_n(U+\tilde{g}, \tilde{P}+\tilde{\pi}) = 0, \quad \tau_T(\tilde{U}) = -\tau_T(\tilde{g}) && \text{on } \Gamma,\\
&\tilde{U}(x,0) = \tilde{U}_0 && \text{on } \Omega,
\end{aligned}$$

where $\tilde{U}_0 = \frac{U_0}{\beta(0)}$, and $\tilde{F} = \tilde{f} - \tilde{g}' - \frac{\beta'}{\beta}\tilde{g} - \beta(\tilde{g}\cdot\nabla)\tilde{g} = \frac{F}{\beta}$.

We will study the well-posedness of $\tilde{U}$ instead of U itself. Set

$$\tilde{K} = \{v \in V \mid v_n + \tilde{g}_n \geq 0 \text{ on } \Gamma\}, \quad \tilde{K}^\sigma = \tilde{K} \cap V^\sigma$$

and consider the following variational problems.

(NS-I$^\sigma$)$\tilde{}$ For a.e. $t \in (0,T)$, find $\tilde{U}(t) \in \tilde{K}^\sigma$ with $\tilde{U}_t(t) \in Q^d$ such that

$$\begin{aligned}
&(\tilde{U}', v-\tilde{U}) + \frac{\beta'}{\beta}(\tilde{U}, v-\tilde{U}) + a(\tilde{U}, v-\tilde{U}) + \beta a_1(\tilde{U}+\tilde{g}, \tilde{U}, v-\tilde{U})\\
&\qquad + \beta a_1(\tilde{U}, \tilde{g}, v-\tilde{U}) \geq (\tilde{F}, v-\tilde{U}) - [[\tau(\tilde{g},\tilde{\pi}), v-\tilde{U}]] \qquad \forall v \in \tilde{K}^\sigma, && \text{(45a)}\\
&\tilde{U}(x,0) = \tilde{U}_0 \qquad \text{on } \Omega. && \text{(45b)}
\end{aligned}$$

(NS$_\varepsilon$-E$^\sigma$)$\tilde{}$ For a.e. $t \in (0,T)$, find $\tilde{U}_\varepsilon(t) \in \tilde{V}^\sigma$ with $\tilde{U}_{\varepsilon,t}(t) \in Q^d$ such that

$$\begin{aligned}
&(\tilde{U}_\varepsilon', v) + \frac{\beta'}{\beta}(\tilde{U}_\varepsilon, v) + a(\tilde{U}_\varepsilon, v) + \beta a_1(\tilde{U}_\varepsilon+\tilde{g}, \tilde{U}_\varepsilon, v)\\
&\qquad + \beta a_1(\tilde{U}_\varepsilon, \tilde{g}, v) - \frac{1}{\varepsilon}\int_\Gamma [\tilde{U}_{\varepsilon n} + \tilde{g}_n]_- v_n \, d\Gamma = (\tilde{F}, v) - [[\tau(\tilde{g},\tilde{\pi}), v]] \qquad \forall v \in V^\sigma, && \text{(46a)}\\
&\tilde{U}(x,0) = \tilde{U}_0 \qquad \text{on } \Omega. && \text{(46b)}
\end{aligned}$$

We see that a solution of (NS$_\varepsilon$-E$^\sigma$) is given as $U_\varepsilon = \beta\tilde{U}_\varepsilon$.

We introduce a regularization of $[\cdot]_-$. For any δ with $0 < \delta \ll 1$, we set

$$\rho_\delta(s) = \begin{cases} 0 & (s \geq 0) \\ \sqrt{s^2+\delta^2} - \delta & (s < 0). \end{cases} \tag{47}$$

We have $\rho_\delta(s) \in C^1(\mathbb{R})$ and

$$\frac{d}{ds}\rho_\delta(s) = \begin{cases} 0 & (s \geq 0) \\ \frac{s}{\sqrt{s^2+\delta^2}} & (s < 0), \end{cases} \quad \frac{d^2}{ds^2}\rho_\delta(s) = \begin{cases} 0 & (s \geq 0) \\ \frac{\delta^2}{(s^2+\delta^2)^{3/2}} & (s < 0). \end{cases} \tag{48}$$

Then we introduce the regularization problem to (NS$_\varepsilon$-E$^\sigma$)':

(NS$_\varepsilon$-E$^\sigma_\delta$)˜ For a.e. $t \in [0,T]$, find $\tilde{U}_{\varepsilon\delta}(t) \in V^\sigma$ with $\tilde{U}'_{\varepsilon\delta}(t) \in Q^d$ such that

$$\begin{aligned} &(\tilde{U}'_{\varepsilon\delta}, v) + \frac{\beta'}{\beta}(\tilde{U}_{\varepsilon\delta}, v) + a(\tilde{U}_{\varepsilon\delta}, v) + \beta a_1(\tilde{U}_{\varepsilon\delta} + \tilde{g}, \tilde{U}_{\varepsilon\delta}, v) \\ &\qquad + \beta a_1(\tilde{U}_{\varepsilon\delta}, \tilde{g}, v) - \frac{1}{\varepsilon}\int_\Gamma \rho_\delta(\tilde{U}_{\varepsilon\delta n} + \tilde{g}_n)v_n \, d\Gamma = (\tilde{F}, v) - [[\tau(\tilde{g}, \tilde{\pi}), v]] \quad \forall v \in V^\sigma, \end{aligned} \tag{49a}$$

$$\tilde{U}_{\varepsilon\delta}(x, 0) = \tilde{U}_0 \qquad \text{on } \Omega. \tag{49b}$$

The regularization problem (NS$_\varepsilon$-E$^\sigma_\delta$)˜ is of use not only for studying the well-posedness of penalty problem (NS$_\varepsilon$-E$^\sigma$)˜ but also for computing numerical solutions.

We show the well-posedness of (NS$_\varepsilon$-E$^\sigma_\delta$)˜ by Galerkin's method. Let $\{w_k\}_{k=1}^\infty \subset V^\sigma$ with $w_1 = \tilde{U}_0$ be linear independent functions such that

$$\bigcup_{m=1}^\infty \overline{\operatorname{span}\{w_k\}_{k=1}^m} \text{ is dense in } V^\sigma. \tag{50}$$

Then we consider the following problems for $m \in \mathbb{N}$.

(NS$_\varepsilon$ E$^\sigma_{\delta m}$)˜ Find

$$\tilde{U}_{\varepsilon\delta m} = \sum_{k=1}^m c_{\varepsilon\delta k}(t) w_k,$$

where $c_{\varepsilon\delta k} \in C^2([0,T])$ such that $\tilde{U}_{\varepsilon\delta m}(0) = U_0$ and, for all $k = 1, \dots, m$,

$$\begin{aligned} &(\tilde{U}'_{\varepsilon\delta m}, w_k) + \frac{\beta'}{\beta}(\tilde{U}_{\varepsilon\delta m}, w_k) + a(\tilde{U}_{\varepsilon\delta m}, w_k) + \beta a_1(\tilde{U}_{\varepsilon\delta m} + \tilde{g}, \tilde{U}_{\varepsilon\delta m}, w_k) \\ &\qquad + \beta a_1(\tilde{U}_{\varepsilon\delta m}, \tilde{g}, w_k) - \frac{1}{\varepsilon}\int_\Gamma \rho_\delta(\tilde{U}_{\varepsilon mn} + \tilde{g}_n) w_{kn} \, d\Gamma = (\tilde{F}, w_k) - [[\tau(\tilde{g}, \tilde{\pi}), w_k]], \end{aligned} \tag{51}$$

where $\tilde{U}_{\varepsilon\delta m}(0) = \tilde{U}_0$, $\tilde{U}_{\varepsilon\delta mn} = \tilde{U}_{\varepsilon\delta m} \cdot n$, and $w_{kn} = w_k \cdot n$.

Below, we prove Propositions 1 and 4 by using several lemmas. We always suppose that (A1)–(A4) are satisfied. Let us denote by c_1 the domain constant appearing in (28), and set

$$c_2(T) = \frac{c_1}{2}\max_{t\in[0,T]} \beta, \quad c_3(T) = \max_{t\in[0,T]} \frac{|\beta'|}{\beta},$$

$$c_4(T, \xi) = \begin{cases} C\xi^{-1}\|\tilde{g}\|_1^2 + c_3(T) & (d = 2) \\ C\xi^{-3}\|\tilde{g}\|_1^4 + c_3(T) & (d = 3) \end{cases} \quad (\xi > 0). \tag{52}$$

Further, we write $C_1(T)$ to express positive constants that depend only on the following quantities:

$$\|\tilde{F}\|_{L^\infty(0,T;Q^d)}, \quad \|\tilde{g}\|_{L^\infty(0,T;H^1(\Omega)^d)}, \quad \|\tilde{g}_t\|_{L^\infty(0,T;H^1(\Omega)^d)},$$
$$\|\tilde{\pi}\|_{L^\infty(0,T;Q)}, \quad \|\tilde{U}(0)\|_{H^2(\Omega)^d}, \quad \alpha, \quad c_1, \quad c_2(T), \quad c_3(T), \quad c_4(T, l\alpha),$$

where α denotes a positive constant appearing in Korn's inequality (25) and l some positive constants.

Lemma 4.1. *Let $m \in \mathbb{N}$. There exists $T_1 \in (0,T]$ such that* $(\mathrm{NS}_\varepsilon\text{-}\mathrm{E}^\sigma_{\delta m})\tilde{}$ *admits a unique solution $\tilde{U}_{\varepsilon\delta m}$ in $0 \le t \le T_1$ satisfying*

$$\|\tilde{U}_{\varepsilon\delta m}\|^2_{L^\infty(0,T_1;Q^d)} + \alpha\|\tilde{U}_{\varepsilon\delta m}\|^2_{L^2(0,T_1;V^\sigma)} + \frac{1}{\varepsilon}\int_0^{T_1}\int_\Gamma \rho_\delta(\tilde{U}_{\varepsilon\delta mn}+\tilde{g}_n)[\tilde{U}_{\varepsilon\delta mn}+\tilde{g}_n]_-\ d\Gamma dt \le C_1(T_1). \quad (53)$$

Proof. Since $\rho_\delta \in C^1(\mathbb{R})$, the system of ordinary differential equations (51) admits a unique solution $c_{\varepsilon\delta k} \in C^2([0,T_1])$ for $k=1,\dots,m$ with some $T_1>0$. We derive the estimation (53). Multiplying the both sides of (51) by $c_{\varepsilon\delta k}(t)$ and taking the summation for k, we have

$$\frac{1}{2}\frac{d}{dt}\|\tilde{U}_{\varepsilon\delta m}\|^2 + \frac{\beta'}{\beta}\|\tilde{U}_{\varepsilon\delta m}\|^2 + \alpha\|\tilde{U}_{\varepsilon\delta m}\|_1^2 + \beta a_1(\tilde{U}_{\varepsilon\delta m}+\tilde{g},\tilde{U}_{\varepsilon\delta m},\tilde{U}_{\varepsilon\delta m}) + \beta a_1(\tilde{U}_{\varepsilon\delta m},\tilde{g},\tilde{U}_{\varepsilon\delta m}) - \frac{1}{\varepsilon}\int_\Gamma \rho_\delta(\tilde{U}_{\varepsilon mn}+\tilde{g}_n)\tilde{U}_{\varepsilon mn}\ d\Gamma = (F,U_{\varepsilon m}) - [[\tau(g,\pi),U_{\varepsilon m}]]. \quad (54)$$

We see

$$\begin{aligned}
-\frac{1}{\varepsilon}\int_\Gamma \rho_\delta(\tilde{U}_{\varepsilon mn}+\tilde{g}_n)\tilde{U}_{\varepsilon mn}\ d\Gamma &= -\frac{1}{\varepsilon}\int_\Gamma \rho_\delta(\tilde{U}_{\varepsilon mn}+\tilde{g}_n)(\tilde{U}_{\varepsilon mn}+\tilde{g}_n-\tilde{g}_n)\ d\Gamma \\
&= \frac{1}{\varepsilon}\int_\Gamma (\rho_\delta(\tilde{U}_{\varepsilon mn}+\tilde{g}_n)[\tilde{U}_{\varepsilon mn}+\tilde{g}_n]_- + \rho_\delta(\tilde{U}_{\varepsilon mn}+\tilde{g}_n)\tilde{g}_n)\ d\Gamma \\
&\ge \frac{1}{\varepsilon}\int_\Gamma \rho_\delta(\tilde{U}_{\varepsilon mn}+\tilde{g}_n)[\tilde{U}_{\varepsilon mn}+\tilde{g}_n]_-\ d\Gamma \ge 0.
\end{aligned}$$

We apply Lemma 2.2 and Remark 2.6 to obtain, for arbitrary $\xi>0$,

$$|\beta a_1(\tilde{U}_{\varepsilon\delta m},\tilde{g},\tilde{U}_{\varepsilon\delta m})| \le \begin{cases} \xi\|\tilde{U}_{\varepsilon\delta m}\|_1^2 + C\xi^{-1}\|\tilde{g}\|_1^2\|\tilde{U}_{\varepsilon\delta m}\|^2 & \text{for } d=2, \\ \xi\|\tilde{U}_{\varepsilon\delta m}\|_1^2 + C\xi^{-3}\|\tilde{g}\|_1^4\|\tilde{U}_{\varepsilon\delta m}\|^2 & \text{for } d=3. \end{cases}$$

On the other hand, again by Lemma 2.2 ,

$$\begin{aligned}
\beta a_1(\tilde{U}_{\varepsilon\delta m}+\tilde{g},\tilde{U}_{\varepsilon\delta m},\tilde{U}_{\varepsilon\delta m}) &= \frac{\beta}{2}\int_\Gamma (\tilde{U}_{\varepsilon\delta mn}+\tilde{g}_n)|\tilde{U}_{\varepsilon\delta m}|^2\ ds \\
&= \frac{\beta}{2}\int_\Gamma [\tilde{U}_{\varepsilon\delta mn}+\tilde{g}_n]_+|\tilde{U}_{\varepsilon\delta m}|^2\ ds - \frac{\beta}{2}\int_\Gamma [\tilde{U}_{\varepsilon\delta mn}+\tilde{g}_n]_-|\tilde{U}_{\varepsilon\delta m}|^2\ ds \\
&\ge -c_1\frac{\beta}{2}\|[\tilde{U}_{\varepsilon\delta mn}+\tilde{g}_n]_-\|_\Gamma\|\tilde{U}_{\varepsilon\delta m}\|_1^2.
\end{aligned}$$

Moreover,

$$\left|(\tilde{F},\tilde{U}_{\varepsilon\delta m}) - [[\tau(\tilde{g},\tilde{\pi}),\tilde{U}_{\varepsilon\delta m}]]\right| \le \xi\|\tilde{U}_{\varepsilon\delta m}\|_1^2 + C\xi^{-1}(\|\tilde{F}\|^2 + \|\tau(\tilde{g},\tilde{\pi})\|_\Gamma^2).$$

Summing up those estimates, we deduce

$$\frac{1}{2}\frac{d}{dt}\|\tilde{U}_{\varepsilon\delta m}\|^2 + \tilde{\alpha}\|\tilde{U}_{\varepsilon\delta m}\|_1^2 + \frac{1}{\varepsilon}\int_\Gamma \rho_\delta(\tilde{U}_{\varepsilon\delta mn}+g_n)[\tilde{U}_{\varepsilon\delta mn}+g_n]_-\ d\Gamma \le C\xi^{-1}(\|\tilde{F}\|^2 + \|\tau(\tilde{g},\tilde{\pi})\|_\Gamma^2) + c_4(T,\xi)\|\tilde{U}_{\varepsilon\delta m}\|^2, \quad (55)$$

where $\tilde{\alpha} = \alpha - 2\xi - c_2\|[\tilde{U}_{\varepsilon\delta mn} + g_n]_-\|_\Gamma$.

At this stage, let $\xi = \alpha/8$. Let T_1 be the maximum time such that

$$c_2\|[\tilde{U}_{\varepsilon\delta mn} + \tilde{g}_n]_-\|_\Gamma \le \frac{\alpha}{4} \quad (t \in [0, T_1]). \tag{56}$$

Consequently,

$$\tilde{\alpha} = \alpha - 2\xi - c_2\|[\tilde{U}_{\varepsilon\delta mn} + \tilde{g}_n]_-\|_\Gamma \ge \frac{\alpha}{2} \quad (t \in [0, T_1]).$$

Since $\tilde{U}_{\varepsilon\delta mn}(0) + \tilde{g}_n(0) = \tilde{U}_0 + \tilde{g}_n \ge 0$, we have $\|[\tilde{U}_{\varepsilon\delta mn}(0) + \tilde{g}_n(0)]_-\|_\Gamma = 0$. Integrating the both sides of (55) with respect to t, we obtain, for any $t \in [0, T_1]$,

$$\begin{aligned}\|\tilde{U}_{\varepsilon\delta m}(t)\|^2 + \alpha \int_0^t \|\tilde{U}_{\varepsilon\delta m}(s)\|_1^2 + \frac{1}{\varepsilon}\int_0^t \int_\Gamma \rho_\delta(\tilde{U}_{\varepsilon\delta mn} + g_n)[\tilde{U}_{\varepsilon\delta mn} + g_n]_- \; d\Gamma ds \\ \le C(\|\tilde{F}\|^2_{L^2(0,t;Q^d)} + \|\tau(\tilde{g}, \tilde{\pi})\|^2_{L^2(0,t;L^2(\Gamma)^d)}) + \|\tilde{U}_0\|^2 + c_4(T, \xi)\int_0^t \|\tilde{U}_{\varepsilon\delta m}(s)\|^2 \; ds.\end{aligned}$$

We apply Gronwall's inequality to obtain

$$\begin{aligned}\|\tilde{U}_{\varepsilon\delta m}(t)\|^2 + \alpha \int_0^t \|\tilde{U}_{\varepsilon\delta m}(s)\|_1^2 + \frac{1}{\varepsilon}\int_0^t \int_\Gamma \rho_\delta(\tilde{U}_{\varepsilon\delta mn} + g_n)[\tilde{U}_{\varepsilon\delta mn} + g_n]_- \; d\Gamma ds \\ \le C_1(T)(\|\tilde{F}\|^2_{L^2(0,t;Q^d)} + \|\tau(\tilde{g}, \tilde{\pi})\|^2_{L^2(0,t;L^2(\Gamma)^d)} + \|\tilde{U}_0\|^2),\end{aligned}$$

which implies (53). □

Lemma 4.2. *Let $m \in \mathbb{N}$. When $d = 2$, the solution $\tilde{U}_{\varepsilon\delta m}$ of* $(\mathrm{NS}_\varepsilon\text{-}\mathrm{E}^\sigma_{\delta m})^\sim$ *satisfies*

$$\begin{aligned}\|\tilde{U}'_{\varepsilon\delta m}\|^2_{L^\infty(0,T_1;Q^d)} + \|\tilde{U}'_{\varepsilon\delta m}\|^2_{L^2(0,T_1;V^\sigma)} \\ + \frac{1}{\varepsilon}\int_0^{T_1}\int_\Gamma \frac{[\tilde{U}_{\varepsilon\delta mn} + \tilde{g}_n]_-}{\sqrt{(\tilde{U}_{\varepsilon\delta mn} + g_n)^2 + \delta^2}} |(\tilde{U}_{\varepsilon\delta mn} + \tilde{g}_n)'|^2 d\Gamma dt \le C(T_1),\end{aligned} \tag{57}$$

where T_1 is the constant appearing in Lemma 4.1. When $d = 3$, there exists $T'_1 \in (0, T_1]$ depending only on $\|\tilde{U}(0)\|_1$ and α such that (57) *holds true with the replacement T_1 by T'_1.*

Proof. First, we consider the case $d = 2$. Differentiating the both side of (51) with respect to t, we have

$$\begin{aligned}(\tilde{U}''_{\varepsilon\delta m}, w_k) + \left(\frac{\beta'}{\beta}\right)'(\tilde{U}_{\varepsilon\delta m}, w_k) + \frac{\beta'}{\beta}(\tilde{U}'_{\varepsilon\delta m}, w_k) + a(\tilde{U}'_{\varepsilon\delta m}, w_k) + \beta' a_1(\tilde{U}_{\varepsilon\delta m} + \tilde{g}, \tilde{U}_{\varepsilon\delta m}, w_k) \\ + \beta a_1(\tilde{U}'_{\varepsilon\delta m} + \tilde{g}, \tilde{U}_{\varepsilon\delta m}, w_k) + \beta a_1(\tilde{U}_{\varepsilon\delta m} + \tilde{g}, \tilde{U}'_{\varepsilon m}, w_k) + \beta' a_1(\tilde{U}_{\varepsilon\delta m}, \tilde{g}, w_k) + \beta a_1(\tilde{U}'_{\varepsilon\delta m}, \tilde{g}, w_k) \\ + \beta a_1(\tilde{U}_{\varepsilon\delta m}, \tilde{g}', w_k) - \frac{1}{\varepsilon}\int_\Gamma \rho_\delta(\tilde{U}_{\varepsilon\delta mn} + \tilde{g}_n)' w_{kn} \; ds = (\tilde{F}', w_k) - [[\tau(\tilde{g}', \pi'), w_k]].\end{aligned}$$

Multiplying the both sides of this equality by $c'_{\varepsilon\delta k}(t)$ and taking the summation for k, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\tilde{U}'_{\varepsilon\delta m}\|^2 + \alpha\|\tilde{U}'_{\varepsilon\delta m}\|_1^2 + \beta a_1(\tilde{U}_{\varepsilon\delta m} + \tilde{g}, \tilde{U}'_{\varepsilon\delta m}, \tilde{U}'_{\varepsilon\delta m}) - \frac{1}{\varepsilon}\int_\Gamma \rho_\delta(\tilde{U}_{\varepsilon\delta mn} + \tilde{g}_n)' \tilde{U}'_{\varepsilon\delta mn} \; ds \\ \le -\left(\frac{\beta'}{\beta}\right)'(\tilde{U}_{\varepsilon\delta m}, \tilde{U}'_{\varepsilon\delta m}) - \frac{\beta'}{\beta}\|\tilde{U}'_{\varepsilon\delta m}\|^2 - \beta' a_1(\tilde{U}_{\varepsilon\delta m} + \tilde{g}, \tilde{U}_{\varepsilon m}, \tilde{U}'_{\varepsilon\delta m}) \\ - \beta a_1(\tilde{U}'_{\varepsilon\delta m} + \tilde{g}', \tilde{U}_{\varepsilon\delta m}, \tilde{U}'_{\varepsilon\delta m}) - \beta' a_1(\tilde{U}_{\varepsilon\delta m}, \tilde{g}, \tilde{U}_{\varepsilon\delta m}) - \beta a_1(\tilde{U}'_{\varepsilon\delta m}, \tilde{g}, \tilde{U}'_{\varepsilon\delta m}) \\ - \beta a_1(\tilde{U}_{\varepsilon\delta m}, \tilde{g}', \tilde{U}'_{\varepsilon\delta m}) + (\tilde{F}', \tilde{U}'_{\varepsilon m}) - [[\tau(\tilde{g}', \tilde{\pi}'), \tilde{U}'_{\varepsilon m}]].\end{aligned} \tag{58}$$

As before, we have

$$\beta a_1(\tilde{U}_{\varepsilon\delta m}+\tilde{g},\tilde{U}'_{\varepsilon\delta m},\tilde{U}'_{\varepsilon\delta m}) \geq -c_2\|[\tilde{U}_{\varepsilon\delta mn}+\tilde{g}_n]_-\|_\Gamma\|\tilde{U}'_{\varepsilon\delta m}\|_1^2 \tag{59}$$

and, for any $\xi>0$,

$$\left|(\tilde{F}',\tilde{U}'_{\varepsilon\delta m})-[[\tau(\tilde{g}',\tilde{\pi}'),\tilde{U}'_{\varepsilon\delta m}]]\right| \leq \xi\|\tilde{U}'_{\varepsilon\delta m}\|_1^2+C\xi^{-1}(\|\tilde{F}'\|^2+\|\tau(\tilde{g}',\tilde{\pi}')\|_\Gamma^2). \tag{60}$$

Since $\tilde{g}=g_0(x)$ on Γ (cf. (4b)), we deduce $\tilde{g}'_n=0$ on Γ. Therefore,

$$\begin{aligned} -\int_\Gamma \rho_\delta(U_{\varepsilon\delta mn}+g_n)'\tilde{U}'_{\varepsilon mn}\,ds &= -\int_\Gamma \rho_\delta(U_{\varepsilon\delta mn}+g_n)'(\tilde{U}_{\varepsilon\delta mn}+\tilde{g}_n)'\,ds \\ &= \int_\Gamma \frac{[\tilde{U}_{\varepsilon\delta mn}+\tilde{g}_n]_-}{\sqrt{(\tilde{U}_{\varepsilon\delta mn}+g_n)^2+\delta^2}}|(\tilde{U}_{\varepsilon\delta mn}+\tilde{g}_n)'|^2 d\Gamma \geq 0. \end{aligned} \tag{61}$$

Moreover, in view of (53), we have, for all $t\in[0,T_1]$,

$$\left|\left(\frac{\beta'}{\beta}\right)'(\tilde{U}_{\varepsilon\delta m},\tilde{U}'_{\varepsilon\delta m})+\frac{\beta'}{\beta}\|\tilde{U}'_{\varepsilon\delta m}\|^2\right| \leq C_1(T)\|\tilde{U}'_{\varepsilon\delta m}\|\|^2+C_1(T). \tag{62}$$

Applying Lemma 2.2, Remark 2.6 and (53), we can perform estimations as, for arbitrary $\xi>0$,

$$\begin{aligned} \left|\beta' a_1(\tilde{U}_{\varepsilon\delta m}+\tilde{g},\tilde{U}_{\varepsilon\delta m},\tilde{U}'_{\varepsilon\delta m})\right| &\leq C\|\tilde{U}_{\varepsilon\delta m}+\tilde{g}\|^{1/2}\|\tilde{U}_{\varepsilon\delta m}+\tilde{g}\|_1^{1/2}\|\tilde{U}_{\varepsilon\delta m}\|_1\|\tilde{U}'_{\varepsilon\delta m}\|^{1/2}\|\tilde{U}'_{\varepsilon\delta m}\|_1^{1/2} \\ &\leq \xi\|\tilde{U}'_{\varepsilon\delta m}\|_1^2+C\eta^{-1/3}(\|\tilde{U}'_{\varepsilon\delta m}\|^2\|\tilde{U}_{\varepsilon\delta m}+\tilde{g}\|_1^2+\|\tilde{U}_{\varepsilon\delta m}\|_1^2); \end{aligned} \tag{63}$$

$$\begin{aligned} \left|\beta a_1(\tilde{U}'_{\varepsilon\delta m}+\tilde{g}',\tilde{U}_{\varepsilon\delta m},\tilde{U}'_{\varepsilon\delta m})\right| &\leq C\|\tilde{U}'_{\varepsilon\delta m}\|\|\tilde{U}_{\varepsilon\delta m}\|_1\|\tilde{U}'_{\varepsilon\delta m}\|_1 \\ &\quad + C\|\tilde{g}'\|^{1/2}\|\tilde{g}'\|_1^{1/2}\|\tilde{U}_{\varepsilon\delta m}\|_1\|\tilde{U}'_{\varepsilon\delta m}\|^{1/2}\|\tilde{U}'_{\varepsilon\delta m}\|_1^{1/2} \\ &\leq \xi\|\tilde{U}'_{\varepsilon\delta m}\|_1^2+C\xi^{-1}\|\tilde{U}'_{\varepsilon\delta m}\|^2\|\tilde{U}_{\varepsilon\delta m}\|_1^2 \\ &\quad + C\xi^{-1/3}(\|\tilde{U}'_{\varepsilon\delta m}\|^2\|\tilde{g}'\|_1^2+\|\tilde{U}_{\varepsilon\delta m}\|_1^2); \end{aligned} \tag{64}$$

$$\left|\beta' a_1(\tilde{U}_{\varepsilon\delta m},\tilde{g},\tilde{U}'_{\varepsilon\delta m})\right| \leq \xi\|\tilde{U}'_{\varepsilon\delta m}\|_1^2+C\xi^{-1/3}(\|\tilde{U}'_{\varepsilon\delta m}\|^2\|\tilde{U}_{\varepsilon\delta m}\|_1^2+\|\tilde{g}\|_1^2); \tag{65}$$

$$\left|\beta a_1(\tilde{U}'_{\varepsilon\delta m},\tilde{g},\tilde{U}'_{\varepsilon\delta m})\right| \leq \xi\|\tilde{U}'_{\varepsilon\delta m}\|_1^2+C\xi^{-1}\|\tilde{U}'_{\varepsilon\delta m}\|^2\|\tilde{g}\|_1^2; \tag{66}$$

and

$$\left|\beta a_1(\tilde{U}_{\varepsilon\delta m},\tilde{g}',\tilde{U}'_{\varepsilon\delta m})\right| \leq \xi\|\tilde{U}'_{\varepsilon\delta m}\|_1^2+C\xi^{-1/3}(\|\tilde{U}'_{\varepsilon\delta m}\|^2\|\tilde{U}_{\varepsilon\delta m}\|_1^2+\|\tilde{g}'\|_1^2). \tag{67}$$

From (58) to (67), we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\tilde{U}'_{\varepsilon\delta m}\|^2+\hat{\alpha}\|\tilde{U}'_{\varepsilon\delta m}\|_1^2+\frac{1}{\varepsilon}\int_0^{T_1}\int_\Gamma \frac{[\tilde{U}_{\varepsilon\delta mn}+\tilde{g}_n]_-}{\sqrt{(\tilde{U}_{\varepsilon\delta mn}+g_n)^2+\delta^2}}|(\tilde{U}_{\varepsilon\delta mn}+\tilde{g}_n)'|^2 d\Gamma dt \\ \leq C_1(T)(\|\tilde{g}\|_1^2+\|\tilde{g}'\|_1^2+\|\tilde{U}_{\varepsilon\delta m}\|_1^2)\|\tilde{U}'_{\varepsilon\delta m}\|^2 \\ + C_1(T)(\|\tilde{F}\|^2+\|\tau(\tilde{g},\tilde{\pi})\|_\Gamma^2)+C_1(T)(\|\tilde{g}'\|_1^2+\|\tilde{U}_{\varepsilon\delta m}\|_1^2), \end{aligned} \tag{68}$$

where $\hat{\alpha}=\alpha-6\xi-c_2\|[\tilde{U}_{\varepsilon\delta mn}+\tilde{g}_n]_-\|_\Gamma$.

Let $\xi = \alpha/12$. In virtue of (56), we see that

$$\hat{\alpha} = \alpha - 6\xi - c_2\|[\tilde{U}_{\varepsilon\delta mn} + \tilde{g}_n]_-\|_\Gamma \ge \frac{\alpha}{2} \quad (t \in [0, T_1]).$$

Applying Gronwall's inequality to (68) and using Lemma 4.1, we obtain

$$\begin{aligned}
&\|\tilde{U}'_{\varepsilon\delta m}\|^2_{L^\infty(0,T_1;Q^d)} + \alpha\|\tilde{U}'_{\varepsilon\delta m}\|^2_{L^2(0,T_1;V^\sigma)} \\
&\quad + \frac{1}{\varepsilon}\int_0^{T_1}\int_\Gamma \frac{[\tilde{U}_{\varepsilon\delta mn} + \tilde{g}_n]_-}{\sqrt{(\tilde{U}_{\varepsilon\delta mn} + g_n)^2 + \delta^2}} |(\tilde{U}_{\varepsilon\delta mn} + \tilde{g}_n)'|^2 d\Gamma dt \le C_1(T_1)(1 + \|\tilde{U}'_{\varepsilon\delta m}(0)\|^2). \quad (69)
\end{aligned}$$

To show the boundedness of $\|\tilde{U}'_{\varepsilon\delta m}(0)\|^2$, we multiply (51) by $c'_{\varepsilon\delta m}(t)$, add the resulting equations, and set $t = 0$. Consequently,

$$\begin{aligned}
&\|\tilde{U}'_{\varepsilon\delta m}(0)\|^2 + a(\tilde{U}_0, \tilde{U}'_{\varepsilon\delta m}(0)) - [[\tau(\tilde{g}, \tilde{\pi})(0), \tilde{U}'_{\varepsilon\delta m}(0)]] - \frac{1}{\varepsilon}\int_\Gamma \rho_\delta(\tilde{U}_0 + \tilde{g}_n(0))\tilde{U}'_{\varepsilon\delta mn}(0)\ ds \\
&= -\frac{\beta'}{\beta}(\tilde{U}_0, \tilde{U}'_{\varepsilon\delta m}(0)) - \beta a_1(\tilde{U}_0 + \tilde{g}(0), \tilde{U}_0, \tilde{U}'_{\varepsilon\delta m}(0)) - \beta a_1(\tilde{U}_0, \tilde{g}(0), \tilde{U}'_{\varepsilon\delta m}(0)) + (\tilde{F}(0), \tilde{U}'_{\varepsilon\delta m}(0)).
\end{aligned}$$

Since $[\tilde{U}_0 + \tilde{g}_n(0)]_- = 0$, we have by (A4)

$$\begin{aligned}
\|\tilde{U}'_{\varepsilon\delta m}(0)\|^2 &\le |a(\tilde{U}_0, \tilde{U}'_{\varepsilon\delta m}(0))| + |(\Delta\tilde{U}_0, \tilde{U}'_{\varepsilon\delta m}(0))| + \left|\frac{\beta'}{\beta}(\tilde{U}_0, \tilde{U}'_{\varepsilon\delta m}(0))\right| \\
&\quad + \left|\beta a_1(\tilde{U}_0 + \tilde{g}(0), \tilde{U}_0, \tilde{U}'_{\varepsilon\delta m}(0))\right| + \left|\beta a_1(\tilde{U}_0, \tilde{g}(0), \tilde{U}'_{\varepsilon\delta m}(0))\right| + \left|(\tilde{F}(0), \tilde{U}'_{\varepsilon\delta m}(0))\right| \\
&\le C\left(\|\tilde{U}_0\| + \|\tilde{U}_0\|_{H^2} + \|\tilde{U}_0 + \tilde{g}(0)\|_{L^\infty}\|\tilde{U}_0\|_1 + \|\tilde{U}_0\|_{L^\infty}\|\tilde{g}(0)\|_1 + \|\tilde{F}(0)\|\right)\|\tilde{U}'_{\varepsilon\delta m}(0)\|, \quad (70)
\end{aligned}$$

which shows $\|\tilde{U}'_{\varepsilon\delta m}(0)\| \le C_1(T)$. This, together with (69), implies (57).
When $d = 3$, the discussion before (63) and the estimation of $\|\tilde{U}'_{\varepsilon\delta m}(0)\|_\Omega$ remain true for $d = 3$. What are changed from the case $d = 2$ are estimations of $\|\tilde{U}'_{\varepsilon\delta m}\|^2_{L^\infty(0,\mathring{T}_1;Q^d)}$ and $\|\tilde{U}'_{\varepsilon\delta m}\|^2_{L^2(0,\mathring{T}_1;V)}$.

In place of (63)–(67), we derive, for arbitrary $\xi > 0$,

$$\begin{aligned}
&\left|\beta' a_1(\tilde{U}_{\varepsilon\delta m} + \tilde{g}, \tilde{U}_{\varepsilon\delta m}, \tilde{U}'_{\varepsilon\delta m})\right| \\
&\quad \le C\|\tilde{U}_{\varepsilon\delta m} + \tilde{g}\|_{L^6}\|\tilde{U}_{\varepsilon\delta m}\|_1\|\tilde{U}'_{\varepsilon\delta m}\|_{L^3} \\
&\quad \le C\|\tilde{U}_{\varepsilon\delta m} + \tilde{g}\|_1\|\tilde{U}_{\varepsilon\delta m}\|_1\|\tilde{U}'_{\varepsilon\delta m}\|^{1/2}\|\tilde{U}'_{\varepsilon\delta m}\|_1^{1/2} \\
&\quad \le \xi\|\tilde{U}'_{\varepsilon\delta m}\|_1^2\|\tilde{U}_{\varepsilon\delta m}\|_1^2 + C\xi^{-1/3}\|\tilde{U}'_{\varepsilon\delta m}\|^{2/3}\|\tilde{U}_{\varepsilon\delta m} + \tilde{g}\|_1^{4/3}\|\tilde{U}_{\varepsilon\delta m}\|_1^{2/3} \\
&\quad \le \xi\|\tilde{U}'_{\varepsilon\delta m}\|_1^2\|\tilde{U}_{\varepsilon\delta m}\|_1^2 + C\xi^{-1/3}\|\tilde{U}'_{\varepsilon\delta m}\|^2\|\tilde{U}_{\varepsilon\delta m}\|_1^2 + C\xi^{-1/3}\|\tilde{U}_{\varepsilon\delta m} + \tilde{g}\|_1^2; \\
&\left|\beta a_1(\tilde{U}'_{\varepsilon\delta m} + \tilde{g}', \tilde{U}_{\varepsilon\delta m}, \tilde{U}'_{\varepsilon\delta m})\right| \\
&\quad \le C\|\tilde{U}'_{\varepsilon\delta m} + \tilde{g}'\|_{L^6}\|\tilde{U}_{\varepsilon\delta m}\|_1\|\tilde{U}'_{\varepsilon\delta m}\|_{L^3} \\
&\quad \le \xi\|\tilde{U}'_{\varepsilon\delta m}\|_1^2(\|\tilde{U}_{\varepsilon\delta m}\|_1 + \|\tilde{U}_{\varepsilon\delta m}\|_1^2) + C\xi^{-3}\|\tilde{U}'_{\varepsilon\delta m}\|^2\|\tilde{U}_{\varepsilon\delta m}\|_1 \\
&\qquad\qquad + C\xi^{-1/3}\|\tilde{U}'_{\varepsilon\delta m}\|^2\|\tilde{U}_{\varepsilon\delta m}\|_1^2 + C\xi^{-1/3}\|\tilde{g}'\|_1^2; \\
&\left|\beta' a_1(\tilde{U}_{\varepsilon\delta m}, \tilde{g}, \tilde{U}'_{\varepsilon\delta m})\right| \le \xi\|\tilde{U}'_{\varepsilon\delta m}\|_1^2\|\tilde{g}\|_1^2 + C\xi^{-1}\|\tilde{U}_{\varepsilon\delta m}\|_1^2 \\
&\left|\beta a_1(\tilde{U}'_{\varepsilon\delta m}, \tilde{g}, \tilde{U}'_{\varepsilon\delta m})\right| \le \xi\|\tilde{U}'_{\varepsilon\delta m}\|_1^2 + C\xi^{-3}\|\tilde{U}'_{\varepsilon\delta m}\|^2\|\tilde{g}\|_1^4;
\end{aligned}$$

and

$$\left|\beta a_1(\tilde{U}_{\varepsilon\delta m}, \tilde{g}', \tilde{U}'_{\varepsilon\delta m})\right| \leq \xi\|\tilde{U}'_{\varepsilon\delta m}\|_1^2\|\tilde{U}_{\varepsilon\delta m}\|_1^2 + C\xi^{-1}\|\tilde{g}'\|_1^2.$$

Hence, in place of (68), we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\tilde{U}'_{\varepsilon\delta m}\|^2 + \bar{\alpha}\|\tilde{U}'_{\varepsilon\delta m}\|_1^2 + \varepsilon^{-1}\|([\tilde{U}_{\varepsilon\delta mn} + \tilde{g}_n]_-)'\|_\Gamma^2 \leq C_1(T)(\|\tilde{F}\|^2 + \|\tau(\tilde{g}, \tilde{\pi})\|_\Gamma^2) \\ + C_1(T)(\|\tilde{g}\|_1^4 + \|\tilde{g}\|_1^2 + \|\tilde{U}_{\varepsilon\delta m}\|_1^2)\|\tilde{U}'_{\varepsilon\delta m}\|^2 + C_1(T)(\|\tilde{g}'\|_1^2 + \|\tilde{U}_{\varepsilon\delta m}\|_1^2), \quad (71)\end{aligned}$$

where $\bar{\alpha} = \alpha - 2\xi - 4\xi\|\tilde{U}_{\varepsilon\delta m}\|_1^2 - \xi\|\tilde{U}_{\varepsilon\delta m}\|_1 - C_1\|[\tilde{U}_{\varepsilon\delta mn} + \tilde{g}_n]_-\|_\Gamma$.

We choose ξ satisfying $2\xi + 4\xi\|\tilde{U}_0\|_1^2 + \xi\|\tilde{U}_0\|_1 \leq \alpha/12$. Let $\hat{T}_1$ be the maximum value of t such that $2\xi + 4\xi\|\tilde{U}_{\varepsilon\delta mn}(t)\|_1^2 + \xi\|\tilde{U}_{\varepsilon\delta mn}(t)\|_1 \leq \alpha/4$. Let $T_1' = \min\{\hat{T}_1, T_1\}$, then $\bar{\alpha} \geq \alpha/2$ for all $t \in [0, T_1']$. Then, applying Gronwall's inequality, we obtain (57) with the replacement T_1 by T_1'. □

Lemma 4.3. *Let $m \in \mathbb{N}$ and $\delta \leq \varepsilon$. The solution $\tilde{U}_{\varepsilon\delta m}$ of* (NS$_\varepsilon$-E$^\sigma_{\delta m}$)$\tilde{}$ *satisfies*

$$\|\tilde{U}_{\varepsilon\delta m}\|^2_{L^\infty(0,T_1;V^\sigma)} + \frac{1}{\varepsilon}\|[\tilde{U}_{\varepsilon\delta mn} + \tilde{g}_n]_-\|^2_{L^\infty(0,T_1;L^2(\Gamma))} \leq C_1(T_1), \quad (72)$$

where T_1 is the constant appearing in Lemma 4.1.

Proof. Multiplying (51) by $c'_{\varepsilon\delta m}(t)$ and taking the summation for k, we have

$$\begin{aligned}\|\tilde{U}'_{\varepsilon\delta m}\|^2 + \frac{1}{2}\frac{d}{dt}a(\tilde{U}_{\varepsilon\delta m}, \tilde{U}_{\varepsilon\delta m}) - \frac{1}{\varepsilon}\int_\Gamma \rho_\delta(\tilde{U}_{\varepsilon\delta mn} + \tilde{g}_n)\tilde{U}'_{\varepsilon\delta mn}\, ds \\ = -\frac{\beta'}{\beta}(\tilde{U}_{\varepsilon\delta m}, \tilde{U}'_{\varepsilon\delta m}) - \beta a_1(\tilde{U}_{\varepsilon\delta m} + \tilde{g}, \tilde{U}_{\varepsilon\delta m}, \tilde{U}'_{\varepsilon\delta m}) \\ - \beta a_1(\tilde{U}_{\varepsilon\delta m}, \tilde{g}, \tilde{U}'_{\varepsilon\delta m}) + (\tilde{F}, \tilde{U}'_{\varepsilon\delta m}) + [\tau(\tilde{g}, \tilde{\pi}), \tilde{U}'_{\varepsilon\delta m}] \equiv \text{RHS}.\end{aligned}$$

Integrating the both sides with respect to t, we have, for $t \in [0, T_1]$,

$$\begin{aligned}\frac{1}{2}a(\tilde{U}_{\varepsilon\delta m}(t), \tilde{U}_{\varepsilon\delta m}(t)) - \frac{1}{\varepsilon}\int_0^t\int_\Gamma \rho_\delta(\tilde{U}_{\varepsilon\delta mn} + \tilde{g}_n)\tilde{U}'_{\varepsilon\delta mn}\, d\Gamma ds \\ = \frac{1}{2}a(\tilde{U}_0, \tilde{U}_0) + \int_0^t \text{RHS}\, ds. \quad (73)\end{aligned}$$

Since $\tilde{g}' = 0$ on Γ,

$$\begin{aligned}& -\int_\Gamma \rho_\delta(\tilde{U}_{\varepsilon\delta mn} + \tilde{g}_n)\tilde{U}'_{\varepsilon\delta mn}\, d\Gamma \\ =\; & -\int_\Gamma \rho_\delta(\tilde{U}_{\varepsilon\delta mn} + \tilde{g}_n)(\tilde{U}_{\varepsilon\delta mn} + \tilde{g}_n)'\, d\Gamma \\ =\; & \int_\Gamma \rho_\delta(\tilde{U}_{\varepsilon\delta mn} + \tilde{g}_n)'(\tilde{U}_{\varepsilon\delta mn} + \tilde{g}_n)\, d\Gamma - \int_\Gamma[\rho_\delta(\tilde{U}_{\varepsilon\delta mn} + \tilde{g}_n)(\tilde{U}_{\varepsilon\delta mn} + \tilde{g}_n)]'\, d\Gamma \\ \equiv\; & I_1 + I_2 \quad (74)\end{aligned}$$

We see that

$$
\begin{aligned}
I_1 &= \int_\Gamma \rho_\delta(\tilde{U}_{\varepsilon\delta mn}+\tilde{g}_n)'([\tilde{U}_{\varepsilon\delta mn}+\tilde{g}_n]_+ - [\tilde{U}_{\varepsilon\delta mn}+\tilde{g}_n)]_-)d\Gamma \\
&= -\int_\Gamma \frac{\tilde{U}_{\varepsilon\delta mn}+\tilde{g}_n}{\sqrt{[\tilde{U}_{\varepsilon\delta mn}+\tilde{g}_n]_-|^2+\delta^2}}[\tilde{U}_{\varepsilon\delta mn}+\tilde{g}_n]_- d\Gamma \\
&= \int_\Gamma \frac{[\tilde{U}_{\varepsilon\delta mn}+\tilde{g}_n]_-^2}{\sqrt{[\tilde{U}_{\varepsilon\delta mn}+\tilde{g}_n]_-|^2+\delta^2}} d\Gamma \geq 0. \qquad (75)
\end{aligned}
$$

Moreover, by using $(\tilde{U}_{\varepsilon\delta mn}+\tilde{g}_n)(0)\geq 0$ and $|\rho_\delta(s)-[s]_-|\leq\delta$ for $s\in\mathbb{R}$, we get, for $t\in[0,T_1]$,

$$
\begin{aligned}
\int_0^t I_2\ ds &= -\int_\Gamma \rho_\delta(\tilde{U}_{\varepsilon\delta mn}(t)+\tilde{g}_n(t))(\tilde{U}_{\varepsilon\delta mn}(t)+\tilde{g}_n(t)) \\
&= \int_\Gamma \rho_\delta(\tilde{U}_{\varepsilon\delta mn}(t)+\tilde{g}_n(t))[\tilde{U}_{\varepsilon\delta mn}(t)+\tilde{g}_n(t)]_- \\
&= \|[\tilde{U}_{\varepsilon\delta mn}(t)+\tilde{g}_n(t)]_-\|_{L^2(\Gamma)}^2 \\
&\quad + \int_\Gamma [\tilde{U}_{\varepsilon\delta mn}(t)+\tilde{g}_n(t)]_-(\rho_\delta(\tilde{U}_{\varepsilon\delta mn}(t)+\tilde{g}_n(t)) - [\tilde{U}_{\varepsilon\delta mn}(t)+\tilde{g}_n(t)]_-)\ d\Gamma \\
&\geq \|[\tilde{U}_{\varepsilon\delta mn}(t)+\tilde{g}_n(t)]_-\|_\Gamma^2 - \delta\int_\Gamma [\tilde{U}_{\varepsilon\delta mn}(t)+\tilde{g}_n(t)]_-\ d\Gamma \\
&\geq \|[\tilde{U}_{\varepsilon\delta mn}(t)+\tilde{g}_n(t)]_-\|_\Gamma^2 - C\delta(\|\tilde{U}_{\varepsilon\delta mn}(t)\|_1+\|\tilde{g}_n(t)\|_1). \qquad (76)
\end{aligned}
$$

Hence, (73) leads to

$$
\begin{aligned}
&\frac{1}{2}a(\tilde{U}_{\varepsilon\delta m}(t),\tilde{U}_{\varepsilon\delta m}(t)) + \frac{1}{\varepsilon}\|[\tilde{U}_{\varepsilon\delta mn}(t)+\tilde{g}_n(t)]_-\|_\Gamma^2 \\
&\qquad\qquad \leq \frac{1}{2}a(\tilde{U}_0,\tilde{U}_0) + \int_0^t \text{RHS}\ ds + C\frac{\delta}{\varepsilon}(\|\tilde{U}_{\varepsilon\delta mn}(t)\|_1+\|\tilde{g}_n(t)\|_1). \qquad (77)
\end{aligned}
$$

In view of (53), (57) and (25), we obtain (72) for $t\in[0,T_1]$. □

Lemma 4.4. *Let $m\in\mathbb{N}$ and suppose $\delta\leq\varepsilon$. When $d=2$,* $(\mathrm{NS}_\varepsilon\text{-}\mathrm{E}_{\delta m}^\sigma)\tilde{}$ *admits a unique solution $\tilde{U}_{\varepsilon\delta m}$ for any $T\in(0,\infty)$ satisfying* (53), (57), *and* (72) *with the replacement T_1 by T.*

Proof. In view of (72), for sufficiently small ε,

$$
\|[\tilde{U}_{\varepsilon\delta mn}+\tilde{g}_n]_-\|_\Gamma \leq C_1(T_1)\sqrt{\varepsilon}\leq C_1(T)\sqrt{\varepsilon} \qquad (t\in[0,T_1]).
$$

Hence, there exists $\varepsilon_2>0$ and $T_2\in(T_1,T]$ such that (56) is satisfied for all $t\in[0,T_2]$ and $\varepsilon\in(0,\varepsilon_2]$. Furthermore, we can replace T_1 in (53), (57) and (72) by T_2. We can continue this process until we reach some $T_k=T$ and (53), (57) and (72) are satisfied with T_1 replaced by $T_k=T$. □

Lemma 4.5. *When $d=2$, for any $T\in(0,\infty)$, there exists $\varepsilon_0>0$ and a solution $\tilde{U}_\varepsilon$ of* $(\mathrm{NS}_\varepsilon\text{-}\mathrm{E}_\delta^\sigma)\tilde{}$ *satisfying*

$$
\|\tilde{U}_{\varepsilon\delta}\|_{L^\infty(0,T;V^\sigma)} + \varepsilon^{-1/2}\|[\tilde{U}_{\varepsilon\delta}+\tilde{g}_n]_-\|_{L^\infty(0,T;L^2(\Gamma))} \leq C_1(T), \qquad (78a)
$$

$$
\|\tilde{U}_{\varepsilon\delta}'\|_{L^\infty(0,T;Q^d)} + \|\tilde{U}_{\varepsilon\delta}'\|_{L^2(0,T;V^\sigma)} \leq C_1(T), \qquad (78b)
$$

if $\varepsilon \in (0, \varepsilon_0]$ *and* $\delta \leq \varepsilon$. *When* $d = 3$, *the same conclusion holds for a smaller time interval* $(0, T_*)$.

Proof. The proof below is valid for both $d = 2, 3$, except that we have to replace T by T_* when $d = 3$. Let $\varepsilon \in (0, \varepsilon_0]$ be fixed. As a consequence of Lemmas 4.1–4.4, there exists some $\bar{U}_{\varepsilon\delta}$ and a subsequence of $\{\tilde{U}_{\varepsilon\delta m}\}_{m=1}^{\infty}$, such that $\bar{U}_{\varepsilon\delta} \in L^\infty(0, T; V^\sigma)$, $\bar{U}'_{\varepsilon\delta} \in L^\infty(0, T; L^2(\Omega)^d) \cap L^2(0, T; V^\sigma)$ and, as $m \to \infty$,

$$
\begin{aligned}
&\tilde{U}_{\varepsilon\delta m} \to \bar{U}_{\varepsilon\delta} && \text{weakly* in } L^\infty(0, T; V^\sigma),\\
&[\tilde{U}_{\varepsilon\delta m} + g_n]_- \to [\bar{U}_{\varepsilon\delta} + g_n]_- && \text{weakly* in } L^\infty(0, T; L^2(\Gamma)),\\
&\tilde{U}'_{\varepsilon\delta m} \to \bar{U}'_{\varepsilon\delta} && \text{weakly* in } L^\infty(0, T; L^2(\Omega)^d),\\
&\tilde{U}'_{\varepsilon\delta m} \to \bar{U}'_{\varepsilon\delta} && \text{weakly in } L^2(0, T; V^\sigma).
\end{aligned}
$$

We show $\bar{U}_{\varepsilon\delta}$ is the solution of (49a). Multiplying (51) by $\phi \in C_0^\infty(0, T)$, and integrating over $(0, T)$, it yields, for all $k = 1, 2, \ldots, m$,

$$
\begin{aligned}
\int_0^T \phi(t) \Big\{ &(\tilde{U}'_{\varepsilon\delta m}, w_k) + \frac{\beta'}{\beta}(\tilde{U}_{\varepsilon\delta m}, \tilde{U}_{\varepsilon\delta m}) + a(\tilde{U}_{\varepsilon\delta m}, w_k) + \beta a_1(\tilde{U}_{\varepsilon\delta m} + \tilde{g}, \tilde{U}_{\varepsilon\delta m}, w_k)\\
&+ \beta a_1(\tilde{U}_{\varepsilon\delta m}, \tilde{g}, w_k) - \frac{1}{\varepsilon}\int_\Gamma [\tilde{U}_{\varepsilon\delta mn} + \tilde{g}_n]_- w_{kn}\, ds - (\tilde{F}, w_k) + [[\tau(\tilde{g}, \tilde{\pi}), w_k]] \Big\}\, dt = 0.
\end{aligned}
$$

It follows from (cf. [23, Theorem 2.1, Chap. 3], [2, Theorem II.5.16] and [18, Theorem 6.1, Corollary 6.2, Chap. 2]) that the embedding

$$\{w \mid w \in L^2(0, T; V),\ w' \in L^2(0, T; L^2(\Omega)^d)\} \subset L^2(0, T; L^4(\Omega)^d)$$

is compact. Hence $\tilde{U}_{\varepsilon\delta m} \to \bar{U}_{\varepsilon\delta}$ strongly in $L^2(0, T; L^4(\Omega)^d)$. Since the trace mapping $H^1(0, T; V) \to L^2(0, T; L^2(\Gamma)^d)$ is compact, we have

$$\tilde{U}_{\varepsilon\delta mn} \to \bar{U}_{\varepsilon\delta n} \text{ strongly in } L^2(0, T; L^2(\Gamma)).$$

Therefore, $\tilde{U}_{\varepsilon\delta mn} \to \bar{U}_{\varepsilon\delta n}$ a.e. on Γ. Moreover, the function $[s]_-$ of $s \in \mathbb{R}$ is continuous so that $[\tilde{U}_{\varepsilon\delta mn} + \tilde{g}_n]_- \to [\bar{U}_{\varepsilon\delta n} + \tilde{g}_n]_-$ a.e. on Γ.

At this stage, letting $m \to \infty$, we obtain, for all $k \in \mathbb{N}$,

$$
\begin{aligned}
\int_0^T \phi(t) \Big\{ &(\bar{U}'_{\varepsilon\delta}, w_k) + \frac{\beta'}{\beta}(\bar{U}_{\varepsilon\delta}, \bar{U}_{\varepsilon\delta}) + a(\bar{U}_{\varepsilon\delta}, w_k) + \beta a_1(\bar{U}_{\varepsilon\delta} + \bar{g}, \bar{U}_{\varepsilon\delta}, w_k)\\
&+ \beta a_1(\bar{U}_{\varepsilon\delta}, \bar{g}, w_k) - \frac{1}{\varepsilon}\int_\Gamma [\bar{U}_{\varepsilon\delta n} + \tilde{g}_n]_- w_{kn}\, ds - (\tilde{F}, w_k) + [[\tau(\tilde{g}, \tilde{\pi}), w_k]] \Big\}\, dt = 0. \qquad (79)
\end{aligned}
$$

In view of (50), we can replace the test function w_k of (79) by arbitrary $v \in V^\sigma$. Consequently, we have proved $\bar{U}_{\varepsilon\delta} = \tilde{U}_{\varepsilon\delta}$ is the solution of (49a) satisfying (80). □

Lemma 4.6. *When* $d = 2$, *for any* $T \in (0, \infty)$, *there exists* $\varepsilon_0 > 0$ *and a solution* $\tilde{U}_\varepsilon$ *of* $(\text{NS}_\varepsilon\text{-E}^\sigma)^\sim$ *satisfying*

$$\|\tilde{U}_\varepsilon\|_{L^\infty(0,T;V^\sigma)} + \varepsilon^{-1/2}\|[\tilde{U}_\varepsilon + \tilde{g}_n]_-\|_{L^\infty(0,T;L^2(\Gamma))} \leq C_1(T), \qquad (80a)$$

$$\|\tilde{U}'_\varepsilon\|_{L^\infty(0,T;Q^d)} + \|\tilde{U}'_\varepsilon\|_{L^2(0,T;V^\sigma)} \leq C_1(T), \qquad (80b)$$

if $\varepsilon \in (0, \varepsilon_0]$. *When* $d = 3$, *the same conclusion holds for a smaller time interval* $(0, T_*)$.

Proof. The proof below is valid for both $d = 2, 3$, except that when $d = 3$, we have to replace T by T_*. As a consequence of Lemma 4.6, there exists some $\bar{U}_\varepsilon$ and a subsequence of $\{\tilde{U}_{\varepsilon\delta_i}\}_{i=1}^{\infty}$, with $\lim_{i\to\infty} \delta_i = 0$ such that $\bar{U}_\varepsilon \in L^\infty(0,T;V^\sigma)$, $\bar{U}'_\varepsilon \in L^\infty(0,T;L^2(\Omega)^d) \cap L^2(0,T;V\sigma)$, and as $i \to \infty$, $\delta_i \to 0$,

$$
\begin{aligned}
&\tilde{U}_{\varepsilon\delta_i} \to \bar{U}_\varepsilon && \text{weakly* in } L^\infty(0,T;V^\sigma),\\
&\rho_{\delta_i}(\tilde{U}_{\varepsilon\delta_i} + g_n) \to [\bar{U}_\varepsilon + g_n]_- && \text{weakly* in } L^\infty(0,T;L^2(\Gamma)),\\
&\tilde{U}'_{\varepsilon\delta_i} \to \bar{U}'_\varepsilon && \text{weakly* in } L^\infty(0,T;L^2(\Omega)^d),\\
&\tilde{U}'_{\varepsilon\delta_i} \to \bar{U}'_\varepsilon && \text{weakly in } L^2(0,T;V^\sigma).
\end{aligned}
$$

It is not difficult to verify that $\bar{U}_\varepsilon$ is the solution to (46). And we proved $\bar{U}_\varepsilon = \tilde{U}_\varepsilon$ is the solution to (46) satisfying (80). $\square$

Lemma 4.7. *When $d = 2$, for any $T \in (0,\infty)$, there exists a solution $\tilde{U}$ of* (NS-I$^\sigma$)$\tilde{}$ *satisfying*

$$\|\tilde{U}\|_{L^\infty(0,T;V^\sigma)} \le C_1(T), \tag{81a}$$

$$\|\tilde{U}'\|_{L^\infty(0,T;L^2(\Omega)^d)} + \|\tilde{U}'\|_{L^2(0,T;V^\sigma)} \le C_1(T). \tag{81b}$$

When $d = 3$, the same conclusion holds for a smaller time interval $(0, T_*)$.

Proof. The proof is valid for both $d = 2, 3$, except we replace T by T_* for the case $d = 3$.

In view of Lemma 4.6, sequences $\|\tilde{U}_\varepsilon\|_{L^\infty(0,T;V^\sigma)}$, $\|\tilde{U}'_\varepsilon\|_{L^\infty(0,T;L^2(\Omega)^d)}$ and $\|\tilde{U}'_\varepsilon\|_{L^2(0,T;V^\sigma)}$ are bounded as $\varepsilon \to 0$ and $\|[\tilde{U}_\varepsilon + \tilde{g}_n]_-\|_{L^\infty(0,T;L^2(\Gamma))} \le C_1(T)\sqrt{\varepsilon}$. Hence, they admit a sequence ε_i ($\varepsilon_i \to 0$ as $i \to \infty$) and $\bar{U} \in L^\infty(0,T;V^\sigma)$ such that $\bar{U}' \in L^\infty(0,T;L^2(\Omega)^d) \cap L^2(0,T;V^\sigma)$ and, as $\varepsilon_i \to 0$,

$$
\begin{aligned}
&\tilde{U}_{\varepsilon_i} \to \bar{U} && \text{weakly* in } L^\infty(0,T;V^\sigma), \quad \text{weakly in } L^2(0,T;V^\sigma),\\
&[\tilde{U}_{\varepsilon_i n} + \tilde{g}_n]_- \to 0 && \text{weakly* in } L^\infty(0,T;L^2(\Gamma)),\\
&\tilde{U}'_{\varepsilon_i} \to \bar{U}' && \text{weakly* in } L^\infty(0,T;L^2(\Omega)^d),\\
&\tilde{U}'_{\varepsilon_i} \to \bar{U}' && \text{weakly in } L^2(0,T;V^\sigma).
\end{aligned}
$$

In the similar manner as the proof of Lemma 4.6, we have, as $\varepsilon_i \to 0$,

$$
\begin{aligned}
&\tilde{U}_{\varepsilon_i} \to \bar{U} && \text{strongly in } L^4(0,T;L^2(\Omega)^d),\\
&\tilde{U}_{\varepsilon_i n} \to \bar{U}_n && \text{strongly in } L^2(0,T;L^2(\Omega)^d),\\
&[\tilde{U}_{\varepsilon_i} + \tilde{g}_n]_- \to [\bar{U}_n + \tilde{g}_n]_- && \text{a.e. on } \Gamma.
\end{aligned}
$$

Hence, $[\bar{U}_n + \tilde{g}_n]_- = 0$ a.e. on Γ, $\bar{U} \in \tilde{K}^\sigma$, and

$$\int_0^T a(\bar{U},\bar{U})dt \le \liminf_{\varepsilon_i\to 0} \int_0^T a(\tilde{U}_\varepsilon,\tilde{U}_\varepsilon)dt.$$

On the other hand, we have from (46)

$$
\begin{aligned}
(\tilde{U}'_\varepsilon, v - \tilde{U}_\varepsilon) + \frac{\beta'}{\beta}(\tilde{U}_\varepsilon, v - \tilde{U}_\varepsilon) + a(\tilde{U}_\varepsilon, v - \tilde{U}_\varepsilon) + \beta a_1(\tilde{U}_\varepsilon, \tilde{g}, v - \tilde{U}_\varepsilon) + \beta a_1(\tilde{U}_\varepsilon + \tilde{g}, \tilde{U}_\varepsilon, v - \tilde{U}_\varepsilon)\\
- \frac{1}{\varepsilon}\int_\Gamma [\tilde{U}_{\varepsilon n} + \tilde{g}_n]_-(v_n - \tilde{U}_{\varepsilon n})ds - (\tilde{F}, v - \tilde{U}_\varepsilon) - [[\tau(\tilde{g},\tilde{\pi}), v - \tilde{U}_\varepsilon]] = 0
\end{aligned}
$$

for any $v \in \tilde{K}^\sigma$ and a sufficiently small ε.

Noting that, for any $v \in \tilde{K}$,

$$\begin{aligned} -[\tilde{U}_{\varepsilon n} + \tilde{g}_n]_-(v_n - \tilde{U}_{\varepsilon n}) &= -[\tilde{U}_{\varepsilon n} + \tilde{g}_n]_-[v_n + \tilde{g}_n - (\tilde{U}_{\varepsilon n} + \tilde{g}_n)] \\ &= -[\tilde{U}_{\varepsilon n} + \tilde{g}_n]_-(v_n + \tilde{g}_n) - |[\tilde{U}_{\varepsilon n} + \tilde{g}_n]_-|^2 \le 0, \end{aligned}$$

we deduce, for all $t \in [0,T]$,

$$\int_0^t \Big\{(\tilde{U}_\varepsilon', v - \tilde{U}_\varepsilon) + (\beta'/\beta)(\tilde{U}_\varepsilon, v - \tilde{U}_\varepsilon) + a(\tilde{U}_\varepsilon, v - \tilde{U}_\varepsilon) + \beta a_1(\tilde{U}_\varepsilon, \tilde{g}, v - \tilde{U}_\varepsilon) \\ + \beta a_1(\tilde{U}_\varepsilon + \tilde{g}, \tilde{U}_\varepsilon, v - \tilde{U}_\varepsilon) - (\tilde{F}, v - \tilde{U}_\varepsilon) - [[\tau(\tilde{g}, \tilde{\pi}), v - \tilde{U}_\varepsilon]]\Big\} \ge 0.$$

Therefore, taking the lower limit as $\varepsilon_i \to 0$, we obtain

$$\int_0^t \{(\bar{U}', v - \bar{U}) + (\beta'/\beta)(\bar{U}, v - \bar{U}) + a(\bar{U}, v - \bar{U}) + \beta a_1(\bar{U}, \tilde{g}, v - \bar{U}) \\ + \beta a_1(\bar{U} + \tilde{g}, \bar{U}, v - \bar{U}) - (\tilde{F}, v - \bar{U}) - [[\tau(\tilde{g}, \tilde{\pi}), v - \bar{U}]]\Big\} \ge 0$$

for any $v \in \tilde{K}^\sigma$. By using this inequality, we conclude that $\bar{U} = \tilde{U}$ is a solution of (45) for a.e. $t \in [0,T]$ in the exactly same way as [7, Paragraph III.3.4.1]. □

Finally, we can state the following proof.

Proof of Propositions 1 and 4. Since $U = \tilde{U}\beta$ and $U_\varepsilon = \tilde{U}_\varepsilon\beta$, Lemmas 4.7 and 4.6 imply Propositions 1 and 4, respectively. □

5 Energy inequalities

In this section, we derive energy inequalities for (NS-E) and (NS$_\varepsilon$-E).

Theorem 4 (Energy inequality for (NS-E))**.** *If there exists*

$$U \in L^\infty(0,T;Q^d) \cap L^2(0,T;V^\sigma), \quad U_t \in L^2(0,T;Q^d) \tag{82}$$

that satisfy (NS-E) *in $0 \le t \le T$ with some $P(t) \in Q$, then we have*

$$\sup_{0 \le t \le T} \|U(t)\|^2 + \int_0^T a(U(t), U(t))\ dt \le C(T), \tag{83}$$

where $C(T)$ denotes a positive constant depending on F, g, U_0, Ω and T.

Theorem 5 (Energy inequality for (NS$_\varepsilon$-E))**.** *Let $\varepsilon > 0$. Suppose that there exists*

$$U_\varepsilon \in L^\infty(0,T;Q^d) \cap L^2(0,T;V^\sigma), \quad U_{\varepsilon,t} \in L^2(0,T;Q^d)$$

that satisfy (NS$_\varepsilon$-E) *in $0 \le t \le T$ with some $P_\varepsilon(t) \in Q$. Moreover, assume that*

$$\|[U_{\varepsilon n} + g_n]_-\|_{L^\infty(0,T;L^2(\Gamma))} \le C_2(T)\sqrt{\varepsilon} \quad (t \in [0,T]). \tag{84}$$

Then, there exists $\varepsilon_1 > 0$ such that we have, for $\varepsilon \in (0, \varepsilon_1]$,

$$\sup_{0 \le t \le T} \|U_\varepsilon(t)\|^2 + \int_0^T a(U_\varepsilon(t), U_\varepsilon(t))\ dt \le C(T). \tag{85}$$

Therein, ε_1, $C_2(T)$ and $C(T)$ denote positive constants depending on F, g, U_0, Ω and T.

Remark 5.1. As is described in the previous section, the existence proof of U_ε depends on the inequality (84) (cf. Lemmas 4.3, 4.4 and 4.6). Hence, it is not restrictive that we assume (84) as long as the solution exists.

We finally state the following proofs.

Proof of Theorem 4. Substituting U into (18a), we have

$$\frac{1}{2}\frac{d}{dt}\|U\|^2 + a(U,U) + a_1(U+g,U,U) = (F,U) - a_1(U,g,U).$$

We apply Lemma 2.2 and Remark 2.6 to deduce, for any $\xi > 0$,

$$|a_1(U,g,U)| \le \begin{cases} \xi\|U\|_1^2 + C\xi^{-1}\|g\|_1^2\|U\|^2 & (d=2) \\ \xi\|U\|_1^2 + C\xi^{-3}\|g\|_1^4\|U\|^2 & (d=3); \end{cases}$$
$$|(F,U)| \le C\|F\|\|U\|_1 \le \xi\|U\|_1^2 + C\xi^{-1}\|F\|^2.$$

On the other hand, since $U_n + g_n \ge 0$ on Γ,

$$a_1(U+g,U,U) = \frac{1}{2}\int_\Gamma (U_n + g_n)|U|^2 d\Gamma \ge 0.$$

Summing up, we have, for any $\xi > 0$,

$$\frac{1}{2}\frac{d}{dt}\|U\|^2 + a(U,U) - 2\xi\|U\|_1^2 \le \begin{cases} \xi^{-1}\|g\|_1^2\|U\|^2 + C\xi^{-1}\|F\|^2 & (d=2) \\ C\xi^{-3}\|g\|_1^4\|U\|^2 + C\xi^{-1}\|F\|^2 & (d=3). \end{cases} \tag{86}$$

From Korn's inequality (25), the left-hand side of (86) is bounded from below by

$$\frac{1}{2}\frac{d}{dt}\|U\|^2 + \left(1 - \frac{2\xi}{\alpha}\right) a(U,U).$$

Supposing $\xi = \alpha/4$ and applying Gronwall's inequality to (86), we obtain (83). □

Proof of Theorem 5. Substituting $v = U_\varepsilon$ into (21a), we get

$$\frac{1}{2}\frac{d}{dt}\|U_\varepsilon\|^2 + a(U_\varepsilon,U_\varepsilon) + a_1(U_\varepsilon + g, U_\varepsilon, U_\varepsilon) - \frac{1}{\varepsilon}\int_\Gamma [U_{\varepsilon n} + g_n]_- U_{\varepsilon n}\, d\Gamma$$
$$= (F, U_\varepsilon) - [[\tau(g,\pi), U_\varepsilon]] - a_1(U_\varepsilon, g, U_\varepsilon).$$

We argue as in the proof of Lemma 4.1 and obtain, for any $\xi > 0$,

$$\frac{1}{2}\frac{d}{dt}\|U_\varepsilon\|^2 + \left[1 - \frac{2\xi}{\alpha} - \frac{c_1}{2\alpha}\|[U_{\varepsilon n} + g_n]_-\|_\Gamma\right] a(U_\varepsilon,U_\varepsilon) + \frac{1}{\varepsilon}\int_\Gamma [U_{\varepsilon n} + g_n]_- g_n\, d\Gamma$$
$$\le C\frac{1}{\xi}(\|F\|^2 + \|\tau(g,\pi)\|_\Gamma^2) + c_4(T,\xi)\|U_\varepsilon\|^2,$$

where c_1 denotes the domain constant appearing in (28) and c_4 is defined as (52) with the replacement of $\tilde{g}$ by g.

At this stage, we choose as $\xi = \alpha/8$ and let $\sqrt{\varepsilon_1} = \alpha/(2c_1C_2(T))$. Then, for $\varepsilon \in (0, \varepsilon_1]$,

$$\frac{1}{2}\frac{d}{dt}\|U_\varepsilon\|^2 + \frac{1}{2}a(U_\varepsilon,U_\varepsilon) \le C(\|F\|^2 + \|\tau(g,\pi)\|_\Gamma^2) + c_4(T,\xi)\|U_\varepsilon\|^2.$$

Applying Gronwall's inequality implies (85). □

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