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by

Shigeo KUSUOKA and Yusuke MORIMOTO

UNIVERSITY OF TOKYO

GRADUATE SCHOOL OF MATHEMATICAL SCIENCES

KOMABA, TOKYO, JAPAN

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Shigeo KUSUOKA *and Yusuke MORIMOTO †

Abstract

Least square regression methods are Monte Carlo methods to solve non-liear problems related to Markov processes and are widely used in practice. In these methods, first we choose a system of functions to approximate value functions. So one of questions on these methods is what kinds of systems of functions one has to take to get a good approximation. In the present paper, we will discuss on this problem.

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1 Introduction

Least square regression methods are Monte Carlo methods to solve non-liear problems related to Markov processes. These methods were introduced by Longstaff-Schwartz [9] and Tsitsiklis-Van Roy[11] and are widely used in practice. There are many works related to this methods. Concerning the applications for pricing Bermudan derivatives, the convergence to a real price was proved by Clement-Lamberton-Protter [4] and rate of convergence was studied by Belomestny [2]. In these methods, first we choose a system of functions to approximate value functions. So one of questions on these methods is what kinds of systems of functions one has to take to get a good approximation. In the present paper, we will discuss on this problem. Related topics have been discussed by Gobet-Lemor-Warin [5] and Bally-Pagés [1].

Let $(\Omega, \mathcal{F}, P)$ be a probability space, $M \geqq 1$, and $\{\mathcal{G}_m\}_{m=0}^M$ be a filtration on $(\Omega, \mathcal{F}, P)$. Let $(E, \mathcal{B})$ a measurable space and $m(E)$ be the set of Borel measurable functions on E. Let $p_m : E \times \mathcal{B} \to [0,1]$, $m = 0, \ldots, M-1$, be such that $p_m(x, \cdot) : \mathcal{B} \to [0,1]$ is a probability measure on E for any $x \in E$, and $p_m(\cdot, A) : E \to [0,1]$ is $\mathcal{B}$-measurable for any $A \in \mathcal{B}$. Let $x_0 \in E$ and fix it throughout. Let $X : \{0, 1, \ldots, M\} \times \Omega \to E$ be an E-valued process such that $X_0 = x_0$, $X_m : \Omega \to E$ is $\mathcal{G}_m$-measurable, $m = 0, \ldots, M$, and

$$P(X_{m+1} \in A | \mathcal{G}_m) = p_m(X_m, A) \ a.s. \qquad A \in \mathcal{B},\ m = 0, \ldots, M-1.$$

*Graduate School of Mathematical Sciences, The University of Tokyo, Komaba 3-8-1, Meguro-ku, Tokyo 153-8914, Japan
†Graduate School of Mathematical Sciences, The University of Tokyo, Komaba 3-8-1, Meguro-ku, Tokyo 153-8914, Japan, Bank of Tokyo Mitsubishi UFJ

So X is a Markov process starting from x_0 whose transition probability is given by $p_m(x, dy)$.

Let ν_m, $m = 1, \dots, M$, be the probability law of X_m, $m = 0, 1, \dots, M$. Then ν_0 is the probability measure concentrated in x_0, and

$$\nu_{m+1}(A) = \int_E p_m(x, A)\nu_m(dx), \qquad y \in E, m = 0, 1, \dots, M-1.$$

Let $P_m : L^2(E; d\nu_{m+1}) \to L^2(E; d\nu_m)$, $m = 0, 1, \dots, M-1$, be a linear operator given by

$$(P_m f)(x) = \int_E p_m(x, dy) f(y), \qquad f \in L^2(E; d\nu_{m+1}).$$

Now let $f_m \in L^4(E; d\nu_m)$, $m = 1, 2, \dots, M$. We define $\tilde{f}_m$, $\tilde{f}^*_m \in L^4(E; d\nu_m)$, $m = 0, 1, 2, \dots, M$, inductively by th following.

$$\tilde{f}_M = f_M,$$

and

$$\tilde{f}^*_m = \tilde{f}_m \vee f_m, \quad \tilde{f}_{m-1} = P_m(\tilde{f}_m \vee f_m), \qquad m = M, M-1, \dots, 1.$$

Then it is well-known that

$$\tilde{f}_0 = \sup\{E[f_\tau(X_\tau)]; \ \ \tau \text{ is a } \{\mathcal{G}_m\}_{m=0}^M\text{-stopping time with } \tau \in \{1, 2, \dots, M\} \ a.s.\}.$$

$\tilde{f}_0$ is the price of a Bermudan derivative for which exercisable times are $1, \dots, M$, and pay-off at each time is $f_m(X_m)$, $m = 1, \dots, M$. Our concern is to compute $\tilde{f}_0$ numerically.

Let $\mathcal{V}$ denote the set of finite dimensional vector subspaces of $m(E)$. For any probability measure ν on $(E, \mathcal{B})$, let $\mathcal{V}(\nu)$ denote the subset of $\mathcal{V}$ such that $V \in \mathcal{V}(\nu)$, if and only if V satisfies the following two conditions.

(1) If $g \in V$, then $\int_E g(x)^4 \nu(dx) < \infty$.

(2) If $g \in V$ and $g(x) = 0$ $\nu - a.e.x$, then $g \equiv 0$.

For any probability measure ν on $(E, \mathcal{B})$ and $V \in \mathcal{V}(\nu)$, we define $\lambda_0(V, \nu)$ and $\lambda_1(V, \nu)$ by the following.

$$\lambda_0(V, \nu) = \sup\{\frac{\int_E g(x)^4 \nu(dx)}{(\int_E g(x)^2 \nu(dx))^2}; \ g \in V \setminus \{0\}\}$$

$$\lambda_1(V; \nu) = \inf\{\int_E (\sum_{r=1}^{\dim V} e_r(x)^2)^2 \nu(dx); \ \{e_r\}_{r=1}^{\dim V} \text{ is an orthonormal basis of } V \text{ as a subspace of } L^2(E; d\nu) \ \}.$$

We will show in Proposition 4 that

$$\lambda_1(V; \nu) \leqq (\dim V)^2 \lambda_0(V; \nu) \text{ and } \lambda_0(V; \nu) \leqq \lambda_1(V; \nu).$$

Now let $(X_0^{(\ell)}, X_1^{(\ell)}, \dots, X_M^{(\ell)})$, $\ell = 1, 2, \dots$, be independent identically distributed E^{M+1}-valued random variables such that the law of $(X_0^\ell, X_1^\ell, \dots, X_M^\ell)$, $\ell = 1, 2, \dots$, is the same as the law of $(X_0, X_1, \dots, X_M)$ under P.

For any $m = 0, 1, \ldots, M-1$, and $L \geqq 1$, we define $D_m^{(L)} : m(E) \times m(E) \times \Omega \to [0, \infty)$ by

$$D_m^{(L)}(g,f)(\omega) = (\frac{1}{L}\sum_{\ell=1}^{L}(g(X_m^{(\ell)}(\omega) - f(X_{m+1}^{(\ell)}(\omega))^2)^{1/2}, \qquad g, f \in m(E).$$

Let $V_m^{(k)}$, $k = 1, 2, \ldots$, be a sequence of strictly increasing vector spaces in $\mathcal{V}(\nu_m)$ such that $\bigcup_{k=1}^{\infty} V_m^{(k)}$ is dense in $L^2(E; d\nu_m)$ for $m = 1, \ldots, M-1$.

Now we assume that $g_m^{(L)} : \Omega \to V_m^{(L)}$, $m = 0, 1, \ldots, M-1$, $L = 1, 2, \ldots$, satisfy the following.

$$D_{m-1}(g_{m-1}^{(L)}(\omega), g_m^{(L)}(\omega) \vee f_m)(\omega) = \inf\{D_{m-1}(h, g_m^{(L)}(\omega) \vee f_m);\ h \in V_m^{(L)}(\omega)\} \qquad (1)$$

for $m = 1, 2, \ldots, M$. Here we let $g_M^{(L)} = f_M$.

We will show that such $g_m^{(L)}$'s always exist.

Then we will prove the following.

Theorem 1 *Suppose that $\lambda_1(V_m^{(L)}; \nu_m)/L \to 0$, as $L \to \infty$ for $m = 1, \ldots, M-1$. Then there are $\Omega_L \in \mathcal{F}$, $L = 1, 2, \ldots$, and random variables Z_L, $L = 1, 2, \ldots$, such that*

$$P(\Omega_L) \to 1, \text{ as } L \to \infty,$$

$$|\tilde{f}_0 - g_0^{(L)}(\omega)| \leqq Z_L(\omega), \qquad L \geqq 1,\ \omega \in \Omega_L,$$

and

$$E[Z_L^2, \Omega_L]^{1/2} \to 0, \text{ as } L \to \infty.$$

Morover, we have

$$E[Z_L^2, \Omega_L]^{1/2}$$

$$\leqq 6\sum_{m=1}^{M-1}\frac{1}{L^{1/2}}\lambda_1(V_m^{(L)}, \nu_m)^{1/4}(1 + \lambda_0(V_m^{(L)}, \nu_m))^{1/4}||P_m\tilde{f}_{m+1}^*||_{L^4(E;d\nu_m)}$$

$$+5\sum_{m=1}^{M-1}||P_m\tilde{f}_{m+1}^* - \pi_{m,V_m^{(L)}}P_m\tilde{f}_{m+1}^*||_{L^2(E;d\nu_m)}.$$

Here $\pi_{m,V_m^{(L)}}$ is the orthogonal projection in $L^2(E, d\nu_m)$ onto $V_m^{(L)}$, $m = 1, \ldots, M$.

So roughly speaking, $g_0^{(L)} \to f_0$ in probability as $L \to \infty$ in a certain rate.

It is obvious that $\lambda_0(V; \nu_m) \geqq 1$ and $\lambda_1(V; \nu_m) \geqq \dim V$ for any $V \in \mathcal{V}_m$, $m = 1, 2, \ldots, M$. So the above theorem raises the following question. Can one estimate $\lambda_0(V; \nu)$ and $||P_m\tilde{f}_{m+1}^* - \pi_{m,V}P_m\tilde{f}_{m+1}^*||_{L^2(E;d\nu_m)}$ for $V \in \mathcal{V}(\nu_m)$? If we can do it, we may find a sequence $V_m^{(k)} \in \mathcal{V}(\nu_m)$ such that the convergence rate is good.

We give an estimate when an underlying process is a 1-dimensional Brownian motion and V is a space of polynomials in Section 6. Also, we introduce a random systems of piece-wise polynomials in Section 8, and we give some estimates when an underlying process is a Hörmander type diffusion process as discussed in [7]. As far as we judge from these estimates, a usual polynomial system is not good, and such a random system of piece-wise polynomials is better.

2 Preliminary results

Let $\mathcal{P}_f(E \times E)$ be the set of probability measures on $(E \times E, \mathcal{B} \times \mathcal{B})$ whose supports are finite subsets of $E\times E$. Let $\pi_i : E\times E$, $i = 1, 2$, be natural projections given by $\pi_1(x, y) = x$, $\pi_2(x, y) = y$, $x, y \in E$. For any $\rho \in \mathcal{P}_f(E \times E)$, let $S(\cdot, *; \rho) : m(E) \times m(E) \to \mathbf{R}$ be given by

$$S(g, f; \rho) = \int_{E\times E} (g(x) - f(y))^2 \rho(dx, dy), \qquad g, f \in m(E). \tag{2}$$

Then we have the following.

Proposition 2 *Let $\rho \in \mathcal{P}_f(E \times E)$. For any $f \in m(E)$ and $V \in \mathcal{V}$, let*

$$s_*(f; V, \rho) = \inf\{S(g, f; \rho);\ g \in V\}$$

and

$$\Gamma(f; V, \rho) = \{g \in V;\ S(g, f; \rho) = s_*(f, V, \rho)\}.$$

Then we have the following.
(1) *$\Gamma(f; V, \rho)$ is not empty for any $f \in m(E)$ and $V \in \mathcal{V}$.*
(2) *Let $V \in \mathcal{V}$. If $f \in m(E)$ and $g \in \Gamma(f; V, \rho)$, then*

$$\int_{E\times E} h(x)(f(y) - g(x))\rho(dx, dy) = 0 \text{ for any } h \in V.$$

Moreover, if $f_1, f_2 \in m(E)$, $g_i \in \Gamma(f_i; V, \rho)$, $i = 1, 2$, then

$$S(g_1 - g_2, 0; \rho) \leqq S(0, f_1 - f_2; \rho).$$

(3) *If $f \in m(E)$, $g \in \Gamma(f; V, \rho)$ and $\tilde{g} \in V$, then*

$$S(g - \tilde{g}, 0; \rho)^{1/2} = \sup\{|\int_{E\times E} h(x)(f(y) - \tilde{g}(x))\rho(dx, dy)|;\ h \in V,\ S(h, 0; \rho) = 1\}.$$

Proof. (1) It is easy to see that

$$S(g, f; \rho) \geqq S(0, f; \rho) + S(g, 0; \rho) - 2S(g, 0; \rho)^{1/2}S(0, f; \rho)^{1/2}, \qquad g \in V.$$

Let $V_0 = \{g \in V;\ S(g, 0; \rho) = 0\} = \{g \in V :\ g(x) = 0 \text{ for } \rho\text{ -a.e. } (x, y) \in E \times E\}$. Then it is easy to see that V_0 is a vector subspace of V. So there is a vector subspace V_1 of V such that $V_0 + V_1 = V$ and $V_0 \cap V_1 = \{0\}$. It is easy to see that $g \in V_1 \to S(g, f; A)$ is a continuous function from V_1 to $[0, \infty)$ and that $S(g, f; A) \to \infty$ as $g \to \infty$ in V_1. So we see that there is a minimum point $g_0 \in V_1$. Note that $S(g + h, f; \rho) = S(g, f; \rho)$ for any $g \in V$ and $h \in V_0$. Therefore we see that $S(g_0, f; \rho) = s_*(f; V, \rho)$ and that $\Gamma(f; V, \rho)$ is not empty.
(2) Let $g \in \Gamma(f; V, \rho)$. The first assertion is obvious, since

$$0 = \frac{d}{dt}S(g + th, f; \rho)|_{t=0} = \int_{E\times E} h(x)(f(y) - g(x))\rho(dx, dy)$$

for any $h \in V$.

Let $f_i \in m(E)$, $g_i \in \Gamma(f_i; V, \rho)$, $i = 1, 2$. Then we have

$$\begin{aligned} & S(g_1 - g_2, f_1 - f_2; \rho) \\ = & -S(g_1 - g_2, 0; \rho) + S(0, f_1 - f_2; \rho) \\ & -2\int_{E\times E} (g_1(x) - g_2(x))(f_1(y) - g_1(x) - (f_2(y) - g_2(x)))\rho(dx, dy). \end{aligned}$$

By the first assertion, we see that

$$S(0, f_1 - f_2; \rho) = S(g_1 - g_2, f_1 - f_2; \rho) + S(g_1 - g_2, 0; \rho).$$

So we have the second assertion.
(3) Let $g \in \Gamma(f; V, \rho)$ and $\tilde{g} \in V$. Then we have

$$S(\tilde{g} + h, f; \rho) = S(\tilde{g}, f; \rho) + S(h, 0; \rho) - 2\int_{E\times E} h(x)(f(y) - \tilde{g}(x))\rho(dx, dy).$$

Let

$$c = \sup\{\int_{E\times E} h(x)(f(y) - \tilde{g}(x))\rho(dx, dy);\ h \in V,\ S(h, 0; \rho) = 1\} \geqq 0.$$

Then we see that

$$s_*(f; V, \rho) = S(\tilde{g}, f; \rho) + \inf_{t\geqq 0}(t^2 - 2tc) = S(\tilde{g}, f; \rho) - c^2.$$

Also, we have by Assertion (2)

$$S(\tilde{g}, f, \rho) = S(g + (\tilde{g} - g), f; \rho) = S(g, f; \rho) + S(\tilde{g} - g, 0 : \rho) = s_*(f; A, V) + S(\tilde{g} - g, 0 : \rho).$$

So we see that $c^2 = S(\tilde{g} - g, 0 : \rho)$. This implies our assertion. ∎

For any $m = 1, 2, \ldots, M$, $V \in \mathcal{V}(\nu_m)$, and $\rho \in \mathcal{P}_f(E \times E)$, let

$$\delta_m(V; \rho) = \sup\{|S(h, 0; \rho) - 1|; h \in V,\ \int_E h(x)^2\nu_m(dx) = 1\}.$$

Then we have the following.

Proposition 3 *Let $m = 1, 2, \ldots, M$, $V \in \mathcal{V}(\nu_m)$, and $\rho \in \mathcal{P}_f(E \times E)$. Let $\{e_k;\ k = 1, \ldots, \dim V\}$ be an orthonormal basis of V. Here we regard V as a Hilbert subspace of $L^2(E, \mathcal{B}(E), d\nu_m)$, and so we have*

$$\int_E e_i(x)e_j(x)\nu_m(dx) = \delta_{ij}, \qquad i, j = 1, \ldots, \dim V.$$

Let A be a $(\dim V) \times (\dim V)$-symmetric matrix valued function defined in E given by

$$A(x) = (A_{ij}(x))_{i,j=1}^{\dim V} = (e_i(x)e_j(x))_{i,j=1}^{\dim V}, \qquad x \in E.$$

Then $\delta_m(V; \rho)$ is equal to the operator norm of the $\dim V \times \dim V$-symmetric matrix $\bar{A} - I$. Here I is the identity matrix and $\bar{A} = (\bar{A}_{ij})_{i,j=1}^{\dim V}$, where

$$\bar{A}_{ij} = \int_E e_i(x)e_j(x)\rho(dx, dy), \qquad i, j = 1, \ldots, \dim V.$$

In particular,

$$\delta_m(V; \rho)^2 \leqq \sum_{i,j=1}^{\dim V} (\int_E (e_i(x)e_j(x) - \delta_{ij})\rho(dx, dy))^2.$$

Proof. It is easy to see that

$$\delta_m(V;\rho) = \sup\{|S(\sum_{i=1}^{\dim V} a_i e_i, 0; \rho) - 1|;\ \sum_{i=1}^{\dim V} a_i^2 = 1\}$$

$$= \sup\{|\sum_{i,j=1}^{\dim V} a_i a_j (\bar{A}_{ij} - \delta_{ij})|;\ \sum_{i=1}^{\dim V} a_i^2 = 1\}.$$

Since $\bar{A} - I$ is symmetric, we see our assertion. ∎

Proposition 4 *For any probability measure ν on $(E, \mathcal{B})$, and $V \in \mathcal{V}(\nu)$,*

$$\lambda_1(V,\nu) \leqq (\dim V)^2 \lambda_0(V,\nu)$$

and

$$\lambda_0(V,\nu) \leqq \lambda_1(V,\nu).$$

Proof. Let $\{e_r\}_{r=1}^{\dim V}$ be an orthonormal basis of V. Then we see that

$$\int_E (\sum_{r=1}^{\dim V} e_r(x)^2)^2 \nu(dx) \leqq \int_E (\dim V)(\sum_{r=1}^{\dim V} e_r(x)^4)\nu(dx) \leqq (\dim V)^2 \lambda_0(V,\nu).$$

So we have the first assertion.

Let $g \in V$. Then we have

$$\int_E g(x)^4 \nu(dx) = \int_E (\sum_{r=1}^{\dim V} (g, e_r)_{L^2(d\nu)} e_r(x))^4 \nu(dx)$$

$$\leqq \int_E (\sum_{r=1}^{\dim V} (g, e_r)^2_{L^2(d\nu)})^2 (\sum_{r=1}^{\dim V} e_r(x)^2)^2 \nu(dx).$$

Note that

$$\sum_{r=1}^{\dim V} (g, e_r)^2_{L^2(d\nu)} = \int_E g(x)^2 \nu(dx).$$

So we have the second assertion. ∎

3 random measures

For $m = 1, \ldots, M$, and $L \geqq 1$, let $\rho_m^{(L)}$ be a random probability measure belonging to $\mathcal{P}_f(E \times E)$ given by

$$\rho_m^{(L)}(A) = \frac{1}{L}\#\{\ell \in \{1,\ldots,L\};\ (X_{m-1}^{(\ell)}, X_m^{(\ell)}) \in A\}, \qquad A \in \mathcal{B} \times \mathcal{B}.$$

For any $m = 0, 1, \ldots, M-1$, and $L \geqq 1$, we define $N_m^{(L)} : m(E) \times \Omega \to [0,\infty)$ by

$$N_m^{(L)}(f)(\omega) = (\frac{1}{L}\sum_{\ell=1}^{L} f(X_m^{(\ell)}(\omega))^2)^{1/2}.$$

Then we see that

$$N_{m-1}^{(L)}(g) = S(g,0;\ \rho_m^{(L)}), \qquad g \in m(E),\ m = 1,\ldots,M.$$

Then we have the following.

Proposition 5 *Let $m = 1,\ldots,M-1$, $L \geqq 1$, and $V \in \mathcal{V}(\nu_m)$. Then we have the following.*
(1) If $\delta_m(V;\rho_m^{(L)}) \leqq 1/2$, then

$$\frac{1}{2}N_{m-1}^{(L)}(g)^2 \leqq \int_E g(x)^2\nu_m(dx) \leqq 2N_{m-1}^{(L)}(g)^2, \qquad g \in V.$$

(2)

$$E[\delta_m(V;\rho_m^{(L)})^2] \leqq \frac{1}{L}\lambda_1(V,\nu_m).$$

In particular, we have

$$P(\delta_m(V;\rho_m^{(L)}) > \frac{1}{2}) \leqq \frac{4}{L}\lambda_1(V,\nu_m).$$

Proof. (1) Suppose that $\delta_m(V;\rho_m^{(L)}) \leqq 1/2$. If $h \in V$ and $\int_E h(x)^2\nu_m(dx) = 1$, then from the definition we have

$$\frac{1}{2} \leqq N_{m-1}^{(L)}(h)^2 \leqq 2.$$

So we have our assertion.
(2) Let $\{e_r\}_{r=1}^{\dim V}$ be an orthonormal basis of V. It is easy to see that

$$E[\delta_m(V;\rho_m^{(L)})^2] \leqq \sum_{r,r'=1}^{\dim V} E[(\frac{1}{L}\sum_{\ell=1}^{L}(e_r(X_m^\ell)e_{r'}(X_m^\ell) - \delta_{r,r'}))^2]$$

$$= \frac{1}{L}\sum_{r,r'=1}^{\dim V}\int_E (e_r(x)e_{r'}(x) - \delta_{r,r'})^2\nu_m(dx) \leqq \frac{1}{L}\sum_{r,r'=1}^{\dim V}\int_E e_r(x)^2 e_{r'}(x)^2\nu_m(dx).$$

$$= \frac{1}{L}\int_E (\sum_{r=1}^{\dim V} e_r(x)^2)^2\nu_m(dx).$$

So we have the first part of our assertion . The second part is an easy consequence of Chebyshev's inequality. ∎

For any $m = 1,2,\ldots,M-1$, and $V \in \mathcal{V}(\nu_m)$, let $\hat{\Gamma}_{m,V} : m(E) \times \mathcal{P}_f(E \times E) \to V$ be defined by the following. $g = \hat{\Gamma}_{m,V}(f,\rho)$, $f \in m(E)$, $\rho \in \mathcal{P}_f(E \times E)$, if $g \in \Gamma(f,V;\rho)$ and

$$\int_E g(x)^2\nu_m(dx) = \inf\{\int_E \tilde{g}(x)^2\nu_m(dx);\ \tilde{g} \in \Gamma(f,V;\rho)\}.$$

$\hat{\Gamma}_{m,V}$ is well-defined by Proposition 2 and the definition of $\mathcal{V}(\nu_m)$.

Let $F : E \times \Omega \to \mathbf{R}$ be $\mathcal{B} \times \mathcal{F}$-measurable function. Then it is easy to see that the mapping $\omega \in \Omega \to s_*(F(\cdot,\omega),V,\rho_m^{(L)}(\omega))$ is $\mathcal{F}$-measurable. So we see that the mapping $\omega \in \Omega \to \hat{\Gamma}_{m,V}(F(\cdot,\omega),\rho_m^{(L)}(\omega))$ is also $\mathcal{F}$-measurable (see Castaing [3] for example).

For $V \in \mathcal{V}(\nu_m)$, $m = 1,\ldots,M$, let $\pi_{m,V} : L^2(E;d\nu_m) \to V$ be the orthogonal projection onto V.

Then we have the following.

Proposition 6 *Let $m = 1, \ldots, M-1$, and $L \geqq 1$. Then for $V \in \mathcal{V}_m$ and $f \in L^4(E, \mathcal{B}(E), d\nu_{m+1})$, we have*

$$E[N_m^{(L)}(\pi_{m,V}P_m f - \hat{\Gamma}_{m,V}(f;\rho_m^{(L)}))^2, \delta_m(V,\rho_m^{(L)}) \leqq \frac{1}{2}]$$
$$\leqq \frac{8}{L}(\lambda_1(V,\nu)(1+\lambda_0(V,\nu)))^{1/2}(\int_E f(y)^4 \nu_{m+1}(dy))^{1/2}.$$

Proof. Let $g = \pi_{m,V}P_m f$, and $\{e_r\}_{r=1}^{\dim V}$ be an orthonormal basis of V. Note that

$$E[e_r(X_m^1)(f(X_{m+1}^1) - g(X_m^1))] = \int_{E\times E} e_r(x)(f(y)-g(x))\nu_m(dx)p_m(x,dy)$$
$$= \int_E e_r(x)(P_m f(x) - g(x))\nu_m(dx) = 0, \qquad r = 1, \ldots, \dim V.$$

By Proposition 2(3) we see that

$$E[N_m^{(L)}(g - \hat{\Gamma}_{m,V}(f;\rho_m^{(L)}))^2, \delta_m(V,\rho_m^{(L)}) \leqq \frac{1}{2}]$$
$$\leqq 2E[\sup\{|\int_{E\times E} h(x)(f(y)-g(x))\rho_{m+1}^{(L)}(dx,dy)|^2;\ h \in V,\ \int_E h(x)^2\nu_m(dx) = 1\}]$$
$$= 2E[\sup\{|\sum_{r=1}^{\dim V} a_r \int_{E\times E} e_r(x)(f(y)-g(x))\rho_{m+1}^{(L)}(dx,dy)|^2;\ \sum_{r=1}^{\dim V} a_r^2 = 1\}]$$
$$= 2E[\sum_{r=1}^{\dim V}(\int_{E\times E} e_r(x)(f(y)-g(x))\rho_{m+1}^{(L)}(dx,dy))^2]$$
$$= 2\sum_{r=1}^{\dim V} E[(\frac{1}{L}\sum_{\ell=1}^{L} e_r(X_m^\ell)(f(X_{m+1}^\ell) - g(X_m^\ell)))^2]$$
$$= \frac{2}{L}\sum_{r=1}^{\dim V} E[e_r(X_m^1)^2(f(X_{m+1}^1) - g(X_m^1))^2]$$
$$= \frac{2}{L}\sum_{r=1}^{\dim V}\int_{E\times E} e_r(x)^2(f(y)-g(x))^2\nu_m(dx)p_m(x,dy)$$
$$\leqq \frac{2}{L}(\int_E(\sum_{r=1}^{\dim V} e_r(x)^2)^2\nu_m(dx))^{1/2}(\int_{E\times E}(f(y)-g(x))^4\nu_m(dx)p_m(x,dy))^{1/2}.$$

Note that

$$\int_{E\times E}(f(y)-g(x))^4\nu_m(dx)p_m(x,dy) \leqq 16\int_{E\times E}(f(y)^4 + g(x)^4)\nu_m(dx)p_m(x,dy)$$
$$= 16(\int_E f(y)^4\nu_{m+1}(dy) + \int_E g(x)^4\nu_m(dx)).$$

By Proposition 4, we see that

$$\int_E g(x)^4\nu_m(dx) \leqq \lambda_0(V,\nu_m)(\int_E (P_m f)(x)^2\nu_m(dx))^2 \leqq \lambda_0(V,\nu_m)\int_E f(y)^4\nu_{m+1}(dy).$$

So we have our assertion . ∎

The following is obvious.

Proposition 7 *Let $m = 1, \ldots, M$, and $L \geqq 1$. Then for any $f \in L^2(E, \mathcal{B}(E), d\nu_m)$, we have*

$$E[N_m^{(L)}(f)^2] = \int_E f(x)^2 \nu_m(dx).$$

4 Proof of Theorem 1

Now let us think of the setting in Introduction. Let $\phi_m : E \times \mathbf{R} \to \mathbf{R}$, $m = 1, \ldots, M$, be given by

$$\phi_m(x, z) = f_m(x) \vee z, \qquad x \in E,\ z \in \mathbf{R},\ m = 1, 2, \ldots, M.$$

Then we see that

$$|\phi_m(x, z_1) - \phi_m(x, z_2)| \leqq |z_1 - z_2|, \qquad x \in E,\ z_1, z_2 \in \mathbf{R},\ m = 1, \ldots, M.$$

Note that

$$\tilde{f}_m^*(x) = \phi_m(x, \tilde{f}_m(x)) \text{ and } \tilde{f}_{m-1} = P_{m-1}\tilde{f}_m^*, \qquad m = 1, \ldots, M.$$

Remind that $V_m^{(L)} \in \mathcal{V}(\nu_m)$, $L \geqq 1$, $m = 1, \ldots, M$. Let us take $g_m^{(L)} : \Omega \to V_m^{(L)}$, $m = M, \ldots, 0$, such that

$$g_M^{(L)}(\omega) = f_M,$$

$$g_m^{(L)}(\omega) \in \Gamma(\phi_{m+1}(\cdot, g_{m+1}^{(L)}(\omega)(\cdot)), V_m^{(L)};\ \rho_m^L(\omega)), \qquad m = M-1, \ldots, 0.$$

Then we see that Equation (1) is satisfied. Let $\tilde{Z}_m^{(L)}$, $m = 0, 1, \ldots, M-1$, be given by

$$\tilde{Z}_m^{(L)}$$

$$= N_m^{(L)}(P_m\tilde{f}_{m+1}^* - \pi_{m,V_m^{(L)}} P_m \tilde{f}_{m+1}^*) + N_m^{(L)}(\pi_{m,V_m^{(L)}} P_m \tilde{f}_{m+1}^* - \hat{\Gamma}_{m,V_{m,L}}(\tilde{f}_{m+1}^*;\ \rho_m^L)).$$

Also, let $Z_m^{(L)}$, $m = 0, 1, \ldots, M-1$, be given by

$$Z_0^{(L)} = \sum_{k=0}^{M-1} \tilde{Z}_k^{(L)},$$

and

$$Z_m^{(L)}$$

$$= ||\tilde{f}_m - \pi_{m,V_m^{(L)}}\tilde{f}_m||_{L^2(E,d\mu_m)} + 2N_m(\tilde{f}_m - \pi_{m,V_m^{(L)}}\tilde{f}_m, \omega) + 2\sum_{k=m}^{M-1} \tilde{Z}_k^{(L)}, \qquad m = 1, \ldots, M-1.$$

Finally, let

$$\Omega_L = \bigcap_{m=1}^{M-1} \{\delta_m(V_m^{(L)}; \rho_m^{(L)}) \leqq \frac{1}{2}\}.$$

Then we have the following.

Proposition 8 (1) $|\tilde{f}_0 - g_0^{(L)}(\omega)| \leqq Z_0^{(L)}$.
(2) *For any* $\omega \in \Omega^{(L)}$,

$$||\tilde{f}_m^* - (g_m^{(L)}(\omega) \vee f_m)||_{L^2(E;d\nu_m)} \leqq ||\tilde{f}_m - g_m^{(L)}(\omega)||_{L^2(E;d\nu_m)} \leqq Z_m^{(L)}, \qquad m = 1, \ldots, M.$$

(3)

$$P(\Omega \setminus \Omega_L) \leqq \sum_{k=1}^{M-1} \frac{4}{L} \lambda_1(V_k^{(L)}, \nu_k),$$

and

$$E[|Z_m^{(L)}|^2, \Omega_L]^{1/2}$$

$$\leqq 6 \sum_{k=1}^{M-1} \{(\frac{1}{L} \lambda_1(V_k^{(L)}, \nu_k))^{1/2} (1 + \lambda_0(V_k^{(L)}, \nu_k))^{1/2}\}^{1/2} ||P_k \tilde{f}_{k+1}^*||_{L^4(E;d\nu_k)}$$

$$+5 \sum_{k=1}^{M-1} ||P_k \tilde{f}_{k+1}^* - \pi_{k,V_k^{(L)}} P_k \tilde{f}_{k+1}^*||_{L^2(E;d\nu_k)}, \qquad m = 0, 1, \ldots, M-1.$$

Proof. Note that

$$N_m^{(L)}(\tilde{f}_m - g_m^{(L)}(\omega), \omega)$$

$$\leqq N_m^{(L)}(P_m \tilde{f}_{m+1}^* - \hat{\Gamma}_{m,V_{m,L}}(\tilde{f}_{m+1}^*;\ \rho_m^L), \omega) + N_m^{(L)}(\hat{\Gamma}_{m,V_{m,L}}(\tilde{f}_{m+1}^*;\ \rho_m^L)) - g_m^{(L)}(\omega), \omega).$$

By Proposition 2(2), we have

$$N_m^{(L)}(\hat{\Gamma}_{m,V_{m,L}}(\tilde{f}_{m+1}^*;\ \rho_m^L)) - g_m^{(L)}(\omega), \omega)$$

$$\leqq N_{m+1}^{(L)}(\phi_{m+1}(\cdot, \tilde{f}_{m+1}(\cdot)) - \phi_{m+1}(\cdot, g_{m+1}^{(L)}(\omega)(\cdot)), \omega)$$

$$\leqq N_{m+1}^{(L)}(\tilde{f}_{m+1} - g_{m+1}^{(L)}(\omega)(\cdot), \omega).$$

So we see that

$$N_m^{(L)}(\tilde{f}_m - g_m^{(L)}(\omega), \omega) \leqq \sum_{k=m}^{M-1} N_k^{(L)}(P_k \tilde{f}_{k+1}^* - \hat{\Gamma}_{k,V_{k,L}}(\tilde{f}_{k+1}^*;\ \rho_k^L), \omega).$$

Then we have

$$N_m^{(L)}(\tilde{f}_m - g_m^{(L)}(\omega), \omega) \leqq \sum_{k=m}^{M-1} \tilde{Z}_k^{(L)}.$$

In particular,

$$|\tilde{f}_0 - g_0^{(L)}(\omega)| \leqq \sum_{k=0}^{M-1} \tilde{Z}_k^{(L)} = Z_0^{(L)}.$$

This implies Assertion (1).

Also, we see that if $\omega \in \Omega_L$, then

$$||\tilde{f}_m - g_m^{(L)}(\omega)||_{L^2(E,d\mu_m)}$$

$$\leqq ||\tilde{f}_m - \pi_{m,V_m^{(L)}} \tilde{f}_m||_{L^2(E,d\mu_m)} + ||\pi_{m,V_m^{(L)}} \tilde{f}_m - g_m^{(L)}(\omega)||_{L^2(E,d\mu_m)}$$

$$\leqq ||\tilde{f}_m - \pi_{m,V_m^{(L)}}\tilde{f}_m||_{L^2(E,d\mu_m)} + N_m^{(L)}(\pi_{m,V_m^{(L)}}\tilde{f}_m - g_m^{(L)}(\omega),\omega)$$

$$\leqq ||\tilde{f}_m - \pi_{m,V_m^{(L)}}\tilde{f}_m||_{L^2(E,d\mu_m)}| + 2N_m(\tilde{f}_m - \pi_{m,V_m^{(L)}}\tilde{f}_m,\omega) + 2N_m^{(L)}(\tilde{f}_m - g_m^{(L)}(\omega),\omega).$$

This implies Assertion (2).

The first assertion of (3) is obvious from Propositions 5. By Propositions 6 and 7, we have

$$E[(\tilde{Z}_m^{(L)})^2,\Omega_L]^{1/2}$$

$$\leqq ||\tilde{f}_m - \pi_{m,V_m^{(L)}}P_m\tilde{f}_m||_{L^2(E,d\mu_m)} + 3(\frac{1}{L}(\lambda_1(V,\nu)(1+\lambda_0(V,\nu))^{1/2})^{1/2}||\tilde{f}^*_{m+1}||_{L^4(E;d\nu_{m+1})}.$$

So we have the second assertion of (3). ∎

Theorem 1 follows from Proposition 8 immediately.

The following is an easy consequence of Proposition 8.

Proposition 9 *Asuume that* $\lambda_1(V_m^{(L)};\nu_m)/L \to 0,\ L \to \infty,\ m = 1,\ldots,M-1$. *Let* $\delta \in (0,1)$, *and let*

$$d_L = \sum_{m=0}^{M-1} E[(Z_m^{(L)})^2,\Omega_L]^{1/2}, \qquad L \geqq 1,$$

and let $\tilde{\Omega}_L^\delta \in \mathcal{F},\ L \geqq 1$, *be given by*

$$\tilde{\Omega}_L^\delta = \Omega_L \cap \bigcap_{m=1}^{M-1}\{Z_m^{(L)} \leqq d_L^{1-\delta}\}.$$

Then $d_L \to 0$, *and* $P(\tilde{\Omega}_L^\delta) \to 1,\ L \to \infty$. *Also, we have*

$$||\tilde{f}_m - g_m(\omega)||_{L^2(E;d\nu_m)} \leqq d_L^{1-\delta}, \qquad m = 1\ldots,M,\ \omega \in \tilde{\Omega}_L^\delta,\ L \geqq 1.$$

5 re-simulation

Let us be back to the situation in Introduction. Let $h_m \in L^2(E;d\nu_m)$, $m = 1,2,\ldots,M$, with $h_M = f_M$. Let σ a stopping time given by $\sigma = \min\{k = 0,1,\ldots,M;\ f_k(X_k) \geqq h_k(X_k)\}$, and let

$$c_0 = c_0(\{h_m\}_{m=1}^{M-1}) = E[f_\sigma(X_\sigma)].$$

Then we have the following.

Proposition 10 *Let* $\beta \geqq 0$. *Assume that there is a* $C_0 > 0$ *such that*

$$\nu_m(\{|f_m - \tilde{f}_m| \leqq \varepsilon\}) \leqq C_0\varepsilon^\beta, \qquad \varepsilon > 0,\ m = 1,2,\ldots,M.$$

Then we have

$$|\tilde{f}_0 - c_0| \leqq (C_0+1)\sum_{m=1}^{M-1}||\tilde{f}_m - h_m||_{L^2(E;d\nu_m)}^{1+\beta/(2+\beta)}.$$

Proof. Let $\hat{h}_m$, $m = M, M-1, \dots, 0$, be inductively given by

$$\hat{h}_M = f_M = h_M,$$

$$\hat{h}_{m-1} = P_{m-1}(1_{\{f_m \geqq h_m\}} f_m + 1_{\{f_m < h_m\}} \hat{h}_m), \qquad m = M, M-1, \dots, 1.$$

Then we see that $c_0 = \hat{h}_0$.

Note that

$$\tilde{f}_{m-1} = P_{m-1}(1_{\{f_m \geqq \tilde{f}_m\}} f_m + 1_{\{f_m < \tilde{f}_m\}} \tilde{f}_m), \qquad m = M, M-1, \dots, 1.$$

Therefore we have

$$\begin{aligned} &\tilde{f}_{m-1} - \hat{h}_{m-1} \\ &= P_{m-1}(1_{\{f_m < \tilde{f}_m \wedge h_m\}}(\tilde{f}_m - \hat{h}_m) + 1_{\{h_m \leqq f_m < \tilde{f}_m\}}(\tilde{f}_m - f_m) + 1_{\{\tilde{f}_m \leqq f_m < h_m\}}(f_m - \hat{h}_m)) \\ &= P_{m-1}(1_{\{f_m < h_m\}}(\tilde{f}_m - \hat{h}_m) + 1_{\{h_m \leqq f_m < \tilde{f}_m\}}(\tilde{f}_m - f_m) + 1_{\{\tilde{f}_m \leqq f_m < h_m\}}(f_m - \tilde{f}_m)), \end{aligned}$$

and so we see that

$$\begin{aligned} &|\tilde{f}_{m-1} - \hat{h}_{m-1}| \\ &\leqq P_{m-1}(|\tilde{f}_m - \hat{h}_m|) + P_{m-1}(1_{\{|f_m - \tilde{f}_m| \leqq |\tilde{f}_m - h_m|\}}|f_m - \tilde{f}_m|) \\ &\leqq P_{m-1}(|\tilde{f}_m - \hat{h}_m|) + P_{m-1}(1_{\{|f_m - \tilde{f}_m| \leqq \varepsilon\}}|f_m - \tilde{f}_m|) + P_{m-1}(1_{\{\varepsilon < |\tilde{f}_m - h_m|\}}|\tilde{f}_m - h_m|) \end{aligned}$$

So we have

$$\begin{aligned} &||\tilde{f}_{m-1} - \hat{h}_{m-1}||_{L^1(E; d\nu_{m-1})} \\ &\leqq ||\tilde{f}_m - \hat{h}_m||_{L^1(E; d\nu_m)} + \varepsilon \nu_m(\{|f_m - \tilde{f}_m| \leqq \varepsilon\}) + \varepsilon^{-1}||\tilde{f}_m - h_m||^2_{L^2(E; d\nu_{m-1})} \\ &\leqq ||\tilde{f}_m - \hat{h}_m||_{L^1(E; d\nu_m)} + C_0 \varepsilon^{1+\beta} + \varepsilon^{-1}||\tilde{f}_m - h_m||^2_{L^2(E; d\nu_m)} \end{aligned}$$

So letting

$$\varepsilon = ||\tilde{f}_m - h_m||^{2/(2+\beta)}_{L^2(E; d\nu_m)},$$

we have

$$||\tilde{f}_{m-1} - \hat{h}_{m-1}||_{L^1(E; d\nu_{m-1})} \leqq ||\tilde{f}_m - \hat{h}_m||_{L^1(E; d\nu_m)} + (C_0 + 1)||\tilde{f}_m - h_m||^{1+\beta/(2+\beta)}_{L^2(E; d\nu_m)}.$$

Since $\tilde{f}_M = \hat{h}_M = h_M = f_M$, we have our assertion. ∎

Now let $\tilde{X}^n = (\tilde{X}^n_0, \tilde{X}^n_1, \dots, \tilde{X}^n_M)$, $n = 1, 2, \dots$, be independent identically distributed E^{M+1}-valued random variables whose distribution is the same as $(X_0, X_1, \dots, X_M)$ under P. We assume that $\sigma\{X_m;\ m = 0, 1, \dots, M\}$, $\sigma\{X^\ell_m,\ m = 0, 1, \dots, M,\ \ell \geqq 1\}$ and $\sigma\{\tilde{X}^n_m;\ m = 0, 1, \dots, M,\ n \geqq\}$ are independent. Let $g^{(L)}_m(\omega) \in V^{(L)}_m$, $m, L \geqq 1$, as in Introduction. Let

$$\tau_n(\omega) = \min\{m \geqq 0; g_m(\omega)(\tilde{X}^n_m(\omega)) \geqq f_m(\tilde{X}^n_m(\omega))\}, \qquad n \geqq 1,$$

and let

$$\tilde{c}^n_0(\omega) = \frac{1}{n}\sum_{k=1}^n f_{\tau_k(\omega)}(\tilde{X}^k_{\tau_k(\omega)}(\omega))$$

Then by law of large number, we have

$$\tilde{c}_0^n(\omega) \to c_0(\{g_m^{(L)}(\omega)\}_{m=1}^{M-1}) \quad a.s., \qquad n \to \infty.$$

By Proposition 8, we see that

$$|\tilde{f}_0 - g_0^{(L)}(\omega)| \leqq d_L, \qquad \omega \in \Omega_L.$$

But Propositions 9 and 10 imply that

$$|\tilde{f}_0 - c_0(\{g_m(\omega)\}_{m=1}^{M-1})| \leqq C d_L^{(1-\delta)(1+\beta/(2+\beta))}, \qquad \omega \in \tilde{\Omega}_L^\delta,$$

even though β is unknown. So $\tilde{c}_0^n(\omega)$ can be a better estimator of $\tilde{f}_0$.

6 Brownian motion Case

From now on, we try to give estimates for $\lambda_0(V,\nu)$ and $||P_m\tilde{f}_{m+1}^* - \pi_{m,V}P_m\tilde{f}_{m+1}^*||$ for some examples.

Let $\{B_t;\ t \geqq 0\}$ be a standard Brownian motion and $T > 0$. Now let V_n, $n \geqq 1$, be the space of polynomials of degree less than or equal to n. Let P_t, $t \geqq 0$, be the diffusion operators for the standard Brownian motion, i.e.,

$$(P_t g)(x) = (\frac{1}{2\pi t})^{1/2} \int_{\mathbf{R}} g(y) \exp(-\frac{(x-y)^2}{2t}) dy, \qquad g \in m(\mathbf{R}).$$

Let ν be a probability law of B_T. So we have

$$\nu(dx) = \frac{1}{\sqrt{2\pi T}} \exp(-\frac{x^2}{2T}) dx.$$

Then we have the following.

Proposition 11 *We have*

$$\lim_{n\to\infty} \frac{1}{n} \log \lambda_0(V_n,\nu) = \lim_{n\to\infty} \frac{1}{n} \log \lambda_1(V_n,\nu) = \log 9.$$

Also, let $f : \mathbf{R} \to [0,\infty)$ be given by $f(x) = x \vee 0$, $x \in \mathbf{R}$. Then there is a $C_0 > 0$ such that

$$||P_t f - \pi_n P_t f||_{L^2(d\nu)} \geqq C_0 n^{-3/4}(1 + t/T)^{-n/2}, \qquad n \geqq 1.$$

Here π_n is the orthogonal projection in $L^2(\mathbf{R}, d\nu)$ onto V_n.

Proof. Let

$$H_n(x;v) = \exp(\frac{x^2}{2v}) \frac{d^n}{dx^n} \exp(-\frac{x^2}{2v}), \qquad x \in \mathbf{R}^N,\ v > 0,\ n \geqq 0.$$

Then we have

$$\sum_{n=0}^\infty \frac{t^n}{n!} H_n(x;v) = \exp(\frac{x^2}{2v}) \exp(-\frac{(x+t)^2}{2v}) = \exp(-\frac{xt}{v} - \frac{t^2}{2v}),$$

and

$$\sum_{n,m=0}^{\infty} \frac{t^n}{n!} H_n(x;v) \frac{s^m}{n!} H_m(x;v) = \exp(-\frac{x(t+s)}{v} - \frac{t^2+s^2}{2v}).$$

So we have

$$\sum_{n,m=0}^{\infty} \frac{t^n}{n!} \frac{s^m}{n!} \int_{\mathbf{R}} H_n(x;T) H_m(x;T) \nu(dx)$$

$$= \exp(\frac{(t+s)^2}{2T} - \frac{t^2+s^2}{2T}) = \exp(\frac{ts}{T}) = \sum_{n=0}^{\infty} \frac{t^n s^n}{n! T^n},$$

and

$$\int_{\mathbf{R}} H_n(x;T) H_m(x;T) \nu(dx) = \delta_{nm} \frac{n!}{T^n}.$$

So we see that $e_n(x;T) = (\frac{T^n}{n!})^{1/2} H_n(x;T)$, $n = 1, 2, \ldots$, is an orthonormal basis in $L^2(\mathbf{R}, d\nu)$.

Note that

$$\sum_{n_1,n_2,n_3,n_4=0}^{\infty} (\prod_{i=1}^{4} \frac{t_i^{n_i}}{n_i!}) \prod_{i=1}^{4} H_{n_i}(x;v) = \exp(-\frac{x(\sum_{i=1}^4 t_i)}{v} - \frac{\sum_{i=1}^4 t_i^2}{2v}).$$

and so

$$\sum_{n_1,n_2,n_3,n_4=0}^{\infty} \prod_{i=1}^{4} \frac{t_i^{n_i}}{n_i!} \int_{\mathbf{R}} \prod_{i=1}^{4} H_{n_i}(x;T) \nu(dx)$$

$$= \exp(\frac{(\sum_{i=1}^4 t_i)^2}{2T} - \frac{\sum_{i=1}^4 t_i^2}{2T}) = \exp(\frac{1}{T} \sum_{1 \leqq i < j \leqq 4} t_i t_j).$$

So we have

$$\int_{\mathbf{R}} H_n(x;T)^4 \nu(dx) = \frac{1}{(2n)!} \frac{d^n}{dt_4^n} \cdots \frac{d^n}{dt_1^n} (\frac{1}{T^{2n}} (\sum_{1 \leqq i < j \leqq 4} t_i t_j)^{2n}))|_{t_1 = \cdots = t_4 = 0}.$$

Note that

$$\sum_{1 \leqq i < j \leqq 4} t_i t_j = t_1(t_2 + t_3 + t_4) + t_2(t_3 + t_4) + t_3 t_4$$

and so we have

$$\frac{d^n}{dt_1^n}((\sum_{1 \leqq i < j \leqq 4} t_i t_j)^{2n})|_{t_1=0} = \frac{(2n)!}{n!} (t_2 + t_3 + t_4)^n (t_2(t_3 + t_4) + t_3 t_4)^n,$$

$$\frac{d^n}{dt_2^n} \frac{d^n}{dt_1^n}((\sum_{1 \leqq i < j \leqq 4} t_i t_j)^{2n})|_{t_1=t_2=0}$$

$$= \frac{(2n)!}{n!} \sum_{k=0}^{n} \binom{n}{k} \frac{n!}{k!} (t_3 + t_4)^k \frac{n!}{(n-k)!} (t_3 + t_4)^k (t_3 t_4)^{n-k}$$

$$= (2n)! \sum_{k=0}^{n} \binom{n}{k}^2 (t_3 + t_4)^{2k} (t_3 t_4)^{n-k}.$$

So we have

$$\frac{d^n}{dt_4^n} \cdots \frac{d^n}{dt_1^n} \Big(\sum_{1 \leqq i < j \leqq 4} t_i t_j)^{2n} \Big)|_{t_1 = \cdots = t_4 = 0} = (2n)!(n!)^2 \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k}.$$

Therefore we see that

$$\int_{\mathbf{R}} e_n(x;T)^4 \nu(dx) = (\frac{T^n}{n!})^2 \int_{\mathbf{R}} H_n(x;T)^4 \nu(dx) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k}.$$

Let

$$a_n = \log(\frac{n!}{n^{n-1/2} e^{-n}}), \qquad n \geqq 0.$$

Then it is well known that $\{a_n\}_{n=1}^{\infty}$ is bounded.

Since

$$\log(n!) = n \log n - n - \frac{1}{2} \log n + a_n,$$

we have

$$\frac{1}{n} \log(\binom{n}{k}^2 \binom{2k}{k}) = 2\frac{1}{n} \log \binom{n}{k} + \frac{1}{n} \log \binom{2k}{k}$$

$$= 2h(\frac{k}{n}) + \frac{1}{n}(-\log n + \log(n-k) + \log k + 2a_n - 2a_{n-k} - 2a_k)$$

$$+ \frac{2k}{n} \log 2 + \frac{1}{n}(-\frac{1}{2} \log(2k) + \log k + a_{2k} - 2a_{2k}),$$

where

$$h(x) = -(x \log x + (1-x) \log(1-x)), \qquad x \in [0,1].$$

Also,we have

$$\max_{k=0,1,\ldots,n} \frac{1}{n} \log(\binom{n}{k}^2 \binom{2k}{k}) \leqq \frac{1}{n} \log(\sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k})$$

$$\leqq \max_{k=0,1,\ldots,n} \frac{1}{n} \log(\binom{n}{k}^2 \binom{2k}{k}) + \frac{1}{n} \log(n+1).$$

So we have

$$\frac{1}{n} \log(\sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k}) \to \max_{x \in [0,1]} (2h(x) + 2x \log 2) = \log 9, \qquad n \to \infty.$$

Therefore we have by Proposition 4

$$\frac{1}{n} \log(\int_{\mathbf{R}} e_n(x;T)^4 \nu(dx)) \to \log 9, \qquad n \to \infty.$$

Since

$$\int_{\mathbf{R}} e_n(x;T)^4 \nu(dx) \leqq \lambda_0(V_n, \nu)$$

and

$$\lambda_0(V_n,\nu) \leqq \lambda_1(V_n,\nu) \leqq (n+1)\sum_{k=0}^{n} \int_{\mathbf{R}} e_k(x;T)^4\nu(dx) \leqq (n+1)^2 \max_{k=0,\ldots,n} \int_{\mathbf{R}} e_k(x;T)^4\nu(dx),$$

we see that

$$\lim_{n\to\infty} \frac{1}{n}\log\lambda_0(V_n,\nu) = \lim_{n\to\infty}\frac{1}{n}\log\lambda_1(V_n,\nu) = \log 9.$$

Note that $\frac{d^2}{dx^2}f(x) = \delta(x)$. So we have

$$\frac{d^2}{dx^2}(P_tf)(x) = \frac{1}{\sqrt{2\pi t}}\exp(-\frac{x^2}{2t}).$$

Then we have

$$\begin{aligned}
&\sum_{n=0}^{\infty}\frac{s^n}{n!}\int_{\mathbf{R}} H_{n+2}(x;T)(P_tf)(x)\nu(dx)\\
&= \sum_{n=0}^{\infty}\frac{s^n}{n!}\frac{1}{\sqrt{2\pi T}}\int_{\mathbf{R}}\frac{d^{n+2}}{dx^{n+2}}(\exp(-\frac{x^2}{2T}))(P_tf)(x)dx\\
&= \sum_{n=0}^{\infty}\frac{s^n}{n!}\frac{1}{\sqrt{2\pi T}}\int_{\mathbf{R}}\frac{d^{n}}{dx^{n}}(\exp(-\frac{x^2}{2T}))\frac{d^2}{dx^2}(P_tf)(x)dx\\
&= \frac{1}{\sqrt{2\pi t}}\frac{1}{\sqrt{2\pi T}}\int_{\mathbf{R}}\exp(-\frac{sx}{T}-\frac{s^2}{2T})\exp(-\frac{x^2}{2T})\exp(-\frac{x^2}{2t})dx\\
&= \frac{1}{\sqrt{2\pi(T+t)}}\exp(\frac{tTs^2}{2T^2(T+t)}-\frac{s^2}{2T}) = \frac{1}{\sqrt{2\pi(T+t)}}\exp(-\frac{s^2}{2(T+t)}).
\end{aligned}$$

So we have

$$\int_{\mathbf{R}} H_{2m+2}(x;T)(P_tf)(x)\nu(dx) == \frac{1}{\sqrt{2\pi(T+t)}}\frac{(2m)!}{m!}(-\frac{1}{2(T+t)})^m$$

and so

$$\int_{\mathbf{R}} e_{2m+2}(x;T)(P_tf)(x)\nu(dx) = (\frac{T^{2m+2}}{(2m+2)!})^{1/2}\frac{1}{\sqrt{2\pi(T+t)}}\frac{(2m)!}{m!}(-\frac{1}{2(T+t)})^m$$

$$= \frac{1}{\sqrt{2\pi(T+t)}}(\frac{1}{(2m+1)(2m+2)})^{1/2}T\frac{(2m)^m e^{-m}(2m)^{-1/4}\exp(a_{2m}/2)}{2^m m^m e^{-m}m^{-1/2}\exp(a_m)}(-1)^m(1+\frac{t}{T})^{-m}.$$

So we see that

$$\lim_{m\to\infty} m^{3/4}(1+\frac{t}{T})^m|\int_{\mathbf{R}} e_{2m+2}(x;T)P_tf(x)\nu(dx)|$$

exists and is positive. Since we see that

$$|\int_{\mathbf{R}} e_{2m+2}(x;T)P_tf(x)\nu(dx)|^2 \leqq ||P_tf-\pi_{2m}P_tf||^2_{L^2(d\nu)},$$

we have our assertion.

7 A remark on Hörmander type diffusion processes

Let $N, d \geqq 1$. Let $W_0 = \{w \in C([0,\infty); \mathbf{R}^d);\ w(0) = 0\}$, $\mathcal{F}$ be the Borel algebra over W_0 and μ be the Wiener measure on $(W_0, \mathcal{F})$. Let $B^i : [0,\infty) \times W_0 \to \mathbf{R}$, $i = 1, \ldots, d$, be given by $B^i(t, w) = w^i(t)$, $(t, w) \in [0,\infty) \times W_0$. Then $\{(B^1(t), \ldots, B^d(t); t \in [0,\infty)\}$ is a d-dimensional Brownian motion under μ. Let $B^0(t) = t$, $t \in [0,\infty)$. Let $V_0, V_1, \ldots, V_d \in C_b^\infty(\mathbf{R}^N; \mathbf{R}^N)$. Here $C_b^\infty(\mathbf{R}^N; \mathbf{R}^n)$ denotes the space of $\mathbf{R}^n$-valued smooth functions defined in $\mathbf{R}^N$ whose derivatives of any order are bounded. We regard elements in $C_b^\infty(\mathbf{R}^N; \mathbf{R}^N)$ as vector fields on $\mathbf{R}^N$.

Now let $X(t, x)$, $t \in [0,\infty)$, $x \in \mathbf{R}^N$, be the solution to the Stratonovich stochastic integral equation

$$X(t, x) = x + \sum_{i=0}^{d} \int_0^t V_i(X(s, x)) \circ dB^i(s). \qquad (3)$$

Then there is a unique solution to this equation. Moreover we may assume that $X(t, x)$ is continuous in t and smooth in x and $X(t, \cdot) : \mathbf{R}^N \to \mathbf{R}^N$, $t \in [0,\infty)$, is a diffeomorphism with probability one.

Let $\mathcal{A} = \{\emptyset\} \cup \bigcup_{k=1}^\infty \{0, 1, \ldots, d\}^k$ and for $\alpha \in \mathcal{A}$, let $|\alpha| = 0$ if $\alpha = \emptyset$, let $|\alpha| = k$ if $\alpha = (\alpha^1, \ldots, \alpha^k) \in \{0, 1, \ldots, d\}^k$, and let $\| \alpha \| = |\alpha| + \mathrm{card}\{1 \leqq i \leqq |\alpha|;\ \alpha^i = 0\}$. Let $\mathcal{A}^*$ and $\mathcal{A}^{**}$ denote $\mathcal{A} \setminus \{\emptyset\}$ and $\mathcal{A} \setminus \{\emptyset, 0\}$, respectively. Also, for each $m \geqq 1$, $\mathcal{A}^{**}_{\leqq m}$, $\{\alpha \in \mathcal{A}^{**};\ \| \alpha \| \leqq m\}$.

We define vector fields $V_{[\alpha]}$, $\alpha \in \mathcal{A}$, inductively by

$$V_{[\emptyset]} = 0, \qquad V_{[i]} = V_i, \quad i = 0, 1, \ldots, d,$$

$$V_{[\alpha * i]} = [V_{[\alpha]}, V_i], \qquad i = 0, 1, \ldots, d.$$

Here $\alpha * i = (\alpha^1, \ldots, \alpha^k, i)$ for $\alpha = (\alpha^1, \ldots, \alpha^k)$ and $i = 0, 1, \ldots, d$.

We say that a system $\{V_i; i = 0, 1, \ldots, d\}$ of vector fields satisfies the following condition (UFG).

(UFG) There are an integer ℓ_0 and $\varphi_{\alpha,\beta} \in C_b^\infty(\mathbf{R}^N)$, $\alpha \in \mathcal{A}^{**}$, $\beta \in \mathcal{A}^{**}_{\leqq \ell_0}$, satisfying the following.

$$V_{[\alpha]} = \sum_{\beta \in \mathcal{A}^{**}_{\leqq \ell_0}} \varphi_{\alpha,\beta} V_{[\beta]}, \qquad \alpha \in \mathcal{A}^{**}.$$

Let $A(x) = (A^{ij}(x))_{i,j=1,\ldots,N}$, $t > 0$, $x \in \mathbf{R}^N$, be a $N \times N$ symmetric matrix given by

$$A^{ij}(x) = \sum_{\alpha \in \mathcal{A}^{**}_{\leqq \ell_0}} V^i_{[\alpha]}(x) V^j_{[\alpha]}(x), \qquad i, j = 1, \ldots, N.$$

Let $h(x) = \det A(x), x \in \mathbf{R}^N$, and $E = \{x \in \mathbf{R}^N;\ h(x) > 0\}$. By Kusuoka-Stroock [8], we see that if $x \in E$, the distribution law of $X(t, x)$ under μ has a smooth density function $p(t, x, \cdot) : \mathbf{R}^N \to [0,\infty)$ for $t > 0$.

By Kusuoka-Morimoto [7] Propositions 3, 8 and 9, we see the following.

Proposition 12 *For any $p > 1$ and $T > 0$, there is a $C \in (0,\infty)$ such that*

$$\int_E p(t, x, y) h(y)^{-p} dy \leqq C h(x)^{-p}, \qquad x \in E,\ t \in (0, T].$$

Proposition 13 *For any $T > 0$, there are $C \in (0,\infty)$ and $\delta_0 > 0$ such that*

$$p(t,x,y) \leqq Ct^{-(N+1)\ell_0/2} h(x)^{-2(N+1)\ell_0} \exp(-\frac{2\delta_0}{t}|y-x|^2), \qquad t \in (0,T],\ x,y \in E,$$

and

$$p(t,x,y) \leqq Ct^{-(N+1)\ell_0/2} h(y)^{-2(N+1)\ell_0} \exp(-\frac{2\delta_0}{t}|y-x|^2), \qquad t \in (0,T],\ x,y \in E.$$

Proposition 14 *Let $\delta \in (0,1/N)$, $\alpha,\beta \in \mathbf{Z}^N_{\geqq 0}$ and $T > 0$. Then there are $C \in (0,\infty)$ such that*

$$|\partial_x^\alpha \partial_y^\beta p(t,x,y)| \leqq Ct^{-(|\alpha|+|\beta|+1)\ell_0/2} h(x)^{-2(|\alpha|+|\beta|+1)\ell_0} p(t,x,y)^{1-\delta}, \qquad x,y \in E,\ t \in (0,T],$$

and

$$|\partial_x^\alpha \partial_y^\beta p(t,x,y)| \leqq Ct^{-(|\alpha|+|\beta|+1)\ell_0/2} h(y)^{-2(|\alpha|+|\beta|+1)\ell_0} p(t,x,y)^{1-\delta}, \qquad x,y \in E,\ t \in (0,T].$$

Then we have the following.

Proposition 15 *For any $m \geqq 1$ and $T > 0$, there is a $C \in (0,\infty)$ such that*

$$p(t,x,y) \leqq Ct^{-N\ell_0} h(x)^{-(4N\ell_0+m+1)} h(y)^m, \qquad x,y \in E,\ t \in (0,T].$$

Proof. Note that for any $\varepsilon > 0$ we have

$$|\frac{\partial}{\partial y_i}(p(t,x,y)(\varepsilon+h(y))^{-m})|^{N+1}$$
$$\leqq 2^{N+1}|\frac{\partial}{\partial y_i} p(t,x,y))|^{N+1}(\varepsilon+h(y))^{-m(N+1)}$$
$$+2^{N+1}m^{N+1}p(t,x,y)^{N+1}(\varepsilon+h(y))^{-(m+1)(N+1)}|\frac{\partial h}{\partial y_i}(y)|^{N+1}.$$

By Proposition 12 and 13, we see that

$$\sup\{t^{N(N+1)\ell_0/2} h(x)^{2N(N+1)\ell_0+m(N+1)} \int_{\mathbf{R}^N} |p(t,x,y)(\varepsilon+h(y))^{-m}|^{N+1} dy;$$
$$t \in [0,T],\ x \in E,\ \varepsilon > 0\} < \infty.$$

Also letting $\delta = 1/(N+1)$ in Proposition 14, we see by Proposition 12 that

$$\sup\{t^{(N-1)(N+1)\ell_0} h(x)^{4(N-1)(N+1)\ell_0+(m+1)(N+1)} \sum_{i=1}^N \int_{\mathbf{R}^N} |\frac{\partial}{\partial y_i}(p(t,x,y)(\varepsilon+h(y))^{-m}))|^{N+1} dy;$$
$$t \in [0,T],\ x \in E,\ \varepsilon > 0\} < \infty.$$

These and Sobolev's inequality imply that there is a $C > 0$ such that

$$t^{N\ell_0} h(x)^{4N\ell_0+m+1} p(t,x,y)(\varepsilon+h(y))^{-m} \leqq C, \qquad x \in E,\ y \in \mathbf{R}^N,\ t \in (0,T],\ \varepsilon > 0.$$

This proves our assertion. ∎

Let P_t, $t \geqq 0$, be a diffusion operator defined in $C_b^\infty(\mathbf{R}^N)$ given by

$$(P_t f)(x) = E[f(X(t,x))], \qquad f \in C_b^\infty(\mathbf{R}^N).$$

Then we see that

$$(P_t f)(x) = \int_E p(t,x,y) f(y) dy, \qquad x \in E.$$

Then we have the following.

Proposition 16 *For any $T > 0$ and $\alpha \in \mathbf{Z}_{\geqq 0}^N$, there is a $C \in (0,\infty)$ such that*

$$|\frac{\partial^\alpha}{\partial x^\alpha}(P_t f)(x)| \leqq C t^{-(|\alpha|+N+2)\ell_0/2} h(x)^{-2(|\alpha|+N+2)\ell_0} (P_t(|f|^2)(x))^{1/2}$$

for any $t \in (0,T]$, $x \in E$ and $f \in C_b^\infty(\mathbf{R}^N)$.

Proof. By Proposition 14, we see that there is a $C_1 \in (0,\infty)$ such that for any $f \in C_b^\infty(\mathbf{R}^N)$

$$|\frac{\partial^\alpha}{\partial x^\alpha}(P_t f)(x)| \leqq \int_E |\frac{\partial^\alpha p}{\partial x^\alpha}(t,x,y)||f(y)|dy$$

$$\leqq C_1 t^{-(|\alpha|+1)\ell_0/2} h(x)^{-2(|\alpha|+1)\ell_0} \int_E p(t,x,y)^{2N/(2N+1)} |f(y)| dy$$

$$\leqq C_1 t^{-(|\alpha|+1)\ell_0/2} h(x)^{-2(|\alpha|+1)\ell_0} |\int_E f(y)^2 p(t,x,y) dy|^{1/2} |\int_E p(t,x,y)^{(2N-1)/(4N+2)} dy|^{1/2}.$$

By Proposition 13, we see that there is a $C_2 > 0$ such that

$$\int_E p(t,x,y)^{(2N-1)/(4N+2)} dy \leqq C_2 t^{-(N+1)\ell_0/4} h(x)^{-(N+1)\ell_0}, \qquad x \in E, t \in (0,T].$$

So we have our assertion. ∎

The following is an easy consequence of Proposition 14.

Proposition 17 *For any $\beta \in (0,1/N)$ and $T > 0$, there is a $C > 0$ such that*

$$|\frac{\partial}{\partial y^i}(p(t,x,y)^\beta)| \leqq C t^{-\ell_0} h(x)^{-4\ell_0}, \qquad x \in E,\ t \in (0,T].$$

8 A random system of piece-wise polynomials

Let ν be a probability measure on $\mathbf{R}^N$.

For any $m \geqq 2$, let

$$D_{\vec{k}}^{(m)} = \prod_{i=1}^N [\frac{(2(k_i-1)-m)}{m}\log m, \frac{(2k_i-m)}{m}\log m), \qquad \vec{k} = (k_1,\ldots,k_N) \in \{1,\ldots,m\}^N.$$

Let $\mathcal{D}_m = \{D_{\vec{k}}^{(m)};\ \vec{k} \in \{1,\ldots,m\}^N\}$. Then we have $\bigcup \mathcal{D}_m = [-\log m, \log m)^N$.

Let $X_1, X_2, \dots,$ i.i.d. random variables defined on a probability space $(\Omega, \mathcal{F}, P)$ whose distributions are ν. Let $\mathcal{D}_{m,n}(\omega)$, $m, n \geqq$, $\omega \in \Omega$, be a random sub-family of $\mathcal{D}_m$ given by

$$\mathcal{D}_{m,n}(\omega) = \{D \in \mathcal{D}_m;\ \text{there is a } k \in \{1, \dots, n\} \text{ such that } X_k(\omega) \in D\ \}.$$

Let $\mathcal{P}_r$, $r = 0, 1, 2, \dots,$ be the set of polynomials on $\mathbf{R}^N$ of degree less than or equal to r. Now let $V_{n,m,r}(\omega)$, $m, n \geqq 2$, $r \geqq 0$, $\omega \in \Omega$, be a finite dimensional vector subspace of $m(\mathbf{R}^N)$ hulled by $f 1_D$, $f \in \mathcal{P}_r$, $D \in \mathcal{D}_{m,n}(\omega)$. It is obvious that $\dim V_{n,m,r}(\omega) \leqq N^m (N+1)^r$.

Now let us use the notation in the previous section. Let $X(t, x)$, $t \in [0, \infty)$, $x \in \mathbf{R}^N$, be the solution to the SDE (3) and we assume the (UFG) condition holds. Let $x_0 \in \mathbf{R}^N$ such that $h(x_0) > 0$, and so $x_0 \in E$. Let $T_0 > 0$ and $\rho(x) = p(T_0, x_0, x)$, $x \in \mathbf{R}^N$. We think of the case that $\nu(dx) = \rho(x) dx$.

Then we have the following.

Theorem 18 *Let $r \geqq 0$, $\delta > 0$, $\gamma > 0$, and $T > 0$, and let n_m, $m =, 2, \dots,$ be integers satisfying $m^{N+\gamma} \leqq n_m < 2m^{N+\gamma}$. Then there are $\Omega_m \in \mathcal{F}$, $m = 1, 2, \dots,$ and $C \in (0, \infty)$ satisfying the following.*
(1) $P(\Omega_m) \to 1$, $m \to \infty$.
(2) *For any $\omega \in \Omega_m$,*

$$\inf_{x \in D} \rho(x) \geqq \frac{1}{2} \sup_{x \in D} \rho(x),$$

and

$$\nu(D) \geqq C^{-1} m^{-(2N+\gamma+\delta)}$$

for any $D \in \mathcal{D}_{m,n_m}(\omega)$ and $m \geqq 2$.
(3) *For any $\omega \in \Omega_m$, $\lambda_0(V_{m,n_m,r}, \nu) \leqq C m^{2N+\gamma+\delta}$.*
(4) *For any $\omega \in \Omega_m$, $f \in C_b^\infty(\mathbf{R}^N)$ and $t \in (0, T]$,*

$$||P_t f - \pi_{V_{m,n_m,r}} P_t f||_{L^2(d\nu)}$$

$$\leqq C(t^{-(r+2N+3)\ell_0} m^{-(r+1)+\delta} + m^{-\gamma/4+\delta})(\int_{\mathbf{R}^N} f(y)^4 p(T_0 + t, x_0, y) dy)^{1/4}.$$

Here $\pi_{V_{m,n,r}}$ is the orthogonal projection in $L^2(E; d\nu)$ onto $V_{m,n,r}(\omega)$.

We make some preparations to prove Theorem 18.

Proposition 19 *For any $r \geqq 0$, there is a $C_r > 0$ such that*

$$(\int_{(-\varepsilon,\varepsilon)^N} f(y)^4\, dy)^{1/4} \leqq C_r \varepsilon^{-N/4} (\int_{(-\varepsilon,\varepsilon)^N} f(y)^2\, dy)^{1/2}$$

for any $\varepsilon > 0$ and $f \in \mathcal{P}_r$.

Proof. Let us fix $n \geqq 0$. Since $\mathcal{P}_r$ is a finite dimensional vector space, any norms on $\mathcal{P}_r$ are equivalent. So we see that there is a $C_r > 0$ such that

$$(\int_{(-1,1)^N} |f(x)|^4\, dx)^{1/4} \leqq C_r (\int_{(-1,1)^N} |f(x)|^2\, dx)^{1/2}, \qquad f \in \mathcal{P}_r.$$

Then we see that

$$(\int_{(-\varepsilon,\varepsilon)^N} f(x)^4\,dx)^{1/4} = \varepsilon^{N/4}(\int_{(-1,1)^N} f(\varepsilon x)^4\,dx)^{1/4}$$

$$\leqq C_r\varepsilon^{N/4}(\int_{(-1,1)^N} f(\varepsilon x)^2\,dx)^{1/2} = C_r\varepsilon^{-N/4}(\int_{(-\varepsilon,\varepsilon)^N} f(x)^2\,dx)^{1/2}.$$

This implies our assertion. ∎

For any Borel subset A in $\mathbf{R}^N$ and n, let $N_n(A)$ be $N_n(A) = \sum_{k=1}^n 1_A(X_i)$.

Let $\gamma > 0$ and $\delta \in (0, \gamma/2)$, and fix them. Let $\gamma_0 = N+\gamma-\delta/3$ and $\gamma_1 = 2N+\gamma+\delta/3$. Now let $\mathcal{D}_m^{(0)}$ and $\mathcal{D}_m^{(1)}$ be subsets of $\mathcal{D}_m$, $m \geqq 1$, given by

$$\mathcal{D}_m^{(0)} = \{D \in \mathcal{D}_m;\ \nu(D) \geqq m^{-\gamma_0}\},$$

and

$$\mathcal{D}_m^{(1)} = \{D \in \mathcal{D}_m;\ \nu(D) \geqq m^{-\gamma_1}\}.$$

Then it is obvious that $\mathcal{D}_m^{(0)} \subset \mathcal{D}_m^{(1)}$.

Then we have the following.

Proposition 20 (1) *Let $\Omega_{0,m,n}$, $m \geqq 2$, $n \geqq 1$, be the set of $\omega \in \Omega$ such that $\mathcal{D}_m^{(0)} \subset \mathcal{D}_{m,n}(\omega)$. Then we have*

$$P(\Omega \setminus \Omega_{0,m,n}) \leqq m^N \exp(-nm^{-(N+\gamma)}m^{\delta/3}), \qquad n \geqq 1,\ m \geqq 2.$$

(2) *Let $\Omega_{1,m,n}$, $m \geqq 2$, $n \geqq 1$, be the set of $\omega \in \Omega$ such that $\mathcal{D}_{m,n}(\omega) \subset D_m^{(1)}$. Then there is an $m_1 \geqq 1$ such that*

$$P(\Omega \setminus \Omega_{1,m,n}) \leqq (2\log 2)nm^{-(N+\gamma)}m^{-\delta/3} \qquad n \geqq 1,\ m \geqq m_1.$$

Proof. Since $(1-1/x)^x$, $x \in (1,\infty)$ is increasing in x, we see that

$$\frac{1}{4} \leqq (1-\frac{1}{x})^x \leqq e^{-1}, \qquad x \geqq 2.$$

For $D \in \mathcal{D}_m$ we have

$$P(N_n(D) = 0) = (1-\nu(D))^n = ((1-\nu(D))^{1/\nu(D)})^{n\nu(D)}.$$

Thus we see that

$$P(N_n(D) = 0) \leqq \exp(-n\nu(D))$$

for any $D \in \mathcal{D}_m$, and

$$2^{-2n\nu(D)} \leqq P(N_n(D) = 0)$$

for any $D \in \mathcal{D}_m$ with $\nu(D) \in [0, 1/2]$. So we see that for any $D \in \mathcal{D}_m$ with $\nu(D) \in [0, 1/2]$,

$$P(N_n(D) \geqq 1) \leqq 1 - \exp(-(2\log 2)n\nu(D)) \leqq (2\log 2)n\nu(D).$$

Note that

$$\nu(D) \leqq (2m^{-1}\log m)^N \sup_{x\in\mathbf{R}^N} \rho(x).$$

So there is an $m_1 \geqq 1$ such that $\nu(D) \leqq 1/2$ for $D \in \mathcal{D}_m$, $m \geqq m_1$.

Therefore we see that

$$P(\Omega \setminus \Omega_{0,m,n}) \leqq \sum_{D \in \mathcal{D}_m^{(0)}} P(N_n(D) = 0) \leqq m^N \exp(-nm^{-\gamma_0}),$$

and

$$P(\Omega \setminus \Omega_{1,m,n}) \leqq \sum_{D \in \mathcal{D}_m \setminus \mathcal{D}_m^{(0)}} P(N_n(D) \geqq 1) \leqq (2\log 2) n m^{N-\gamma_1} \qquad m \geqq m_1.$$

So we have our assertions. ∎

Proposition 21 *There is an $m_2 \geqq 1$ satisfying the following.*
If $D \in \mathcal{D}_m^{(1)}$, then

$$\inf_{x \in D} \rho(x) \geqq \frac{1}{2} \sup_{x \in D} \rho(x) \geqq m^{-(N+\gamma+2\delta/3)}, \qquad m \geqq m_2.$$

Proof. Assume that $D \in \mathcal{D}_m^{(1)}$. Let $x_1 \in \bar{D}$ be a maximal point of $\rho(x)$, $x \in \bar{D}$. Then we see that $\rho(x_1) \geqq (2\log m)^N m^{-N-\gamma-\delta/3}$. Appliing Proposition 17 for $\beta = 1/(2(N+\gamma+\delta/3)) > 0$, we see that there is a $C_0 > 0$ such that

$$|\rho(x)^\beta - \rho(y)^\beta| \leqq C_0 |x - y|, \qquad x, y \in \mathbf{R}^N.$$

So we see that

$$|\rho(x)^\beta - \rho(x_1)^\beta| \leqq C_0 \frac{2N \log m}{m}. \qquad x \in D,$$

and so

$$\rho(x)^\beta \geqq \rho(x_1)^\beta - C_0 \frac{2N \log m}{m}$$
$$\geqq (\frac{1}{2}\rho(x_1))^\beta + (1 - 2^{-\beta})(2\log m)^{-N\beta} m^{-1/2} - C_0 \frac{2N \log m}{m}$$

So we see that if m is sufficiently large

$$\inf_{x \in D} \rho(x) \geqq \frac{1}{2} \sup_{x \in D} \rho(x) \geqq m^{-(N+\gamma+2\delta/3)}.$$

Thus we have our assertion.

Proposition 22 *There is an $m_3 \geqq 1$ satisfying the following. If $\omega \in \Omega_{1,n,m}$ and $m \geqq m_3$, then*

$$\lambda_0(V_{m,n,r}(\omega); \nu) \leqq m^{2N+\gamma+\delta}.$$

Proof. Let $m_2 \geqq 1$ be as in Proposition 21. Suppose that $\omega \in \Omega_{1,n,m}$ and $m \geqq m_2$. Then $\mathcal{D}_{m,n}(\omega) \subset \mathcal{D}_m^{(1)}$.

Let $f \in V_{m,n,r}(\omega)$. Then there are $f_D \in \mathcal{P}_r$, $D \in \mathcal{D}_{m.n}(\omega)$, such that

$$f = \sum_{D \in \mathcal{D}_{m.n}(\omega)} f_D 1_D.$$

Then we see that

$$\int_{\mathbf{R}^N} f(x)^4 \nu(dx) = \sum_{D\in\mathcal{D}_{m.n}(\omega)} \int_D f_D(x)^4 \nu(dx) \leqq \sum_{D\in\mathcal{D}_{m.n}(\omega)} \sup_{x\in D} \rho(x) \int_D f_D(x)^4 dx$$

$$\leqq 2 \sum_{D\in\mathcal{D}_{m.n}(\omega)} \inf_{x\in D} \rho(x) C_r^4 (2m^{-1}\log m)^{-N} (\int_D f_D(x)^2 dx)^2$$

$$\leqq 2 \sum_{D\in\mathcal{D}_{m.n}(\omega)} \frac{1}{\inf_{x\in D}\rho(x)} C_r^4 (2m^{-1}\log m)^{-N} (\int_D f_D(x)^2 \nu(dx))^2$$

$$\leqq m^{2N+\gamma+\delta}(2^{N+1}C_r^4 m^{-\delta/3}(\log m)^{-N})(\int_{\mathbf{R}^N} f(x)^2\nu(dx))^2.$$

This implies our assertion. ∎

Proposition 23 *For any $r \geqq 0$, there is a $C \in (0,\infty)$ satisfying the following.*

$$\inf\{(\int_{(-\varepsilon,\varepsilon)^N} |f(x)-g(x)|^2 dx)^{1/2};\ g\in\mathcal{P}_r\}$$

$$\leqq C\varepsilon^{r+1} \sum_{\alpha\in\mathbf{Z}^N_{\geqq 0}, r+1\leqq|\alpha|\leqq r+N+1} (\int_{(-\varepsilon,\varepsilon)^N} |\frac{\partial^\alpha f}{\partial x^\alpha}(x)|^2 dx)^{1/2}$$

for any $f\in C^\infty(\mathbf{R}^N)$ and $\varepsilon\in(0,1]$.

Proof. By Sobolev's inequality, we see that there is a $C_0 >$ such that

$$\sup_{x\in(-1,1)^N} |f(x)| \leqq C_0 \sum_{\alpha\in\mathbf{Z}^N_{\geqq 0},|\alpha|\leqq N} (\int_{(-1,1)^N} |\frac{\partial^\alpha f}{\partial x^\alpha}(x)|^2 dx)^{1/2}, \qquad f\in C^\infty(\mathbf{R}^N).$$

So we see that

$$\sup_{x\in(-\varepsilon,\varepsilon)^N} |f(x)| \leqq C_0 \sum_{\alpha\in\mathbf{Z}^N_{\geqq 0},|\alpha|\leqq N} |\int_{(-1,1)^N} |\frac{\partial^\alpha}{\partial x^\alpha}(f(\varepsilon x))|^2 dx)^{1/2}$$

$$\leqq C_0 \sum_{\alpha\in\mathbf{Z}^N_{\geqq 0},|\alpha|\leqq N} \varepsilon^{|\alpha|-N/2} (\int_{(-\varepsilon,\varepsilon)^N} |\frac{\partial^\alpha f}{\partial x^\alpha}(x)|^2 dx)^{1/2}.$$

For any $f\in C^\infty(\mathbf{R}^N)$,

$$|f(x) - \sum_{\alpha\in\mathbf{Z}^N_{\geqq 0},|\alpha|\leqq r} \frac{1}{\alpha!}\frac{\partial^\alpha f}{\partial x^\alpha}(0)x^\alpha| \leqq \int_0^t \frac{(1-t)^r}{r!} |\frac{d^{r+1}}{dt^{r+1}} f(tx)| dt$$

$$\leqq |x|^{r+1} \sum_{\alpha\in\mathbf{Z}^N_{\geqq 0},|\alpha|=r+1} \sup_{t\in[0,1]} |\frac{\partial^\alpha f}{\partial x^\alpha}(tx)|,$$

and so we have

$$\inf\{(\int_{(-\varepsilon,\varepsilon)^N}|f(x)-g(x)|^2dx)^{1/2};\ g\in\mathcal{P}_r\}\leqq(2N\varepsilon)^{r+1+N/2}\sum_{\alpha\in\mathbf{Z}^N_{\geqq 0},|\alpha|=r+1}\sup_{x\in(-\varepsilon,\varepsilon)^N}|\frac{\partial^\alpha f}{\partial x^\alpha}(x)|$$

$$\leqq\varepsilon^{r+1}C_0(2N)^{r+1+N/2}\sum_{\alpha,\beta\in\mathbf{Z}^N_{\geqq 0},|\alpha|=r+1,|\beta|\leqq N}(\int_{((-\varepsilon,\varepsilon)^N}|\frac{\partial^{\alpha+\beta}f}{\partial x^{\alpha+\beta}}(x)|^2dx)^{1/2}.$$

This implies our assertion ∎

Proposition 24 *For any $T>0$ there is an $m_4\geqq 1$ such that for any $D\in\mathcal{D}^{(1)}_m$, $m\geqq m_4$,*

$$\inf\{\int_D|P_tf(x)-g(x)|^2\nu(dx);\ g\in\mathcal{P}_r\}$$

$$\leqq m^{-2(r+1)+2\delta/3}t^{-(r+2N+3)\ell_0/2}\int_D P_t(|f|^2)(x)\nu(dx),\qquad t\in(0,T],\ f\in C^\infty_b(\mathbf{R}^N).$$

Proof. Let $m_2\geqq 1$ be as in Proposition 21. Then

$$\rho(x)\geqq m^{-(N+\gamma+2\delta/3)},\qquad x\in D,\ D\in\mathcal{D}^{(1)}_m$$

for any $m\geqq m_2$. By Proposition 15, there is a $C_0>0$ such that

$$h(x)\geqq C_0m^{-\delta/(8(r+2N+3)\ell_0)},\qquad x\in D,\ D\in\mathcal{D}^{(1)}_m,\ m\geqq m_2.$$

Then by Proposition 16 we see that there is a $C_1>0$ such that

$$\sum_{\alpha\in\mathbf{Z}^N_{\geqq 0},r+1\leqq|\alpha|\leqq N+r+1}|\frac{\partial^\alpha}{\partial x^\alpha}P_tf(x)|\leqq C_1m^{\delta/4}t^{-(r+2N+3)\ell_0/2}(P_t(|f|^2)(x))^{1/2},$$

for any $x\in D$, $D\in\mathcal{D}^{(1)}_m$, $m\geqq m_2$, and $f\in C^\infty_b(\mathbf{R}^N)$. Then by Propositions 23 we see that there is a $C_2>0$ such that for $D\in\mathcal{D}^{(1)}_m$, $m\geqq m_2$,

$$\inf\{(\int_D|P_tf(x)-g(x)|^2\nu(dx))^{1/2};\ g\in\mathcal{P}_r\}$$

$$\leqq(\sup_{x\in D}\rho(x))^{1/2}\inf\{(\int_D|P_tf(x)-g(x)|^2dx)^{1/2};\ g\in\mathcal{P}_r\}$$

$$\leqq 2(\inf_{x\in D}\rho(x))^{1/2}C_2(2m^{-1}\log m)^{r+1}\sum_{\alpha\in\mathbf{Z}^N_{\geqq 0},r+1\leqq|\alpha|\leqq r+N}(\int_D|\frac{\partial^\alpha}{\partial x^\alpha}P_tf(x)|^2dx)^{1/2}$$

$$\leqq 2C_2(2m^{-1}\log m)^{r+1}C_1m^{\delta/4}t^{-(r+2N+3)\ell_0/2}(\int_D(P_t(|f|^2))(x)\nu(dx))^{1/2}.$$

So we have our assertion. ∎

Proposition 25 *Let $A_{0,m}=\bigcup\mathcal{D}^{(0)}_m$. Then there is an $m_5\geqq 1$ such that*

$$\nu(\mathbf{R}^N\setminus A_{0,m})\leqq m^{-\gamma+\delta},\qquad m\geqq m_5.$$

Proof. We see by Proposition 13 that

$$\nu(\mathbf{R}^N \setminus A_{0,m}) = \nu([-\log m, \log m)^N \setminus A_{0,m}) + \nu(\mathbf{R}^N \setminus [-\log m, \log m)^N)$$

$$= \sum_{D\in\mathcal{D}_m\setminus\mathcal{D}_m^{(0)}} \nu(D) + \int_{\mathbf{R}^N\setminus[-\log m,\log m)^N)} p(T_0, x_0, x)dx$$

$$\leqq m^{N-\gamma_0} + CT_0^{-(N+1)\ell_0/2} h(x_0)^{-2(N+1)\ell_0} \int_{\mathbf{R}^N\setminus[-\log m,\log m)^N)} \exp(-\frac{2\delta_0|x-x_0|^2}{T_0})dx.$$

This implies our assertion. ∎

Proposition 26 *Let $r \geqq 0$, and $T > 0$. There is an $m_6 \geqq 2$ satisfying the following. For any $\omega \in \Omega_{0,m,n}$, $m \geqq m_6$, $n \geqq 1$,*

$$||P_t f - \pi_{V_{m,n,r}} P_t f||_{L^2(d\nu)}$$

$$\leqq (t^{-(r+2N+3)\ell_0/2} m^{-(r+1)+\delta/2} + m^{-\gamma/4+\delta/2})(\int_{\mathbf{R}^N} f(y)^4 p(T_0+t, x_0, y)dy)^{1/4}$$

for any $t \in (0, T]$, and $f \in C_b^\infty(\mathbf{R}^N)$.

Proof. Let $m_4, m_5 \geqq 2$ be as in Propositions 24 and 25. Let $\omega \in \Omega_{0,m,n}$, and $m \geqq m_4 \vee m_5$. Then we see that $\mathcal{D}_{m,n}(\omega) \supset \mathcal{D}_m^{(0)}$ and so we see that

$$\inf\{\int_{\mathbf{R}^N} |P_t f(x) - g(x)|^2 \nu(dx);\ g \in \mathcal{P}_r\}$$

$$= \sum_{D\in\mathcal{D}_{m,n}(\omega)} \inf\{\int_D |(P_t f)(x) - g(x)|^2 \nu(dx);\ g \in \mathcal{P}_r\} + \int_{\mathbf{R}^N\setminus\bigcup\mathcal{D}_{m,n}(\omega)} |P_t f(x)|^2 \nu(dx)$$

$$\leqq \sum_{D\in\mathcal{D}_{m,n}(\omega)} m^{-2(r+1)+2\delta/3} t^{-(r+2N+3)\ell_0} \int_D P_t(|f|^2)(x)\nu(dx)$$

$$+\nu(\mathbf{R}^N \setminus A_{0,m})^{1/2} (\int_{\mathbf{R}^N} |P_t f(x)|^4 \nu(dx))^{1/2}$$

$$\leqq m^{-2(r+1)+2\delta/3} t^{-(r+2N+3)\ell_0} \int_{\mathbf{R}^N} f(y)^2 p(T_0+t, x_0, y)dy$$

$$+m^{-(\gamma-\delta)/2} (\int_{\mathbf{R}^N} f(y)^4 p(T_0+t, x_0, y)dy)^{1/2}.$$

So this and Proposition 25 imply our assertion. ∎

Now we have Theorem 18 from Propositions 20, 21, 22 and 26, letting $\Omega_m = \Omega_{0,m,n_m} \cap \Omega_{1,m,n_m}$.

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ADDRESS:
Graduate School of Mathematical Sciences, The University of Tokyo
3–8–1 Komaba Meguro-ku, Tokyo 153, JAPAN
TEL +81-3-5465-7001 FAX +81-3-5465-7012