

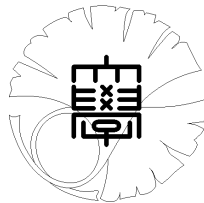
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**Homogenization of the Hele-Shaw problem in
periodic spatiotemporal media**

by

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ABSTRACT. We consider the homogenization of the Hele-Shaw problem in periodic media that are inhomogeneous both in space and time. After extending the theory of viscosity solutions into this context, we show that the solutions of the inhomogeneous problem converge in the homogenization limit to the solution of a homogeneous Hele-Shaw-type problem with a general, possibly nonlinear dependence of the free boundary velocity on the gradient. Moreover, the free boundaries converge locally uniformly in Hausdorff distance.

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1. INTRODUCTION

Let $n \geq 2$ be the dimension and let $\Omega \subsetneq \mathbb{R}^n$ be a domain (open, connected) with a non-empty compact Lipschitz boundary ($\partial\Omega \in C^{0,1}$). For given $\varepsilon > 0$, we shall consider the following Hele-Shaw-type problem on a parabolic cylinder $Q := \Omega \times (0, T]$ for some $T > 0$: find $u^\varepsilon : Q \rightarrow [0, \infty)$ that formally satisfies

$$(1.1) \quad \begin{cases} -\Delta u(x, t) = 0 & \text{in } \{u > 0\}, \\ V_\nu(x, t) = g\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}\right) |Du^+(x, t)| & \text{on } \partial\{u > 0\}, \end{cases}$$

with some boundary data to be specified later (Theorem 1.1), where $D = D_x$ and $\Delta = \Delta_x$ are respectively the gradient and the Laplace operator in the x variable, and V_ν is the normal velocity of the *free boundary* $\partial\{u > 0\}$. Assuming that the solution is sufficiently smooth, the free boundary $\partial\{u > 0\}$ is a level set of u and its

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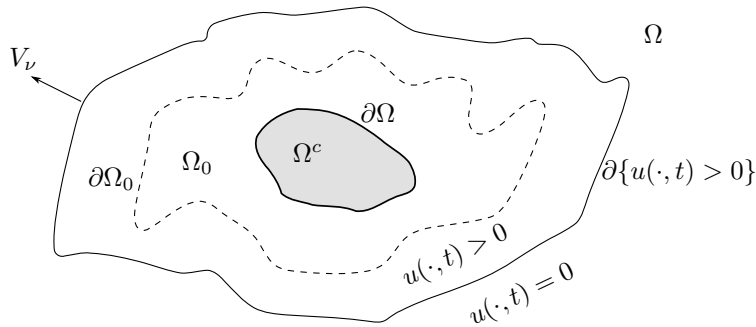


FIGURE 1. The Hele-Shaw problem in a plane

normal velocity can be expressed as

$$V_\nu = \frac{u_t^+}{|Du^+|}.$$

Here u_t^+ and Du^+ formally represent the limits of the derivatives from the positive set of u . The well-posedness of problem (1.1) requires some basic regularity of g . In the following we shall assume that g satisfies

$$(1.2) \quad 0 < m \leq g \leq M$$

for some positive constants m, M , and that g is L -Lipschitz both in x and t , that is,

$$(1.3) \quad |g(x, t) - g(y, s)| \leq L(|x - y| + |t - s|) \quad \text{for all } (x, t), (y, s) \in \mathbb{R}^n \times \mathbb{R}.$$

We are interested in the behavior of the solutions of (1.1) in the *homogenization limit* $\varepsilon \rightarrow 0+$. Since we want to observe an averaging behavior, we further assume that g is a \mathbb{Z}^{n+1} -periodic function, i.e.,

$$(1.4) \quad g(x + k, t + l) = g(x, t) \quad \text{for all } (x, t) \in \mathbb{R}^n \times \mathbb{R}, (k, l) \in \mathbb{Z}^n \times \mathbb{Z}.$$

In the following we define

$$g^\varepsilon(x, t) := g\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}\right).$$

Note that g^ε is an $\varepsilon\mathbb{Z}^{n+1}$ -periodic $\frac{L}{\varepsilon}$ -Lipschitz function which satisfies (1.2).

Problem (1.1) with $g \equiv \text{const}$ is the standard Hele-Shaw problem with no surface tension. In two dimensions, it was introduced in [13] as a model of a slow movement of a viscous fluid injected in between two nearby parallel plates that form the so-called Hele-Shaw cell. This problem naturally generalizes to all dimensions $n \geq 1$. In particular, in three dimensions it serves as a model of a pressure-driven flow of an incompressible fluid through a porous medium. Following this motivation, problem (1.1) with general g describes a pressure-driven flow of an incompressible fluid in an inhomogeneous, time-dependent medium. Free boundary problems with similar velocity laws have various applications in the plastics industry and other fields [30, 31, 36]. The Hele-Shaw-type problem (1.1) can also be viewed as a quasi-stationary limit of the one-phase Stefan problem with a latent heat of phase transition depending on position and time [1, 26, 29, 33].

Homogenization overview. There is a large amount of literature on homogenization which is beyond the scope of this discussion and thus we refer the reader to [15,37] and the references therein.

In the context of viscosity solutions for fully nonlinear problems of the first and second order, the now standard approach to homogenization using correctors that are the solutions of an appropriate cell problem was pioneered by Lions, Papanicolau & Varadhan [25] for first order equations, and later by Evans [12] for second order equations. Unfortunately, this approach does not apply to problems like (1.1) because the zero level of solutions has a special significance and the perturbation of a test function by a global periodic solution of some cell problem, i.e., the corrector, does not seem feasible.

The idea of using obstacle problems to recover the homogenized operator without the need for a cell problem was developed by Caffarelli, Souganidis & Wang [7] for the stochastic homogenization of fully-nonlinear second order elliptic equations. It was then applied to the homogenization of the Hele-Shaw problem with spatial periodic inhomogeneity by Kim [18], using purely the methods of the theory of viscosity solutions. This approach was later extended to a model of contact angle dynamics [19], and an algebraic rate of the convergence of free boundaries was obtained [20].

Kim & Mellet [21,22] later succeeded at applying a combination of viscosity and variational approaches and obtained a homogenization result in the setting of spatial stationary ergodic random media. A related question of long-time asymptotics of the spatially inhomogeneous Hele-Shaw problem was addressed by the author [28]. This technique relies on the special structure of problem (1.1) with time-independent g which allows us to rewrite the problem as a variational inequality of a certain obstacle problem. We refer the reader to [32] and the references therein for an exposition of obstacle problems and their homogenization.

The homogenization results for spatial media do not necessarily translate directly to the spatiotemporal homogenization. In the context of Hamilton-Jacobi equations, for instance, a lack of uniform estimates in the spatiotemporal case was encountered by Schwab [34] who proved the homogenization in spatiotemporal stationary ergodic random media, which was before established by Souganidis [35] for spatial random media. The situation seems to be more extreme in the case of the Hele-Shaw problem since the homogenization in spatiotemporal media is qualitatively different even in the periodic case.

This difference can be already observed in the homogenization of a single ordinary differential equation of the type $x'_\varepsilon(t) = f(x_\varepsilon(t)/\varepsilon, t/\varepsilon, x_\varepsilon(t))$. In fact, the Hele-Shaw problem (1.1) reduces to this type of ODE in one dimension; see Section 3.1 for further discussion. It is known that, under some assumptions on f , $x_\varepsilon \rightrightarrows \bar{x}$ locally uniformly, where \bar{x} is the solution of a homogenized ODE [14,27]. However, the form of the homogenized problem depends on f . If, for illustration, $f(y, s, x) = f(y)$ is a \mathbb{Z} -periodic Lipschitz function, then the homogenized problem has the form $\bar{x}'(t) = \bar{f}$ where \bar{f} is the constant

$$(1.5) \quad \bar{f} = \left(\int_0^1 (f(y))^{-1} dy \right)^{-1}.$$

On the other hand, if $f(y, s, x) = f(y, s)$ has a nontrivial dependence on both y and s , then no such explicit formula exists and, in fact, the right-hand side of the

homogenized problem, while still a constant, can possibly have a value in the range $[\min f, \max f]$.

Another explanation of the qualitative difference, slightly more intuitive, is the interpretation of the Hele-Shaw problem (1.1) as the quasi-stationary limit of the Stefan problem. Then the quantity $(g(x/\varepsilon, t/\varepsilon))^{-1}$ can be interpreted as the *latent heat of phase transition*, that is, the “energy” necessary to change a unit volume of the dry region into the wet region and advance the free boundary [28]. The “energy flux” is proportional to the gradient of the pressure. If g does not depend on time, the homogenized latent heat is simply the average, i.e., $\int_{[0,1]^n} (g(x))^{-1} dx$, giving the homogenized velocity recovered in [21] (of the form (1.5)).

This formula was rigorously justified in [21] using the variational formulation via an obstacle problem, which allows to solve for the shape of the wet region at a given time without the need to solve the problem at previous times. The variational formulation was introduced in [11] using a transformation due to [3]. In a sense, for the evolution overall, the free boundary feels the same influence of the medium no matter how it passes through it. Or, in other words, the amount of the energy required to fill a given region is always the same.

This drastically changes when the latent heat depends on both space and time. In this case, the energy required to advance the boundary depends on the specific history of the motion of the free boundary through the space-time. The variational formulation does not apply anymore. What is more, it is no longer obvious that the problem should homogenize. As the one-dimensional situation indicates—Section 3.1—the homogenized velocity might have a complicated dependence on the gradient, with velocity pinning and directional dependence. Some of these features appear in the spatial homogenization of non-monotone problems [19, 20].

Our approach can be characterized as geometric, relying on maximum principle arguments. A similar approach to homogenization, albeit of (local) geometric motions, was recently pursued by Caffarelli & Monneau [5]. The geometric approach to the Hele-Shaw problem is complicated by the nonlocal nature of the problem. Indeed, (1.1) can be interpreted as a geometric motion of the free boundary with the velocity given by a nonlocal operator based on the Dirichlet-to-Neumann map. Thus, the domain is of crucial importance.

Main results. We present new well-posedness and homogenization results for the Hele-Shaw-type problem (1.1).

The well-posedness result is a generalization and an improvement of the previous results in [16, 18]. In full generality, we consider the Hele-Shaw type problem

$$(1.6) \quad \begin{cases} -\Delta u(x, t) = 0 & \text{in } \{u > 0\} \cap Q, \\ u_t^+(x, t) = f(x, t, Du) |Du^+(x, t)|^2 & \text{on } \partial\{u > 0\} \cap Q. \end{cases}$$

Let us introduce a number of assumptions on the function $f(x, t, p) : Q \times \mathbb{R}^n \rightarrow \mathbb{R}$. In the following, we use the semi-continuous envelopes f_* and f^* of f on $Q \times \mathbb{R}^n$ (see (2.1) for definition). Let us point out that we do not require continuity of $f(x, t, p)$ in p . We shall need this generality to handle the homogenized problem.

- (A1) (non-degeneracy) There exist constants m and M such that $0 < m \leq f(x, t, p) \leq M$ for all $(x, t, p) \in Q \times \mathbb{R}^n$.
- (A2) (Lipschitz continuity) There exists a constant $L > 0$ such that f is L -Lipschitz in x and t for all p .

(A3) (monotonicity) $f^*(x, t, a_1 p) |a_1 p| \leq f_*(x, t, a_2 p) |a_2 p|$ for any $(x, t, p) \in Q \times \mathbb{R}^n$ and $0 < a_1 < a_2$.

The assumption (A3) implies that the free boundary velocity is monotone with respect to the gradient, while allowing for certain jumps.

We have the following existence theorem for viscosity solutions that are introduced in Section 2.

Theorem 1.1 (Existence). *Let $Q := \Omega \times (0, T]$ where Ω is a domain that satisfies the assumptions above and $T > 0$. Assume that either $f(x, t, p) = f(x, t)$ or $f(x, t, p) = f(p)$ and that f satisfies (A1)–(A3). Then for any positive function $\psi \in C(\partial\Omega \times [0, T])$ strictly increasing in time, and for any open set $\Omega_0 \subset \mathbb{R}^n$ with smooth boundary, $\partial\Omega_0 \in C^{1,1}$, such that $\Omega^c \subset \Omega_0$ and $\Omega_0 \cap \Omega$ is bounded (see Figure 1), there exists a unique bounded viscosity solution $u : \bar{Q} \rightarrow [0, \infty)$ of the Hele-Shaw-type problem (1.6) such that $u^* = u_* = u$ on $\partial_P Q := \bar{Q} \setminus Q$ with boundary data $u(x, t) = \psi(x, t)$ on $\partial\Omega \times [0, T]$ and initial data $u(\cdot, 0) > 0$ in $\Omega \cap \Omega_0$ and $u(\cdot, 0) = 0$ in $\Omega \setminus \Omega_0$. The solution is unique in the sense that $u_* = v_*$ and $u^* = v^*$ for any two viscosity solutions u, v with the given boundary data.*

In the context of a flow in porous media, Ω^c represents the source of a liquid with the prescribed pressure ψ on its boundary, and Ω_0 is the initial wet region. The situation is depicted in Figure 1.

However, the main result of this paper concerns the homogenization of (1.1). Note that because the solution of the homogenized problem might be discontinuous, we cannot expect uniform convergence of solutions in general. Furthermore, we do not know if the homogenized velocity (r in Theorem 1.2 below) is continuous.

Theorem 1.2 (Homogenization). *Suppose that $g \in C(\mathbb{R}^n \times \mathbb{R})$ satisfies (1.2)–(1.4). Then there exists a function $r : \mathbb{R}^n \rightarrow \mathbb{R}$ such that*

$$f(x, t, p) := \frac{r(p)}{|p|}$$

satisfies (A1)–(A3), and, for any Q and initial and boundary data Ω_0, ψ that satisfy the assumptions in Theorem 1.1 the following results hold:

- (a) *the unique solutions u^ε of (1.1) with data Ω_0, ψ converge in the sense of half-relaxed limits as $\varepsilon \rightarrow 0$ (Definition 4.1) to the unique solution u of (1.6) with $f(x, t, p) = r(p)/|p|$ and the same data Ω_0, ψ ;*
- (b) *if u is also continuous on a compact set $K \subset \bar{Q}$ then $u^\varepsilon \rightrightarrows u$ converge uniformly on K ;*
- (c) *the free boundaries $\partial\{(u^\varepsilon)_* > 0\}$ converge uniformly to the free boundary $\partial\{u_* > 0\}$ with respect to the Hausdorff distance (Definition 4.5).*

Sketch of the proof. Let us give an overview of the main ideas in the paper. Since the variational formulation via an obstacle problem, discussed above, is not available, we have to rely solely on the technically heavy tools of the viscosity theory. The time-dependence of g poses significant new challenges which require a rather nontrivial extension of the previous results.

The first step is the identification of the homogenized problem. Since the solutions of (1.1) are harmonic in space in their positive sets on any scale ε , their limit as $\varepsilon \rightarrow 0$ should also be harmonic in space. The free boundary velocity of the homogenized problem is, however, unknown. Following the ideas from [7, 18], we identify the

correct homogenized free boundary velocity by solving an obstacle problem and study its behavior as $\varepsilon \rightarrow 0$. To motivate our approach, suppose that the solutions u^ε in the limit $\varepsilon \rightarrow 0$ converge to a solution u of the homogenized Hele-Shaw problem with the free boundary law given as

$$V_\nu = r(Du) \quad \text{on } \partial\{u > 0\},$$

where $a \mapsto r(aq)$ is an increasing function on \mathbb{R}_+ for any $q \in \mathbb{R}^n$. The crucial observation is that this problem has traveling wave solutions; these are the planar solutions of the form $P_{q,r}(x, t) = (|q|rt + x \cdot q)_+$, with the particular choice $r = r(q)$. Here $(\cdot)_+$ stands for the positive part, i.e., $s_+ := \max(s, 0)$. If $r > r(q)$ then $P_{q,r}$ is a supersolution of the homogenized problem, and if $r < r(q)$ it is a subsolution. This observation allows us to identify the correct velocity $r(q)$ by solving the ε -problem (1.1) for each $\varepsilon > 0$ and comparing the solution with the planar solution $P_{q,r}$ for given r and q . Loosely speaking, if r is too large for a given q , the solutions u^ε should evolve slower than $P_{q,r}$ for small ε and, similarly, if r is too small then u^ε should evolve faster than $P_{q,r}$ for small ε .

To make this idea rigorous, we solve the following obstacle problems for each fixed ε : given a fixed domain Q , r and q , find the largest subsolution $\bar{u}_{\varepsilon;q,r}$ of the ε -problem that stays under $P_{q,r}$ in Q and the smallest supersolution $\underline{u}_{\varepsilon;q,r}$ of the ε -problem that stays above $P_{q,r}$ in Q . Following this reasoning, we find two candidates for the correct homogenized velocity $r(q)$ by “properly measuring” how much contact there is as $\varepsilon \rightarrow 0$ between the obstacle $P_{q,r}$ and the largest subsolution $\bar{u}_{\varepsilon;q,r}$, yielding $\bar{r}(q)$, and the smallest supersolution $\underline{u}_{\varepsilon;q,r}$, yielding $\underline{r}(q)$. Since the free boundary velocity law is nonlocal, given by a Dirichlet-to-Neumann map, a good choice of the domain Q for this procedure is important.

For homogenization to occur, it is necessary that both candidates $\bar{r}(q)$ and $\underline{r}(q)$ yield the same limit problem (1.6). We need to give a proper meaning to the intuitive idea of evolving slower or faster than the obstacle $P_{q,r}$. The selection of a quantity that not only “properly measures” the amount of contact between the obstacle $P_{q,r}$ and the solutions of the obstacle problem in the homogenization limit $\varepsilon \rightarrow 0$ but is also convenient to work with is far from obvious. In particular, we want to take advantage of the natural monotonicity (Birkhoff property) of the solutions of the obstacle problem, which is a consequence of the periodicity of the medium.

In [7] as well as in [18–20], the authors consider the coincidence set of the solutions of the obstacle problem and the obstacle $P_{q,r}$ as their quantity of choice. This choice is motivated by the fact that if there is a contact on a sufficiently large set, then the solution must be close to the obstacle everywhere. This kind of estimate can be established using the Alexandroff-Bakelman-Pucci estimate in the case of fully nonlinear elliptic problems in [7]. Since ABP-type estimates are not available for the Hele-Shaw problem, the closeness to the obstacle must be derived by other means [18].

The disadvantage of this approach in the context of free boundary problems, in particular for the homogenization of (1.1), stems from the restriction that it imposes on the directions in which one can shift the solutions of the obstacle problem to take advantage of their natural monotonicity property, and to recover the monotonicity of the contact set. This creates technical difficulties and requires a separate treatment of rational and irrational directions. It seems that, to overcome these difficulties, it is necessary to scale the solutions in time for the arguments to work. However, a scaling in time is not available in the time-dependent medium.

The main new idea in this paper is the introduction of *flatness* and its critical value ε^β (Section 3.3). More specifically, to recover the correct homogenized boundary velocity for a given gradient q we test if, for some fixed $\beta \in (0, 1)$, the free boundary of the solution of the obstacle problem stays ε^β -close to the obstacle for a unit time on an arbitrary small scale ε . This choice is motivated by the new cone flatness property (Proposition 3.20): the boundary of the solution of the obstacle problem stays between two cones that are $\sim \varepsilon |\ln \varepsilon|^{1/2}$ apart. Therefore if, for $\varepsilon \ll 1$, the boundary is farther than $\varepsilon^\beta \gg \varepsilon |\ln \varepsilon|^{1/2}$, it will be detached from the obstacle on a large set. Moreover, the improvement of the *local comparison principle* (Theorem 3.16) for ε^β -flat solutions to allow for $\beta \in (4/5, 1)$ is a necessary ingredient to close the argument. This approach allows us to prove that the candidates $\bar{r}(q)$ and $r(q)$ almost coincide, i.e., $\bar{r} = r^*$, and that $f(x, t, p) = \bar{r}(p)/|p|$ satisfies the assumptions (A1)–(A3). Moreover, if the medium is time-independent, that is, $g(x, t) = g(x)$, we are able to show that $\bar{r} = r$ are continuous and one-homogeneous (Section 3.8), essentially recovering the result of [18].

Once the homogenized velocity has been identified, we can prove that the half-relaxed limits satisfy the homogenized problem (Section 4). The perturbed test function method cannot be applied for free boundary problems and a different, more geometric argument based on a comparison of the solutions of the ε -problem with the solutions of the obstacle problems must be engaged. Finally, since the comparison with barriers guarantees that the limits have the correct boundary data, the comparison principle for the limit problem establishes that the upper and lower half-relaxed limits coincide with the unique solution of the homogenized problem.

Open problems. Let us conclude the introduction by mentioning some of the open problems. We have been only able to show that the homogenized velocity r is semi-continuous. However, it seems quite reasonable to expect continuity, or even Hölder continuity. As the one-dimensional case suggests in Section 3.1, this is the highest regularity one may hope for in a general situation. Another open problem is the question of a convergence rate of free boundaries in the homogenization limit. An algebraic rate was obtained in the periodic spatial homogenization case [20]. These issues will be addressed in future work.

A related open problem is the homogenization in random media when the free boundary problem does not have a variational structure, unlike [21, 22]. This problem has not yet been solved since there is no obvious subadditive quantity to which the subadditive ergodic theorem can be applied to overcome the lack of compactness of a general probability space.

Finally, in the current paper, we use the monotone propagation property of the free boundary—that is, that the wet region cannot recede—to obtain some of the important estimates. The presented method, however, seems robust enough to handle non-monotone problems such as the model of contact angle dynamics as in [19, 20]. This will also be a subject of future work.

Outline. The exposition of the proof of the homogenization result was split in a number of steps. First we give the necessary definitions of solutions in Section 2, together with some preliminary results including the comparison principle and a well-posedness result. These are used in Section 3, where we identify a candidate for the limit velocity. Finally, Section 4 is devoted to showing that the solutions

converge in the homogenization limit. In addition, the appendices contain some auxiliary results used throughout the text.

2. VISCOSITY SOLUTIONS

In this section, we briefly revisit the theory of viscosity solutions of the Hele-Shaw problem (1.1). We generalize the definitions introduced in [16, 18], and outline the proof of the comparison principle in our settings, with an aim at establishing Theorem 1.1. Let us mention that viscosity solutions seem to be the natural class of solutions of (1.1) since the problem has a maximum principle structure. Moreover, due to the possible topological changes and merging of free boundaries, the solutions might be discontinuous. The standard notion of solutions due to [11] does not apply when g in (1.1) depends on time.

Since viscosity solutions are the only weak notion of solutions used throughout the paper, we will often refer to them simply as *solutions*.

Before we give the definitions of viscosity solutions, we need to introduce some notation. For given radius $\rho > 0$ and center $(x, t) \in \mathbb{R}^n \times \mathbb{R}$, we define the open balls

$$B_\rho(x, t) := \left\{ (y, s) \in \mathbb{R}^n \times \mathbb{R} : |y - x|^2 + |s - t|^2 < \rho^2 \right\},$$

$$B_\rho(x) := \{y \in \mathbb{R}^n : |y - x| < \rho\}.$$

Let $E \subset \mathbb{R}^d$ for some $d \geq 1$. Then $USC(E)$ and $LSC(E)$ are respectively the sets of all upper semi-continuous and lower semi-continuous functions on E . For a locally bounded function u on E we define the semi-continuous envelopes $u^{*,E} \in USC(\mathbb{R}^d)$ and $u_{*,E} \in LSC(\mathbb{R}^d)$ as

$$(2.1) \quad u^{*,E} := \inf \{v \in USC(\mathbb{R}^d) : v \geq u\}, \quad u_{*,E} := \sup \{v \in LSC(\mathbb{R}^d) : v \leq u\}.$$

Note that $u^{*,E} : \mathbb{R}^d \rightarrow [-\infty, \infty)$ and $u_{*,E} : \mathbb{R}^d \rightarrow (-\infty, \infty]$ are finite on \overline{E} . We simply write u^* and u_* if the set E is understood from the context. The envelopes can be also expressed as

$$u^{*,E}(x) = \limsup_{\delta \rightarrow 0} \sup \{u(y) : y \in E, |y - x| < \delta\} \quad \text{for } x \in \overline{E}, \quad u_{*,E} = -(-u)^{*,E}.$$

It will be also useful to use a shorthand notation for the set of positive values of a given function $u : E \rightarrow [0, \infty)$, defined on a set $E \subset \mathbb{R}^n \times \mathbb{R}$,

$$\Omega(u; E) := \{(x, t) \in E : u(x, t) > 0\}, \quad \Omega^c(u; E) := \{(x, t) \in E : u(x, t) = 0\},$$

and the closure $\overline{\Omega}(u; E) := \overline{\Omega(u; E)}$. For $t \in \mathbb{R}$, the time-slices $\overline{\Omega}_t(u; E)$, $\Omega_t(u; E)$ and $\Omega_t^c(u; E)$ are defined in the obvious way, i.e.,

$$\overline{\Omega}_t(u; E) = \{x : (x, t) \in \overline{\Omega}(u; E)\}, \quad \text{etc.}$$

We shall call the boundary of the positive set in E the *free boundary* of u and denote it $\Gamma(u; E)$, i.e.,

$$\Gamma(u; E) = (\partial\Omega(u; E)) \cap E.$$

If the set E is understood from the context, we shall simply write $\Omega(u)$, etc.

For given constant $\tau \in \mathbb{R}$ we will often abbreviate

$$\{t \leq \tau\} := \{(x, t) \in \mathbb{R}^n \times \mathbb{R} : t \leq \tau\}, \quad \text{etc.}$$

It will be convenient to define viscosity solutions on general parabolic neighborhoods; we refer the reader to [38] for a more general definition.

Definition 2.1 (Parabolic neighborhood and boundary).

A nonempty set $E \subset \mathbb{R}^n \times \mathbb{R}$ is called a parabolic neighborhood if $E = U \cap \{t \leq \tau\}$ for some open set $U \subset \mathbb{R}^n \times \mathbb{R}$ and some $\tau \in \mathbb{R}$. We say that E is a parabolic neighborhood of $(x, t) \in \mathbb{R}^n \times \mathbb{R}$ if $(x, t) \in E$. Let us define $\partial_P E := \overline{E} \setminus E$, the parabolic boundary of E .

Remark 2.2. Since $\text{int } E = U \cap \{t < \tau\}$, one can observe that $\partial_P E = \partial E \setminus (U \cap \{t = \tau\})$.

Following [16, 18], we define the viscosity solutions for (1.6). In the following definitions, $Q \subset \mathbb{R}^n \times \mathbb{R}$ is an arbitrary parabolic neighborhood in the sense of Definition 2.1 and $f(x, t, p) : Q \times \mathbb{R}^n \rightarrow \mathbb{R}$ satisfies assumptions (A1)–(A3) (Section 1). We do *not* assume that f is continuous in p .

Definition 2.3. We say that a locally bounded, non-negative function $u : Q \rightarrow [0, \infty)$ is a viscosity subsolution of (1.6) on Q if

(i) (continuous expansion)

$$\overline{\Omega}(u; Q) \cap Q \cap \{t \leq \tau\} \subset \overline{\Omega}(u; Q) \cap \{t < \tau\} \quad \text{for every } \tau > 0,$$

(ii) (maximum principle)

for any $\phi \in C^{2,1}$ such that $u^* - \phi$ has a local maximum at $(x_0, t_0) \in Q \cap \overline{\Omega}(u; Q)$ in $\overline{\Omega}(u; Q) \cap \{t \leq t_0\}$, we have

(ii-1) if $u^*(x_0, t_0) > 0$ then $-\Delta\phi(x_0, t_0) \leq 0$,

(ii-2) if $u^*(x_0, t_0) = 0$ then either $-\Delta\phi(x_0, t_0) \leq 0$ or $D\phi(x_0, t_0) = 0$

$$\text{or } [\phi_t - f^*(x_0, t_0, D\phi(x_0, t_0)) |D\phi|^2](x_0, t_0) \leq 0.$$

Remark 2.4. The condition (i) in Definition 2.3 is necessary to prevent a scenario where a ‘bubble’ closes instantly; more precisely, a subsolution cannot become instantly positive on an open set surrounded by a positive phase, or cannot fill the whole space instantly.

The definition of a viscosity supersolution is similar.

Definition 2.5. We say that a locally bounded, non-negative function $u : Q \rightarrow [0, \infty)$ is a viscosity supersolution of (1.6) on Q if

(i) (monotonicity of support)

if $(\xi, \tau) \in \Omega(u_*; Q)$ then $(\xi, t) \in \Omega(u_*; Q)$ for all $(\xi, t) \in Q$, $t \geq \tau$.

(ii) (maximum principle)

for any $\phi \in C^{2,1}$ such that $u_* - \phi$ has a local minimum at $(x_0, t_0) \in Q \cap \{t \leq t_0\}$, we have

(ii-1) if $u_*(x_0, t_0) > 0$ then $-\Delta\phi(x_0, t_0) \geq 0$,

(ii-2) if $u_*(x_0, t_0) = 0$ then either $-\Delta\phi(x_0, t_0) \geq 0$ or $D\phi(x_0, t_0) = 0$

$$\text{or } [\phi_t - f_*(x_0, t_0, D\phi(x_0, t_0)) |D\phi|^2](x_0, t_0) \geq 0.$$

Remark 2.6. The assumption Definition 2.5(i) states that the support of a supersolution is nondecreasing. It is a purely technical assumption which simplifies the proof of the comparison theorem by preventing an instantaneous disappearance of a component of a positive phase. Indeed, it can be shown that all solutions of the Hele-Shaw problem have nondecreasing support. This assumption can be removed by using the tools developed in [24].

Finally, we define a viscosity solution by combining the two previous definitions.

Definition 2.7. A function $u : Q \rightarrow [0, \infty)$ is a viscosity solution of (1.6) on Q if u is both a viscosity subsolution and a viscosity supersolution on Q .

It will be useful to state an equivalent definition of viscosity solutions via a comparison with barriers, which in our case are strict classical sub- and supersolutions. This definition seems more natural, as it follows a more common pattern appearing in the treatment of free boundary problems; see [2, 6, 8, 23, 24].

Definition 2.8. For a given closed set $K \in \mathbb{R}^n \times \mathbb{R}$, we say that $\phi \in C_{x,t}^{2,1}(K)$ if there exists open set $U \supset K$ and $\psi \in C_{x,t}^{2,1}(U)$, i.e. a function twice continuously differentiable in space and once in time, such that $\phi = \psi$ on K .

Definition 2.9. Given a nonempty open set $U \subset \mathbb{R}^n \times \mathbb{R}$, function $\phi \in C(\bar{U}) \cap C_{x,t}^{2,1}(\bar{\Omega}(\phi; U))$ is called a superbarrier of (1.6) in U if there exists a positive constant c such that

- (i) $-\Delta\phi < -c$ on $\Omega(\phi; U)$, and
- (ii) $|D\phi^+| > c$ and $\phi_t^+ - f_*(\cdot, \cdot, D\phi^+) |D\phi^+|^2 < -c$ on $\Gamma(\phi; U)$.

Here $D\phi^+$ and ϕ_t^+ denote the limits of $D\phi$ and ϕ_t , respectively, on $\Gamma(\phi; U)$ from $\Omega(\phi; U)$.

Definition 2.10. Given a nonempty open set $U \subset \mathbb{R}^n \times \mathbb{R}$, function $\phi \in C(\bar{U}) \cap C_{x,t}^{2,1}(\bar{\Omega}(\phi; U))$ is called a subbarrier of (1.6) in U if there exists a positive constant c such that

- (i) $-\Delta\phi > c$ on $\Omega(\phi; U)$, and
- (ii) $|D\phi^+| > c$ and $\phi_t^+ - f^*(\cdot, \cdot, D\phi^+) |D\phi^+|^2 > c$ on $\Gamma(\phi; U)$.

A notion of *strict separation* is a crucial concept in the theory and will be used multiple times (our definition differs slightly from the one introduced in [16]).

Definition 2.11 (Strict separation). Let $E \subset \mathbb{R}^n \times \mathbb{R}$ be a parabolic neighborhood, and $u, v : E \rightarrow \mathbb{R}$ be bounded nonnegative functions on E , and let $K \subset \bar{E}$. We say that u and v are strictly separated on K with respect to E , and we write $u \prec v$ in K w.r.t. E , if

$$u^{*,E} < v_{*,E} \quad \text{in} \quad K \cap \bar{\Omega}(u; E).$$

A definition of viscosity solutions using barriers follows.

Definition 2.12. We say that a locally bounded, non-negative function $u : Q \rightarrow [0, \infty)$ is a viscosity subsolution of (1.6) on Q if for every bounded parabolic neighborhood $E = U \cap \{t \leq \tau\}$, $E \subset Q$, and every superbarrier ϕ on U such that $u \prec \phi$ on $\partial_P E$ w.r.t. E , we also have $u \prec \phi$ on \bar{E} w.r.t. E .

Similarly, a locally bounded, non-negative function $u : Q \rightarrow [0, \infty)$ is a viscosity supersolution if Definition 2.5(i) holds and for any subbarrier ϕ on U such that $\phi \prec u$ on $\partial_P E$ w.r.t. E , we also have $\phi \prec u$ on \bar{E} w.r.t. E .

Finally, u is a viscosity solution if it is both a viscosity subsolution and a viscosity supersolution.

We finish this section by stating the equivalence of the two definitions of viscosity solutions.

Proposition 2.13. The definitions of viscosity subsolutions (resp. supersolutions) in Definition 2.3 (resp. 2.5) and in Definition 2.12 are equivalent.

Before proceeding with the proof, let us state a useful property of a strict separation on time dependent sets.

Lemma 2.14. *Suppose that E is a bounded parabolic neighborhood and u, v are nonnegative locally bounded functions on E . The set*

$$(2.2) \quad \Theta_{u,v;E} := \{\tau : u \prec v \text{ in } \overline{E} \cap \{t \leq \tau\} \text{ w.r.t. } E\}$$

is open and $\Theta_{u,v;E} = (-\infty, T)$ for some $T \in (-\infty, \infty]$.

Proof. Let us choose $\tau \in \Theta_{u,v;E}$. Clearly $s \in \Theta_{u,v;E}$ for all $s \leq \tau$. Therefore, we only need to show that there exists $s > \tau$ such that $s \in \Theta_{u,v;E}$. We can assume that $\{t \leq \tau\} \cap \overline{E} \neq \emptyset$, otherwise the claim is trivial since \overline{E} is compact. Suppose that $u \not\prec v$ on $\overline{E} \cap \{t \leq s\}$ w.r.t. E for all $s > \tau$. Hence there is a sequence $(x_k, t_k) \in \overline{\Omega}(u; E)$ such that $t_k \searrow \tau$ and $u(x_k, t_k) \geq v(x_k, t_k)$. By compactness of \overline{E} , we can assume that $(x_k, t_k) \rightarrow (\xi, \tau)$, and we see that $(\xi, \tau) \in \overline{\Omega}(u; E) \cap \{t \leq \tau\}$. By semi-continuity we have $u(\xi, \tau) - v(\xi, \tau) \geq 0$, a contradiction with $u \prec v$ in $\overline{E} \cap \{t \leq \tau\}$. \square

Proof of Proposition 2.13. Let us prove the statement for subsolutions. Suppose that u is a subsolution on some parabolic neighborhood Q in the sense of Definition 2.3. By taking the semi-continuous envelope, we can assume that $u \in USC(\overline{Q})$.

Let $E = U \cap \{t \leq \tau\} \subset Q$ be a bounded parabolic neighborhood and let ϕ be a superbarrier on U such that $u \prec \phi$ on $\partial_P E$ w.r.t. E . Consider the set $\Theta_{u,\phi,E}$, introduced in (2.2). Let us define $\hat{t} = \sup \Theta_{u,\phi,E}$. Since E is bounded, we have $\hat{t} > -\infty$. To show that u is a subsolution in the sense of Definition 2.12, it is enough to show that $u \prec \phi$ in \overline{E} , which is equivalent to $\hat{t} = +\infty$.

Hence suppose that $\hat{t} < +\infty$. Lemma 2.14 implies that

- (a) $u \not\prec \phi$ in $\overline{E} \cap \{t \leq \hat{t}\}$ w.r.t. E ,
- (b) $u \prec \phi$ in $\overline{E} \cap \{t \leq s\}$ for all $s < \hat{t}$, and, finally,
- (c) $\overline{\Omega}(u; E) \cap \{t \leq \hat{t}\} \subset \overline{\Omega}(\phi; E)$ due to (b) and the continuous expansion in Definition 2.3(i).

By semicontinuity and compactness, $u - \phi$ has a local maximum at some point (ξ, σ) in $\overline{\Omega}(u; E) \cap \{t \leq \hat{t}\}$. Then (a) implies that $u(\xi, \sigma) - \phi(\xi, \sigma) \geq 0$. Consequently, since (b) holds, we must have $\sigma = \hat{t}$, and $u \prec \phi$ on $\partial_P E$ implies $(\xi, \hat{t}) \in E$. Finally, using (c) we conclude that $(\xi, \hat{t}) \in E \cap \overline{\Omega}(u; E) \cap \overline{\Omega}(\phi; E)$. Therefore either:

- $u(\xi, \hat{t}) > 0$, and that is a contradiction with Definition 2.3(ii-1); or
- $u(\xi, \hat{t}) = 0$, but then $\phi(\xi, \hat{t}) = 0$ and consequently $(\xi, \hat{t}) \in \Gamma(\phi; E)$, which yields a contradiction with Definition 2.3(ii-2).

The proof for supersolutions is analogous, but more straightforward since the support of u_* cannot decrease due to Definition 2.5(i).

The direction from Definition 2.12 to Definitions 2.3 and 2.5 is quite standard. In particular, Definition 2.3(i) can be shown by a comparison with barriers as in the proof of Corollary A.6. Furthermore, any test function ϕ such that $u - \phi$ has a local maximum at (\hat{x}, \hat{t}) and violates either of Definition 2.3(ii) can be used for a construction of a superbarrier in a parabolic neighborhood of (\hat{x}, \hat{t}) by considering $\tilde{\phi}(x, t) = (\phi(x, t) + |x - \hat{x}|^4 + |t - \hat{t}|^2 - \phi(\hat{x}, \hat{t}))_+$. Once again, the proof for supersolutions is similar. \square

Definition 2.15. *For a given function f and an nonempty parabolic neighborhood $Q \subset \mathbb{R}^n \times \mathbb{R}$, we define the following classes of functions:*

- $\overline{\mathcal{S}}(f, Q)$, the set of all viscosity supersolutions of the Hele-Shaw problem (1.6) on Q ;

- $\underline{\mathcal{S}}(f, Q)$, the set of all viscosity subsolutions of (1.6) on Q ;
- $\mathcal{S}(f, Q) = \overline{\mathcal{S}}(f, Q) \cap \underline{\mathcal{S}}(f, Q)$, the set of all viscosity solutions of (1.6) on Q .

We have the following obvious inclusions:

Lemma 2.16. *Let $Q \subset \mathbb{R}^n \times \mathbb{R}$ be a nonempty parabolic neighborhood. If $f \leq g$ for given functions $f, g : Q \times \mathbb{R}^n \rightarrow (0, \infty)$ then*

$$\overline{\mathcal{S}}(f, Q) \supset \overline{\mathcal{S}}(g, Q) \quad \text{and} \quad \underline{\mathcal{S}}(f, Q) \subset \underline{\mathcal{S}}(g, Q).$$

Finally, we observe that subsolutions and supersolutions can be “stitched” together.

Lemma 2.17. *Let $Q_1, Q_2 \subset \mathbb{R}^n \times \mathbb{R}$ be two parabolic neighborhoods such that $Q_1 \subset Q_2$ and let $u_i \in \underline{\mathcal{S}}(f, Q_i)$, $i = 1, 2$, for some $f(x, t, p) : Q_2 \times \mathbb{R}^n \rightarrow (0, \infty)$. We define*

$$u = \begin{cases} \max(u_1, u_2) & \text{in } Q_1, \\ u_2 & \text{in } Q_2 \setminus Q_1. \end{cases}$$

If $u_1^{*, Q_1} \leq u_2^{*, Q_2}$ on $(\partial_P Q_1) \cap Q_2$ and

$$(2.3) \quad \overline{\Omega}(u_1; Q_1) \cap \partial_P Q_1 \subset \overline{\Omega}(u_2; Q_2)$$

then $u \in \underline{\mathcal{S}}(f, Q_2)$.

Let now $v_i \in \overline{\mathcal{S}}(f, Q_i)$, $i = 1, 2$, and we define

$$v = \begin{cases} \min(v_1, v_2) & \text{in } Q_1, \\ v_2 & \text{in } Q_2 \setminus Q_1. \end{cases}$$

If $(v_1)_{*, Q_1} \geq (v_2)_{*, Q_2}$ on $(\partial_P Q_1) \cap Q_2$, then $v \in \overline{\mathcal{S}}(f, Q_1)$.

Proof. Let us show the proof for subsolutions. We use Definition 2.3. Clearly $\overline{\Omega}(u; Q_2) = \overline{\Omega}(u_1; Q_1) \cup \overline{\Omega}(u_2; Q_2)$ and therefore, by (2.3), we have

$$\begin{aligned} \overline{\Omega}(u; Q_2) &= \overline{\Omega}(u_1; Q_1) \cup \overline{\Omega}(u_2; Q_2) \\ &= (\overline{\Omega}(u_1; Q_1) \cap Q_1) \cup (\overline{\Omega}(u_1; Q_1) \cap \partial_P Q_1) \cup \overline{\Omega}(u_2; Q_2) \\ &= (\overline{\Omega}(u_1; Q_1) \cap Q_1) \cup \overline{\Omega}(u_2; Q_2). \end{aligned}$$

Thus Definition 2.3(i) is clear. To prove (ii), suppose that $u^* - \phi$ has a local maximum at $(\hat{x}, \hat{t}) \in \overline{\Omega}(u; Q_2)$ in $\overline{\Omega}(u; Q_2) \cap \{t \leq \hat{t}\}$. We observe that, by hypothesis, $u^* = \max(u_1^{*, Q_1}, u_2^{*, Q_2})$ on Q_1 and $u^* = u_2^{*, Q_2}$ on $Q_2 \setminus Q_1$. Consequently, if $(\hat{x}, \hat{t}) \in Q_1$ then $u_1^* - \phi$ and/or $u_2^* - \phi$ have a local maximum in the appropriate set, and if $(\hat{x}, \hat{t}) \in Q_2 \setminus Q_1$ then $u_2^* - \phi$ has a local maximum. Definition 2.3(ii) follows from this observation.

The proof for supersolutions is straightforward since there is no restriction on their supports. \square

2.1. Comparison principle. A crucial result for a successful theory viscosity solutions is a comparison principle. In this section we establish the comparison principle between strictly separated viscosity solutions on an arbitrary parabolic neighborhood. We only give an outline of the proof and point out the differences from previous results in [16, 18, 24].

Theorem 2.18. *Let Q be a bounded parabolic neighborhood and let either $f(x, t, p) = g(x, t)$ or $f(x, t, p) = f(p)$ that satisfy the assumptions (A1)–(A3). If $u \in \underline{\mathcal{S}}(f, Q)$ and $v \in \overline{\mathcal{S}}(f, Q)$ such that $u \prec v$ on $\partial_P Q$ w.r.t. Q then $u \prec v$ in \overline{Q} w.r.t. Q .*

Proof. We can assume that $u \in USC(\overline{Q})$ and $v \in LSC(\overline{Q})$. Similarly to the proof of Proposition 2.13, we consider the set $\Theta_{u,v;Q}$, defined in (2.2), and set $\hat{t}_0 = \sup \Theta_{u,v;Q} > -\infty$. Then $u \prec v$ on \overline{Q} w.r.t. Q is equivalent to $\hat{t}_0 = +\infty$.

Therefore suppose that $\hat{t}_0 < +\infty$. Let us introduce the sup- and inf-convolutions

$$\begin{aligned} Z(x, t) &= \left(1 + \frac{3Lr}{m}\right) \sup_{\Xi_r(x,t)} u(x, t) \\ W(x, t) &= \left(1 - \frac{3Lr}{m}\right) \inf_{\Xi_{r-\delta t}} v(x, t), \end{aligned}$$

with $r > 0$ and $\delta = \frac{r}{2T}$ where $T := \inf \{\tau : Q \subset \{t \leq \tau\}\}$. The open set $\Xi_r(x, t)$ is defined as

$$\Xi_r(x, t) = \left\{ (y, s) : (|y - x| - r)_+^2 + |s - t|^2 < r^2 \right\}.$$

We refer the reader to [16, 24] for a detailed discussion of the properties of Z and W . In particular, Z and W are well-defined on the closure of the parabolic neighborhood Q_r ,

$$Q_r = \{(x, t) : \Xi_r(x, t) \subset Q\},$$

$Z \in USC(\overline{Q}_r)$ and $W \in LSC(\overline{Q}_r)$.

Moreover, $Z \in \underline{\mathcal{S}}(f, Q_r)$ and $W \in \overline{\mathcal{S}}(f, Q_r)$. This is straightforward if $f(x, t, p) = f(p)$. However, if f depends on (x, t) as in $f(x, t, p) = g(x, t)$ we have to use Proposition B.1 with factors $a = (1 \pm 3Lr/m)$, yielding the same conclusion.

Finally, and this is the main motivation for the choice of the convolutions, both Z and W have an important interior/exterior ball property of their level sets both in space for each time and in spacetime; see [24].

Since $u \prec v$ on $\partial_P Q$ w.r.t. Q , we claim that if r is chosen small enough we have $Z \prec W$ on $\partial_P Q_r$ w.r.t. Q_r . Suppose that this is not true for any $r > 0$. Let $r_k \rightarrow 0$ as $k \rightarrow \infty$. There exists a sequence of $(x_k, t_k) \in \partial_P Q_{r_k} \cap \overline{\Omega}(Z_{r_k}; Q_r)$ such that $Z_{r_k}(x_k, t_k) \geq W_{r_k}(x_k, t_k)$. By definition of Z and W and semi-continuity, there exist $(\xi_k, \sigma_k), (\zeta_k, \tau_k) \in \Xi_{r_k}(x_k, t_k)$ such that $(\xi_k, \sigma_k) \in \overline{\Omega}(u; Q)$ and $u(\xi_k, \sigma_k) \geq v(\zeta_k, \tau_k)$. By compactness of $\overline{\Omega}(u; Q)$, for a subsequence, still denoted by k , (ξ_k, σ_k) converges to a point (ξ, σ) , which must lie on $\partial_P Q \cap \overline{\Omega}(u; Q)$, and a simple observation shows that $(\zeta_k, \tau_k) \rightarrow (\xi, \sigma)$. This is a contradiction with the semi-continuity of u and v and the fact $u(\xi, \sigma) < v(\xi, \sigma)$.

We claim that W is a supersolution of (1.6) with $f^\delta(x, t, p) = f(x, t, p) + \delta|p|$. Indeed, suppose that ϕ is a strict classical supersolution and $W - \phi$ has a strict local maximum 0 at (ξ, σ) in $\{t \leq \sigma\}$. Then there is $(y, s) \in \Xi_{r-\delta\sigma}$ and $v(\xi + y, \sigma + s) = W(\xi, \sigma)$. We observe that due to the fact that the ‘radius’ of the set $\Xi_{r-\delta t}$ is decreasing, we can conclude that

$$(2.4) \quad \phi(x, t) \leq W(x, t) \leq \left(1 - \frac{3Lr}{m}\right) v(x + y + z, t + s),$$

for $(y + z, s) \in \Xi_{r-\delta t}$, in particular if $z \in B_{\delta(\sigma-t)}(0)$. Let us set $\nu = -\frac{D\phi}{|D\phi|}(\xi, \sigma)$, the unit outer normal to $\Omega_\sigma(\phi)$ at ξ . Observe that we can take $z = -\delta(\sigma - t)\nu$, and

we can rewrite (2.4) as

$$\psi(x, t) := \phi(x - y + \delta(\sigma - t)\nu, t) \leq \left(1 - \frac{3Lr}{m}\right) v(x, t).$$

By the definition of supersolutions, we must have at $(\zeta, \mu) = (\xi + y, \sigma + s)$

$$\begin{aligned} [\phi_t - \delta |D\phi|](\xi, \sigma) &= \psi_t(\zeta, \mu) \\ &\geq \left(1 - \frac{3Lr}{m}\right)^{-1} f_* \left(\zeta, \mu, \left(1 - \frac{3Lr}{m}\right)^{-1} D\psi(\zeta, \mu) \right) |D\psi(\zeta, \mu)|^2. \end{aligned}$$

If $f(x, t, p) = g(x, t)$, then we use the assumptions (A1)–(A2) that imply $g(\zeta, \mu) \geq \left(1 - \frac{3Lr}{m}\right) g(\xi, \sigma)$, and if $f(x, t, p) = f(p)$ we use the assumption (A3) to conclude that

$$[\phi_t - \delta |D\phi|](\xi, \sigma) \geq f_*(\xi, \sigma) |D\phi(\xi, \sigma)|^2.$$

Therefore $W \in \overline{\mathfrak{S}}(f^\delta, Q_r)$. This will provide one perturbation later in the proof.

We proceed with the proof of the comparison theorem. Let us define

$$\hat{t} = \sup \Theta_{Z, W; Q_r}$$

and observe that $\hat{t} \leq \hat{t}_0$.

As in the proof Proposition 2.13, we have $\overline{\Omega}(Z) \cap \{t \leq \hat{t}\} \subset \overline{\Omega(W) \cap \{t < \hat{t}\}}$, and there exists $(\xi, \hat{t}) \in Q_r \cap \overline{\Omega}(Z) \cap \overline{\Omega}(W)$ such that (ξ, \hat{t}) is a point of maximum of $Z - W$ in $\overline{\Omega}(Z) \cap \{t \leq \hat{t}\}$, and $Z(\xi, \hat{t}) - W(\xi, \hat{t}) \geq 0$. Since the set $\Omega_i^c(W)$ has the interior ball property of radius $r/2$, we can show by a barrier argument that in fact $Z = 0$ on $\Omega^c(W) \cap \{t \leq \hat{t}\}$, see [16, 24] for details. Most of the arguments are analogous since $Z \in \underline{\mathfrak{S}}(m, Q_r)$, $W \in \underline{\mathfrak{S}}(M, Q_r)$. This together with the assumption that $\Omega(W)$ cannot decrease in time due to Definition 2.5(i) yields $\Omega(Z) \cap \{t \leq \hat{t}\} \subset \Omega(W)$.

Let us denote U the connected component of $\Omega_i(W)$ that contains (ξ, \hat{t}) . Suppose that $Z(\xi, \hat{t}) > 0$. Since $Z(\cdot, \hat{t})$ is subharmonic in the (open) set U , we must have $\overline{\Omega}_i(Z) \cap \partial U \neq \emptyset$. Hence there is a point on $\partial U \cap \partial \Omega_r$ where $0 < Z < W$, by the strict separation. Consequently, a straightforward argument of adding a small harmonic function to Z , using the comparison principle for subharmonic and superharmonic functions, yields $Z(\xi, \hat{t}) < W(\xi, \hat{t})$, a contradiction.

Therefore $Z(\xi, \hat{t}) = W(\xi, \hat{t}) = 0$ and $Z \leq W$ on $\{t \leq \hat{t}\}$. We arrive at the last step of the comparison theorem, where we need to find some weak ordering of gradients and the corresponding velocities, as in [24].

Hopf's lemma implies that the "gradients" of Z and W are strictly ordered. Indeed, let ν be the unique unit outer normal of $\Omega_i(Z)$ and $\Omega_i(W)$ at ξ . Since $Z(\cdot, \hat{t})$ is subharmonic in U and W is superharmonic in U , and there is an interior ball of $\Omega_i(Z)$ at ξ , and $W(\cdot, \hat{t}) \not\equiv Z(\cdot, \hat{t})$ in U , we can apply the Hopf's lemma in U at $\xi \in \partial U$ and conclude that

$$(2.5) \quad \alpha := \limsup_{h \rightarrow 0} \frac{Z(\xi - h\nu, \hat{t})}{h} < \liminf_{h \rightarrow 0} \frac{W(\xi - h\nu, \hat{t})}{h} =: \beta.$$

The assumption on f implies

$$(2.6) \quad f^*(\xi, \hat{t}, -\alpha\nu) |-\alpha\nu| \leq f^*(\xi, \hat{t}, -\beta\nu) |-\beta\nu|.$$

It can be showed as in [16] (see also [23, 24]) that the sets $\overline{\Omega}(Z)$ and $\Omega^c(W)$ have interior space-time balls at (ξ, \hat{t}) with space-time slopes that represent the velocities

of the free boundaries at (ξ, \hat{t}) which we denote as m_Z and m_W . A barrier argument yields $0 < \delta \leq m_W \leq m_Z < \infty$.

If $f(x, t, p) = g(x, t)$ then we proceed as in [24]: there are points (ξ_u, t_u) and (ξ_v, t_v) on the free boundaries of u and v , respectively, such that $(\xi_u, t_u), (\xi_v, t_v) \in \Xi_r(\xi, \hat{t})$. A barrier argument similar to the one in [24, Lemma 3.23] yields

$$\begin{aligned} m_Z &\leq g(\xi_u, t_u) \limsup_{h \rightarrow 0} \frac{u(\xi_u - hv, t_u)}{h} \leq \frac{g(\xi_u, t_u)}{1 + \frac{3Lr}{m}} \limsup_{h \rightarrow 0} \frac{Z(\xi - hv, \hat{t})}{h} \\ &\leq g(\xi, \hat{t}) |-\alpha\nu|. \end{aligned}$$

The argument is analogous when $f(x, t, p) = f(p)$. In either case, we use (2.5) or (2.6) and we arrive at

$$m_W \leq m_Z \leq f^*(\xi, \hat{t}, -\alpha\nu) |-\alpha\nu| \leq f_*(\xi, \hat{t}, -\beta\nu) |-\beta\nu| \leq m_W - \delta,$$

a contradiction. Therefore Z and W cannot cross, which in turns implies that u and v cannot cross and hence $u \prec v$ in \bar{Q} w.r.t. Q . \square

2.2. Well-posedness. We will use the comparison principle, Theorem 2.18, to establish the well-posedness of the Hele-Shaw-type problem (1.6), Theorem 1.1. Since Theorem 2.18 requires strictly separated boundary data, it does not provide uniqueness of solutions for general boundary data. However, it is possible to prove uniqueness when the boundary data features some special structure; see [16, 18]. We show the well-posedness theorem for boundary data strictly increasing in time.

Proof of Theorem 1.1. In this proof we return to the parabolic cylinder $Q = \Omega \times (0, T]$ introduced in Section 1.

Bounded support. First observe that exactly one of Ω and Ω_0 is bounded and that both $\partial\Omega$ and $\partial\Omega_0$ are bounded.

Let u be a bounded viscosity solution on Q with the correct boundary data. We will show that $\bar{\Omega}(u; Q) \subset B_R(0) \times [0, T]$ for some large R . If Ω is bounded, this is obvious. Therefore assume that Ω is unbounded and therefore Ω_0 is bounded. Let $d = \text{diam } \Omega_0 < \infty$, set $K := \sup_Q u < \infty$ (by assumption) and recall that M is the upper bound on f . We define $\mu := 2\sqrt{2nKM T}$ and $R := d + \mu$. The claim now follows from Corollary A.6 applied with $E = \bar{\Omega}_0^c$. We set $\Sigma := B_R(0) \cap \Omega$, which is clearly bounded. Note that $\partial\Omega_0 \subset B_R(0)$.

Barriers. Before proceeding with the proof of existence, let us first construct barriers for the Perron's method. We write $\Omega_0^\rho := \Omega_0 + B_\rho$, $\Omega_0^0 := \Omega_0$. Since $\partial\Omega_0$ is $C^{1,1}$ and compact, it is a set of a positive reach and there exists $\rho_0 > 0$ such that $\partial\Omega_0^\rho \in C^{1,1}$ and $\partial\Omega_0^\rho \subset B_R(0)$ for $\rho \in [0, \rho_0]$.

For $\rho \in [0, \rho_0]$, $t \in [0, T]$, let us define $u_t^\rho \in C(\Omega)$, such that $\Delta u_t^\rho = 0$ in $\Omega_0^\rho \cap \Omega$ with boundary data $u_t^\rho(x) = \psi(x, t)$ on $\partial\Omega$, $u_t^\rho = 0$ in $\Omega \setminus \Omega_0^\rho$.

Since $\partial\Omega \subset B_R(0)$ and $\partial\Omega \in C^{0,1}$, we can extend ψ to $C(\mathbb{R}^n \times [0, T])$ as $\Delta\psi = 0$ in $(B_R(0) \setminus \partial\Omega) \times [0, T]$ and $\psi = 0$ in $B_R^c(0) \times [0, T]$. Observe that $\psi \in \mathcal{S}(M, \Sigma \times [0, T])$.

Now we introduce the barriers at the initial time $t = 0$. By the Hopf's lemma and comparison for harmonic functions, and the fact that ψ is bounded from above and from zero on $\partial\Omega \times [0, T]$, there exists a constant $c > 0$ such that $0 < c^{-1} \leq |D(u_t^\rho)^+| \leq c$ on $\partial\Omega_0^\rho$ for all $\rho \in [0, \rho_0/2]$, $t \in [0, T]$. Therefore we can find $\omega > 0$ large enough so that $U(x, t) := u_t^{\omega^{-1}t}(x)$ is a subsolution $U \in \underline{\mathcal{S}}(m/2, \Omega \times (0, \delta])$, and $V(x, t) := u_t^{\omega t}(x)$ is a supersolution $V \in \bar{\mathcal{S}}(M, \Omega \times (0, \delta])$

for some $\delta \in (0, \rho_0/\omega)$. Moreover, $U, V \in C(\Omega \times [0, \delta])$. Let us extend U and V by defining $U(x, t) = u_t^{\omega^{-1}\delta}(x)$ and $V(x, t) = \psi(x, t)$ for $t \in (\delta, T]$ (recall the extension of ψ above). It is straightforward to show that $U \in \underline{\mathcal{S}}(m/2, Q)$, $V \in \overline{\mathcal{S}}(M, Q)$ and that U and V satisfy the boundary condition on $\partial_P Q$ in the correct sense. That is, $U^* = U_* = V^* = V_* = \psi$ on $\partial\Omega \times [0, T]$ and $U^* = U_* = V^* = V_* = u_0^0$.

Uniqueness. Let u and v be two solutions with the correct boundary data.

We first check the value of u and v at $t = 0$. In general, for any solution u , we have $u^*(\cdot, t)$ is subharmonic in Ω and $u_*(\cdot, t)$ is superharmonic in $\Omega_t(u_*; Q)$ (see [24] for a detailed proof). Thus a simple argument, using the facts that a support of a subsolution must expand continuously, and that a support of a supersolution cannot decrease, shows that $u_*(\cdot, 0) = u^*(\cdot, 0) = v_*(\cdot, 0) = v^*(\cdot, 0) = u_0^0$ on Ω .

Now we observe that the support of a supersolution u is strictly increasing at $t = 0$, i.e., $\overline{\Omega}_0(u_*; \overline{Q}) = \overline{\Omega} \cap \Omega_0 \subset \Omega_t(u_*; \overline{Q})$ for any $t > 0$. This follows from the comparison with a perturbation of U . Specifically, fix a point $\zeta \in B_R(0) \setminus \overline{\Omega}$ and define $U_\kappa(x, t) = (U(x, t) + \kappa|x - \zeta|^2 - 4\kappa R^2)_+$. Clearly $U_\kappa \rightrightarrows U$ uniformly on \overline{Q} as $\kappa \rightarrow 0$ and U_κ is a subbarrier on $\Omega \times (0, \delta]$ for $\kappa > 0$ small enough (recall that $U \in \underline{\mathcal{S}}(m/2, Q)$ with $m/2 < m$). Moreover, the support of U strictly increases in the above sense. Since $U_\kappa \prec u$ on $\partial_P Q$ for any $\kappa > 0$, we conclude that the support of u_* must strictly increase by sending $\kappa \rightarrow 0$.

Because the comparison principle requires strictly separated boundary data, we have to perturb the solutions and create this separation. We use a perturbation in time. Such perturbation is more involved because of the time-dependence of $f(x, t, p)$, and in that case we will use the nonlinear perturbation from Appendix B. Thus let

- $\theta_\eta(t) = t - \eta$ if $f(x, t, p) = f(p)$, or let
- $\theta_\eta(t)$ be the function constructed in Section B.1 with $\tau = -\eta$ and $\rho = 0$ if $f(x, t, p) = f(x, t)$.

Observe that, in either case, θ_η is well-defined on $[0, \infty)$, $\theta'_\eta \in (0, 1]$ and therefore it is invertible, $\theta_\eta(0) = -\eta$ and $\theta_\eta(t) \rightarrow t$ as $\eta \rightarrow 0$, locally uniformly in t .

We define $w(x, t) = u(x, \theta_\eta(t))$. As we showed above, the support of v_* is strictly increasing at $t = 0$. Furthermore, the boundary data ψ on $\partial\Omega_0$ is strictly increasing in time and $\theta_\eta(t) < t$ for all $t \geq 0$. Since $\theta_\eta^{-1}(0) > 0$, let us define $Q_\eta := \Sigma \times (\theta_\eta^{-1}(0), T]$, where R was chosen above, and we have for all small $\eta > 0$

$$w \prec v \quad \text{on } \partial_P Q_\eta.$$

The comparison theorem 2.18 then implies $w \prec v$ on \overline{Q}_η , and thus $u^*(x, \theta_\eta(t)) \leq v_*(x, t)$. By sending $\eta \rightarrow 0$, we conclude $u_* \leq (u^*)_* \leq v_*$ on Q . Furthermore, clearly

$$u^*(x, t) \leq v_*(x, \theta_\eta^{-1}(t)) \quad \text{on } \overline{Q} \times [0, \theta_\eta^{-1}(T)],$$

which yields $u^* \leq (v_*)^* \leq v^*$ on Q after sending $\eta \rightarrow 0$. The argument repeated with u and v swapped implies that the solution is unique.

Existence. We use the classical Perron-Ishii method; see [10, 16]. Let

$$u = \sup \{w \in \underline{\mathcal{S}}(f, Q) : w \leq V\}, \quad v = \inf \{w \in \overline{\mathcal{S}}(f, Q) : w \geq U\}.$$

Clearly $u \geq U$ and $v \leq V$. Since $(V^*)_* = V$ and $(U_*)^* = U$, a standard argument yields that $u, u^*, v, v_* \in \mathcal{S}(f, Q)$. Moreover, because of the continuity of U and V at $\partial_P Q$ we conclude that all these solutions have the correct boundary data.

Therefore, by uniqueness, if $w \in \mathcal{S}(f, Q)$ with the correct boundary data, we have $w_* = u_* = v_* = (u^*)_*$ and $w^* = u^* = v^* = (v_*)^*$, and thus

$$(w^*)_* = w_*, \quad (w_*)^* = w^*,$$

which completes the proof. \square

The last step of the previous proof provides the following regularity.

Corollary 2.19. *Let u be the unique solution of the problem (1.6) with data satisfying the assumptions of Theorem 1.1. Then*

$$(2.7) \quad (u_*)^* = u^* \quad \text{and} \quad (u^*)_* = u_*.$$

and

$$(2.8) \quad \bar{\Omega}(u_*; Q) = \bar{\Omega}(u; Q) = \bar{\Omega}(u^*; Q).$$

Proof. As was indicated above, (2.7) was shown in the last step of the proof of Theorem 1.1.

Let us point out that $\bar{\Omega}(u) = \bar{\Omega}(u^*)$ is always true. It is enough to prove that $\bar{\Omega}(u^*) \subset \bar{\Omega}(u_*)$ since the other direction is obvious. Suppose that $(\xi, \sigma) \in \bar{\Omega}(u^*) \setminus \bar{\Omega}(u_*)$. Then there exists $\delta > 0$ such that $B_\delta(\xi, \sigma) \cap \bar{\Omega}(u_*) = \emptyset$ and therefore $u_* = 0$ on $B_\delta(\xi, \sigma)$. However, Corollary 2.19 implies that $u^* = (u_*)^* = 0$ on $B_\delta(\xi, \sigma)$, a contradiction with the choice of (ξ, σ) . \square

3. IDENTIFICATION OF THE LIMIT PROBLEM

3.1. Homogenization in one dimension. Let us first explore the situation in one dimension. For simplicity, we take $\Omega = (0, \infty)$, $\Omega_0 = (-\infty, x_0)$ for some $x_0 > 0$. Furthermore, $g(x, t) : \mathbb{R} \times \mathbb{R} \rightarrow [m, M]$ is a positive, L -Lipschitz, \mathbb{Z}^2 -periodic function. The boundary data on $\partial\Omega = \{0\}$ is $\psi(0, t) = \psi(t) > 0$, $\psi \in C([0, \infty))$.

We observe that, in this simple settings, $\Omega(u; \Omega) = (0, x(t))$, where $x(t)$ is the position of the free boundary $\partial\Omega(u)$ at time t , i.e., $\partial\Omega(u) = \{(x(t), t) : t \geq 0\}$. Therefore the normal velocity is simply $V_\nu = x' = x_t$. Moreover, the harmonic function $u(\cdot, t)$ must be linear with slope $|Du(x(t), t)| = \psi(t)/x(t)$. Therefore, for every fixed $\varepsilon > 0$, the free boundary condition in (1.1) is equivalent to a simple ODE:

$$(3.1) \quad \begin{cases} x'_\varepsilon(t) = g\left(\frac{x_\varepsilon(t)}{\varepsilon}, \frac{t}{\varepsilon}\right) \frac{\psi(t)}{x_\varepsilon(t)}, \\ x(0) = x_0. \end{cases}$$

Due to the result of [14] (see also [27]), $x_\varepsilon \rightarrow x$ locally uniformly, where $x(t)$ is the solution of

$$\begin{cases} x'(t) = f(x(t), t), \\ x(0) = x_0. \end{cases}$$

The arguments in this paper then imply that f is of the form

$$f(x(t), t) = r(\psi(t)/x(t)).$$

The function r can be found numerically by scaling (3.1) and thus solving

$$(3.2) \quad \begin{cases} x'(t) = qg(x(t), t), \\ x(0) = x_0. \end{cases}$$

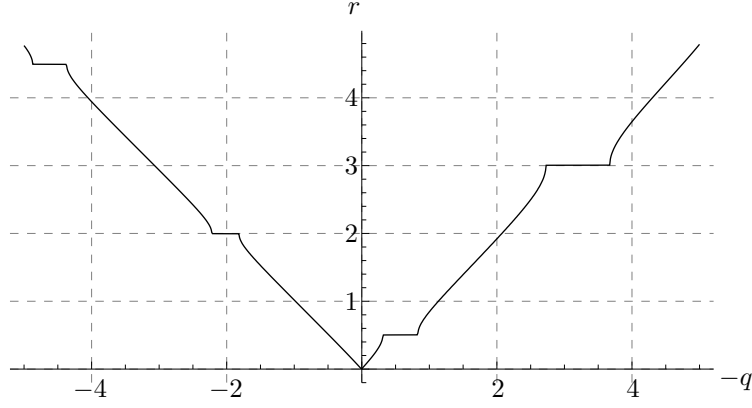


FIGURE 2. The graph of $r(q)$ for $g(x, t) = \sin(2\pi(x-3t)) \sin(2\pi(2t+x)) + 11/10$

in some $t \in [0, T]$ for T large. Then $r(q)$ can be approximated by $x(T)/T$. The limit $\lim_{T \rightarrow \infty} \frac{x(T)}{T}$ is independent of x_0 since we can squeeze any solution for $x_0 \neq 0$ between $x(t)$ and $x(t) + 1$ using the comparison principle.

Consider the medium

$$g(x, t) = \sin^2(\pi(x - t)) + 1.$$

Then for $q \in [1/2, 1]$, $x(t) = t + \pi^{-1} \arccos \sqrt{q^{-1} - 1}$ is a solution of (3.2). Thus we see that $r(q) = 1$ for $q \in [1/2, 1]$. If the boundary propagates in the opposite direction, i.e., when $g = \sin^2(\pi(-x - t)) + 1$, this effect does not appear and $r(q)$ is strictly increasing. The situation can be even more complicated; see Figure 2 for an example of this phenomenon.

3.2. Obstacle problem. In this section, we introduce a proxy obstacle problem that will be later used to identify the candidate for the homogenized velocity $r(Du)$ in Theorem 1.2.

3.2.1. Planar solutions. The homogenized Hele-Shaw problem with boundary velocity law $V_\nu = r(Du)$ admits simple traveling wave solutions. We call them planar solutions since each is a positive part of a linear function with gradient $q \in \mathbb{R}^n \setminus \{0\}$ moving in the direction $\nu := -q/|q|$ with the normal velocity $V_\nu = r(q)$. These solutions will serve as the obstacles for the obstacle problem that will be used to find a candidate of the homogenized free boundary velocity $r(q)$.

For any $q \in \mathbb{R}^n \setminus \{0\}$ and $r > 0$ we define

$$P_{q,r}(x, t) := (|q|rt + x \cdot q)_+ = |q|(rt - x \cdot \nu)_+, \quad \nu := \frac{-q}{|q|},$$

where $(s)_+ := \max(s, 0)$ is the positive part. We interpret r as the velocity of the free boundary

$$\Gamma(P_{q,r}) = \{(x, t) : rt = x \cdot \nu\}$$

and $q \neq 0$ as the gradient $q = DP_{q,r}$. Thus ν is the unit outer normal vector to the free boundary at every time.

The following observation is a trivial consequence of the nondegeneracy assumption (1.2).

Proposition 3.1. $P_{q,r} \in \underline{\mathcal{S}}(g^\varepsilon)$ if

$$r \leq m |q|$$

and $P_{q,r} \in \overline{\mathcal{S}}(g^\varepsilon)$ if

$$r \geq M |q|.$$

Proof. Use (1.2). □

In view of Proposition 3.1, we will often impose the following restriction on the values of r and q throughout the paper:

$$(3.3) \quad q \neq 0, \quad m \leq \frac{r}{|q|} \leq M.$$

Proposition 3.2. For given $q \neq 0$ and $r > 0$ we have

$$P_{q,r}(x - y, t - \tau) \leq P_{q,r}(x, t) \quad \text{if and only if} \quad y \cdot \nu \leq r\tau,$$

and

$$P_{q,r}(x - y, t - \tau) \geq P_{q,r}(x, t) \quad \text{if and only if} \quad y \cdot \nu \geq r\tau.$$

Proof. A simple calculation yields

$$P_{q,r}(x - y, t - \tau) = |q| (r(t - \tau) - (x - y) \cdot \nu) = P_{q,r}(x, t) + |q| (y \cdot \nu - r\tau)$$

and the conclusion immediately follows. □

3.2.2. Domain. In this section, we construct a class of spacetime cylinders Q_q , which will serve as the domains for the obstacle problem used later in the paper. This particular choice of geometry will let us control how fast the solution of the obstacle problem detaches from the obstacle at the lateral boundary of the domain. In fact, this is the main motivation for this technical construction. The ability to control the boundary behavior will be necessary for the extension of the monotonicity property of the obstacle problem beyond the straightforward inclusion of domains. One of the main consequences of this extra monotonicity is the cone flatness property, Proposition 3.20.

Throughout the rest of this section, we fix $q \in \mathbb{R}^n \setminus \{0\}$ and $r > 0$ and let $\nu = \frac{-q}{|q|}$. For simplicity, we shall denote

$$\Gamma_t = \Gamma_t(P_{q,r}).$$

Since the homogenization result is trivial when $m = M$, we may assume throughout the rest of the paper that $0 < m < M$. We define angles $\theta, \theta^+ \in (0, \frac{\pi}{2})$, and $\theta^- \in (\theta, \frac{\pi}{2})$ as

$$(3.4a) \quad \theta = \arccos \sqrt{\frac{m}{M}}$$

$$(3.4b) \quad \theta^+ = \frac{\pi}{2} - \theta$$

$$(3.4c) \quad \theta^- = \frac{\pi}{2} + \theta - \varphi^-, \quad \text{where } \varphi^- := \arccos \frac{m}{M}.$$

Note that indeed $\theta^- \in (\theta, \frac{\pi}{2})$ since $\varphi^- \in (\theta, \frac{\pi}{2})$ by $\cos \varphi^- = \frac{m}{M} < \sqrt{\frac{m}{M}} = \cos \theta$.

Remark 3.3. Since the position of vertices V_t^\pm of cones C_t^\pm clearly depend linearly on time, we can extend their definition to all $t \in \mathbb{R}$ as

$$V_t^\pm = V_0^\pm + r_V^\pm t \nu \quad t \in \mathbb{R},$$

where r_V^\pm are constants and we call them the velocity of V_t^\pm . Note that $V = V_{-r-1}^+ = V_{-r-1}^- = -\nu$ due to (3.5) and the fact that $V = -\nu \in \Gamma_t(P_{q,r})$ at $t = -\frac{1}{r}$. The velocities of r_V^\pm can be found explicitly. Since the cones share a base, (3.5), we infer that (see Figure 3)

$$(r_V^+ - r) \tan \theta^+ = r \tan \theta = (r - r_V^-) \tan \theta^-.$$

Since $\theta^+ = \frac{\pi}{2} - \theta$, $\tan \theta^+ = \frac{1}{\tan \theta}$ holds and we have

$$r_V^+ = r(1 + \tan^2 \theta) = \frac{r}{\cos^2 \theta} = \frac{M}{m} r > r.$$

We can similarly express the velocity r_V^- as

$$r_V^- = \left(1 - \frac{\tan \theta}{\tan \theta^-}\right) r = c_{\frac{M}{m}} r \in (0, r).$$

The reason for this choice of domain Q_q is the result in Lemma 3.7 below. But we begin with the following geometrical result. Let us introduce the set of all directions of rays in $\partial\Omega_q$:

$$(3.6) \quad \Xi = \{\xi \in \mathbb{R}^n : |\xi| = 1, \quad \xi \cdot \nu = \cos \theta\}.$$

Proposition 3.4. *For any given $x \in \partial\Omega_q$, $x \neq V$, there exists unique $\xi \in \Xi$ such that $x \in L_\xi \subset \partial\Omega_q$, where L_ξ is the ray*

$$(3.7) \quad L_\xi := \{\sigma \xi + V : \sigma \geq 0\}.$$

Furthermore, there exist unique ξ^- and ξ^+ such that $|\xi^\pm| = 1$,

$$L_{\xi^\pm, t} := \{\sigma \xi^\pm + V_t^\pm : \sigma \geq 0\} \subset \partial C_t^\pm$$

and

$$L_{\xi^\pm, t} \cap L_\xi = \{I_{\xi, t}\} \quad \text{for all } t \geq 0$$

for some point $I_{\xi, t}$. Let η_ξ^\pm be the unit normal to ∂C_t^\pm on $L_{\xi^\pm, t}$ with $\eta^\pm \cdot \nu > 0$. There exist unique constants μ^\pm , r^\pm and T^\pm depending only on (q, r, m, M) , and independent of ξ , such that

$$(3.8) \quad \Gamma_t(R_\xi^\pm) \cap \partial C_t^\pm = L_{\xi^\pm, t} \quad \text{for all } t \geq 0,$$

$$(3.9) \quad R_\xi^\pm = P_{q,r} \quad \text{on } L_\xi \times [0, \infty)$$

where

$$(3.10) \quad R_\xi^\pm(x, t) := P_{-\mu^\pm \eta_\xi^\pm, r^\pm}(x, t - T^\pm).$$

Proof. The existence of ξ, ξ^+, ξ^- is straightforward.

μ^\pm and r^\pm can be expressed using the following geometric considerations.

By definition,

$$\cos \theta = \xi \cdot \nu.$$

We introduce $\varphi^\pm \in (0, \pi/2)$ via $\cos \varphi^\pm = \xi \cdot \eta^\pm$ and note that

$$(3.11) \quad \theta^+ = \frac{\pi}{2} + \varphi^+ - \theta, \quad \theta^- = \frac{\pi}{2} + \theta - \varphi^-.$$

Observe that the point $I_{\xi,t} \in L_\xi \cap \Gamma_t$ moves in the direction ξ along L_ξ with the velocity r_I given as

$$r_I = \frac{r}{\cos \theta}.$$

Since R_ξ^\pm propagates in the direction η^\pm , $|\eta^\pm| = 1$, and

$$\{I_{\xi,t}\} = L_{\xi,t}^\pm \cap L_\xi = \Gamma_t(R_\xi^\pm) \cap \partial C_t^\pm \cap L^\xi$$

by condition (3.8), we must have $I_{\xi,t} \in \Gamma_t(R_\xi^\pm)$ for all $t \geq 0$ and therefore

$$(3.12) \quad r^\pm = r_I(\xi \cdot \eta^\pm) = r_I \cos \varphi^\pm = r \frac{\cos \varphi^\pm}{\cos \theta}.$$

The slope μ_L of $P_{q,r}$ on L_ξ is

$$\mu_L = |q| \cos \theta.$$

In (3.9) we require that this is also the slope of R_ξ^\pm on L_ξ , i.e.,

$$(3.13) \quad \mu^\pm = \frac{\mu_L}{\cos \varphi^\pm} = \frac{|q| \cos \theta}{\cos \varphi^\pm}.$$

Finally, the constants T^\pm are then fixed by requiring $V_{T^\pm}^\pm = 0$, and therefore, by Remark 3.3,

$$T^\pm = \frac{1}{r_V^\pm} - \frac{1}{r}.$$

It is straightforward to check that R_ξ^\pm defined with such unique choice of μ^\pm , r^\pm and T^\pm satisfy (3.8) and (3.9). \square

We complete the family of R_ξ^\pm defined in (3.10) by introducing two special planar functions:

$$(3.14) \quad \begin{aligned} R_0^+(x, t) &= P_{q, \max(M|q|, r)}(x, t), \\ R_0^-(x, t) &= P_{q, \min(m|q|, r)}(x, t). \end{aligned}$$

Corollary 3.5. *We have*

$$R_\xi^- \leq P_{q,r} \leq R_\xi^+ \quad \text{on } Q_q \text{ for all } \xi \in \Xi \cup \{0\},$$

with equality

$$\begin{aligned} R_\xi^\pm &= P_{q,r} && \text{on } L_\xi \times [0, \infty) \subset \partial Q_q \text{ for } \xi \in \Xi, \\ R_0^\pm &= P_{q,r} && \text{at } t = 0, \end{aligned}$$

where L_ξ was introduced in (3.7).

Moreover,

$$(3.15) \quad \sup_{\xi \in \Xi} R_\xi^- \geq P_{q,r} \geq \inf_{\xi \in \Xi} R_\xi^+ \quad \text{on } \Omega_q^c \times [0, \infty).$$

Finally,

$$(3.16) \quad \bigcap_{\xi \in \Xi} \Omega_t(R_\xi^+) = C_t^+, \quad \bigcap_{\xi \in \Xi} \Omega_t^c(R_\xi^-) = \overline{C_t^-} \quad \text{for all } t \geq 0.$$

Proof. Since the statement is obvious for $\xi = 0$ by the definition of R_0^\pm in (3.14), we fix $\xi \in \Xi$. Let η be the unit normal vector to $\partial\Omega_q$ on $L_\xi \subset \partial\Omega_q$. Clearly $\eta \cdot \xi = 0$. Define the half-space $H := \{x : (x - V) \cdot \eta > 0\}$ and observe that $\Omega_q \subset H$ and $\partial\Omega_q \cap \partial H = L_\xi$.

Let us first show that $P_{q,r} = R_\xi^\pm$ on ∂H . We note that if $x \in \partial H$ we can express $x - V = \sigma\xi + y$, where $\sigma \in \mathbb{R}$ and $y \cdot \xi = y \cdot \eta = 0$. Since $\{\nu, \eta^+, \eta^-\} \subset \text{span}\{\xi, \eta\}$, and $P_{q,r} = R_\xi^\pm$ on L_ξ by Proposition 3.4, linearity implies

$$\begin{aligned} P_{q,r}(x, t) &= P_{q,r}(V + \sigma\xi + y, t) = P_{q,r}(V + \sigma\xi, t) \\ &= R_\xi^\pm(V + \sigma\xi, t) = R_\xi^\pm(x, t) \quad (x, t) \in \partial H \times \mathbb{R}. \end{aligned}$$

Additionally, by linearity, since $V_t^\pm \in H$ for all $t \geq 0$, and

$$V_t^+ \in \overline{\Omega}_t(R_\xi^+) \setminus \overline{\Omega}_t(P_{q,r}), \quad V_t^- \in \overline{\Omega}_t(P_{q,r}) \setminus \overline{\Omega}_t(R_\xi^-),$$

we must have

$$R_\xi^- \leq P_{q,r} \leq R_\xi^+ \quad \text{on } H \times [0, \infty) \supset Q_q$$

and

$$R_\xi^- \geq P_{q,r} \geq R_\xi^+ \quad \text{on } H^c \times [0, \infty).$$

Since for every $(x, t) \in \Omega_q^c \times [0, \infty)$ there exists $\xi \in \Xi$ such that $x \in H^c$, (3.15) is also proved.

To show one inclusion (\supset) of (3.16), we recall that by (3.8) and the choice $\eta^\pm \cdot \nu > 0$, we clearly have

$$C_t^+ \subset \Omega_t(R_\xi^+), \quad \overline{C_t^-} \subset \Omega_t^c(R_\xi^-), \quad t \geq 0.$$

To show the other inclusion (\subset) of (3.16), we only need to show that for any $x \in \mathbb{R}^n$ there exists $\xi \in \Xi$ that generates ξ^\pm, η^\pm as in Proposition 3.4 and x can be expressed as

$$(3.17) \quad x = \alpha^\pm \xi^\pm + \beta^\pm \eta^\pm + V_t^\pm,$$

and therefore clearly for such ξ we have

$$\begin{aligned} x \notin C_t^+ &\Leftrightarrow \beta^+ \geq 0 \Leftrightarrow R_\xi^+(x, t) = 0 \Leftrightarrow x \notin \Omega_t(R_\xi^+), \\ x \notin \overline{C_t^-} &\Leftrightarrow \beta^- < 0 \Leftrightarrow R_\xi^-(x, t) > 0 \Leftrightarrow x \notin \Omega_t^c(R_\xi^-). \end{aligned}$$

To find such ξ for a given $x \in \mathbb{R}^n$, first observe that there exists $y \in \partial\Omega_q$ such that $x = y + a\nu$ for some $a \in \mathbb{R}^n$. Since $y \in \partial\Omega_q$, there exists $\xi \in \Xi$ such that $y \in L_\xi$ and therefore $x = \sigma\xi + a\nu + V$. Finally, we can write x as (3.17) since $\{\xi, \nu, V_t^\pm - V\} \subset \text{span}\{\xi^\pm, \eta^\pm\}$. This concludes the proof. \square

The following geometrical observation, which shows a connection between the cones C_t^\pm and the planar functions R_ξ^\pm , will be useful later in deriving monotonicity properties of solutions of the obstacle problems.

Proposition 3.6. *For fixed $a > 0$, the following are equivalent (all with superscript either + or -):*

- (a) $\pm a R_\xi^\pm(a^{-1}x, a^{-1}t) \leq \pm R_\xi^\pm(x - y, t - \tau)$ for all $\xi \in \Xi$, $(x, t) \in \mathbb{R}^n \times \mathbb{R}$;
- (b) $a C_{a^{-1}t}^\pm \subset C_{t-\tau}^\pm + y$ for some $t \in \mathbb{R}$;
- (c) $y \in \text{Cone}_{\pm\nu, \theta^\pm}(r_V^\pm \tau\nu + (a-1)V_0^\pm)$ where r_V^\pm were introduced in Remark 3.3.

Proof. Clearly $aR_\xi^\pm(a^{-1}x, a^{-1}t)$ is just a translation of $R_\xi^\pm(x, t)$ by scaling invariance, therefore we can compare the planar solutions in (a) by comparing their supports. In fact, we have

$$aV_{a^{-1}t}^\pm \in \Gamma_t(aR_\xi^\pm(a^{-1}\cdot, a^{-1}\cdot))$$

by construction and

$$\Omega_t(R_\xi^\pm(\cdot - y, \cdot - \tau)) = \Omega_{t-\tau}(R_\xi^\pm) + y,$$

for all $\xi \in \Xi$. Therefore by Corollary 3.5 it is straightforward that (a) is equivalent to

$$aV_{a^{-1}t}^\pm \in \overline{C_{t-\tau}^\pm} + y \quad \text{for some } t \in \mathbb{R}.$$

This is clearly equivalent to (b) since $aV_{a^{-1}t}^\pm$ is the vertex of $aC_{a^{-1}t}^\pm$. And it is also equivalent to (c) if we take $t = 0$ and compute

$$(3.18) \quad aV_0^\pm \in \overline{C_{-\tau}^\pm} + y = \overline{\text{Cone}_{\mp\nu, \theta^\pm}(V_{-\tau}^\pm + y)}.$$

Since $V_{-\tau}^\pm = V_0^\pm - r_{\nabla^\pm}^\pm \tau \nu$ by Remark 3.3, we can rewrite (3.18) as (c). \square

The main motivation for such an involved choice of the domain for the obstacle problem is the following observation on the properties of functions R_ξ^\pm .

Lemma 3.7. *Suppose that q and r satisfy the condition (3.3). Then the choice of the angles θ, θ^\pm in (3.4), depending only on $\frac{m}{M}$, guarantees that*

$$R_\xi^+ \in \overline{\mathcal{S}}(M) \subset \overline{\mathcal{S}}(g^\varepsilon) \quad \text{and} \quad R_\xi^- \in \underline{\mathcal{S}}(m) \subset \underline{\mathcal{S}}(g^\varepsilon) \quad \text{for all } \xi \in \Xi \cup \{0\},$$

where R_ξ^\pm were introduced in (3.10) and (3.14).

Proof. The statement is obvious for R_0^\pm , by definition. For any $\xi \in \Xi$, by Proposition 3.1, we only need to verify $r^+ \geq M\mu^+$ and $r_- \leq m\mu^-$. Recall that q and r satisfy (3.3).

We first observe that $\cos \varphi^+ = 1$ by (3.11) and (3.4b). Then, in order, (3.12), (3.13), (3.4a), and finally (3.3) lead to

$$\frac{r^+}{\mu^+} = \frac{r \cos^2 \varphi^+}{|q| \cos^2 \theta} = \frac{r}{|q|} \frac{M}{m} \geq M,$$

which verifies that $R_\xi^+ \in \overline{\mathcal{S}}(M)$.

Similarly, (3.12), (3.13), (3.4a), (3.4c) and (3.3) yield

$$\frac{r^-}{\mu^-} = \frac{r \cos^2 \varphi^-}{|q| \cos^2 \theta} = \frac{r}{|q|} \frac{m}{M} \leq m,$$

which verifies that $R_\xi^- \in \underline{\mathcal{S}}(m)$. \square

3.2.3. Obstacle problem formulation. For every $\varepsilon > 0$, $q \in \mathbb{R}^n \setminus \{0\}$ and $r > 0$, we define the solutions of an obstacle problem on $\mathbb{R}^n \times \mathbb{R}$:

$$(3.19a) \quad \underline{u}_{\varepsilon; q, r} = \left(\inf \{ u : u \geq P_{q, r} \text{ on } \mathbb{R}^n \times \mathbb{R}, u \in \overline{\mathcal{S}}(g^\varepsilon, Q_q) \} \right)_*,$$

$$(3.19b) \quad \overline{u}_{\varepsilon; q, r} = \left(\sup \{ u : u \leq P_{q, r} \text{ on } \mathbb{R}^n \times \mathbb{R}, u \in \underline{\mathcal{S}}(g^\varepsilon, Q_q) \} \right)^*.$$

The basic properties of solutions of the obstacle problem are gathered in the following proposition.

Proposition 3.8. *For any $q \in \mathbb{R}^n \setminus \{0\}$, $r > 0$ satisfying (3.3) and any $\varepsilon > 0$, the following statements apply.*

- (a) $\underline{u}_{\varepsilon;q,r} \in \overline{\mathcal{S}}(g^\varepsilon, Q_q)$ and $\underline{u}_{\varepsilon;q,r} \in \underline{\mathcal{S}}(g^\varepsilon, Q_q \setminus (\Gamma(\underline{u}_{\varepsilon;q,r}) \cap \Gamma_{q,r}))$;
- (b) $\overline{u}_{\varepsilon;q,r} \in \underline{\mathcal{S}}(g^\varepsilon, Q_q)$ and $\overline{u}_{\varepsilon;q,r} \in \overline{\mathcal{S}}(g^\varepsilon, Q_q \setminus (\Gamma(\overline{u}_{\varepsilon;q,r}) \cap \Gamma_{q,r}))$;
- (c) $(\underline{u}_{\varepsilon;q,r})^* \geq \underline{u}_{\varepsilon;q,r} \geq P_{q,r}$ and $(\overline{u}_{\varepsilon;q,r})^* \leq \overline{u}_{\varepsilon;q,r} \leq P_{q,r}$ in $\mathbb{R}^n \times \mathbb{R}$, with equality on $(\mathbb{R}^n \times \mathbb{R}) \setminus Q_q$;
- (d) $\Omega_t(\underline{u}_{\varepsilon;q,r}) \subset C_t^+ \cup \Omega_t(P_{q,r})$ and $\Omega_t^c(\overline{u}_{\varepsilon;q,r}) \subset \overline{C_t^-} \cup \Omega_t^c(P_{q,r})$;
- (e) $\underline{u}_{\varepsilon;q,r} \in \underline{\mathcal{S}}(M, \mathbb{R}^n \times \mathbb{R})$ and $\overline{u}_{\varepsilon;q,r} \in \overline{\mathcal{S}}(m, \mathbb{R}^n \times \mathbb{R})$;
- (f) $\underline{u}_{\varepsilon;q,r} \leq \inf_{\xi \in \Xi \cup \{0\}} R_\xi^+$ and $\overline{u}_{\varepsilon;q,r} \geq \sup_{\xi \in \Xi \cup \{0\}} R_\xi^-$ on Q_q .

Proof. (a) and (b) are standard; see [18, Lemma 4].

To prove (c), we recall the definition of R_ξ^\pm in (3.10) and (3.14). Lemma 3.7 and Corollary 3.5 yield that $\max(P_{q,r}, R_\xi^+) \in \overline{\mathcal{S}}(g^\varepsilon, Q_q)$ and $\min(P_{q,r}, R_\xi^-) \in \underline{\mathcal{S}}(g^\varepsilon, Q_q)$ and thus, by definition,

$$(3.20) \quad \underline{u}_{\varepsilon;q,r} \leq \max(P_{q,r}, R_\xi^+), \quad \overline{u}_{\varepsilon;q,r} \geq \min(P_{q,r}, R_\xi^-) \quad \text{for all } \xi \in \Xi \cup \{0\},$$

which proves (f). Therefore Corollary 3.5 implies that

$$(\underline{u}_{\varepsilon;q,r})^* = \underline{u}_{\varepsilon;q,r} = (\overline{u}_{\varepsilon;q,r})^* = \overline{u}_{\varepsilon;q,r} = P_{q,r} \quad \text{on } (\mathbb{R}^n \times \mathbb{R}) \setminus Q_q,$$

and (c) is proved.

The result of (d) follows from (3.20), using the support property of R_ξ^\pm in (3.16).

Since q and r satisfy (3.3), (e) follows from (a)–(b) and Proposition 3.1. \square

Remark 3.9. Fix $T > 0$ and a pair q, r that satisfies (3.3). We shall show that $\underline{u}_{\varepsilon;q,r}$ coincides with the function defined on $Q^T = Q_q \cap \{t < T\}$,

$$v = \left(\inf \left\{ u \in \overline{\mathcal{S}}(g^\varepsilon, Q^T) : u \geq P_{q,r} \text{ on } Q^T \right\} \right)_{*, Q^T},$$

which is the solution of the obstacle problem considered in [18]. An analogous result holds for $\overline{u}_{\varepsilon;q,r}$.

Indeed, $v \leq \underline{u}_{\varepsilon;q,r}$ trivially, but also, since $v \leq R_\xi^+$ on \overline{Q}^T , we can define for any $s < T$ the function

$$w(x, t) = \begin{cases} v(x, t) & t \leq s, \\ R_0^+(x, t) & t > s. \end{cases}$$

Clearly $w \in \overline{\mathcal{S}}(g^\varepsilon, Q_q)$ and hence, by definition, $\underline{u}_{\varepsilon;q,r} \leq w = v$ for $t \leq s$ for any $s < T$.

The solutions of the obstacle problem admit the following crucial monotonicity property with respect to scaling and translation.

Proposition 3.10 (Monotonicity). *Let q and r satisfy (3.3), and let $a > 0$. Then*

$$\underline{u}_{\varepsilon;q,r}(x, t) \leq a^{-1} \underline{u}_{a\varepsilon;q,r}(a(x-y), a(t-\tau))$$

for all $(x, t) \in \mathbb{R}^n \times \mathbb{R}$ and for all $(y, \tau) \in \varepsilon \mathbb{Z}^{n+1}$ such that

$$(3.21) \quad y \cdot \nu \geq \max(r\tau, M|q|\tau), \quad \text{and} \quad y \in \overline{\text{Cone}_{\nu, \theta^+}(r_V^+ \tau \nu + (a-1)V_0^+)},$$

where r_V^\pm , V_0^\pm and θ^\pm were defined in Section 3.2.2.

Similarly,

$$\overline{u}_{\varepsilon;q,r}(x, t) \geq a^{-1} \overline{u}_{a\varepsilon;q,r}(a(x-y), a(t-\tau))$$

for all $(x, t) \in \mathbb{R}^n \times \mathbb{R}$ and for all $(y, \tau) \in \varepsilon \mathbb{Z}^{n+1}$ such that

$$(3.22) \quad y \cdot \nu \leq \min(r\tau, m|q|\tau), \quad \text{and} \quad y \in \overline{\text{Cone}_{-\nu, \theta^-}(r_V^- \tau \nu + (a-1)V_0^-)}.$$

Proof. We shall show the ordering for $\bar{u}_{\varepsilon;q,r}$, the proof for $\underline{u}_{\varepsilon;q,r}$ is analogous.

Fix $a > 0$ and (y, τ) that satisfies the hypothesis (3.21) and define the cylinder

$$\Sigma = Q_q \cap (a^{-1}Q_q + (y, \tau)).$$

Let us, for simplicity, denote

$$v^a(x, t) := a^{-1}\bar{u}_{a\varepsilon;q,r}(a(x - y), a(t - \tau)).$$

Due to the $\varepsilon\mathbb{Z}^{n+1}$ -periodicity of g^ε , we observe that $v^a \in \underline{\mathcal{S}}(g^\varepsilon, a^{-1}Q_q + (y, \tau))$. We immediately have

$$(3.23) \quad P_{q,r}(x, t) \geq P_{q,r}(x - y, t - \tau) = a^{-1}P_{q,r}(a(x - y), a(t - \tau)) \geq v^a(x, t)$$

by (3.22), the scale invariance and Proposition 3.2.

Our goal is to show that

$$(3.24) \quad \bar{u}_{\varepsilon;q,r} \geq v^a \text{ on } \Sigma^c \quad \text{and} \quad \bar{\Omega}(v^a) \cap \Sigma^c \subset \bar{\Omega}(\bar{u}_{\varepsilon;q,r})$$

since then we can apply Lemma 2.17 with $Q_1 = \Sigma$, $Q_2 = Q_q$, $u_1 = v^a$ and $u_2 = \bar{u}_{\varepsilon;q,r}$ to conclude that $u \in \underline{\mathcal{S}}(g^\varepsilon, Q_q)$, where u is defined in Lemma 2.17. Because clearly $u \leq P_{q,r}$ by (3.23), we have by definition of $\bar{u}_{\varepsilon;q,r}$ that $\bar{u}_{\varepsilon;q,r} \geq u \geq v^a$ on Σ and therefore $\bar{u}_{\varepsilon;q,r} \geq v^a$ on $\mathbb{R}^n \times \mathbb{R}$.

Let us prove (3.24). We can write $\Sigma^c = A_1 \cup A_2 \cup A_3$ where $A^1 := (Q_q)^c$, $A_2 := Q_q \cap ((a^{-1}\Omega_q + y)^c \times [\tau, \infty))$ and $A_3 := Q_q \cap \{t < \tau\}$. We therefore prove $\bar{u}_{\varepsilon;q,r}(x, t) \geq v^a(x, t)$ for all $(x, t) \in \Sigma^c$ by considering the following three cases:

- $(x, t) \in A_1$: This case is very simple since Proposition 3.8(c) and (3.23), yield

$$\bar{u}_{\varepsilon;q,r}(x, t) = P_{q,r}(x, t) \geq v^a(x, t).$$

- $(x, t) \in A_2$: Observe that (3.22) implies Proposition 3.6(c) and therefore Proposition 3.6(a) holds. We use Proposition 3.8(f), Proposition 3.6(a) and (3.15), all properly rescaled, and estimate

$$\begin{aligned} \bar{u}_{\varepsilon;q,r}(x, t) &\geq \sup_{\xi \in \Xi} R_\xi^-(x, t) \geq \sup_{\xi \in \Xi} a^{-1}R_\xi^-(a(x - y), a(t - \tau)) \\ &\geq a^{-1}P_{q,r}(a(x - y), a(t - \tau)) \geq v^a(x, t). \end{aligned}$$

- $(x, t) \in A_3$: Since $0 \leq t \leq \tau$, Proposition 3.8(f) and (3.22) yield

$$\begin{aligned} \bar{u}_{\varepsilon;q,r}(x, t) &\geq R_0^-(x, t) = |q| (m |q| t - x \cdot \nu)_+ \\ &\geq |q| (r(t - \tau) - (x - y) \cdot \nu)_+ = a^{-1}P_{q,r}(a(x - y), a(t - \tau)) \\ &\geq v^a(x, t). \end{aligned}$$

Therefore we have shown the first part of (3.24), and the second part can be inferred from the above estimates as well. Consequently, the argument using Lemma 2.17 described above applies and we can conclude that $\bar{u}_{\varepsilon;q,r} \geq v^a$ on $\mathbb{R}^n \times \mathbb{R}$. \square

We finish this section by stating an almost obvious fact that the solutions of the obstacle problems are increasing in time. However, in contrast to [18], Proposition 3.10 only implies monotonicity in time for multiplies of ε due to the time-dependence of g . Here we present an argument using a nonlinear scaling of the solution, which also provides a lower bound on the speed of the free boundary.

Lemma 3.11 (Monotonicity in time). *Let r and q satisfy (3.3) and let $\varepsilon > 0$. Then*

$$\begin{aligned}\underline{u}_{\varepsilon;q,r}(x-y, t_1) &\leq \underline{u}_{\varepsilon;q,r}(x, t_2), \\ \bar{u}_{\varepsilon;q,r}(x-y, t_1) &\leq \bar{u}_{\varepsilon;q,r}(x, t_2)\end{aligned}$$

for any $t_1 < t_2$ and $x \in \mathbb{R}^n$, and $y \in \mathbb{R}^n$ such that $|y| \leq \rho$ for some positive constant ρ depending only on $m, L, \varepsilon, t_1, t_2$.

Proof. Let us fix r, q and ε . We first prove the result for $\underline{u} := \underline{u}_{\varepsilon;q,r}$. Since we have to compare solutions that are not necessarily shifted by a multiply of ε , we cannot use the monotonicity of Proposition 3.10 directly. Instead, we will compare \underline{u} with its nonuniform perturbation using Proposition B.1.

First, assume that $0 < t_1 < t_2 < t_1 + \gamma$ where $\gamma := \frac{m}{L}$. We shall define

$$(3.25) \quad v(x, t) = \inf_{y \in \bar{B}_\rho(0)} \underline{u}_{\varepsilon;q,r}(x-y, \theta(t)),$$

where ρ will be a positive constant and $\theta(t) := \theta(t; \bar{\lambda}) = \bar{\lambda} + f(t; \bar{\lambda})$ for some $\bar{\lambda}$ determined below. Function $f(t; \lambda)$ is the function constructed in Section B.2 for given parameters γ (defined above), $a = 1$, and $\lambda > 0$. Because $D\rho \equiv 0$ we set $\alpha = a = 1$.

With these parameters, the expression for $\theta(t; \lambda)$ is simply

$$\theta(t; \lambda) = t - \gamma W \left(-\frac{\lambda}{\gamma} e^{-\frac{\lambda}{\gamma}} e^{\frac{t}{\gamma}} \right),$$

where W is the Lambert W function. Therefore $\theta(t; \lambda)$ is well-defined on $t \in [0, t_\lambda]$ and $\lambda \in [0, \gamma]$, where $t_\lambda := \log \frac{\gamma}{\lambda} + \frac{\lambda}{\gamma} - 1$ (we set $t_0 = +\infty$). Observe that t_λ is strictly decreasing in λ and positive for $\lambda \in (0, \gamma]$ with $t_\lambda = 0$ at $\lambda = \gamma$ and $t_\lambda \rightarrow +\infty$ as $\lambda \rightarrow 0+$. Therefore for every $t > 0$ there exists a unique $\lambda =: \lambda_t$ such that $t_\lambda = t$. From the properties of W , we observe that $\theta(t; 0) = t$ and $\theta(t; \lambda_t) = t + \gamma$ for any $t > 0$. By continuity and the assumption $t_2 \in (t_1, t_1 + \gamma)$, there exists $\bar{\lambda} \in (0, \lambda_{t_1})$ such that $\theta(t_1; \bar{\lambda}) = t_2$. Since $\bar{\lambda} < \lambda_{t_1}$ we must have $t_{\bar{\lambda}} > t_1$.

Let us thus take $\theta(t) = \theta(t; \bar{\lambda})$ and consider $\underline{u}_{\varepsilon;q,r}(x-y, \theta(t))$ for some $y \in \mathbb{R}^n$ and $(x, t) \in \{t \leq t_{\bar{\lambda}}\}$. It was shown in the proof of Proposition 3.10 that $\underline{u}_{\varepsilon;q,r}(x, t) \leq \underline{u}_{\varepsilon;q,r}(x-y, t-\tau)$ on $(Q_q \cap (Q_q + (y, \tau))^c$ for all (y, τ) satisfying (3.21). Let us set $\tau = -\bar{\lambda}$. There exists $\rho > 0$ such that (y, τ) satisfies (3.21) for all $y \in \bar{B}_\rho(0)$. Since $t + \bar{\lambda} \leq \theta(t)$ for all $t \in [0, t_{\bar{\lambda}}]$, the same argument yields $\underline{u}_{\varepsilon;q,r}(x, t) \leq \underline{u}_{\varepsilon;q,r}(x-y, \theta(t))$ on $(Q_q \cap (Q_q + (y, \tau))^c \cap \{t \leq t_{\bar{\lambda}}\}$ for $(y, \tau) \in \bar{B}_\rho(0) \times \{-\bar{\lambda}\}$. Hence the argument using Lemma 2.17 as in the proof of Proposition 3.10 (with Remark 3.9 for extension of v for $t > t_{\bar{\lambda}}$) yields that $\underline{u}_{\varepsilon;q,r}(x, t) \leq \underline{u}_{\varepsilon;q,r}(x-y, \theta(t))$ for $(x, t) \in \{t < t_{\bar{\lambda}}\}$. In particular $u(x, t_1) \leq u(x-y, \theta(t_1)) = u(x-y, t_2)$ for all $y \in \bar{B}_\rho(0)$, and that is what we wanted to prove.

The proof for general $0 < t_1 < t_2$ can be done iteratively by splitting $[t_1, t_2]$ into subintervals of length smaller than γ .

We skip the proof for $\bar{u}_{\varepsilon;q,r}$, which follows the same idea, but is simpler. In particular, it is unnecessary to restrict $t_2 < t_1 + \gamma$ since f constructed in Section B.1 can handle arbitrary shifts. \square

For completeness, we also show that the solutions are harmonic in their positive sets.

Proposition 3.12. *Let r and q satisfy (3.3) and let $\varepsilon > 0$. The functions $\underline{u}_{\varepsilon;q,r}(\cdot, t)$ and $(\bar{u}_{\varepsilon;q,r})_*(\cdot, t)$ are harmonic in $\Omega_t(\underline{u}_{\varepsilon;q,r}) \cap \Omega_q$ and $\Omega_t((\bar{u}_{\varepsilon;q,r})_*) \cap \Omega_q$, respectively, for every t .*

Proof. Since the claim is trivially true for $t \leq 0$, we fix $\tau > 0$. We first show that $\underline{u}_{\varepsilon;q,r}(\cdot, \tau)$ is harmonic in the open set $E := \Omega_\tau(\underline{u}_{\varepsilon;q,r}) \cap \Omega_q$. For any $w \in LSC(\bar{E})$, superharmonic in E , such that $w \geq \underline{u}_{\varepsilon;q,r}(\cdot, \tau)$ on ∂E , we define a new function on $Q_q \cap \{t \leq \tau\}$

$$v(x, t) = \begin{cases} \min(\underline{u}_{\varepsilon;q,r}(x, t), w(x)) & x \in E, \\ \underline{u}_{\varepsilon;q,r}(x, t) & \text{otherwise.} \end{cases}$$

By definition of w , we observe that $v \in LSC(Q_q \cap \{t \leq \tau\})$ and $v \in \bar{\mathcal{S}}(g^\varepsilon, Q_q \cap \{t \leq \tau\})$. Therefore $\underline{u}_{\varepsilon;q,r}(x, \tau) \leq v(x, \tau) \leq w(x)$ in E by definition of $\underline{u}_{\varepsilon;q,r}$ and Remark 3.9. We recall that $\underline{u}_{\varepsilon;q,r}(\cdot, \tau)$ is superharmonic in E , and therefore it is the smallest superharmonic function, i.e., it must be harmonic.

Now we show that $(\bar{u}_{\varepsilon;q,r})_*(\cdot, \tau)$ is harmonic in the open set $E = \Omega_\tau((\bar{u}_{\varepsilon;q,r})_*) \cap \Omega_q$. First we recall that $(\bar{u}_{\varepsilon;q,r})_*(\cdot, \tau)$ is superharmonic since $\bar{u}_{\varepsilon;q,r} \in \bar{\mathcal{S}}(m, \mathbb{R}^n \times \mathbb{R})$ by Proposition 3.8. Let $w \in LSC(\bar{E})$ be a superharmonic function in E , such that $w \geq (\bar{u}_{\varepsilon;q,r})_*(\cdot, \tau)$ on ∂E . By the monotonicity in time, Lemma 3.11, and in particular the positive speed of expansion, we observe that $\bar{u}_{\varepsilon;q,r}(x, t) = 0$ for $x \in \Gamma_\tau((\bar{u}_{\varepsilon;q,r})_*)$ and $t < \tau$. Since $\bar{u}_{\varepsilon;q,r}(\cdot, t)$ is subharmonic in Ω_q for all t , we must have $\bar{u}_{\varepsilon;q,r}(x, t) \leq w(x)$ for $t < \tau$. Therefore, $(\bar{u}_{\varepsilon;q,r})_*(x, \tau) \leq w(x)$. Again, $(\bar{u}_{\varepsilon;q,r})_*(\cdot, \tau)$ is the smallest superharmonic function and thus harmonic. \square

3.3. Flatness. In this section we introduce *flatness*, the quantity that will be a measure of how good is the guess of the homogenized velocity $r(q)$.

For $\eta \in \mathbb{R}$, denote

$$P_{q,r}^\eta(x, t) := P_{q,r}(x - \eta\nu, t).$$

Note that $P_{q,r}^\eta$ represents a translation of $P_{q,r}$ in the direction ν by distance η .

Definition 3.13. *For $q \neq 0$, $r > 0$ and $\varepsilon > 0$, we define the upper flatness $\Phi_{\varepsilon;q,r}(\tau)$, i.e., the flatness of $\underline{u}_{\varepsilon;q,r}$ up to time $\tau \in \mathbb{R}$ as*

$$\Phi_{\varepsilon;q,r}(\tau) = \inf \{ \eta : \underline{u}_{\varepsilon;q,r} \leq P_{q,r}^\eta \text{ in } \{t \leq \tau\} \}.$$

Similarly, the lower flatness $\bar{\Phi}_{\varepsilon;q,r}(\tau)$ is defined as the flatness of $\bar{u}_{\varepsilon;q,r}$ up to time $\tau \in \mathbb{R}$,

$$\bar{\Phi}_{\varepsilon;q,r}(\tau) = \inf \{ \eta : \bar{u}_{\varepsilon;q,r} \geq P_{q,r}^{-\eta} \text{ in } \{t \leq \tau\} \}.$$

We say that $\underline{u}_{\varepsilon;q,r}$ (resp. $\bar{u}_{\varepsilon;q,r}$) is η -flat up to time τ if $\Phi_{\varepsilon;q,r}(\tau) \leq \eta$ (resp. $\bar{\Phi}_{\varepsilon;q,r}(\tau) \leq \eta$).

The flatness is Lipschitz and monotone:

Proposition 3.14. *Let $q \in \mathbb{R}^n \setminus \{0\}$, $r > 0$ and $\varepsilon > 0$. Then for any $t \in \mathbb{R}$, $h > 0$,*

$$\begin{aligned} \Phi_{\varepsilon;q,r}(t) &\leq \Phi_{\varepsilon;q,r}(t+h) \leq \Phi_{\varepsilon;q,r}(t) + h(M|q| - r)_+, \\ \bar{\Phi}_{\varepsilon;q,r}(t) &\leq \bar{\Phi}_{\varepsilon;q,r}(t+h) \leq \bar{\Phi}_{\varepsilon;q,r}(t) + h(r - m|q|)_+. \end{aligned}$$

In particular, $\bar{\Phi}_{\varepsilon;q,r}$ and $\Phi_{\varepsilon;q,r}$ are non-decreasing and Lipschitz continuous.

Proof. $\bar{\Phi}_{\varepsilon;q,r}$ and $\underline{\Phi}_{\varepsilon;q,r}$ are clearly nondecreasing from their definition.

To show the upper bound for $\underline{\Phi}_{\varepsilon;q,r}(\tau + h)$ for any $\tau \in \mathbb{R}$, $h > 0$, we may assume that $r < M|q|$, otherwise $\underline{u}_{\varepsilon;q,r} = P_{q,r}$ and so $\underline{\Phi}_{\varepsilon;q,r}(\tau + h) = \underline{\Phi}_{\varepsilon;q,r}(\tau) = 0$. We only need to compare $\underline{u}_{\varepsilon;q,r}$ with an appropriate translation of $R_0^+ = P_{q,M|q|}$, using the fact that $\underline{u}_{\varepsilon;q,r}(\cdot, \tau) \leq P_{q,r}^\eta(\cdot, \tau)$ and the identity $P_{q,M|q|}(\cdot, h) = P_{q,r}^{(M|q|-r)h}(\cdot, h)$. Similarly, the upper bound for $\bar{\Phi}_{\varepsilon;q,r}(\tau + h)$ follows from comparison with a translation of $R_0^- = P_{q,m|q|}$. \square

Lemma 3.15. *Let $q \in \mathbb{R}^n \setminus \{0\}$, $r > 0$ and $\varepsilon > 0$. Let $\tau \in \mathbb{R}$. Then for any $\eta \in (0, \underline{\Phi}_{\varepsilon;q,r}(\tau)]$ there exists $(\zeta, \sigma) \in \Gamma(\underline{u}_{\varepsilon;q,r})$ such that $\zeta \cdot \nu = r\sigma + \eta$ and $\underline{\Phi}_{\varepsilon;q,r}(t) \leq \eta$ for all $t \leq \sigma$. Similarly, for any $\eta \in (0, \bar{\Phi}_{\varepsilon;q,r}(\tau)]$ there exists $(\zeta, \sigma) \in \Gamma(\bar{u}_{\varepsilon;q,r})$ such that $\zeta \cdot \nu = r\sigma - \eta$ and $\bar{\Phi}_{\varepsilon;q,r}(t) \leq \eta$ for all $t \leq \sigma$.*

Proof. We will show the proof for $\underline{u}_{\varepsilon;q,r}$, the proof for $\bar{u}_{\varepsilon;q,r}$ is similar.

Let us define $T = \sup\{t \leq \tau : \underline{\Phi}_{\varepsilon;q,r}(t) \leq \eta\}$. Clearly $T \in (0, \tau]$. Moreover, by continuity, $\underline{\Phi}_{\varepsilon;q,r}(T) = \eta$. We claim that we can find $(\zeta, \sigma) \in \Omega^c(P_{q,r}^\eta) \cap \bar{\Omega}(\underline{u}_{\varepsilon;q,r}) \cap \{t \leq T\}$. If such point does not exist, we must have $\text{dist}(\Omega^c(P_{q,r}^\eta) \cap \{0 \leq t \leq T\}, \bar{\Omega}(\underline{u}_{\varepsilon;q,r})) > 0$ and therefore there exists $\delta \in (0, \eta)$ such that $\Omega^c(P_{q,r}^{\eta-\delta}) \cap \bar{\Omega}(\underline{u}_{\varepsilon;q,r}) \cap \{t \leq T\} = \emptyset$. Since $P_{q,r}^{\eta-\delta}$ is harmonic in its positive set, it follows that $\underline{u}_{\varepsilon;q,r} \leq P_{q,r}^{\eta-\delta}$ in $\{t \leq T\}$. But that is a contradiction with $\underline{\Phi}_{\varepsilon;q,r}(T) = \eta > \eta - \delta$. Therefore (ζ, σ) exists and we observe that $(\zeta, \sigma) \in \Gamma(\underline{u}_{\varepsilon;q,r}) \cap \Gamma(P_{q,r}^\eta) \cap \{0 < t \leq T\}$. \square

3.4. Local comparison principle. An important tool in the analysis of the behavior in the homogenization limit is the *local comparison principle*. In contrast to the standard comparison principle which requires ordering on the parabolic boundary to guarantee ordering in the whole cylinder, local comparison relies on the extra information about flatness of solutions in the sense of Definition 3.13, and therefore allows for solutions to cross on the lateral boundary but still guarantees the ordering of free boundaries for short time in a small region far from the boundary. This is a generalization of a result that appeared in [18, 19] (with $\beta = 1$).

Theorem 3.16 (Local comparison principle). *Let $\beta \in (\frac{4}{5}, 1)$, $r_1 > r_2 > 0$ and $q_1, q_2 \in \mathbb{R}^n$ such that $a^3 q_1 = q_2$ for some $a > 1$.*

Then there exists $\varepsilon_0 = \varepsilon_0(r_1, r_2, |q_1|, |q_2|, m, M, n, \beta)$ such that

$$\max(\bar{\Phi}_{\varepsilon;q_1,r_1}(1), \underline{\Phi}_{\varepsilon;q_2,r_2}(1)) > \varepsilon^\beta \quad \text{for } \varepsilon < \varepsilon_0.$$

In other words, $\bar{u}_{\varepsilon;q_1,r_1}$ and $\underline{u}_{\varepsilon;q_2,r_2}$ cannot be both ε^β -flat up to time $t = 1$ for any $\varepsilon < \varepsilon_0$.

Before proceeding with the proof of Theorem 3.16, we first present a simple result on the nondegeneracy of $\underline{u}_{\varepsilon;q,r}$. This is an improvement of Lemma A.5 when we also know that $\underline{u}_{\varepsilon;q,r}$ is ε^β -flat.

Lemma 3.17. *For given $q \in \mathbb{R}^n \setminus \{0\}$, $r > 0$, there exists a positive constant $c = c(r, |q|, M, n)$ such that if $\underline{\Phi}_{\varepsilon;q,r}(T) \leq \eta$, for some $\varepsilon, \eta, T > 0$, then*

$$\underline{u}_{\varepsilon;q,r}(x, t) \geq c \frac{\rho^2}{\eta}$$

for every $0 < \rho \leq \eta$ and (x, t) such that $B_\rho(x) \subset \Omega_t(\underline{u}_{\varepsilon;q,r}) \cap \Omega_q$, $t \leq T$.

Proof. Let $c_H = c_H(n)$ be the constant from Harnack inequality such that if $\psi \geq 0$, harmonic in $B_1(0)$ then $\inf_{\overline{B_{1/2}(0)}} \psi \geq c_H \sup_{\overline{B_{1/2}(0)}} \psi$.

Let us denote $u = \underline{u}_{\varepsilon; q, r}$. Fix (ξ, τ) such that $B_\rho(\xi) \subset \Omega_\tau(u) \cap \Omega_q$. Then we have two possible scenarios:

(a) $\xi \cdot \nu \leq r\tau + \frac{\rho}{4}$: We estimate

$$u(\xi - \frac{\rho}{2}\nu, \tau) \geq P_{q,r}(\xi - \frac{\rho}{2}\nu, \tau) \geq |q| \frac{\rho}{4} \geq \frac{|q|}{4} \frac{\rho^2}{\eta}.$$

(b) $\xi \cdot \nu > r\tau + \frac{\rho}{4}$: In this case necessarily $r < M|q|$, otherwise $u = P_{q,r}$, a contradiction. We have $u \in \underline{\mathcal{S}}(M, \mathbb{R}^n \times \mathbb{R})$ by Proposition 3.8 and u is nondecreasing in time by Lemma 3.11. Due to flatness assumption $\underline{\Phi}_{\varepsilon; q, r}(\tau) \leq \eta$, $u(\cdot, t) = 0$ on $\overline{B_{\rho/4}(\xi)}$ for $t \leq \tau - \frac{\eta}{r}$. Consequently, Lemma A.5 implies

$$\sup_{B_{\rho/4}(\xi)} u(\cdot, \tau) \geq \frac{\rho^2}{32nM\frac{\eta}{r}} = \frac{r}{32nM} \frac{\rho^2}{\eta}.$$

Since $u(\cdot, \tau)$ is harmonic in $\Omega_\tau(u) \cap \Omega_q \supset B_\rho(\xi)$, the Harnack inequality, properly rescaled, yields

$$u(\xi, \tau) \geq c_H \min\left(\frac{|q|}{4}, \frac{r}{32nM}\right) \frac{\rho^2}{\eta}. \quad \square$$

An important tool in the proof of the local comparison principle is the nonuniform perturbation in space, see Section B. We will construct a particular radius $\rho(x)$ in the following lemma.

Lemma 3.18. *There exists $R > 0$ and a smooth, radially symmetric function $\rho : \overline{B_{2R}(0)} \setminus B_R(0) \rightarrow [1, 6]$ such that $\rho \Delta \rho \geq (n-1)|D\rho|^2$ in $B_{2R}(0) \setminus \overline{B_R(0)}$, $\rho = 1$ on $\partial B_{2R}(0)$ and $\rho > 5$ on $\overline{B_{R+1}(0)} \setminus B_R(0)$.*

Proof. Let φ be the radially symmetric smooth positive solution of

$$\begin{cases} \Delta \varphi^{2-n} = 0, & \text{in } B_2 \setminus \overline{B_1} \\ \varphi = 1 & \text{on } \partial B_2, \\ \varphi = 6 & \text{on } \partial B_1. \end{cases}$$

It was observed in [4, p. 148] that φ satisfies $\varphi \Delta \varphi \geq (n-1)|D\varphi|^2$. Moreover, note that also the rescaled function $a\varphi(bx)$ satisfies the same inequality for any a, b positive. By continuity, there exists $\delta > 0$ such that $\varphi > 5$ on $\overline{B_{1+\delta}} \setminus B_1$. Define $R = 1/\delta$ and $\rho(x) = \varphi(\delta x)$ on $\overline{B_{2R}} \setminus B_R$. It is straightforward to verify that ρ satisfies the properties asserted in the statement and the proof is finished. \square

The following technical lemma shows that the closure of a time-slice of $\Omega(\overline{u}_{\varepsilon; q, r})$ is the time-slice of the closure, a fact that will be used throughout the proof of the local comparison principle and that allows us to work with simple time-slices of a subsolution with nondecreasing support.

Lemma 3.19. *Let $u \in \underline{\mathcal{S}}(M, Q)$ be a bounded subsolution on Q for some $M > 0$ and $Q = E \times (t_1, t_2)$, $E \subset \mathbb{R}^n$ open, and assume that $\Omega_t(u^*) \subset \Omega_s(u^*)$ for all t, s such that $t_1 < t \leq s < t_2$. Then for any $(x, t) \in Q$ we have*

$$x \in \overline{\Omega_t(u^*; Q)} \quad \text{if and only if} \quad (x, t) \in \overline{\Omega}(u; Q),$$

or, equivalently,

$$\overline{\Omega_t(u^*; Q)} = \overline{\Omega_t(u; Q)}.$$

Proof. Since $\overline{\Omega}(u) = \overline{\Omega}(u^*)$, we can WLOG assume that $u = u^*$. Let $(\xi, \tau) \in Q$. It is clear that if $\xi \in \overline{\Omega_\tau(u)}$ then $(\xi, \tau) \in \overline{\Omega}(u)$. Now suppose that $(\xi, \tau) \in \overline{\Omega}(u) \cap Q$. By Definition 2.3(i) and the monotonicity assumption, $(\xi, \tau) \in \overline{\Omega}(u) \cap \{t < \tau\} \subset \overline{\Omega_\tau(u)} \times \{t \leq \tau\}$. \square

Proof of Theorem 3.16. The proof proceeds in a number of technical steps.

Suppose that such ε_0 does not exist and both $\overline{u}_{\varepsilon; q_1, r_1}$ and $\underline{u}_{\varepsilon; q_2, r_2}$ are ε^β -flat up to time $t = 1$ along some subsequence of ε that converges to 0. We will show that this leads to a contradiction with the comparison principle for the Hele-Shaw problem for ε sufficiently small.

(1) Setup. Let $\alpha := \frac{4}{5}$, and observe that, since $\beta \in (4/5, 1)$,

$$(3.26) \quad 2 - 2\beta \leq 4 - 4\beta < \alpha < \beta.$$

First, for fixed $\varepsilon > 0$, we introduce notation that will be used throughout the proof. Symbol c with a subscript will denote quantities independent of ε .

We shall use multiple perturbations of the subsolution $\overline{u}_{\varepsilon, q_1, r_1}$. Throughout the proof, $\nu = \frac{-q_1}{|q_1|} = \frac{-q_2}{|q_2|}$. Let us denote $\Lambda := 6$ and define the constant

$$(3.27) \quad \gamma = \frac{1}{\Lambda} \min \left(\frac{m}{L} (a^{1/2} - 1), \frac{1}{2} \right)$$

and a shift

$$\xi_0 \in \arg \min_{\substack{\xi \in \varepsilon Z^n \\ \xi \cdot \nu \leq -\varepsilon}} |\xi|.$$

Clearly $|\xi_0| \leq n\varepsilon$. For $\lambda \in [0, \Lambda]$, we define the perturbations

$$u^\lambda(x, t) := \sup_{y \in \overline{B}_{\lambda\gamma\varepsilon}(x)} a \overline{u}_{\varepsilon, q_1, r_1}(y - \xi_0, t),$$

where $\overline{B}_0(x) := \{x\}$ for simplicity. The choice of γ and ξ_0 , and Proposition B.1 imply that $u^\lambda \in \underline{\mathcal{S}}(g^\varepsilon, \tilde{Q})$, where $\tilde{Q} := \{(x, t) : B_{(n+1)\varepsilon}(x) \times \{t\} \subset Q_q\}$. Furthermore, since $\gamma \leq \frac{1}{2\Lambda}$ we note that $u^\lambda \leq P_{a q_1, r_1}$ for all $\lambda \in [0, \Lambda]$.

Next, we introduce the time

$$T := \frac{3\varepsilon^\beta}{r_1 - r_2} = c_T \varepsilon^\beta > 0.$$

If ε is small enough so that $T \leq 1$ then u^λ and v will be $(\varepsilon^\beta + n\varepsilon)$ -flat and ε^β -flat, respectively, up to time $t = T$, and if also $\varepsilon^\beta \geq n\varepsilon \geq |\xi_0 \cdot \nu|$ then the free boundary of u^λ must completely overtake the free boundary of v by time T .

Finally, we choose the constants σ_1, σ_2 as

$$\begin{aligned} \sigma_1 &:= \frac{3a|q_1| + 1}{|q_2| - a|q_1|} \varepsilon^\beta = c_{\sigma_1} \varepsilon^\beta, \\ \sigma_2 &:= T r_1 = c_{\sigma_2} \varepsilon^\beta. \end{aligned}$$

Let us also denote $\sigma := \sigma_1 + \sigma_2$ and remark that $\sigma = c_\sigma \varepsilon^\beta$. The motivation behind this particular choice of σ_1 is the fact that

$$(3.28) \quad v(x, t) - u^\lambda(x, t) \geq P_{q_2, r_2}(x, t) - P_{aq_1, r_1}(x, t) \geq \varepsilon^\beta$$

for $x \cdot \nu \leq -\sigma_1$, $t \in [0, T]$, $\lambda \leq \Lambda$. Similarly, σ_2 is motivated by

$$(3.29) \quad \bar{\Omega}_t(P_{aq_1, r_1}) \cup \bar{\Omega}_t(P_{q_2, r_2}^{\varepsilon^\beta}) \subset \{x : x \cdot \nu \leq \sigma_2\}$$

for $t \in [0, T]$, i.e. the free boundaries of $\bar{u}_{\varepsilon; q_1, r_1}$ and $\underline{u}_{\varepsilon; q_2, r_2}$ will be contained in $\{x : 0 \leq x \cdot \nu \leq \sigma_2\}$ for $t \in [0, T]$.

(2) Fast-shrinking domain. For a point $x \in \mathbb{R}^n$, we define x^\perp to be the component of x orthogonal to ν ,

$$x^\perp = x - (x \cdot \nu)\nu = (I - \nu \otimes \nu)x.$$

By symbol $\text{Cyl}(\rho; \eta_1, \eta_2)$ we shall denote the open space cylinder of radius ρ with axis parallel to ν , between the hyperplanes $\{x : x \cdot \nu = \eta_1\}$ and $\{x : x \cdot \nu = \eta_2\}$,

$$\text{Cyl}(\rho; \eta_1, \eta_2) = \{x : |x^\perp| < \rho, \eta_1 < x \cdot \nu < \eta_2\}.$$

Let us define a “fast-shrinking” domain Σ , whose each time-slice Σ_t is the cylinder

$$\Sigma_t := \begin{cases} \text{Cyl}(\rho_\Sigma(t); -\sigma_1, \sigma_2) & t \geq 0, \\ \emptyset & t < 0. \end{cases}$$

with radius

$$\rho_\Sigma(t) = \frac{\tan \theta}{2} - \varepsilon^{-\alpha} t,$$

where θ is the opening of the cone Ω_q defining Q_q , introduced in (3.4a). In particular, the radius $\rho_\Sigma(t)$ of the cylinder Σ_t shrinks with velocity $\varepsilon^{-\alpha}$. Furthermore, if ε is small enough, we have $\Sigma + (\bar{B}_{(n+1)\varepsilon} \times \{0\}) \subset Q_q$, which also implies $u^\lambda \in \underline{\mathcal{S}}(g^\varepsilon, \Sigma)$ for $\lambda \in [0, \Lambda]$. Additionally, $\Sigma_t \neq \emptyset$ for all $t \in [0, T]$ since, as $\alpha < \beta$ by (3.26),

$$\rho_\Sigma(T) = \frac{\tan \theta}{2} - \varepsilon^{-\alpha} T = \frac{\tan \theta}{2} - c_T \varepsilon^{\beta-\alpha} > 0 \quad \text{for } \varepsilon > 0 \text{ small.}$$

Finally, we need to fix a small time step τ ,

$$(3.30) \quad \tau := \frac{\gamma^2 \varepsilon^2}{4na\sigma |q_1| M} = c_\tau \varepsilon^{2-\beta}.$$

Let us explain how such choice of τ guarantees that, for $\lambda \in [0, \Lambda]$, the support of u^λ cannot expand by more than $\gamma\varepsilon$ from time $\hat{t} - \tau$ to time \hat{t} for any $\hat{t} \in [\tau, T]$ in the sense

$$(3.31) \quad \bar{\Omega}_s(u^\lambda) \subset \bar{\Omega}_{\hat{t}-\tau}(u^\lambda) + \bar{B}_{\gamma\varepsilon} \quad s \in [\hat{t} - \tau, \hat{t}].$$

This follows from a straightforward application of Lemma A.6, on the parabolic cylinder

$$E = \{(x, t) : -\sigma_1 < x \cdot \nu, t > \hat{t} - \tau\},$$

taking advantage of the bound

$$(3.32) \quad u^\lambda(x, t) \leq P_{aq_1, r_1}(x, t) \leq a |q_1| \sigma =: c_K \varepsilon^\beta,$$

valid for $x \cdot \nu \geq -\sigma_1$ and $t \in [0, T]$ due to (3.29).

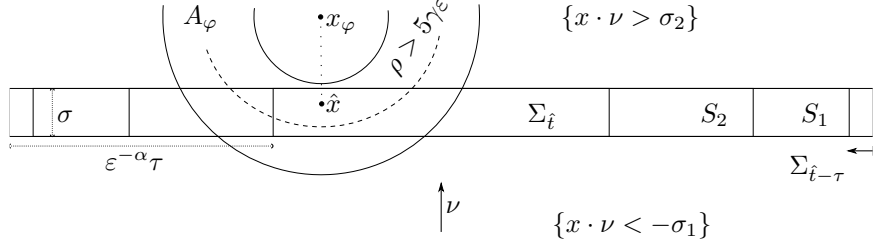


FIGURE 4. Arrangement of cylinders

(3) Boundary crossing point. With these definitions at hand, we proceed with the proof. As we observed above, since $n\varepsilon \leq \varepsilon^\beta$ if ε is small enough, u^λ is $(\varepsilon^\beta + n\varepsilon)$ -flat for any $\lambda \in [0, \Lambda]$, and v is ε^β -flat, the choice of ξ_0 and T guarantees that the supports of u^λ and v are originally strictly ordered, but will cross each other before the time T in the domain Σ . So let \hat{t} be the first time that the boundaries of u^5 and v touch in Σ , i.e.,

$$\hat{t} := \sup \{s : \bar{\Omega}(u^5) \cap \Omega^c(v) \cap \bar{\Sigma} \cap \{t \leq s\} = \emptyset\} < T.$$

Observe that $\hat{t} \geq \tau$ since $\bar{\Omega}_{\hat{t}}(u^5)$ cannot advance more than $\gamma\varepsilon$ in time τ by (3.31).

We claim that the set $\bar{\Omega}_{\hat{t}}(u^5) \cap \Omega_{\hat{t}}^c(v) \cap \bar{\Sigma}_{\hat{t}}$ is nonempty. Indeed, if that were not the case, $\text{dist}(\bar{\Omega}_{\hat{t}}(u^5), \Omega_{\hat{t}}^c(v) \cap \bar{\Sigma}_{\hat{t}}) > 0$ because the first set is closed and the second one is compact. Therefore, there exists $\delta > 0$ such that the distance is positive for $t < \hat{t} + \delta$ by Lemma A.6 and the monotonicity of $\Omega_{\hat{t}}^c(v) \cap \bar{\Sigma}_{\hat{t}}$, which yields a contradiction with the definition of \hat{t} . Let us thus choose

$$\hat{x} \in \bar{\Omega}_{\hat{t}}(u^5) \cap \Omega_{\hat{t}}^c(v) \cap \bar{\Sigma}_{\hat{t}}.$$

We will show that the existence of such point leads to a contradiction with the comparison principle for the Hele-Shaw problem since u^5 and v were strictly separated on a much larger domain $\bar{\Sigma}_{\hat{t}-\tau}$ at time $t = \hat{t} - \tau$.

To make this idea rigorous, we introduce the cylinders

$$S_i = \text{Cyl}(\rho_i; -\sigma_1, \sigma_2) \quad \text{for } i = 1, 2,$$

(see Figure 4) with radii

$$\rho_1 := \rho_\Sigma(\hat{t} - \tau) - 2\gamma\varepsilon, \quad \rho_2 := \rho_1 - \left(\frac{c_K}{c_N} \varepsilon^{2\beta-2} + 2 \right) c_\sigma \varepsilon^\beta,$$

where constant c_N is specified below in (3.35). We point out that if ε is small enough, we have

$$(3.33) \quad \rho_2 > \rho_\Sigma(\hat{t}) + 2R\sigma,$$

where R is the constant from Lemma 3.18, since

$$\rho_\Sigma(\hat{t} - \tau) - \rho_\Sigma(\hat{t}) = \tau\varepsilon^{-\alpha} = c_\tau \varepsilon^{2-\alpha-\beta} \gg \rho_\Sigma(\hat{t} - \tau) - \rho_2 + 2R\sigma \quad \text{if } \varepsilon \ll 1$$

due to (3.26). In particular, we have

$$\Sigma_{\hat{t}-\tau} \supset S_1 \supset S_2 \supset \Sigma_{\hat{t}},$$

see Figure 4.

Now we consider function u^2 . First, clearly $\overline{\Omega}_t(u^5) = \overline{\Omega}_t(u^2) + \overline{B}_{3\gamma\varepsilon}(0)$. The definition of \hat{t} yields $\overline{\Omega}_t(u^5) \cap \Omega_t^c(v) \cap \overline{\Sigma}_t = \emptyset$ for $t < \hat{t}$, and thus, in particular,

$$(\overline{\Omega}_{\hat{t}-\tau}(u^2) + \overline{B}_{3\gamma\varepsilon}(0)) \cap \Omega_{\hat{t}-\tau}^c(v) \cap \overline{\Sigma}_{\hat{t}-\tau} = \emptyset.$$

As $\Omega_{\hat{t}}^c(v)$ cannot expand (due to the monotonicity, Lemma 3.11), and the positive set of u^2 cannot expand by more than $\gamma\varepsilon$ in time τ , in the sense of (3.31), we conclude that

$$(3.34) \quad (\overline{\Omega}_t(u^2) + \overline{B}_{2\gamma\varepsilon}(0)) \cap \Omega_t^c(v) \cap \overline{S}_1 = \emptyset \quad t \in [0, \hat{t}].$$

Let us define the closed sets

$$A_t := (\overline{\Omega}_t(u^2) + \overline{B}_{\gamma\varepsilon}(0)) \cap \overline{S}_1 \quad t \in [0, \hat{t}].$$

Due to the nondegeneracy of v (Lemma 3.17 applied with $\eta = \varepsilon^\beta$ and $\rho = \gamma\varepsilon$), we observe that

$$(3.35) \quad v(x, t) \geq c(r_2, |q_2|, M, n)\gamma^2\varepsilon^{2-\beta} = c_N\varepsilon^{2-\beta} \quad x \in A_t, t \in [0, \hat{t}],$$

where $c(r_2, |q_2|, M, n)$ is the constant from Lemma 3.17. Consider the function $w^t(x) = \frac{u^2(x, t) - v(x, t)}{c_N\varepsilon^{2-\beta}}$ defined on A_t . Function w^t is subharmonic on A_t for all $t \in [0, \hat{t}]$ since $v(\cdot, t)$ is superharmonic in $\text{int } A_t$ as $A_t \subset \Omega_t(v) \cap \Omega_q$, and $u^2(\cdot, t)$ is clearly subharmonic in $\text{int } A_t$.

Our goal is to apply Lemma C.1 to function w^t . Since ε is small, we have $\varepsilon^\beta \geq c_N\varepsilon^{2-\beta}$ and thus $w^t \leq -1$ on $\partial A_t \cap \{x : x \cdot \nu = -\sigma_1\}$ by (3.28). Additionally, $u^2 = 0$ on $\partial A_t \cap S_1$ by definition of A_t and therefore $w^t \leq -1$ on $\partial A_t \cap S_1$. Finally, by (3.32),

$$w^t \leq \frac{u^2}{c_N\varepsilon^{2-\beta}} \leq \frac{c_K}{c_N}\varepsilon^{2\beta-2} \quad \text{on } A_t.$$

Therefore Lemma C.1 implies that $w^t < 0$ on the set $S_1 \cap \{x : |x^\perp| \leq \rho'_1\}$, where ρ'_1 is given by

$$\rho'_1 = \rho_1 - c_T \left(\frac{c_K}{c_N}\varepsilon^{2\beta-2} + 2 \right) c_\sigma \varepsilon^\beta,$$

and $\rho' \geq \rho_2$ by the choice of ρ_1 and ρ_2 . We conclude that $w^t < 0$ in $\overline{S}_2 \cap A_t$ for all $t \in [0, \hat{t}]$, and therefore, with (3.34),

$$(3.36) \quad u^2 \prec v \quad \text{in } \overline{S}_2 \times [0, \hat{t}].$$

Let R and $\rho(x)$ be from Lemma 3.18 and define

$$x_\varphi := \hat{x}^\perp + (\sigma_2 + R\sigma)\nu$$

and

$$\varphi(x) := \gamma\varepsilon\rho\left(\frac{x - x_\varphi}{\sigma}\right) \quad \text{on } A_\varphi := B_{2R\sigma}(x_\varphi) \setminus \overline{B}_{R\sigma}(x_\varphi).$$

Let us set $c_\rho = \max_{\overline{B}_{2R}(0) \setminus B_R(0)} |D\rho| < \infty$. Clearly

$$\max_{A_\varphi} |D\varphi| = \frac{\gamma\varepsilon}{\sigma} c_\rho = \frac{c_\rho}{c_\sigma} \gamma\varepsilon^{1-\beta}.$$

Hence for ε small enough we have

$$(3.37) \quad (1 - \max_{A_\varphi} |D\varphi|)^{-2} \leq a^{1/2}.$$

Moreover, $\sigma_2 < \sigma$ yields $\hat{x} \in \overline{B}_{(R+1)\sigma}(x_\varphi) \setminus B_{R\sigma}(x_\varphi)$ and hence, by the definition of ρ , we also have $\varphi(\hat{x}) > 5\gamma\varepsilon$.

(4) Non-uniform perturbation. In the final step, we consider the nonuniform perturbation u^φ of $\overline{u}_{\varepsilon;q,r}$,

$$u^\varphi(x, t) := \sup_{y \in \overline{B}_\varphi(x)} a\overline{u}_{\varepsilon;aq_1,r_1}(y - \xi_0, t) \quad (x, t) \in A_\varphi \times (0, \hat{t}].$$

This function is a subsolution $u^\varphi \in \underline{\mathcal{S}}(g^\varepsilon, Q_\varphi)$ on $Q_\varphi := A_\varphi \times (0, \hat{t}]$ for all ε small enough so that (3.37) holds, due to Proposition B.1 and the choice of γ in (3.27).

We want to show that $u^\varphi \prec v$ on the parabolic boundary $\partial_P(A_\varphi \times (0, \hat{t}])$ to be able to use the comparison theorem for the Hele-Shaw problem:

- $u^\varphi \prec v$ on $A_\varphi \times \{0\}$ by the choice of ξ_0 , which implies $u^\Lambda \prec v$ on $A_\varphi \times \{0\}$, and $u^\varphi \leq u^\Lambda$.
- $\overline{\Omega}(u^\varphi; Q_\varphi) \cap (\partial B_{R\sigma}(x_\varphi) \times [0, \hat{t}]) = \emptyset$ since $\overline{B}_{R\sigma}(x_\varphi) \subset \{x : x \cdot \nu \geq \sigma_2\}$, recalling (3.32) and (3.29).
- To show $u^\varphi \prec v$ on $\partial B_{2R\sigma}(x_\varphi) \times [0, \hat{t}]$ w.r.t. Q_φ , we split $\partial B_{2R\sigma}(x_\varphi)$ into three parts. By construction of S_2 and (3.33), we have

$$\partial B_{2R\sigma}(x_\varphi) \subset \{x : x \cdot \nu \leq -\sigma_1\} \cup S_2 \cup \{x : x \cdot \nu \geq \sigma_2\}.$$

But the strict separation $u^\varphi \prec v$ follows from $u^\varphi = u^1$ on $\partial B_{2R\sigma}(x_\varphi) \times [0, \hat{t}]$, and then (3.28) for the first part, (3.36) for the second part, and (3.32) with (3.29) for the third part.

Therefore $u^\varphi \prec v$ on $\overline{A}_\varphi \times [0, \hat{t}]$ by the comparison theorem, Theorem 2.18. This is however a contradiction with

$$\hat{x} \in \overline{\Omega}_{\hat{t}}(u^\varphi) \cap \Omega_{\hat{t}}^c(v),$$

since $u^\varphi(\hat{x}, \hat{t}) > 0$ due to $\hat{x} \in \overline{\Omega}_{\hat{t}}(u^\varphi)$ and $\varphi(\hat{x}) > 5\gamma\varepsilon$. This finishes the proof of the local comparison principle. \square

3.5. Cone flatness. In this part we show that long thin fingers of the positive (or zero) set cannot form, and, on the contrary, that the boundary is in fact in between two cones that are $\sim \varepsilon |\ln \varepsilon|^{1/2}$ apart.

The proof proceeds in 3 steps. Let u denote either $\underline{u}_{\varepsilon;q,r}$ or $(\overline{u}_{\varepsilon;q,r})_*$.

- 1) a supersolution $u \in \overline{\mathcal{S}}(m, Q_q)$, harmonic in its positive set can be bound from above using Proposition A.2;
- 2) the monotonicity property, Proposition 3.10, implies that u is in fact small in a whole cone;
- 3) a barrier prevents u from becoming positive too far from the boundary of the cone.

We have the following result:

Proposition 3.20. *Let q and r satisfy (3.3) and let $T > 0$. There exist positive constants K and ε_0 , both depending only on $(n, m, M, |q|, T)$, such that for any $(\zeta, \sigma) \in \Omega^c(\underline{u}_{\varepsilon;q,r})$, $\sigma \leq T$ and $\varepsilon < \varepsilon_0$*

$$\text{Cone}_{\nu, \theta^+}(\zeta + K\varepsilon |\ln \varepsilon|^{1/2} \nu) \times (-\infty, \sigma] \subset \Omega^c(\underline{u}_{\varepsilon;q,r}),$$

and similarly for any $(\zeta, \sigma) \in \Omega^c(\overline{u}_{\varepsilon;q,r})$ and $\varepsilon < \varepsilon_0$

$$\text{Cone}_{\nu, \theta^-}(\zeta + K\varepsilon |\ln \varepsilon|^{1/2} \nu) \times (-\infty, \sigma] \subset \Omega^c(\overline{u}_{\varepsilon;q,r}).$$

(Note the different opening angles θ^+ and θ^- .)

Proof. For the sake of brevity, let us denote $\underline{u} = \underline{u}_{\varepsilon; q, r}$. By Remark C.5 there exists $\lambda = \lambda(n, \theta^+) > 0$ such that

$$(3.38) \quad \text{Cone}_{\nu, \theta^+}(x) \subset (\text{Cone}_{\nu, \theta^+}(x))^{::\varepsilon} + B_{\lambda\varepsilon}(0) \quad \text{for any } x \in \mathbb{R}^n, \varepsilon > 0,$$

where the notation $::\varepsilon$ is introduced in Definition C.2.

Pick $(\zeta, \sigma) \in \Omega^c(\underline{u}) \cap \{t \leq T\}$. Note that \underline{u} is in $\overline{\mathcal{S}}(m, Q_q)$, nondecreasing in time by Lemma 3.11 and harmonic in $\Omega_t(\underline{u}; Q_q)$ by Proposition 3.12. Therefore Proposition A.2 yields

$$(3.39) \quad \underline{u}(z, t) \leq \frac{C}{m} \frac{|z - \zeta|^2}{\sigma - t} \leq \frac{C}{m} \frac{\lambda^2 \varepsilon^2}{\sigma - t} \quad \text{for all } z \in \overline{B}_{\lambda\varepsilon}(\zeta), t < \sigma.$$

By the monotonicity result of Proposition 3.10, we have

$$(3.40) \quad \underline{u}(z + y, t) \leq \underline{u}(z, t - \varepsilon)$$

for all $(z, t) \in \mathbb{R}^n \times \mathbb{R}$ and

$$(3.41) \quad y \in (\text{Cone}_{q, \theta^+}(c\varepsilon\nu))^{::\varepsilon},$$

where $c = \max(M|q|, r_V^+)$.

(3.38) implies $\text{Cone}_{\nu, \theta^+}(c\varepsilon\nu + \zeta) \subset \zeta + (\text{Cone}_{\nu, \theta^+}(c\varepsilon\nu))^{::\varepsilon} + B_{\lambda\varepsilon}(0)$ and therefore for any $x \in \text{Cone}_{\nu, \theta^+}(c\varepsilon\nu + \zeta)$ there exist $z \in B_{\lambda\varepsilon}(\zeta)$ and y satisfying (3.41), and, combining (3.40) and (3.39), we obtain

$$(3.42) \quad \underline{u}(x, t) = \underline{u}(z + y, t) \leq \underline{u}(z, t - \varepsilon) \leq \frac{C\lambda^2}{m} \frac{\varepsilon^2}{\varepsilon + \sigma - t}.$$

Let us now choose μ satisfying

$$\frac{m\mu^2}{2nMC\lambda^2\varepsilon^2} = -2 \ln \varepsilon,$$

i.e.,

$$\mu = \tilde{K} |\ln \varepsilon|^{1/2} \varepsilon.$$

Hence, provided that $\varepsilon < \varepsilon_0 = \varepsilon_0(T)$, we have

$$(3.43) \quad \varepsilon \left(e^{\frac{m\mu^2}{2nMC\lambda^2\varepsilon^2}} - 1 \right) = \varepsilon(\varepsilon^{-2} - 1) > T.$$

We choose K such that for all $\varepsilon < \varepsilon_0$

$$\text{dist} \left(\text{Cone}_{\nu, \theta^+}(K |\ln \varepsilon|^{1/2} \varepsilon \nu), \partial \text{Cone}_{\nu, \theta^+}(c\varepsilon\nu) \right) > \mu.$$

Consequently, whenever $\xi \in \text{Cone}_{\nu, \theta^+}(\zeta + K\varepsilon |\ln \varepsilon|^{1/2} \nu)$ we also have $\overline{B}_\mu(\xi) \subset \text{Cone}_{\nu, \theta^+}(\zeta + c\varepsilon\nu)$ and therefore the bound (3.42) for \underline{u} holds in $\overline{B}_\mu(\xi) \times [0, \sigma]$. Since $\sigma \leq T$ and (3.43) holds, we set $A = \frac{C\lambda^2\varepsilon^2}{m}$ and apply Corollary A.7 to conclude that $\underline{u}(\xi, t) = 0$ for $t \in [0, \sigma]$. This finishes the proof for $\underline{u}_{\varepsilon; q, r}$.

The proof for $\bar{u} = \bar{u}_{\varepsilon; q, r}$ is analogous since $\bar{u} \in \overline{\mathcal{S}}(m, Q_q)$ due to Lemma 3.8(e), $(\bar{u}_{\varepsilon; q, r})_*$ is harmonic in the positive phase by Proposition 3.12, and the correct monotonicity holds due to Proposition 3.10 with a cone $\text{Cone}_{\nu, \theta^-}$. \square

Remark 3.21. The statement of Proposition 3.20 can be reversed in the following sense: For q, r, T, K and ε_0 from Proposition 3.20 we have that if $(\zeta, \sigma) \in \Omega(\underline{u}_{\varepsilon; q, r})$, $\sigma \leq T$ and $\varepsilon < \varepsilon_0$ then

$$(3.44) \quad \text{Cone}_{-\nu, \theta^+}(\zeta - K\varepsilon |\ln \varepsilon|^{\frac{1}{2}} \nu) \times [\sigma, +\infty) \subset \Omega(\underline{u}_{\varepsilon; q, r}).$$

The proof is very simple: suppose that the inclusion (3.44) is violated, i.e., we have $(\zeta, \sigma) \in \Omega(\underline{u}_{\varepsilon; q, r})$ for which there exists (y, s) that belongs to the left-hand side of (3.44) but belongs also to $\Omega^c(\underline{u}_{\varepsilon; q, r})$. Since $s \geq \sigma$, monotonicity in time, Lemma 3.11, implies that $(y, \sigma) \in \Omega^c(\underline{u}_{\varepsilon; q, r})$ and we can apply Proposition 3.20 and conclude that $(\xi, \sigma) \in \Omega^c(\underline{u}_{\varepsilon; q, r})$ since $\xi \in \text{Cone}_{\nu, \theta^+}(y + K\varepsilon |\ln \varepsilon|^{\frac{1}{2}} \nu)$. But that is obviously a contradiction. A similar reflection argument applies for $\bar{u}_{\varepsilon; q, r}$ as well.

Since it is easier to work with a ball instead of a cone, we also formulate the following consequence of Proposition 3.20.

Corollary 3.22. *Let q and r satisfy (3.3) and let $\lambda > 0$ and $\beta \in [0, 1)$. There exists $\varepsilon_0 = \varepsilon_0(n, m, M, q, \lambda, \beta)$ such that for all $\varepsilon < \varepsilon_0$ the following statements hold:*

- if $\underline{\Phi}_{\varepsilon; q, r}(1) > \frac{3}{4}\varepsilon^\beta$ then there exists (ζ, σ) such that $\zeta \cdot \nu = r\sigma + \frac{1}{2}\varepsilon^\beta$, $\underline{\Phi}_{\varepsilon; q, r}(\sigma + \lambda\varepsilon) \leq \varepsilon^\beta$ and

$$B_{\lambda\varepsilon}(\zeta, \sigma) \subset \Omega(\underline{u}_{\varepsilon; q, r}) \cap Q_q \cap C^+ \cap \{t \leq 1\};$$

- if $\bar{\Phi}_{\varepsilon; q, r}(1) > \frac{3}{4}\varepsilon^\beta$ then there exists (ζ, σ) such that $\zeta \cdot \nu = r\sigma - \frac{1}{2}\varepsilon^\beta$, $\bar{\Phi}_{\varepsilon; q, r}(\sigma + \lambda\varepsilon) \leq \varepsilon^\beta$ and

$$B_{\lambda\varepsilon}(\zeta, \sigma) \subset \Omega^c(\bar{u}_{\varepsilon; q, r}) \cap Q_q \cap C^+ \cap \{t \leq 1\}.$$

Proof. Fix q and r . For $T = 1$ we have constants ε_0 and K from Proposition 3.20. Let us define

$$(3.45) \quad \eta := \inf \left\{ s : B_\lambda(s\nu) \subset \text{Cone}_{\nu, \theta^+}(0) \right\} = \frac{\lambda}{\sin \theta^+}.$$

Because $\beta < 1$, we may assume, making ε_0 smaller if necessary, that for all $\varepsilon < \varepsilon_0$

$$(3.46) \quad \max \left\{ r\lambda\varepsilon + \eta\varepsilon + K\varepsilon |\ln \varepsilon|^{\frac{1}{2}}, 2\lambda\varepsilon(M - m)|q| \right\} < \frac{1}{8}\varepsilon^\beta.$$

Lemma 3.15 provides a point $(\hat{\zeta}, \hat{\sigma}) \in \partial\Omega(\underline{u}_{\varepsilon; q, r})$ such that $\hat{\zeta} = r\hat{\sigma} + \frac{5}{8}\varepsilon^\beta$ and $\underline{\Phi}_{\varepsilon; q, r}(\hat{\sigma}) = \frac{5}{8}\varepsilon^\beta$. We define

$$\zeta = \hat{\zeta} + (r\lambda\varepsilon - \frac{1}{8}\varepsilon^\beta)\nu, \quad \sigma = \hat{\sigma} + \lambda\varepsilon.$$

Clearly $\zeta \cdot \nu = r\sigma + \frac{1}{2}\varepsilon^\beta$. Furthermore, due to (3.46), the definition of η in (3.45) and Remark 3.21, we have

$$B_{\lambda\varepsilon}(\zeta, \sigma) \in \text{Cone}_{-\nu, \theta^+}(\hat{\zeta} - K\varepsilon |\ln \varepsilon|^{\frac{1}{2}} \nu) \times [\hat{\sigma}, +\infty) \subset \Omega(\underline{u}_{\varepsilon; q, r}),$$

see Figure 5. Since $B_{\lambda\varepsilon}(\zeta, \sigma) \subset \Omega^c(P_{q, r})$, Proposition 3.8(d) implies $B_{\lambda\varepsilon}(\zeta, \sigma) \subset Q_q \cap C^+$.

Finally, Proposition 3.14 and (3.46) imply that

$$\begin{aligned} \underline{\Phi}_{\varepsilon; q, r}(\sigma + \lambda\varepsilon) &= \underline{\Phi}_{\varepsilon; q, r}(\hat{\sigma} + 2\lambda\varepsilon) \leq \underline{\Phi}_{\varepsilon; q, r}(\hat{\sigma}) + 2\lambda\varepsilon(M|q| - r)_+ \\ &\leq \frac{5}{8}\varepsilon^\beta + 2\lambda\varepsilon(M - m)|q| < \frac{3}{4}\varepsilon^\beta. \end{aligned}$$

In particular, $\underline{\Phi}_{\varepsilon; q, r}(\sigma + \lambda\varepsilon) \leq \varepsilon^\beta$ and $\sigma + \lambda\varepsilon \leq 1$, that is, $B_{\lambda\varepsilon}(\zeta, \sigma) \subset \{t \leq 1\}$. This finishes the proof for $\underline{u}_{\varepsilon; q, r}$. The proof for $\bar{u}_{\varepsilon; q, r}$ is analogous. \square

Take ε_0 from Corollary 3.22. By making ε_0 smaller if necessary, we can assume that

$$\lambda\varepsilon \leq \mu\varepsilon^\beta \quad \text{for } \varepsilon < \varepsilon_0.$$

This guarantees, since $-\gamma E \subset H - \frac{\gamma}{2}\varepsilon^\beta(\nu, 1)$, that

$$(3.49) \quad B_{\gamma\mu\varepsilon^\beta}(r\nu, 1) + B_{\gamma\lambda\varepsilon}(0, 0) - \gamma E = B_{\gamma\mu\varepsilon^\beta + \gamma\lambda\varepsilon}(r\nu, 1) - \gamma E \subset H$$

for all $\gamma > 0$, $\varepsilon < \varepsilon_0$.

We claim that we can find $\gamma \in (0, 1)$ such that, for all $\varepsilon \in (0, \varepsilon_0)$,

$$(3.50) \quad B_{\gamma\mu\varepsilon^\beta}(r\nu, 1) + B_{\gamma\lambda\varepsilon}(0, 0) - \gamma E \subset S_\gamma.$$

This is indeed possible by choosing γ small, because the set on the left-hand side is bounded, contained in a ball $B_{\gamma R}(r\nu, 1)$ for some $R > 0$, independent of $\varepsilon < 1$, and $(r\nu, 1)$ is in the interior of S_γ for all $\gamma < 1$. Such choice of γ and ε_0 that provides (3.49) and (3.50) then assures by Lemma C.3 that

$$(3.51) \quad B_{\gamma\mu\varepsilon^\beta}(r\nu, 1) \subset (S_\gamma \cap H)^{::\gamma\varepsilon} + B_{\gamma\lambda\varepsilon}(0, 0) + \gamma(\zeta, \sigma)$$

for any $\varepsilon < \varepsilon_0$ and $(\zeta, \sigma) \in E$, where the notation $::\gamma\varepsilon$ is introduced in Definition C.2.

By Corollary 3.22 we can find $(\zeta, \sigma) \in E$ such that

$$(3.52) \quad \underline{u}_{\varepsilon; q, r} > 0 \text{ in } B_{\lambda\varepsilon}(\zeta, \sigma).$$

Additionally, any $(y, \tau) \in -(S_\gamma \cap H)^{::\gamma\varepsilon}$ satisfies the condition (3.21) with $a = \gamma$ and also $(y, \tau) \in \gamma\varepsilon\mathbb{Z}^{n+1}$. Hence, by the monotonicity proposition 3.10,

$$\underline{u}_{\gamma\varepsilon; q, r}(x, t) \geq \gamma \underline{u}_{\varepsilon; q, r}\left(\frac{x+y}{\gamma}, \frac{t+\tau}{\gamma}\right) > 0$$

for all (x, t) such that $\gamma^{-1}(x+y, t+\tau) \in B_{\lambda\varepsilon}(\zeta, \sigma)$; this can be rewritten as

$$(x, t) \in (S_\gamma \cap H)^{::\gamma\varepsilon} + B_{\gamma\lambda\varepsilon}(0, 0) + \gamma(\zeta, \sigma).$$

Therefore $\underline{u}_{\gamma\varepsilon; q, r} > 0$ in $B_{\gamma\mu\varepsilon^\beta}(r\nu, 1)$ by (3.51). \square

3.6. Homogenized velocity. For given fixed $\beta \in (4/5, 1)$ and $q \neq 0$, we define the lower and upper homogenized velocities

$$(3.53a) \quad \bar{r}(q) := \sup \left\{ r > 0 : \limsup_{\varepsilon \rightarrow 0} \varepsilon^{-\beta} \bar{\Phi}_{\varepsilon; q, r}(1) < 1 \right\},$$

$$(3.53b) \quad \underline{r}(q) := \inf \left\{ r > 0 : \limsup_{\varepsilon \rightarrow 0} \varepsilon^{-\beta} \underline{\Phi}_{\varepsilon; q, r}(1) < 1 \right\}.$$

If $q = 0$ we set $\bar{r}(0) = \underline{r}(0) = 0$. Heuristically, if $r > \bar{r}(q)$ then $\bar{u}_{\varepsilon; q, r}(\cdot, t)$ does not have ε^β -flat free boundary up to time 1 for a subsequence of ε converging to 0. Similarly, if $r < \underline{r}(q)$ then $\underline{u}_{\varepsilon; q, r}$ does not have ε^β -flat free boundary up to time 1 for a subsequence of ε converging to 0.

The quantities \bar{r} and \underline{r} are two candidates for the homogenized velocity r . In this and in the following subsection we show that they give rise to the same homogenized velocity r (Proposition 3.28).

Proposition 3.24. *For all $q \in \mathbb{R}^n$, the homogenized velocities $\bar{r}(q)$ and $\underline{r}(q)$ introduced in (3.53) are well defined. Moreover,*

$$\bar{r}(q) \in [m|q|, M|q|], \quad \underline{r}(q) \in [m|q|, M|q|].$$

Proof. We will show the statement for $\bar{r}(q)$, the proof for $\underline{r}(q)$ is similar.

First, we consider $r \in (0, m|q|]$. Due to Proposition 3.1, $P_{q,r} \in \underline{\mathcal{S}}(g^\varepsilon, Q_q)$ and therefore $\bar{u}_{\varepsilon;q,r} = P_{q,r}$, by definition. In particular, $\bar{\Phi}_{\varepsilon;q,r} \equiv 0$ and we conclude that $\bar{r}(q) \geq m|q|$.

On the other hand, if $r > M|q|$, we can find r_2, q_2 such that $q_2 = a^3q$ for some $a > 1$ and $M|q_2| = r_2 < r_1$. Since $P_{q_2,r_2} \in \bar{\mathcal{S}}(g^\varepsilon, Q_q)$, we now have $\underline{u}_{\varepsilon;q,r} = P_{q_2,r_2}$, which is ε^β -flat for all $\varepsilon > 0$. Therefore, by the local comparison, Theorem 3.16, there exists ε_0 such that $\bar{u}_{\varepsilon;q,r}$ cannot be ε^β -flat for any $\varepsilon < \varepsilon_0$. We conclude that $\bar{r}(q) \leq M|q|$. \square

\bar{r} and \underline{r} also enjoy the following monotonicity property.

Proposition 3.25. *For any $a > 1$ and $q \in \mathbb{R}^n$ we have*

$$\bar{r}(q) \leq \bar{r}(aq) \qquad \underline{r}(q) \leq \underline{r}(aq).$$

Proof. The statement is trivial for $q = 0$. Hence we assume that $q \neq 0$ and $a > 1$. Suppose that $\bar{r}(q) > \bar{r}(aq)$. By definition of \bar{r} , there exists r with $\bar{r}(q) \geq r > \bar{r}(aq)$ and $\varepsilon_0 > 0$ such that $\limsup_{\varepsilon \rightarrow 0} \varepsilon^{-\beta} \bar{\Phi}_{\varepsilon;q,r}(1) < \theta < 1$ for some θ . Observe that due to Proposition B.1 we have $a\bar{u}_{\varepsilon;q,r} \in \underline{\mathcal{S}}(g^\varepsilon, Q_q)$, and clearly $a\bar{u}_{\varepsilon;q,r} \leq aP_{q,r} = P_{aq,r}$. Therefore, by definition of $\bar{u}_{\varepsilon;aq,r}$ and flatness of $\bar{u}_{\varepsilon;q,r}$,

$$P_{aq,r}^{-\varepsilon^\beta \theta} = aP_{q,r}^{-\varepsilon^\beta \theta} \leq a\bar{u}_{\varepsilon;q,r} \leq \bar{u}_{\varepsilon;aq,r} \quad \text{in } \{t \leq 1\}, \varepsilon \ll 1,$$

and we conclude that $\limsup_{\varepsilon \rightarrow 0} \varepsilon^{-\beta} \bar{\Phi}_{\varepsilon;aq,r}(1) \leq \theta < 1$ and therefore $r \leq \bar{r}(aq)$, a contradiction. The proof for \underline{r} is analogous. \square

However, the local comparison principle, Theorem 3.16, also provides some ordering between \underline{r} and \bar{r} .

Corollary 3.26. *For any $q \in \mathbb{R}^n$ and $a > 1$ we have*

$$\bar{r}(q) \leq \underline{r}(aq).$$

Proof. Again, the ordering is obvious if $q = 0$. Therefore let us assume that $q \neq 0$ and $a > 1$. Suppose that the ordering does not hold, i.e., $\underline{r}(aq) < \bar{r}(q)$.

Set $q_1 = q$ and $q_2 = aq$. By the definition of \bar{r} and \underline{r} , there are r_1 and r_2 with

$$\underline{r}(aq) < r_2 < r_1 < \bar{r}(q)$$

such that

$$\limsup_{\varepsilon \rightarrow 0} \varepsilon^{-\beta} \max(\bar{\Phi}_{\varepsilon;q_1,r_1}(1), \underline{\Phi}_{\varepsilon;q_2,r_2}(1)) < 1.$$

But that is in contradiction with the conclusion of Theorem 3.16. \square

3.7. Semi-continuity. In this section we will prove that $\underline{r}(q) \leq \bar{r}(q)$, and that in fact \bar{r} and \underline{r} are semi-continuous. We start with the following geometric result.

Lemma 3.27. *For every $q \in \mathbb{R}^n$, $\eta > 0$ there exists $\delta > 0$ such that*

$$\underline{r}(p) \leq \bar{r}(q) + \eta, \qquad \underline{r}(q) \leq \bar{r}(p) + \eta,$$

for all $p \in \mathbb{R}^n$, $|p - q| < \delta$.

Proof. By the continuity of \bar{r} and \underline{r} at 0 (Proposition 3.24), we can assume that $q \neq 0$.

We will prove the second inequality. The proof of the first one is analogous. Let us further assume that $\underline{r}(q) - \eta > 0$. We set $r = \underline{r}(q) - \eta/2$ and $\hat{r} = \underline{r}(q) - \eta = r - \eta/2 < r$.

Let $\lambda := \sqrt{n+1}$. By the definition of $\underline{r}(q)$, there exists $\varepsilon > 0$ small enough such that Corollary 3.22 applies and there exists (ζ, σ) such that $B_{\lambda\varepsilon}(\zeta, \sigma) \subset \Omega(\underline{u}_{\varepsilon; q, r}) \cap \{0 \leq t \leq 1\}$ and $\zeta \cdot \nu = r\sigma + \varepsilon^\beta/2$. Moreover, $\underline{\Phi}_{\varepsilon; q, r}(\sigma + \lambda\varepsilon) \leq \varepsilon^\beta$.

Suppose that the conclusion does not hold and there exists a sequence $\{q_k\}_k \subset \mathbb{R}^n$ such that $q_k \rightarrow q$ as $k \rightarrow \infty$ and $\underline{r}(q) > \bar{r}(q_k) + \eta$. By the definition of \bar{r} , there exists $\varepsilon_k \in (0, \varepsilon)$, $\varepsilon_k \rightarrow 0$ such that $\bar{\Phi}_{\varepsilon_k; q_k, \hat{r}} \geq \frac{3}{4}\varepsilon_k^\beta$. We will show that this leads to a contradiction with the comparison principle for k large enough.

Recall that we set $\nu = -q/|q|$ and $\nu_k = -q_k/|q_k|$. By Lemma 3.15, there exists $(\zeta_k, \sigma_k) \in \Gamma(\bar{u}_{\varepsilon_k; q_k, \hat{r}})$ such that $\zeta_k \cdot \nu_k = \hat{r}\sigma_k - \frac{1}{2}\varepsilon_k^\beta$ and $\bar{\Phi}_{\varepsilon_k; q_k, \hat{r}}(\sigma_k) = \frac{1}{2}\varepsilon_k^\beta$.

We introduce the scaling factor $\gamma_k := \varepsilon_k/\varepsilon$. By the choice of λ we can find $(\xi_k, \tau_k) \in \varepsilon\mathbb{Z}^{n+1}$ such that

$$(X_k, T_k) := (\xi_k, \tau_k) + \gamma_k^{-1}(\zeta_k, \sigma_k) \in B_{\lambda\varepsilon}(\zeta, \sigma).$$

By compactness and taking a subsequence, we can assume that $(X_k, T_k) \rightarrow (X, T) \in \bar{B}_{\lambda\varepsilon}(\zeta, \sigma)$. Let v_k be the rescaled solution

$$v_k(x, t) = \gamma_k^{-1}\bar{u}_{\varepsilon_k; q_k, \hat{r}}(\gamma_k(x - \xi_k), \gamma_k(t - \tau_k)).$$

Observe that $(X_k, T_k) \in \Gamma(v_k)$ and $(X_k, T_k) \in B_{\lambda\varepsilon}(\zeta, \sigma) \subset \Omega(\underline{u}_{\varepsilon; q, r})$. If we prove that $\underline{u}_{\varepsilon; q, r} \leq v_k$ in $B_{\lambda\varepsilon}(\zeta, \sigma)$ for large k , we get a contradiction.

To that end, we will use the monotonicity of the obstacle problem and the fact that $\bar{u}_{\varepsilon_k; q_k, \hat{r}}$ is a solution away from the boundary of the obstacle. Let $\Sigma_k := \gamma_k^{-1}Q_{q_k} + (\xi_k, \tau_k)$. Furthermore, let us define the translated planar solutions

$$P_k(x, t) := P_{q_k, \hat{r}}(x - \xi_k, t - \tau_k) \quad P_k^{-\rho_k} := P_{q_k, \hat{r}}^{-\rho_k}(x - \xi_k, t - \tau_k),$$

where $\rho_k := \frac{1}{2}\gamma_k^{-1}\varepsilon_k^\beta = \frac{1}{2}\varepsilon_k^{\beta-1}$. Note that $\rho_k \rightarrow \infty$ since $\varepsilon_k \rightarrow 0$ and $\beta < 1$.

We observe that $v_k \in \mathcal{S}(g^\varepsilon, \Sigma_k \setminus \Gamma(P_k))$. Therefore it is enough to show that

- (a) $v_k \geq P_{q, r}$ in $Q_q \cap \{t \leq T_k\}$,
- (b) $\bar{\Omega}(\underline{u}_{\varepsilon; q, r}) \cap \{t \leq T_k\} \cap \Gamma(P_k) = \emptyset$, and
- (c) $\underline{u}_{\varepsilon; q, r} \leq (v_k)_*$ on $\partial_P(Q_q \cap \Sigma_k \cap \{t \leq T_k\})$.

By the properties of the obstacle problem, Proposition 3.8, we have

$$\underline{u}_{\varepsilon; q, r} = P_{q, r} \quad \text{on } \partial Q_q \quad (v_k)_* = P_k \quad \text{on } \partial \Sigma_k.$$

Moreover, the flatness assumptions provide the bounds

$$\underline{u}_{\varepsilon; q, r} \leq P_{q, r}^{\varepsilon^\beta} \quad v_k \geq P_k^{-\rho_k}, \quad \text{both in } \{t \leq T_k\}.$$

Therefore, since $\partial_P(Q_q \cap \Sigma_k \cap \{t \leq T_k\}) \subset (\partial Q_q \cup \partial \Sigma_k) \cap \bar{Q}_q$, we only have to show that, for large k ,

$$(3.54) \quad P_k > P_{q, r}^{\varepsilon^\beta} \quad \text{in } \bar{Q}_q \cap \{t \leq T_k\} \cap \bar{\Omega}(P_{q, r}^{\varepsilon^\beta})$$

and

$$(3.55) \quad P_k^{-\rho_k} > P_{q, r} \quad \text{in } \bar{Q}_q \cap \{t \leq T_k\} \cap \bar{\Omega}(P_{q, r}),$$

and (a)–(c) shall follow. A straightforward calculation yields

$$P_k^{-\rho_k}(x, t) = P_{q_k, \hat{r}}(x - X_k, t - T_k) \rightarrow P_{q, \hat{r}}(x - X, t - T), \quad k \rightarrow \infty$$

where the convergence is locally uniform in $\mathbb{R}^n \times \mathbb{R}$. A similar calculation yields

$$P_k(x, t) = P_{q_k, \hat{r}}^{\rho_k}(x - X_k, t - T_k) \rightarrow \infty, \quad k \rightarrow \infty$$

locally uniformly, since $\rho_k \rightarrow \infty$. Finally, we realize that

$$P_{q, \hat{r}}(x - X, t - T) > P_{q, r} \quad \text{in } \overline{Q}_q \cap \{t \leq \sigma + \lambda\varepsilon\} \cap \overline{\Omega}(P_{q, r}),$$

due to $\hat{r} < r$ and $X \cdot \nu \geq rT + \varepsilon^\beta/4$, and we conclude that the orderings (3.54) and (3.55) hold for sufficiently large k . Therefore by the stitching lemma, Lemma 2.17, and obstacle monotonicity we have that $\underline{u}_{\varepsilon; q, r} \leq v_k$ in $B_{\lambda\varepsilon}(\zeta, \sigma)$ for large k but that is a contradiction with the facts that $\underline{u}_{\varepsilon; q, r} > 0$ on $B_{\lambda\varepsilon}(\zeta, \sigma)$ and $(X_k, T_k) \in B_{\lambda\varepsilon}(\zeta, \sigma) \cap \Gamma(v_k)$. The proof of the lemma is finished. \square

The important consequence of the previous Lemma 3.27 is the semi-continuity of \bar{r} and \underline{r} .

Proposition 3.28. *The functions \bar{r} and \underline{r} are semi-continuous on \mathbb{R}^n and*

$$\bar{r} = \underline{r}^* \in USC(\mathbb{R}^n), \quad \underline{r} = \bar{r}_* \in LSC(\mathbb{R}^n).$$

Furthermore, $a \mapsto \bar{r}(aq)$ and $a \mapsto \underline{r}(aq)$ are increasing on $(0, \infty)$ for any $q \in \mathbb{R}^n$. (Recall that we set $\bar{r}(0) = \underline{r}(0) = 0$.)

Proof. Let us fix $q \in \mathbb{R}^n$ and $\eta > 0$, and let $\delta > 0$ be from Lemma 3.27. We will show that \bar{r} is USC at q . First, we observe that whenever $p \in \mathbb{R}^n$ and $|p - q| < \delta$, there exists $a > 1$ such that $|ap - q| < \delta$. Therefore, for such p and a , Corollary 3.26 and Lemma 3.27 yield

$$\bar{r}(p) - \eta \leq \underline{r}(ap) - \eta \leq \bar{r}(q).$$

Since $\delta > 0$ exists for arbitrary $\eta > 0$, we conclude that \bar{r} is USC at q for all $q \in \mathbb{R}^n$, and thus $\bar{r} \in USC(\mathbb{R}^n)$. We can similarly show that $\underline{r} \in LSC(\mathbb{R}^n)$.

Furthermore, we have $\underline{r} \leq \bar{r}$ on \mathbb{R}^n by Lemma 3.27 which together with the semi-continuity implies

$$\underline{r} \leq \bar{r}_*, \quad \underline{r}^* \leq \bar{r}.$$

On the other hand, Corollary 3.26 yields $\bar{r}(a^{-1}q) \leq \underline{r}(q) \leq \bar{r}(q) \leq \underline{r}(aq)$ for any $a > 1$ and therefore

$$\bar{r}_*(q) \leq \liminf_{a \rightarrow 1^+} \bar{r}(a^{-1}q) \leq \underline{r}(q) \leq \bar{r}(q) \leq \limsup_{a \rightarrow 1^+} \underline{r}(aq) \leq \underline{r}^*(q) \quad q \in \mathbb{R}^n.$$

Finally, the monotonicity of $\bar{r}(aq)$ and $\underline{r}(aq)$ in a is a simple consequence of

$$\underline{r}(a_1q) \leq \bar{r}(a_1q) \leq \underline{r}(a_2q) \leq \bar{r}(a_2q) \quad 0 < a_1 < a_2, \quad q \in \mathbb{R}^n.$$

\square

3.8. Time-independent medium.

In this section, we show that in a time-independent medium, i.e., when $g(x, t) = g(x)$, we can in fact prove that $\bar{r} = \underline{r} \in C(\mathbb{R}^n)$ and that $\bar{r}(q) = \underline{r}(q) = |q|r(q/|q|)$, and thus obtain the same results as in [18]. This fact is the consequence of the key difference from time-dependent medium, that is, the availability of scaling in time independently of space. Indeed, it is trivial to check that if $u \in \mathcal{S}(g^\varepsilon)$ then also $[(x, t) \mapsto au(x, at)] \in \mathcal{S}(g^\varepsilon)$ for any $\varepsilon > 0$ and $a > 0$.

Let us first show the continuity. We will need the following scaling result.

Lemma 3.29. *Suppose that $g(x, t) = g(x)$. Then for any $q \in \mathbb{R}^n$ and $\varepsilon, r > 0$ we have*

$$(3.56) \quad \underline{\Phi}_{\varepsilon; aq, ar}(t) \leq a \underline{\Phi}_{a^{-1}\varepsilon; q, r}(t) \quad \text{for all } a > 1 \text{ and } t \in \mathbb{R}.$$

Proof. Since the statement is trivial for $q = 0$, we assume $q \neq 0$. Fix $a > 1$ and define

$$v(x, t) = a^2 \underline{u}_{a^{-1}\varepsilon; q, r}(a^{-1}x, t).$$

Since $\underline{u}_{a^{-1}\varepsilon; q, r} \in \overline{\mathcal{S}}(g^{a^{-1}\varepsilon}, Q_q)$, the scaling of the problem and the independence of g on time yield that $v \in \overline{\mathcal{S}}(g^\varepsilon, aQ_q)$. Moreover,

$$v(x, t) \geq a^2 P_{q, r}(a^{-1}x, t) = P_{aq, ar}(x, t).$$

Therefore, by the definition of $\underline{u}_{\varepsilon; aq, ar}$ and $Q_q \subset aQ_q$, we get

$$\underline{u}_{\varepsilon; aq, ar} \leq v \quad \text{in } \mathbb{R}^n \times \mathbb{R}.$$

By the definition of flatness, we have for any $\tau \in \mathbb{R}$ and $t \leq \tau$

$$\begin{aligned} \underline{u}_{\varepsilon; aq, ar}(x, t) &\leq v(x, t) \leq a^2 P_{q, r}(a^{-1}x - \underline{\Phi}_{a^{-1}\varepsilon; q, r}(\tau)\nu, t) \\ &= P_{aq, ar}(x - a \underline{\Phi}_{a^{-1}\varepsilon; q, r}(\tau)\nu, t). \end{aligned}$$

Thus (3.56) follows. \square

Proposition 3.30. *Suppose that $g(x, t) = g(x)$. Then*

$$\bar{r} = \underline{r} \in C(\mathbb{R}^n).$$

Proof. Fix $q \in \mathbb{R}^n$. Since $\bar{r}(0) = \underline{r}(0)$ by definition, we shall assume that $q \neq 0$. By Lemma 3.27 and Corollary 3.26 we infer

$$\underline{r}(q) \leq \bar{r}(q) \leq \underline{r}(aq) \quad \text{for all } a > 1.$$

Now fix $\delta > 0$. By definition of $\underline{r}(q)$, there exists $r \in [\underline{r}(q), \underline{r}(q) + \delta/2]$ such that $\limsup_{\varepsilon \rightarrow 0} \varepsilon^{-\beta} \underline{\Phi}_{\varepsilon; q, r}(1) = \theta < 1$. Now Lemma 3.29 for any $a \in (1, \theta^{1/(\beta-1)})$ yields

$$\limsup_{\varepsilon \rightarrow 0} \varepsilon^{-\beta} \underline{\Phi}_{\varepsilon; aq, ar}(1) \leq \limsup_{\varepsilon \rightarrow 0} a^{1-\beta} (a^{-1}\varepsilon)^{-\beta} \underline{\Phi}_{a^{-1}\varepsilon; q, r}(1) = a^{1-\beta} \theta < 1.$$

Therefore if also $a < 1 + \delta/2r$, the definition of $\underline{r}(aq)$ implies

$$\underline{r}(aq) \leq ar \leq r + \delta/2 \leq \underline{r}(q) + \delta.$$

Since $\delta > 0$ was arbitrary, we have $\bar{r}(q) = \underline{r}(q)$. The semi-continuity result of Proposition 3.28 implies $\bar{r} = \underline{r} \in C(\mathbb{R}^n)$. \square

For the proof of the one-homogeneity, we will use the following scaling result.

Lemma 3.31. *Suppose that $g(x, t) = g(t)$. Then for any $q \in \mathbb{R}^n$ and $\varepsilon, r > 0$ we have*

$$(3.57) \quad \underline{\Phi}_{\varepsilon; q, r}(t) = \underline{\Phi}_{\varepsilon; aq, ar}(a^{-1}t) \quad \text{for all } a > 0, t \in \mathbb{R}.$$

The same holds for $\bar{\Phi}$.

Proof. Let us first observe that a simple calculation yields

$$(3.58) \quad a P_{q, r}^\eta(x, at) = P_{aq, ar}^\eta(x, t) \quad \text{for all } q \in \mathbb{R}^n, a, r > 0.$$

Fix $a > 0$ and consider $v(x, t) = a\underline{u}_{\varepsilon; q, r}(x, at)$. Since g is time-independent, we have $v \in \overline{\mathcal{S}}(g^\varepsilon, Q_q)$ and (3.58) yields $v \geq P_{aq, ar}$. Therefore, by definition of $\underline{u}_{\varepsilon; aq, ar}$, we obtain

$$(3.59) \quad \underline{u}_{\varepsilon; aq, ar}(x, t) \leq v(x, t) = a\underline{u}_{\varepsilon; q, r}(x, at).$$

Similarly, we set $w(x, t) = a^{-1}\underline{u}_{\varepsilon; aq, ar}(x, a^{-1}t)$ and observe that $w \in \overline{\mathcal{S}}(g^\varepsilon, Q_q)$ and $w \geq P_{q, r}$. Therefore

$$(3.60) \quad \underline{u}_{\varepsilon; q, r}(x, t) \leq w(x, t) = a^{-1}\underline{u}_{\varepsilon; aq, ar}(x, a^{-1}t).$$

Since $a > 0$ was arbitrary, (3.59) and (3.60) yield together

$$\underline{u}_{\varepsilon; q, r}(x, t) = a^{-1}\underline{u}_{\varepsilon; aq, ar}(x, a^{-1}t).$$

The scaling of the planar solutions in (3.58) and the definition of flatness thus yield (3.57).

The same statement for $\overline{\Phi}$ can be obtained analogously. \square

Proposition 3.32 (Homogeneity). *Suppose that $g(x, t) = g(x)$. Then*

$$\bar{r}(q) = \underline{r}(q) = |q| \underline{r}\left(\frac{q}{|q|}\right) \quad \text{for all } q \in \mathbb{R}^n \setminus \{0\}.$$

Proof. Let us fix $q \in \mathbb{R}^n \setminus \{0\}$ and $a \in (0, 1)$. Then Proposition 3.14 and Lemma 3.31 yield

$$\overline{\Phi}_{\varepsilon; aq, ar}(1) \leq \overline{\Phi}_{\varepsilon; aq, ar}(a^{-1}) \leq \overline{\Phi}_{\varepsilon; q, r}(1).$$

Thus the definition of \bar{r} implies

$$\bar{r}(aq) \geq a\bar{r}(q),$$

and an analogous argument with $\underline{\Phi}$ yields

$$\underline{r}(aq) \leq a\underline{r}(q).$$

Since $\underline{r} = \bar{r}$ by Proposition 3.30, we can conclude that

$$(3.61) \quad \underline{r}(aq) = \bar{r}(aq) = a\underline{r}(q) = a\bar{r}(q) \quad \text{for all } a \in (0, 1).$$

Since $q \in \mathbb{R}^n \setminus \{0\}$ was arbitrary, we can define $q' = aq$ and show

$$a^{-1}\underline{r}(q') = a^{-1}\bar{r}(q') = \underline{r}(a^{-1}q') = \bar{r}(a^{-1}q') \quad \text{for all } a \in (0, 1).$$

Therefore (3.61) holds for all $q \in \mathbb{R}^n$ and $a > 0$. In particular, if $q \neq 0$, for $a = 1/|q|$ which concludes the proof. \square

4. CONVERGENCE IN THE HOMOGENIZATION LIMIT

In the view of Proposition 3.28, we define the homogenized Hele-Shaw problem

$$(4.1) \quad \begin{cases} -\Delta u = 0, & \text{in } \{u > 0\} \cap Q, \\ V = \bar{r}(Du), & \text{on } \partial\{u > 0\} \cap Q \end{cases}$$

with the appropriate domain $Q := \Omega \times (0, T]$ and the initial and boundary data Ω_0, ψ as in Theorem 1.1. The viscosity solutions of this problem were introduced in Section 2.

We show that the solutions of the inhomogeneous Hele-Shaw problem (1.1) converge as $\varepsilon \rightarrow 0$ to the unique solution of the homogenized problem (4.1) with the same initial and boundary data, in the sense of half-relaxed limits. Since the

solution of the homogenized problem might be discontinuous, a half-relaxed limit is the standard mode of convergence in such setting.

Definition 4.1. *Suppose that $\{v_\varepsilon\}_{\varepsilon>0}$ is a family of functions $v_\varepsilon : E \rightarrow \mathbb{R}$ on some set $E \subset \mathbb{R}^n \times \mathbb{R}$. For $(x, t) \in \overline{E}$ we define the upper and lower half-relaxed limits*

$$\begin{aligned} \star\text{-limsup}_{\varepsilon \rightarrow 0} v_\varepsilon(x, t) &:= \limsup_{\eta \rightarrow 0} \{v_\varepsilon(y, s) : (y, s) \in E, |y - x| + |s - t| + \varepsilon \leq \eta\} \\ \star\text{-liminf}_{\varepsilon \rightarrow 0} v_\varepsilon(x, t) &:= \liminf_{\eta \rightarrow 0} \{v_\varepsilon(y, s) : (y, s) \in E, |y - x| + |s - t| + \varepsilon \leq \eta\}. \end{aligned}$$

Let us recall that $v_\varepsilon \rightrightarrows v$ uniformly as $\varepsilon \rightarrow 0$ on a compact set $K \subset E$, where $v \in C(K)$, if $\star\text{-limsup}_{\varepsilon \rightarrow 0} v_\varepsilon = \star\text{-liminf}_{\varepsilon \rightarrow 0} v_\varepsilon = v$ on K .

We proceed with proving the homogenization result. First, we have to address the convergence of the support which is important for a stability of subsolutions of one-phase problems such as (1.6). Nondecreasing-in-time subsolutions in fact satisfy $\overline{\Omega}(\bar{v}) = \limsup_{\varepsilon \rightarrow 0} \overline{\Omega}(v_\varepsilon)$, see [19, Lemma 4.1], but we will only need the following (we give the proof here for completeness).

Lemma 4.2. *Let $v^\varepsilon \in \mathcal{S}(M, Q)$ be a locally uniformly bounded sequence of subsolutions on the spacetime cylinder $Q = \Omega \times (0, T]$ for open domain Ω and $T > 0$, and let $\bar{v} = \star\text{-limsup}_{\varepsilon \rightarrow 0} v^\varepsilon$. Moreover, suppose that $(v^\varepsilon)^*$ are nondecreasing in time and $\overline{\Omega}_0(v^\varepsilon; Q) \subset \overline{\Omega}_0(\bar{v}; Q)$ for all $\varepsilon > 0$.*

If $\varepsilon_k \rightarrow 0$ as $k \rightarrow \infty$, $(x_k, t_k) \in \overline{\Omega}(v_{\varepsilon_k}; Q)$ for all k and $(x_k, t_k) \rightarrow (\hat{x}, \hat{t})$ then $(\hat{x}, \hat{t}) \in \overline{\Omega}(\bar{v}; Q) \cup \partial_P Q$.

Proof. Suppose that $\varepsilon_k \rightarrow 0$ as $k \rightarrow \infty$ and $(x_k, t_k) \in \overline{\Omega}(v_{\varepsilon_k})$ such that $(x_k, t_k) \rightarrow (\hat{x}, \hat{t}) \in \overline{Q}$.

We argue by contradiction and assume that $(\hat{x}, \hat{t}) \notin \overline{\Omega}(\bar{v}) \cup \partial_P Q$. Let us denote $H_\rho(\xi, \sigma) = B_\rho(\xi, \sigma) \cap \{t \leq T\}$ in the following. In particular, there exists $\delta > 0$ such that $\overline{H}_{2\delta}(\hat{x}, \hat{t}) \subset Q \setminus \overline{\Omega}(\bar{v})$. Since $(v_\varepsilon)^*$ is nondecreasing in time, so is \bar{v} and thus

$$\overline{B}_\delta(x_k) \cap \overline{\Omega}_0(v_\varepsilon) \subset \overline{B}_{2\delta}(\hat{x}) \cap \overline{\Omega}_0(\bar{v}) = \emptyset$$

for k large so that $\overline{B}_\delta(x_k) \subset B_{2\delta}(\hat{x})$. Hence the nondegeneracy property of subsolutions, Corollary A.5, applies and, used with the monotonicity in time, yields the estimate

$$\sup_{H_{2\delta}(\hat{x}, \hat{t})} v_{\varepsilon_k} \geq \sup_{H_\delta(x_k, t_k)} v_{\varepsilon_k} \geq \frac{\delta^2}{2nMt_k} \rightarrow \frac{\delta^2}{2nM\hat{t}} > 0 \quad \text{as } k \rightarrow \infty.$$

But this is a contradiction with the property of $\star\text{-limsup}$, namely that v_{ε_k} converges to 0 uniformly on $\overline{H}_{2\delta}(\hat{x}, \hat{t})$ as $k \rightarrow \infty$. The proof is complete. \square

Corollary 4.3. *Let v_ε be a sequence of subsolutions as in Lemma 4.2. Then $\bar{v} = \star\text{-limsup}_{\varepsilon \rightarrow 0} v_\varepsilon$ satisfies Definition 2.3(i).*

Proof. Suppose that \bar{v} does not satisfy Definition 2.3(i) and there exists

$$(\xi, \tau) \in (\overline{\Omega}(\bar{v}) \cap Q) \setminus \overline{\Omega(\bar{v}) \cap \{t < \tau\}}.$$

Then for some $\delta > 0$ we have

$$(4.2) \quad \overline{B}_{2\delta}(\xi, \tau) \cap \{t \leq T\} \subset Q \setminus \overline{\Omega(\bar{v}) \cap \{t < \tau\}}.$$

Let $K = \sup_{\Omega \times [0, \tau]} v_\varepsilon < \infty$ and find $h \in (0, \delta)$ such that

$$\sqrt{2nKMh} < \delta.$$

By the definition of \star -limsup there exists sequences $\varepsilon_k \rightarrow 0$ and $(\xi_k, \tau_k) \rightarrow (\xi, \tau)$ as $k \rightarrow \infty$ such that $(\xi_k, \tau_k) \in \Omega(v_{\varepsilon_k})$. Therefore by Corollary A.6 we conclude that there exist $(x_k, t_k) \in \overline{\Omega}(v_{\varepsilon_k}) \cap \overline{B}_{2\delta}(\xi, \tau) \cap \{t \leq \tau - h\}$. By compactness, $(x_k, t_k) \rightarrow (\hat{x}, \hat{t}) \subset \overline{B}_{2\delta}(\xi, \tau) \cap \{t \leq \tau - h\}$ as $k \rightarrow \infty$ up to a subsequence, and Lemma 4.2 implies that $(\hat{x}, \hat{t}) \in \overline{\Omega}(\bar{v})$. But that is a contradiction with (4.2). \square

In the following proposition we show that the half-relaxed limits of the solutions of the inhomogeneous Hele-Shaw problem (1.1) are solutions of the homogeneous problem (4.1).

Proposition 4.4. *Let v_ε be a locally uniformly bounded sequence of subsolutions of (1.1) on the spacetime cylinder $Q = \Omega \times (0, T]$ satisfying also the hypothesis of Lemma 4.2. Then the upper half-relaxed limit*

$$\bar{v} = \star\text{-limsup}_{\varepsilon \rightarrow 0} v_\varepsilon$$

is a viscosity subsolution of the homogenized problem (4.1) on Q .

Similarly, if v_ε is a locally uniformly bounded sequence of supersolutions of (1.1) on Q then the lower half-relaxed limit

$$\underline{v} = \star\text{-liminf}_{\varepsilon \rightarrow 0} v_\varepsilon$$

is a supersolution of (4.1) on Q .

Proof. We will prove that \bar{v} is a subsolution of (4.1). A parallel, simpler argument yields that \underline{v} is a supersolution; in this case there is no need to study the convergence of supports.

We shall use Definition 2.3. Its part (i) follows from Corollary 4.3.

To prove Definition 2.3(ii) we shall argue by contradiction. If \bar{v} fails to be a subsolution at some point P of its free boundary, it is possible to compare v_ε with the subsolution of the obstacle problem $\bar{u}_{\varepsilon; q, r}$ for some small $\varepsilon, \hat{\varepsilon}$ and appropriate q, r , in the neighborhood of P and it can be shown that this subsolution must be detached enough from its obstacle to prevent the free boundary from reaching P in the limit $\varepsilon \rightarrow 0$.

First, we can assume that v_ε are *USC* since that does not change \star -limsup. Let us therefore suppose that $\bar{v} - \phi$ has a local maximum at (\hat{x}, \hat{t}) in $\overline{\Omega}(\bar{v}) \cap \{t \leq \hat{t}\}$ for some $\phi \in C_{x, t}^{2,1}$. If $\bar{v}(\hat{x}, \hat{t}) > 0$ then a standard stability argument yields that $-\Delta\phi(\hat{x}, \hat{t}) \leq 0$ and thus \bar{v} satisfies Definition 2.3(ii-1).

Hence suppose that \bar{v} does not satisfy Definition 2.3(ii-2) and $\bar{v}(\hat{x}, \hat{t}) = 0$, $-\Delta\phi(\hat{x}, \hat{t}) > 0$, $q := D\phi(\hat{x}, \hat{t}) \neq 0$ and $\phi_t > \bar{r}(D\phi)|D\phi|$ at (\hat{x}, \hat{t}) . In this case we can find $r \in \mathbb{R}$ such that

$$\frac{\phi_t}{|D\phi|} > r > \bar{r}(q) = \bar{r}(D\phi) \quad \text{at } (\hat{x}, \hat{t}).$$

Let us first observe that we can assume that $\phi \in C^\infty$. In fact, let us take

$$\psi(x, t) = |q| r(t - \hat{t}) + q \cdot (x - \hat{x}) + \frac{1}{2} A(x - \hat{x}) \cdot (x - \hat{x}),$$

where $A = D^2\phi(\hat{x}, \hat{t}) + \eta I$, with $\eta > 0$ chosen small so that $-\Delta\psi = -\text{tr } A = -\Delta\phi(\hat{x}, \hat{t}) - n\eta > 0$. By the Taylor's theorem, there exists $\delta > 0$ such that $\phi(x, t) - \psi(x, t) \leq \psi(x, t)$ for $|x - \hat{x}| \leq \delta$, $\hat{t} - \delta \leq t \leq \hat{t}$, and the equality holds only at (\hat{x}, \hat{t}) . Therefore $\bar{v} - \psi$ has a strict maximum 0 at (\hat{x}, \hat{t}) in $\overline{\Omega}(\bar{v}) \cap \{t \leq \hat{t}\} \cap \overline{B}_\delta(\hat{x}, \hat{t})$.

Consequently, along a subsequence of $\varepsilon \rightarrow 0$, using Lemma 4.2, there exist $(x_\varepsilon, t_\varepsilon)$ such that $(x_\varepsilon, t_\varepsilon) \rightarrow (\hat{x}, \hat{t})$ as $\varepsilon \rightarrow 0$ and $v_\varepsilon - \psi$ has a local maximum at $(x_\varepsilon, t_\varepsilon)$ in $\overline{\Omega}(v_\varepsilon) \cap \{t \leq t_\varepsilon\} \cap \overline{B}_\delta(\hat{x}, \hat{t})$. To simplify the notation in the rest of the proof, ε only takes on the values from this subsequence. Since $-\Delta\psi(x_\varepsilon, t_\varepsilon) > 0$, we must have $v_\varepsilon(x_\varepsilon, t_\varepsilon) = 0$ and therefore

$$(4.3) \quad v_\varepsilon(x, t) \leq [\psi(x, t) - \psi(x_\varepsilon, t_\varepsilon)]_+ \quad \text{on } \overline{B}_\delta(\hat{x}, \hat{t}) \cap \{t \leq t_\varepsilon\}.$$

By the definition of $\bar{r}(q)$ and the choice of $r > \bar{r}(q)$, there exists arbitrarily small ε' such that $\overline{\Phi}_{\varepsilon', q, r}(1) > \frac{3}{4}\varepsilon'^\beta$. Hence Lemma 3.23 implies that there exists arbitrarily small $\hat{\varepsilon} = \gamma\varepsilon'$ such that $\bar{u}_{\hat{\varepsilon}; q, r} = 0$ in $B_{c\hat{\varepsilon}^\beta}(r\nu, 1)$, where c is a positive constant independent of $\hat{\varepsilon}$. Let us fix one such $\hat{\varepsilon}$ small enough so that

$$(4.4) \quad c\hat{\varepsilon}^{\beta-1} > r + 2\sqrt{n} + 1,$$

and set $h := \varepsilon/\hat{\varepsilon}$.

Denoting $q_\varepsilon = D\psi(x_\varepsilon, t_\varepsilon)$, we can estimate

$$(4.5) \quad \begin{aligned} \psi(x, t) - \psi(x_\varepsilon, t_\varepsilon) &= \psi_t(x_\varepsilon, t_\varepsilon)(t - t_\varepsilon) + D\psi(x_\varepsilon, t_\varepsilon) \cdot (x - x_\varepsilon) \\ &\quad + \frac{1}{2}D^2\psi(x_\varepsilon, t_\varepsilon)(x - x_\varepsilon) \cdot (x - x_\varepsilon) \\ &\leq |q| r(t - t_\varepsilon) + q \cdot (x - x_\varepsilon) \\ &\quad + |q_\varepsilon - q| |x - x_\varepsilon| + \|A\| |x - x_\varepsilon|^2. \end{aligned}$$

We introduce $E := Q_q \cap C^+ \cap \{t \leq 2\}$ and its diameter $d := \text{diam } E$, and set (recall that $\nu := -q/|q|$)

$$(4.6) \quad \begin{aligned} \tilde{y}_\varepsilon &= x_\varepsilon - hr\nu + \|A\| h^2 d^2 |q|^{-1} \nu + |q_\varepsilon - q| hd |q|^{-1} \nu + (r + \sqrt{n})\varepsilon\nu, \\ \tilde{\tau}_\varepsilon &= t_\varepsilon - h. \end{aligned}$$

Then we choose $(y_\varepsilon, \tau_\varepsilon) \in \varepsilon\mathbb{Z}^{n+1}$ such that

$$(y_\varepsilon, \tau_\varepsilon) \in \arg \min_{(x, t) \in \varepsilon\mathbb{Z}^{n+1}} [|x - \tilde{y}_\varepsilon| + |t - \tilde{\tau}_\varepsilon|].$$

Clearly

$$(4.7) \quad |y_\varepsilon - \tilde{y}_\varepsilon| \leq \sqrt{n}\varepsilon, \quad |\tau_\varepsilon - \tilde{\tau}_\varepsilon| \leq \varepsilon$$

and therefore, recalling (4.4) and $h = \varepsilon/\hat{\varepsilon}$, we have for all ε sufficiently small

$$\begin{aligned} \text{dist}((y_\varepsilon + hr\nu, \tau_\varepsilon + h), (x_\varepsilon, t_\varepsilon)) &\leq (r + 2\sqrt{n} + 1)\varepsilon \\ &\quad + |q_\varepsilon - q| hd |q|^{-1} + \|A\| h^2 d^2 |q|^{-1} \\ &\leq c\hat{\varepsilon}^{\beta-1}\varepsilon = ch\hat{\varepsilon}^\beta. \end{aligned}$$

since $q_\varepsilon \rightarrow q$ as $\varepsilon \rightarrow 0$. In particular,

$$(4.8) \quad (x_\varepsilon, t_\varepsilon) \in B_{ch\hat{\varepsilon}^\beta}(y_\varepsilon + hr\nu, \tau_\varepsilon + h) \subset hE + (y_\varepsilon, \tau_\varepsilon)$$

for sufficiently small ε . From this and the definition of $d = \text{diam } E$, we infer that $\text{dist}((x, t), (x_\varepsilon, t_\varepsilon)) \leq hd$ for all $(x, t) \in hE + (y_\varepsilon, \tau_\varepsilon)$. Thus we estimate, using (4.5),

(4.6) and (4.7),

$$\begin{aligned}
|q| (r(t - \tau_\varepsilon) - (x - y_\varepsilon) \cdot \nu) &= |q| r(t - t_\varepsilon + t_\varepsilon - \tilde{\tau}_\varepsilon + \tilde{\tau}_\varepsilon - \tau_\varepsilon) \\
&\quad + q \cdot (x - x_\varepsilon + x_\varepsilon - \tilde{y}_\varepsilon + \tilde{y}_\varepsilon - y_\varepsilon) \\
&\geq \psi(x, t) - \psi(x_\varepsilon, t_\varepsilon) - \|A\| |x - x_\varepsilon|^2 - |q_\varepsilon - q| |x - x_\varepsilon| \\
&\quad + |q| r(-h) + hr |q| \\
&\quad + \|A\| h^2 d^2 + |q_\varepsilon - q| hd + (r + \sqrt{n}) \varepsilon |q| \\
&\quad - (|q| r + |q| \sqrt{n}) \varepsilon \\
&\geq \psi(x, t) - \psi(x_\varepsilon, t_\varepsilon) \quad \text{for all } (x, t) \in hE + (y_\varepsilon, \tau_\varepsilon).
\end{aligned}$$

And therefore (4.3) yields

$$v_\varepsilon(x, t) \leq [\psi(x, t) - \psi(x_\varepsilon, t_\varepsilon)]_+ \leq P_{q,r}(x - y_\varepsilon, t - \tau_\varepsilon) \quad \text{on } (hE + (y_\varepsilon, \tau_\varepsilon)) \cap \{t \leq t_\varepsilon\}.$$

Let us define

$$w_\varepsilon(x, t) = \begin{cases} h^{-1} v_\varepsilon(hx + y_\varepsilon, ht + \tau_\varepsilon) & (x, t) \in Q_q \cap C^+, \\ 0 & (x, t) \in Q_q \setminus C^+. \end{cases}$$

We observe that $w_\varepsilon \in \underline{\mathcal{S}}(g^\varepsilon/h, Q_q)$ and $w_\varepsilon \leq P_{q,r}$ on $Q_q \cap \{t \leq 1 + h^{-1}(\tilde{\tau}_\varepsilon - \tau_\varepsilon)\}$. Therefore, by definition, $w_\varepsilon \leq \bar{u}_{\varepsilon/h; q, r} = \bar{u}_{\hat{\varepsilon}; q, r}$ in $Q_q \cap \{t \leq 1 + h^{-1}(\tilde{\tau}_\varepsilon - \tau_\varepsilon)\}$. But by the choice of $\hat{\varepsilon}$ we have that $\bar{u}_{\hat{\varepsilon}; q, r} = 0$ in $B_{c\hat{\varepsilon}\beta}(r\nu, 1)$, and by scaling back, we have

$$v_\varepsilon = 0 \quad \text{in } B_{ch\hat{\varepsilon}\beta}(y_\varepsilon + hr\nu, \tau_\varepsilon + h) \cap \{t \leq t_\varepsilon\}.$$

When we apply Definition 2.3(i) and (4.8), we obtain $(x_\varepsilon, t_\varepsilon) \notin \bar{\Omega}(v_\varepsilon)$ and that is a contradiction with the choice of $(x_\varepsilon, t_\varepsilon)$. \square

In particular, if the solution of the homogenized problem (4.1) is unique, then the limit is the unique solution. Before stating this result in Theorem 4.7, we introduce the notion of convergence in Hausdorff distance.

Definition 4.5. *Let $E_\varepsilon, E \subset \mathbb{R}^n \times \mathbb{R}$ for $\varepsilon > 0$ be compact sets. We say that E_ε converges to E uniformly in Hausdorff distance as $\varepsilon \rightarrow 0$ if for every $\delta > 0$ there exists $\varepsilon_0 > 0$ such that*

$$E \subset E_\varepsilon + B_\delta(0, 0), \quad E_\varepsilon \subset E + B_\delta(0, 0) \quad \text{for } \varepsilon < \varepsilon_0.$$

The following characterization is well known [9].

Proposition 4.6. *Compact sets $E_\varepsilon \subset \mathbb{R}^n \times \mathbb{R}$ converge to a compact set $E \subset \mathbb{R}^n \times \mathbb{R}$ uniformly in Hausdorff distance if and only if the following two statements are true:*

- (i) *if $(x_\varepsilon, t_\varepsilon) \in E_\varepsilon$ for all $\varepsilon > 0$, then every limit point of $\{(x_\varepsilon, t_\varepsilon)\}_{\varepsilon > 0}$ lies in E ;*
- (ii) *for every $(x, t) \in E$ there exists $(x_\varepsilon, t_\varepsilon) \in E_\varepsilon$ such that $(x_\varepsilon, t_\varepsilon) \rightarrow (x, t)$ as $\varepsilon \rightarrow 0$.*

Theorem 4.7 (Homogenization). *Suppose that the domain $Q := \Omega \times (0, T]$ and the initial and boundary data satisfy the assumptions of Theorem 1.1. Let $u^\varepsilon \in \mathcal{S}(g^\varepsilon, Q)$ be the unique solution of problem (1.1) for $\varepsilon > 0$ and let $u \in \mathcal{S}(\bar{r}(p)/|p|, Q)$ be the unique solution of (4.1).*

Then $u_ = \star\text{-}\liminf_{\varepsilon \rightarrow 0} u^\varepsilon$ and $u^* = \star\text{-}\limsup_{\varepsilon \rightarrow 0} u^\varepsilon$. Moreover, $\partial\Omega((u^\varepsilon)_*; Q)$ converge to $\partial\Omega(u_*; Q)$ uniformly in Hausdorff distance.*

Proof. We shall denote $\bar{u} = \star\text{-limsup}_{\varepsilon \rightarrow 0} u^\varepsilon$ and $\underline{u} = \star\text{-liminf}_{\varepsilon \rightarrow 0} u^\varepsilon$.

Let $U \in \underline{\mathcal{S}}(m/2, Q)$ and $V \in \overline{\mathcal{S}}(M, Q)$ be the boundary barriers constructed in the proof of Theorem 1.1. From that proof we know that $U_* \leq u^\varepsilon \leq V^*$ for all $\varepsilon > 0$ and hence $U_* \leq \underline{u} \leq \bar{u} \leq V^*$ on Q . This implies that \underline{u} and \bar{u} have the correct boundary data in the sense of Theorem 1.1. Since $\bar{u} \in \underline{\mathcal{S}}(\bar{\tau}(p)/|p|, Q)$ and $\underline{u} \in \overline{\mathcal{S}}(\bar{\tau}(p)/|p|, Q)$ by Proposition 4.4, the proof of uniqueness in Theorem 1.1 yields $u_* \leq \underline{u} \leq \bar{u} \leq u^*$, which implies $\underline{u} = u_*$, $\bar{u} = u^*$ using Corollary 2.19.

Let us proceed with the proof of the convergence of free boundaries. Denote $E_\varepsilon = \partial\Omega((u^\varepsilon)_*; Q)$ and $E = \partial\Omega(u_*; Q)$. Clearly E, E_ε are compact subsets of \bar{Q} (see also the proof of Theorem 1.1). We point out that due to the choice of initial and boundary data, we have that

$$(4.9) \quad E = \Gamma(u_*; Q) \cup (\partial\Omega \times [0, T]) \cup ((\Omega \cap \Omega_0) \times \{0\}) \subset \Gamma(u_*; Q) \cup \partial_P Q,$$

and similarly for E_ε (see the proof of Theorem 1.1). Let us also recall that $\bar{\Omega}(u_*) = \bar{\Omega}(u)$ by Corollary 2.19. Therefore $\partial\Omega(u_*) = \bar{\Omega}(u) \setminus \Omega(u_*)$. The same is true for u^ε .

We shall establish the uniform convergence $E_\varepsilon \rightarrow E$ in Hausdorff distance using the characterization in Proposition 4.6. Let us denote

$$H_\rho(\xi, \sigma) := B_\rho(\xi, \sigma) \cap \{t \leq T\}, \quad (\xi, \sigma) \in \mathbb{R}^n \times \mathbb{R}, \rho > 0,$$

in the following.

(i) Suppose that $\varepsilon_k \rightarrow 0$ as $k \rightarrow \infty$ and $(x_k, t_k) \in E_{\varepsilon_k}$ such that $(x_k, t_k) \rightarrow (\hat{x}, \hat{t})$. Clearly $\hat{t} \leq T$. Lemma 4.2 yields that $(\hat{x}, \hat{t}) \in \bar{\Omega}(u)$. Furthermore, due to the behavior of u^ε at the parabolic boundary of $\partial_P Q$, we have $\hat{x} \in \Omega$, and $(\hat{x}, \hat{t}) \in E$ if also $\hat{t} = 0$. Let us therefore assume that $\hat{t} > 0$ and $(\hat{x}, \hat{t}) \in \Omega(u_*)$. There exists $\delta > 0$ such that $\bar{H}_\delta(\hat{x}, \hat{t}) \subset \Omega(u_*)$. Semicontinuity yields $\eta = \min_{\bar{H}_\delta(\hat{x}, \hat{t})} u_* > 0$. The property of $\star\text{-liminf}$ then implies that $\min_{\bar{H}_\delta(\hat{x}, \hat{t})} (u^{\varepsilon_k})_* > \eta/2 > 0$ for k sufficiently large. But this is a contradiction with $(x_k, t_k) \in H_\delta(\hat{x}, \hat{t}) \cap \partial\Omega((u^\varepsilon)_*)$ for large k .

(ii) Now suppose that there exist $(\hat{x}, \hat{t}) \in E$ and $\delta > 0$ such that for some $\varepsilon_k \rightarrow 0$ as $k \rightarrow \infty$ we have $E_{\varepsilon_k} \cap \bar{H}_\delta(\hat{x}, \hat{t}) = \emptyset$ for all k . Again, since u^ε and u satisfy the same boundary conditions, we must have $(\hat{x}, \hat{t}) \in Q$ by (4.9). Thus we can assume that $\bar{H}_\delta(\hat{x}, \hat{t}) \subset Q$.

Suppose that $\bar{H}_\delta(\hat{x}, \hat{t}) \cap \bar{\Omega}(u^{\varepsilon_k}) = \emptyset$ for infinitely many k . In this case $u = 0$ in $\bar{H}_\delta(\hat{x}, \hat{t})$ and we get a contradiction with the choice of $(\hat{x}, \hat{t}) \in \partial\Omega(u_*)$.

Hence we must have $\bar{H}_\delta(\hat{x}, \hat{t}) \subset \Omega((u^{\varepsilon_k})_*)$ for all but finitely many k . We can assume for simplicity that this inclusion holds for all k . We recall that the comparison with the barrier U in the proof of Theorem 1.1 yields that the support of u strictly expands at $t = 0$. Consequently, $\hat{x} \notin \bar{\Omega}_0(u; Q)$. By taking δ smaller if necessary, we can assume that $\bar{B}_\delta(\hat{x}) \cap \bar{\Omega}_0(u; Q) = \emptyset$. Therefore $\bar{B}_\delta(\hat{x}) \cap \bar{\Omega}_0(u^\varepsilon; Q) = \emptyset$ and u^ε is nondegenerate in the sense of Corollary A.5. Since $(u^\varepsilon)_* = ((u^\varepsilon)_*)^*$ is nondecreasing in time and $(\hat{x}, \hat{t} - \delta/2) \in \bar{\Omega}((u^{\varepsilon_k})_*) = \bar{\Omega}(u^{\varepsilon_k})$, we have

$$\sup_{x \in B_{\delta/2}(\hat{x})} (u^{\varepsilon_k})_*(x, t) \geq 2nM \frac{\delta^2}{\hat{t} - \delta/2} \quad \text{for } t \in I := [\hat{t} - \delta/2, \hat{t} + \delta/2] \cap (0, T].$$

Furthermore, we observe that $B_{\sqrt{3}\delta/2}(\hat{x}) \in \Omega_t((u^{\varepsilon_k})_*)$ for $t \in I$ and, since $(u^\varepsilon)_*$ is harmonic in $\Omega_t((u^\varepsilon)_*)$ due the same argument as in the proof of Proposition 3.12, the Harnack inequality yields $\inf_{x \in B_{\delta/2}(\hat{x})} (u^{\varepsilon_k})_*(x, t) \geq \eta > 0$ for $t \in I$, in particular $\inf_{H_{\delta/2}(\hat{x}, \hat{t})} u^{\varepsilon_k} \geq \eta$ with η independent of k . Therefore, by regularity, Corollary 2.19,

we have

$$u_* = (u^*)_* = \left(\star\text{-}\limsup_{\varepsilon \rightarrow 0} u^\varepsilon \right)_* \geq \eta > 0 \quad \text{in } H_{\delta/2}(\hat{x}, \hat{t}).$$

But that is a contradiction with the choice of $(\hat{x}, \hat{t}) \in \partial\Omega(u_*)$.

The proof of uniform convergence in Hausdorff distance is finished. \square

APPENDIX A. HELE-SHAW PROBLEM

In this section we derive bounds on solutions of the Hele-Shaw problem which can be obtained by a comparison with radially symmetric barriers. Some of these estimates appeared previously in various forms in [16, 18, 21]. But the estimate in Proposition A.2, for example, appears to be new.

Lemma A.1. *Given constants $m > 0$, $K > 0$, $A \in (0, 1)$ there exists a self-similar (w.r.t. the parabolic scaling), radially symmetric, exterior solution $\varphi \in \mathcal{S}(m, \mathbb{R}^n \setminus \{0\} \times (0, \infty))$ of the form $\varphi(x, t) = \psi(x/\rho(t))$ such that $\varphi(x, t) > 0$ for $0 < |x| < \rho(t)$ and $\varphi(x, t) = 0$ for $\rho(t) \leq |x|$, and $\varphi(x, t) = K$ for $|x| = A\rho(t)$, $\lim_{t \rightarrow 0^+} \rho(t) = 0$ and $\rho(t) = \sqrt{\alpha K m t}$ where $\alpha = \alpha(n, A)$.*

Proof. The solution φ has the form $\varphi(x, t) = \psi(x/\rho(t))$, where (formulas written in columns for $n \geq 3$ and $n = 2$)

$$\begin{array}{ll} n \geq 3 & n = 2 \\ \psi(x) = K \frac{\left(|x|^{2-n} - 1\right)_+}{A^{2-n} - 1} & \psi(x) = K \frac{\left(\ln \frac{1}{|x|}\right)_+}{\ln \frac{1}{A}}, \\ \rho(t) = \sqrt{\frac{2(n-2)}{A^{2-n} - 1} K m t} & \rho(t) = \sqrt{\frac{2}{-\ln A} K m t}. \end{array}$$

One can easily verify that $\varphi \in \mathcal{S}(m, \mathbb{R}^n \setminus \{0\} \times (0, \infty))$, checking the free boundary condition $\frac{d\rho}{dt} = V_\nu = m |D\varphi^+|_{|x|=\rho}$. \square

The solutions constructed in Lemma A.1 allow us to obtain upper bound on supersolutions that are harmonic in their positive sets.

Proposition A.2. *There exists a dimensional constant $C = C(n)$ such that for any space-time cylinder $Q = E \times (t_1, t_2)$, where $E \subset \mathbb{R}^n$ is an open set and $t_1 < t_2$, and any supersolution $u \in \overline{\mathcal{S}}(m, Q)$ for some $m > 0$ such that $u \in LSC(\overline{Q})$, u is nondecreasing in time and harmonic in $\Omega_t(u; Q)$ for all $t \in (t_1, t_2)$, we have*

$$u(x, t) \leq \inf \left\{ \frac{C}{m} \frac{|x - \zeta|^2}{\sigma - t} : (\zeta, \sigma) \in Q, u(\zeta, \sigma) = 0, t < \sigma, |\zeta - x| < \text{dist}(x, \partial E) \right\}$$

for all $(x, t) \in Q$.

Proof. The idea of the proof is straightforward: if $u(x, t)$ violates the bound for some (x, t) , $(\zeta, \sigma) \in Q$, it is possible to put a barrier under u that will become positive at (ζ, σ) , yielding a contradiction.

Set $C = \frac{1}{c_H \alpha}$, where $\alpha = \alpha(n, A)$ is the constant from Lemma A.1 with $A = 1/4$, and $c_H = c_H(n) > 0$ is the constant from the standard Harnack inequality: for any positive harmonic function ψ on B_1

$$(A.1) \quad \inf_{B_{1/2}} \psi \geq c_H \sup_{B_{1/2}} \psi.$$

Suppose that there exist $(\zeta, \sigma), (\xi, \tau) \in Q$ and $K > 0$ such that $u(\zeta, \sigma) = 0$, $\tau < \sigma$, $|\zeta - x| < \text{dist}(x, \partial E)$ and

$$(A.2) \quad u(\xi, \tau) > \frac{K}{c_H} > \frac{C}{m} \frac{|\xi - \zeta|^2}{\sigma - \tau}.$$

We can in fact assume, by translation invariance, that $(\xi, \tau) = (0, 0)$.

Let $\varphi \in \mathcal{S}(m, (\mathbb{R}^n \setminus \{0\}) \times (0, \infty))$ and $\rho : [0, \infty) \rightarrow [0, \infty)$ be the self-similar, radially symmetric solution and radius of its free boundary, respectively, from Lemma A.1 with parameters m , K and $A = 1/4$. Since $\rho(t) = \sqrt{\alpha K m t}$, it follows from the second inequality in (A.2) that $\rho(\sigma) > |\zeta|$, and, consequently, $\varphi(\zeta, \sigma) > 0$.

We want to show that $u(\zeta, \sigma) > \varphi(\zeta, \sigma)$ to get a contradiction. Because $u \in LSC$ and u is nondecreasing in time, there exists $\lambda > 0$ such that $u > \frac{K}{c_H}$ in $\overline{B}_{2\lambda}(0) \times [0, t_2]$. Clearly $\lambda < |\zeta|$. Let us find $\sigma_0 > 0$ such that $\rho(\sigma_0) < \lambda$ and let $k \in \mathbb{N}$ be the smallest number such that $\rho\left(\left(\frac{9}{4}\right)^k \sigma_0\right) \geq |\zeta|$. Set $\sigma_j = \left(\frac{9}{4}\right)^j \sigma_0$ for $1 \leq j < k$ and find σ_k so that $\rho(\sigma_k) = |\zeta|$. We chose σ_j in this way because now we have

$$(A.3) \quad A\rho(t) = \frac{1}{4}\rho(t) \leq \frac{3}{8}\rho(\sigma_{j-1}) < \frac{1}{2}\rho(\sigma_{j-1}) \quad t \in [\sigma_{j-1}, \sigma_j], \quad j = 1, \dots, k.$$

Therefore, if $1 \leq j \leq k$ and $u(x, \sigma_{j-1}) > 0$ for all x such that $|x| \leq \rho(\sigma_{j-1})$, then, by monotonicity in time and Harnack inequality (A.1),

$$(A.4) \quad u(x, t) \geq u(x, \sigma_{j-1}) \geq \inf_{|y| < \frac{1}{2}\rho(\sigma_{j-1})} u(y, \sigma_{j-1}) \geq c_H \sup_{|y| < \frac{1}{2}\rho(\sigma_{j-1})} u(y, \sigma_{j-1}) > K$$

for $|x| \leq A\rho(t) < \frac{1}{2}\rho(\sigma_{j-1})$ and $t \in [\sigma_{j-1}, \sigma_j]$.

Define $Q_j = \{(x, t) \in Q : |x| > A\rho(t), t \in (\sigma_{j-1}, \sigma_j]\}$. By construction, $\varphi \leq K$ on \overline{Q}_j . Our goal is to show inductively that

$$(A.5) \quad \varphi \prec u \quad \text{on } \overline{Q}_k \text{ w.r.t. } Q_k,$$

which implies $u(\zeta, \sigma) \geq u(\zeta, \sigma_k) > \varphi(\zeta, \sigma_k) > 0$, yielding a contradiction.

We will show the inductive step: if $2 \leq j \leq k$ and $\varphi \prec u$ on \overline{Q}_i w.r.t. Q_i for all $i = 1, \dots, j-1$ then $\varphi \prec u$ on \overline{Q}_j w.r.t. Q_j .

The goal is to apply the comparison theorem, and therefore we need to show $\varphi \prec u$ on $\partial_P Q_j$ w.r.t. Q_j . First, observe that (for any $j = 1, \dots, k$) we have $\partial_P Q_j = C_j^1 \cup C_j^2 \cup C_j^3$, where C_j^i are the closed sets

$$\begin{aligned} C_j^1 &:= \partial E \times [\sigma_{j-1}, \sigma_j], && \text{outer boundary} \\ C_j^2 &:= \{(x, \sigma_{j-1}) \in \overline{Q} : |x| \geq A\rho(\sigma_{j-1})\}, && \text{initial boundary} \\ C_j^3 &:= \{(x, t) : |x| = A\rho(t), t \in [\sigma_{j-1}, \sigma_j]\}. && \text{inner boundary} \end{aligned}$$

(1) It is simple to see that $\varphi \prec u$ on C_j^1 w.r.t. Q_j for $1 \leq j \leq k$ because $\rho(t) \leq \rho(\sigma_k) = |\zeta| < \text{dist}(0, \partial E)$ for $t \in [\sigma_{j-1}, \sigma_j]$ and therefore $\overline{\Omega}(\varphi; Q_j) \cap C_j^1 = \emptyset$.

(2) Since the support of φ expands continuously,

$$(A.6) \quad \overline{\Omega}(\varphi; Q_i) \cap C_i^2 = \{(x, \sigma_{i-1}) : A\rho(\sigma_{i-1}) \leq |x| \leq \rho(\sigma_{i-1})\} \quad i = 1, \dots, k,$$

and

$$\overline{\Omega}(\varphi; Q_{i-1}) \cap C_i^2 = \overline{\Omega}(\varphi; Q_i) \cap C_i^2 \quad i = 2, \dots, k.$$

In particular, $\varphi \prec u$ in \overline{Q}_{j-1} w.r.t. Q_{j-1} implies $\varphi \prec u$ in C_j^2 w.r.t. Q_j since $C_j^2 \subset \overline{Q}_{j-1}$.

(3) The induction hypothesis and (2) implies $\varphi \prec C_i^2$ w.r.t. Q_i for $i = 1, \dots, j$, and thus (A.6) and the monotonicity of u yield

$$u(x, t) \geq u(x, \sigma_{i-1}) > 0 \quad t \in [\sigma_{i-1}, t_2], \quad A\rho(\sigma_{i-1}) \leq |x| \leq \rho(\sigma_{i-1}).$$

Since $A\rho(\sigma_i) < \rho(\sigma_{i-1})$ for $i = 1, \dots, k$ and $\rho(\sigma_0) < \lambda$, we conclude that $u(x, t) > 0$ for $|x| \leq \rho(\sigma_{j-1})$ and $t \in [\sigma_{j-1}, t_2]$. Therefore by (A.4) we have $u > K \geq \varphi$ on C_j^3 .

We have showed in (1)–(3) that $\varphi \prec u$ on $\partial_P Q_j$ w.r.t. Q_j and therefore the comparison principle, Theorem 2.18, yields $\varphi \prec u$ on \bar{Q}_j w.r.t. Q_j , finishing the proof of the inductive step.

Finally, to close the induction argument, we need to show the base case, i.e., that $\varphi \prec u$ on \bar{Q}_1 w.r.t. Q_1 . Part (1) still applies for $j = 1$ as well, and $\varphi \prec u$ on $C_1^2 \cup C_1^3$ w.r.t. Q_1 follows from the choice $A\rho(\sigma_1) < \rho(\sigma_0) < \lambda$. Therefore the comparison theorem applies on Q_1 . Induction yields (A.5), finishing the proof of the theorem. \square

Now we turn our attention to obtaining bounds for subsolutions. The first step is a construction of barriers.

Lemma A.3. *Let M and μ be positive constants and let $\chi \in C((-\infty, 0))$, $\chi(t) > 0$. Define $K(t) = \int_0^t \chi(s) ds$ and*

$$t_0 = \inf \left\{ t < 0 : K(t) > -\frac{\mu^2}{2nM} \right\} \in [-\infty, 0).$$

There exists a classical, radially symmetric solution $\varphi \in \mathcal{S}(M, Q)$ on a space-time cylinder $Q = B_\mu \times (t_0, 0)$, with boundary data $\varphi(x, t) = \chi(t)$ for $|x| = \mu$, such that $\varphi(x, t) > 0$ for $\rho(t) < |x| \leq \mu$, and $\varphi(x, t) = 0$ for $|x| \leq \rho(t)$, where $\rho(t)$ is the radius of free boundary of φ . Moreover, ρ is strictly decreasing and $\rho(t) \in (0, \mu)$ for all $t \in (t_0, 0)$. Finally, $\lim_{t \rightarrow 0^-} \rho(t) = 0$.

Proof. The solution φ is given by the formulas (separately for $n \geq 3$ and $n = 2$)

$$(A.7) \quad \begin{array}{ll} n \geq 3 & n = 2 \\ \varphi(x, t) = \chi(t) \frac{(\rho(t)^{2-n} - |x|^{2-n})_+}{\rho(t)^{2-n} - \mu^{2-n}}, & \varphi(x, t) = \chi(t) \frac{\left(\ln \frac{|x|}{\rho(t)}\right)_+}{\ln \frac{\mu}{\rho(t)}}, \end{array}$$

and $\rho(t)$ is the unique solution of

$$(A.8) \quad \frac{\mu^2}{2-n} \left(\frac{\rho(t)^2}{2\mu^2} - \frac{\rho(t)^n}{n\mu^2} \right) = MK(t), \quad \frac{\rho(t)^2}{2} \left(\ln \frac{\rho(t)}{\mu} - \frac{1}{2} \right) = MK(t).$$

The equations for $\rho(t)$ are derived by integrating $V_\nu = -\frac{d\rho}{dt} = |D\varphi^+|_{|x|=\rho}$, using the condition $\rho(0) = 0$ and the expressions

$$(A.9) \quad |D\varphi^+|_{|x|=\rho} = \chi \frac{(n-2)\rho^{1-n}}{\rho^{2-n} - \mu^{2-n}}, \quad |D\varphi^+|_{|x|=\rho} = \chi \frac{1}{\rho \ln \frac{\mu}{\rho}},$$

The left-hand side of the equation (A.8) is strictly increasing for $\rho \in (0, \mu)$, with limit values $-\frac{\mu^2}{2n}$ as $\rho \rightarrow \mu$ and 0 as $\rho \rightarrow 0$, and therefore the equation has a unique solution in the set $\rho \in (0, \mu)$ as long as $MK(t) \in (-\frac{\mu^2}{2n}, 0)$, which holds for $t \in (t_0, 0)$ by the definition of t_0 . It is straightforward to check that $\varphi \in \mathcal{S}(M, Q)$. \square

Now we will use the radially symmetric barriers of Lemma A.3 to obtain various estimates on subsolutions.

Proposition A.4. *Let M, μ be positive constants and let $\chi \in C([t_1, t_2])$ for some $t_1 < t_2$, with $\chi > 0$. Suppose that $\int_{t_1}^{t_2} \chi(s) ds < \frac{\mu^2}{2nM}$.*

If $u \in \underline{\mathcal{S}}(M, Q)$, where Q is the cylinder $Q = B_\mu(\zeta) \times (t_1, t_2)$ for some $\zeta \in \mathbb{R}^n$, such that $\overline{\Omega}_{t_1}(u; Q) = \emptyset$ and $u^{, \overline{Q}}(x, t) \leq \chi(t)$ for $(x, t) \in \partial B_\mu(\zeta) \times [t_1, t_2]$, then*

$$(\zeta, t) \notin \overline{\Omega}(u; Q) \quad \text{for all } t \in [t_1, t_2].$$

Proof. Let us extend χ to \mathbb{R} by defining $\chi(t) = \chi(t_2)$ for $t > t_2$ and $\chi(t) = \chi(t_1)$ for $t < t_1$. By continuity, we can find $\delta > 0$ small enough such that

$$\int_{t_1}^{t_2+\delta} \chi(s) + \delta ds < \frac{\mu^2}{2n(M + \delta)}.$$

Let $\varphi \in \mathcal{S}(M + \delta, B_\mu(0) \times (t_0, 0))$ be the contracting solution provided by Lemma A.3 with $\tilde{\chi}(t) = \chi(t + t_2 + \delta) + \delta$ and $\tilde{M} = M + \delta$, where $t_0 < t_1 - t_2 - \delta$, and let $\rho(t)$ be the radius of its free boundary. Define

$$\psi(x, t) = \varphi(x - \zeta, t - t_2 - \delta) - \kappa(|x - \zeta|^2 - \rho^2(t - t_2 - \delta))_+,$$

where $\kappa \in (0, \delta/\mu^2)$ is chosen small enough so that ψ is a superbarrier on Q and $\psi > M$ on $(\partial B_\mu(\zeta)) \times [t_1, t_2]$. Observe that $u \prec \psi$ on $\partial_P Q$ w.r.t. Q . and hence Definition 2.12 implies that $u \prec \psi$ on \overline{Q} w.r.t. Q and the result follows since $(\zeta, t) \notin \overline{\Omega}(\psi; Q)$ for all $t \in [t_1, t_2]$ because $\rho(t - t_2 - \delta) \geq \rho(-\delta) > 0$. \square

Proposition A.4 has the following consequence for the bound of a supersolution on a cylinder.

Corollary A.5 (Nondegeneracy). *Let $u \in \underline{\mathcal{S}}(M, Q)$, where Q is the cylinder $Q := B_\mu(\zeta) \times (t_1, t_2)$ for some $\zeta \in \mathbb{R}^n$, $\mu > 0$ and $t_1 < t_2$, such that $(\zeta, t_2) \in \overline{\Omega}(u; Q)$ and $\overline{\Omega}_{t_1}(u; Q) = \emptyset$. Then*

$$\sup_Q u \geq \frac{\mu^2}{2nM(t_2 - t_1)}.$$

Proof. If $K := \sup_Q u < \frac{\mu^2}{2nM(t_2 - t_1)}$, we can apply Proposition A.4 on Q with $\theta(t) = K$, yielding $(\zeta, t_2) \notin \overline{\Omega}(u; Q)$, a contradiction. \square

The support of subsolutions expands with a speed that can be controlled with the help of Proposition A.4.

Corollary A.6. *If $u \in \underline{\mathcal{S}}(M, Q)$, where $Q = E \times (t_1, t_2)$ for some open set $E \subset \mathbb{R}^n$ and $t_1 < t_2$, and $u \leq K$ on Q for some $K > 0$, then*

$$\overline{\Omega}_t(u; Q) \subset \overline{\Omega}_{t_1}(u; Q) \cup E^c + \overline{B}_{\rho(t)} \quad \text{for all } t \in [t_1, t_2],$$

where

$$\rho(t) = \sqrt{2nKM(t - t_1)}.$$

Proof. The statement will be proved if we show that $(\zeta, \sigma) \notin \overline{\Omega}(u; Q)$ for any $(\zeta, \sigma) \in \overline{Q}$ such that

$$(A.10) \quad \overline{B}_{\rho(\sigma)}(\zeta) \subset E \quad \text{and} \quad \overline{B}_{\rho(\sigma)}(\zeta) \cap \overline{\Omega}_{t_1}(u; Q) = \emptyset.$$

Fix one such (ζ, σ) . By compactness of $\overline{B}_{\rho(\sigma)}(\zeta)$ there exists $\mu > \rho(\sigma)$ such that (A.10) holds with μ instead of $\rho(\sigma)$. Set $\theta(t) = K$, and note that

$$\int_{t_1}^{\sigma} K \, ds = (\sigma - t_1)K = \frac{\rho(\sigma)^2}{2nM} < \frac{\mu^2}{2nM}.$$

Therefore Proposition A.4 yields $(\zeta, \sigma) \notin \overline{\Omega}(u; Q)$. \square

Corollary A.7. *Suppose that positive constants M, μ, σ, A and ε satisfy*

$$(A.11) \quad \sigma < \varepsilon \left(e^{\frac{\mu^2}{2nMA}} - 1 \right).$$

If $u \in \underline{\mathcal{S}}(M, Q)$, where $Q = B_{\mu}(\zeta) \times (0, \sigma)$, $\overline{\Omega}_0(u; Q) = \emptyset$ and

$$(A.12) \quad u(x, t) \leq \frac{A}{\varepsilon + \sigma - t} \quad \text{in } Q,$$

then $(\zeta, t) \notin \overline{\Omega}(u; Q)$ for $t \in (0, \sigma]$.

Proof. Set $\theta(t) = \frac{A}{\varepsilon + \sigma - t}$ and $t_1 = 0, t_2 = \sigma$. An easy computation yields, using the bound (A.11) in the last step,

$$\begin{aligned} \int_{t_1}^{t_2} \theta(s) \, ds &= \int_0^{\sigma} \frac{A}{\varepsilon + \sigma - s} \, ds = -A \ln \frac{\varepsilon}{\varepsilon + \sigma} \\ &= A \ln \left(1 + \frac{\sigma}{\varepsilon} \right) < \frac{\mu^2}{2nM}. \end{aligned}$$

The conclusion follows from Proposition A.4. \square

APPENDIX B. NONUNIFORM PERTURBATION

When the free boundary velocity law in the Hele-Shaw problem (1.1) depends on time, the solutions cannot be easily scaled in time. In particular, a linear scaling in time preserves subsolutions or supersolutions only for short time intervals of length $\frac{m\varepsilon}{L}$, where m and L are constants from (1.2) and (1.3). In this section we devise a nonlinear scaling in time that can provide perturbed solutions on arbitrary long intervals.

Proposition B.1. *Suppose that $\rho \in C^2(E)$ and $\theta \in C^1(I)$ on some open sets $E \subset \mathbb{R}^n$ and $I \subset \mathbb{R}$, and ρ, θ satisfy, for all $x \in E, t \in I$,*

- (i) $\theta' > 0$, and either $\rho > 0$ or $\rho \equiv 0$,
- (ii) $|D\rho| \leq 1/2$,
- (iii) $\rho\Delta\rho \geq (n-1)|D\rho|^2$,
- (iv) for some positive constants a, L, m ,

$$a \geq \theta'(t)(1 - |D\rho(x)|)^{-2} \left[1 + \frac{L}{m} (|t - \theta(t)| + \rho(x)) \right].$$

Then if $u \in \underline{\mathcal{S}}(g, Q)$, a subsolution of the Hele-Shaw problem with $V_{\nu} = g(x, t) |Du|$ on an open set $Q \subset E \times I$, where g is L -Lipschitz and $g \geq m > 0$, then

$$v(x, t) := \sup_{y \in \overline{B}_{\rho(x)}(x)} a u(y, \theta(t)).$$

is a subsolution of the same problem on the set

$$\tilde{Q} = \{(x, t) : t \in \theta(I), B_{\rho(x)}(x) \times \{\theta^{-1}(t)\} \subset Q\},$$

i.e., $v \in \underline{\mathcal{S}}(g, \tilde{Q})$.

Similarly, if we replace condition (iv) with (iv') for some positive constants a, L, m ,

$$a \leq \theta'(t)(1 + |D\rho(x)|)^{-2} \left[1 - \frac{L}{m} (|t - \theta(t)| + \rho(x)) \right],$$

we have that if $u \in \overline{\mathcal{S}}(g, Q)$ then

$$w(x, t) := \inf_{y \in \overline{B}_{\rho(x)}(x)} a u(y, \theta(t)).$$

is a supersolution, $w \in \overline{\mathcal{S}}(g, \tilde{Q})$.

The proof of a similar statement for time independent Hele-Shaw problem was previously given in, for instance, [17, Lemma 3.40] with the help of [4, Lemma 9] (see also [18, proof of Lemma 12]). Here we present a slightly more explicit, compact proof for reader's convenience, including the new nonlinear deformation in time, and without the assumption that u is harmonic or continuous in its positive phase.

Proof. First, we proof the statement for subsolutions. We can assume that u is USC, observing that the sup-convolution commutes with taking the upper semi-continuous envelope, and therefore we assume that v is USC. Suppose that $\phi \in C^{2,1}$ and that $v - \phi$ has a local maximum at (\hat{y}, \hat{s}) in $\overline{\{v > 0\}}$. Then, by definition of v , there exists $(\hat{x}, \hat{t}) \in \overline{\{u > 0\}}$ such that $|\hat{x} - \hat{y}| \leq \rho(\hat{y})$, and $v(\hat{y}, \hat{s}) = au(\hat{x}, \hat{t})$ for $\hat{s} = \theta^{-1}(\hat{t})$. We point out here that θ is invertible by assumption (i).

Let us set $h := \hat{y} - \hat{x}$ for convenience. We observe that $au(x, \theta(t)) \leq v(y, t)$ as long as

$$(B.1) \quad \rho(y) \geq |y - x|.$$

With this fact in mind, we shall find a smooth function $y(x)$ so that $y(\hat{x}) = \hat{y}$ and (B.1) holds in a neighborhood of \hat{x} , and therefore $u(x, t) - a^{-1}\phi(y(x), \theta^{-1}(t))$ has a local max at (\hat{x}, \hat{t}) in $\overline{\{u > 0\}}$.

If $|h| < \rho(\hat{y})$, then $y(x) = x + h$ is sufficient due to the continuity of ρ . A more delicate situation arises when $|h| = \rho(\hat{y})$. We define $y(x)$ in such a way that the gradients of both sides in (B.1) are equal, while the Hessians are strictly ordered. The differentiation of $\rho(y(x))$ yields

$$(B.2) \quad \frac{\partial}{\partial x_i} \rho(y(x)) = \sum_k D_k \rho \frac{\partial y_k}{\partial x_i}, \quad \frac{\partial^2}{\partial x_i \partial x_j} \rho(y(x)) = \sum_{k,l} D_{kl} \rho \frac{\partial y_k}{\partial x_i} \frac{\partial y_l}{\partial x_j} + \sum_k D_k \rho \frac{\partial^2 y_k}{\partial x_i \partial x_j}.$$

Let us choose a coordinate system (e_1, \dots, e_n) so that $e_n = h/|h|$ and $D\rho(\hat{y}) = \alpha e_1 + \beta e_n$, and set $\gamma^2 = (1 - \beta)^2 + \alpha^2$. Let us recall that $|D\rho| \leq 1/2$ and therefore $\gamma \geq 1/2$. We define $y(x)$ in the form

$$(B.3) \quad y(x) = \hat{y} + A(x - \hat{x}) + e_n \left[\frac{1}{2} B(x - \hat{x}) \cdot (x - \hat{x}) \right],$$

where B is a symmetric matrix with $\text{tr } B \geq 0$ to be specified later and A is the rotation and scaling matrix

$$A = \frac{1 - \beta}{\gamma^2} e_1 \otimes e_1 + \frac{1 - \beta}{\gamma^2} e_n \otimes e_n + \frac{\alpha}{\gamma^2} e_n \otimes e_1 - \frac{\alpha}{\gamma^2} e_1 \otimes e_n + \frac{1}{\gamma} \sum_{k=2}^{n-1} e_k \otimes e_k.$$

Using the formulas (B.2) and (B.3), we can express the derivatives at $x = \hat{x}$ as

$$(B.4) \quad D(\rho(y(x)))|_{x=\hat{x}} = A^t[D\rho(\hat{y})], \quad D^2(\rho(y(x)))|_{x=\hat{x}} = A^t[D^2\rho(\hat{y})]A + \beta B,$$

and similarly

$$\begin{aligned} D(|y(x) - x|)|_{x=\hat{x}} &= (A - I)^t e_n, \\ D^2(|y(x) - x|)|_{x=\hat{x}} &= \frac{1}{|h|}(A - I)^t(I - e_n \otimes e_n)(A - I) + B. \end{aligned}$$

A quick computation shows that the choice of A guarantees that $A^t[D\rho(\hat{y})] = (A - I)^t e_n$, with the equality of gradients in (B.1) as a consequence. To get the strict ordering of Hessians $D^2(\rho(y(x)))|_{x=\hat{x}} > D^2(|y(x) - x|)|_{x=\hat{x}}$, we need to find a symmetric matrix B with $\text{tr } B \geq 0$ such that

$$M := A^t[D^2\rho(\hat{y})]A - \frac{1}{|h|}(A - I)^t(I - e_n \otimes e_n)(A - I) > (1 - \beta)B.$$

Since $|\beta| < 1$, this will be possible if and only if the trace of the left-hand side M is strictly greater than 0. We observe that $AA^t = \gamma^{-2}I$ and $(A - I)(A - I)^t = (\gamma^{-2} + 1)I - A - A^t$, which is a diagonal matrix, and therefore we can evaluate

$$\begin{aligned} \text{tr } M &= \gamma^{-2}\Delta\rho(\hat{y}) - \frac{1}{|h|}[(n-1)(\gamma^{-2} + 1) - 2(n-2)\gamma^{-1} - 2\gamma^{-2}(1 - \beta)] \\ &= \gamma^{-2} \left[\Delta\rho(\hat{y}) - \frac{1}{|h|}[(n-2)(1 - \gamma)^2 + \alpha^2 + \beta^2] \right]. \end{aligned}$$

Recall that $|h| = \rho(\hat{y})$, and we can also estimate $(n-2)(1 - \gamma)^2 + \alpha^2 + \beta^2 \leq (n-2)\beta^2 + \alpha^2 + \beta^2 \leq (n-1)|D\rho(\hat{y})|^2$. This implies that $\text{tr } M > 0$ due to the assumption (iii) on ρ .

The above computation proves that $u - \tilde{\phi}$ has a local max at (\hat{x}, \hat{t}) in $\overline{\{u > 0\}}$, where $\tilde{\phi}(x, t) := a^{-1}\phi(y(x), \theta^{-1}(t))$. As always, two cases must be addressed:

- (a) $u(\hat{x}, \hat{t}) > 0$: We first observe that $D\phi(\hat{y}, \hat{s}) \cdot e_n \leq 0$. Indeed, since $|D\rho| < 1$, we see that $v(\hat{y}, \hat{s}) = au(\hat{x}, \hat{t}) \leq v(\hat{y} - \sigma e_n, \hat{s})$ for all $\sigma \geq 0$ small. Since u is a subsolution of the Hele-Shaw problem, (B.2) and $\text{tr } B \geq 0$ yield the inequality

$$0 \geq -\Delta\tilde{\phi}(\hat{x}, \hat{t}) = -\text{tr } A^t [D^2\phi(\hat{y}, \hat{s})] A - [D\phi(\hat{y}, \hat{s}) \cdot e_n] \text{tr } B \geq -\gamma^{-2}\Delta\phi(\hat{y}, \hat{s}).$$

- (b) $u(\hat{x}, \hat{t}) = 0$: First, we set $\xi := a^{-1}\gamma^{-2}\theta'(\hat{s}) \leq a^{-1}\theta'(\hat{s})(1 - |D\rho(\hat{y})|)^{-2}$ and, using the assumption (iv) and the assumptions on g , we estimate

$$(B.5) \quad \xi g(\hat{x}, \hat{t}) \leq \xi g(\hat{y}, \hat{s}) \left(1 + \frac{L}{m}(|\hat{s} - \theta(\hat{s})| + |h|) \right) \leq g(\hat{y}, \hat{s}),$$

recalling that $|h| \leq \rho(\hat{y})$. Additionally, differentiation of $\tilde{\phi}$ in time yields

$$\tilde{\phi}_t(x, t) = \frac{\partial}{\partial t}\phi(y(x), \theta^{-1}(t)) = \phi_t(y(x), \theta^{-1}(t)) \frac{1}{\theta'(\theta^{-1}(t))}.$$

The formula (B.4) for the gradient, recalling that A is a rotation and scaling by a factor γ^{-1} , as well as the estimate (B.5) allow us to transform the

viscosity inequality and obtain

$$\begin{aligned}
0 &\geq \tilde{\phi}_t(\hat{x}, \hat{t}) - g(\hat{x}, \hat{t}) \left| D\tilde{\phi}(\hat{x}, \hat{t}) \right|^2 \\
&= a^{-1}[\theta'(\hat{s})]^{-1} \phi_t(\hat{y}, \hat{s}) - a^{-2} \gamma^{-2} g(\hat{x}, \hat{t}) |D\phi(\hat{y}, \hat{s})|^2 \\
&= a^{-1}[\theta'(\hat{s})]^{-1} \left[\phi_t(\hat{y}, \hat{s}) - \xi g(\hat{x}, \hat{t}) |D\phi(\hat{y}, \hat{s})|^2 \right] \\
&\geq a^{-1}[\theta'(\hat{s})]^{-1} \left[\phi_t(\hat{y}, \hat{s}) - g(\hat{y}, \hat{s}) |D\phi(\hat{y}, \hat{s})|^2 \right].
\end{aligned}$$

Hence v is a viscosity subsolution of the HS problem and the proof is finished.

The proof for supersolutions follows same idea, but, in contrast to the proof for subsolutions, we consider a strict local minimum of $w - \phi$ at (\hat{y}, \hat{s}) . It is also necessary to make the obvious changes in the directions of the inequality signs. Finally, the estimate (B.5) is replaced by

$$\xi g(\hat{x}, \hat{t}) \geq \xi g(\hat{y}, \hat{s}) \left(1 - \frac{L}{m} (|\hat{s} - \theta(\hat{s})| + |h|) \right) \geq g(\hat{y}, \hat{s}),$$

which applies due to the assumption (iv'), since now we estimate $\gamma^{-2} \geq (1 + |D\rho(\hat{y})|)^{-2}$. The rest is analogous. \square

B.1. Subsolutions. In this section, we will give an expression for a particular choice of θ so that Proposition B.1 applies for subsolutions, given a, m, L and ρ . The notation throughout this section is consistent with the statement of Proposition B.1.

Let us assume that ρ and $D\rho$ are bounded, and define

$$\alpha := a \min_E (1 - |D\rho|)^2, \quad \gamma := \frac{m}{L}, \quad \lambda := |\tau| + \max_E \rho.$$

We will look for θ of the form $\theta(t) = f(t; \alpha, \gamma, \lambda) + \tau$, for some constant $\tau \neq 0$ and $f \in C^1([0, \infty))$, with $f(0) = 0$ and $0 < f' < 1$.

The upper bound on the derivative yields $f(t) \leq t$ for all $t \geq 0$, hence the estimate $|t - \theta(t)| \leq t - f(t) + |\tau|$ holds. The condition (iv) in Proposition B.1 can be rewritten, after multiplication by γ , as

$$(B.6) \quad \alpha\gamma \geq f'(t)(\gamma + \lambda + t - f(t)) \geq \theta'(t)(\gamma + \rho + |t - \theta(t)|).$$

Let us now assume that $\alpha\gamma < \gamma + \lambda$. In fact, if this condition is violated, we can set $f(t) = t$ and the condition (iv) is satisfied.

To find f satisfying (B.6), we substitute $h(t) = t - f(t) + \gamma + \lambda$, and after a short calculation, using $f' = 1 - h'$, h must satisfy

$$(B.7) \quad \frac{h}{h - \alpha\gamma} h' \geq 1.$$

With the initial data $h(0) = \gamma + \lambda$, the solution of the equality can be expressed in terms of the principal branch of the Lambert W function,

$$h(t) = \alpha\gamma \left(1 + W \left(\frac{\xi}{\alpha\gamma} e^{\frac{t+\xi}{\alpha\gamma}} \right) \right),$$

where $\xi = \gamma + \lambda - \alpha\gamma$, positive by assumption.

Thus the function f can be expressed in terms of h as

$$f(t; \alpha, \gamma, \lambda) = t + \xi - \alpha\gamma W \left(\frac{\xi}{\alpha\gamma} e^{\frac{t+\xi}{\alpha\gamma}} \right), \quad \xi = \gamma + \lambda - \alpha\gamma.$$

Lemma B.2. *The function $f(t; \alpha, \gamma, \lambda)$, for α, γ, λ positive such that $\gamma + \lambda - \alpha\gamma > 0$, has the following properties:*

- (a) $f(0) = 0$,
- (b) $f(t) \rightarrow t$ locally uniformly on $[0, \infty)$ as $(\alpha, \lambda) \rightarrow (1, 0)$,
- (c) $f' = \frac{\alpha\gamma}{h}$, $f'' < 0$, $f'(0) = \frac{\alpha\gamma}{\gamma + \lambda}$.

Proof. For (a), use the formula $W(xe^x) = x$. To show (b), note that $f(t) \rightarrow t$ locally uniformly on $[0, \infty)$ as $(\alpha, \lambda) \rightarrow (1, 0)$ is equivalent to the locally uniform convergence $h(t) \rightarrow \gamma$. Since $W(0) = 0$ and W is continuous at 0, the result follows from the fact that the argument of W goes to 0 locally uniformly in t as $(\alpha, \lambda) \rightarrow (1, 0)$. (c) is a simple consequence of the properties of Lambert W function. \square

B.2. Supersolutions. We follow the idea from §B.1 to construct a function $\theta(t) = f(t) + \tau$ such that condition (iv') in Proposition B.1 holds.

We define $\alpha = a \max(1 + |D\rho|)^2$ now with γ, λ as before, and look for $f \in C^1([0, T])$, for $T > 0$, $f(0) = 0$ and $f' > 1$. The condition on the derivative yields $f(t) \geq t$ for $t \geq 0$ and thus $|t - \theta(t)| \geq f(t) - t - |\tau|$. The condition (iv') can be rewritten as in (B.6),

$$\alpha\gamma \leq f'(t)(\gamma - \lambda + t - f(t)) \leq f'(t)(\gamma - \rho - |t - \theta(t)|).$$

Here it is necessary to assume $\gamma > \lambda$. Furthermore, we assume that $\alpha\gamma > \gamma - \lambda$, otherwise $f(t) = t$ satisfies the inequality and we are done.

After a substitution $h(t) = \gamma - \lambda + t - f(t)$, we discover that h satisfies (B.7). This time, however, h is decreasing, and will become zero in finite time. Therefore the inequality does not have a global solution. With initial data $h(0) = \gamma - \lambda$, the solution can be written as

$$h(t) = \alpha\gamma \left(1 + W \left(\frac{\eta}{\alpha\gamma} e^{\frac{t+\eta}{\alpha\gamma}} \right) \right),$$

where $\eta = \gamma - \lambda - \alpha\gamma$, negative by assumption. The singularity occurs when the argument of W becomes $-\frac{1}{e}$.

The function f has now the form

$$f(t; \alpha, \gamma, \lambda) = t + \eta - \alpha\gamma W \left(\frac{\eta}{\alpha\gamma} e^{\frac{t+\eta}{\alpha\gamma}} \right), \quad \eta = \gamma - \lambda - \alpha\gamma.$$

Lemma B.3. *For every $T > 0$, there exists $\alpha_T > 1$ and $\lambda_T \in (0, \gamma)$ such that the function f is well defined on $[0, T]$ for $\alpha \in (0, \alpha_T)$ and $\lambda \in (0, \lambda_T)$ with $\gamma - \lambda - \alpha\gamma < 0$, and has the following properties:*

- (a) $f(0) = 0$,
- (b) $f(t) \rightarrow t$ uniformly on $[0, T]$ as $\alpha \rightarrow 1+$ and $\lambda \rightarrow 0+$,
- (c) $f' = \frac{\alpha\gamma}{h}$, $f'' > 0$, $f'(0) = \frac{\alpha\gamma}{\gamma - \lambda}$.

Proof. The existence of α_T and λ_T follows after realizing that, as $(\alpha, \lambda) \rightarrow (1, 0)$, $\frac{\eta}{\alpha\gamma} e^{\frac{\eta}{\alpha\gamma}} \rightarrow 0$ while $e^{\frac{t}{\alpha\gamma}}$ stays uniformly bounded on $t \in [0, T]$. Consequently, it is possible to choose $\alpha_T > 1$ and $\lambda_T \in (0, \gamma)$ which guarantee that the argument of W is larger than $-\frac{1}{e}$ for $t \in [0, T]$. As in Lemma B.2, the above observation also implies (b). \square

APPENDIX C. TECHNICAL RESULTS

C.1. Harmonic function on a thin cylinder. The idea of the following lemma is motivated by the observation that the values of a harmonic function on a thin cylinder of a larger radius are not strongly influenced by its values on the side of the cylinder.

Lemma C.1 (Subharmonic function on a thin cylinder). *There is a dimensional constant $c = c(n) > 0$ such that whenever u is a subharmonic function on Ω , $\Omega \subset C$ open, where*

$$C = \{x = (x', x_n) : |x'| < R, 0 < x_n < \delta\}$$

for some $R > 0$ and $\delta > 0$, and furthermore for some $K > 0$

$$\begin{aligned} u &\leq K && \text{on } \overline{C}, \\ u &\leq -1 && \text{on } \partial\Omega \cap \{|x'| < R\}, \end{aligned}$$

then

$$u \leq -\frac{1}{2} \quad \text{on } \{x \in \Omega : |x'| \leq R', 0 < x_n < \delta\},$$

where

$$R' = R - c(K + 2)\delta.$$

Proof. We shall compare the function u on $\overline{\Omega}$ with a translate of the superharmonic function

$$\varphi(x) = \sqrt{1 + \frac{|x'|^2}{n}} \cdot \cos x_n - \frac{3}{2}.$$

Indeed, writing $|x'| = r$, a straightforward computation yields

$$\Delta\varphi = \left(\partial_{x_n x_n} + \partial_{rr} + \frac{n-2}{r} \partial_r \right) \varphi = -\frac{(n + 2r^2 + nr^2 + r^4) \cos x_n}{n^2 \left(\frac{n+r^2}{n} \right)^{3/2}}.$$

This quantity is negative if $|x_n| < \pi/2$.

Define $\tilde{\varphi}(x) = \varphi(\frac{\pi}{3\delta}x)$. Observe that for $0 \leq x_n \leq \delta$

- (a) $-\Delta\tilde{\varphi} > 0$,
- (b) $\tilde{\varphi} \geq -1$,

- (c) $\tilde{\varphi} \geq K$ for $|x'| \geq \frac{6\sqrt{n}}{\pi}(K + 2)\delta$. This follows from estimating $\sqrt{1 + \frac{|x'|^2}{n}} \geq \frac{|x'|}{\sqrt{n}}$ and $\cos x_n \geq \cos \frac{\pi}{3}\delta = \cos \frac{\pi}{3} = \frac{1}{2}$.

So if we set $c := \frac{6\sqrt{n}}{\pi}$, we can compare a translate $\tilde{\varphi}(\cdot - x_0)$, $x_0 = (x'_0, 0)$ with u on \overline{C} whenever $|x'_0| \leq R - c(K + 2)\delta$ and the maximum principle lets us conclude that $u \leq \varphi(\cdot - x_0)$ on \overline{C} . That shows that

$$u \leq -\frac{1}{2} \quad \text{on } \{x : |x'| \leq R - c(K + 2)\delta, 0 \leq x_n \leq \delta\}.$$

□

C.2. Integer approximation. In this section we consider the approximation of a set $A \subset \mathbb{R}^d$ by the set $A \cap \mathbb{Z}^d$. First, we introduce some useful notation.

Definition C.2. For a set $A \subset \mathbb{R}^d$, $d \geq 1$, we shall denote

$$A^{\dagger} = A \cap \mathbb{Z}^d, \quad A^{\dagger;\varepsilon} = A \cap \varepsilon\mathbb{Z}^d.$$

The following two lemmas provide sufficient conditions for approximation of A by A^{\dagger} .

Lemma C.3. Suppose that $A, E \subset \mathbb{R}^d$ and $\lambda > \frac{1}{2}\sqrt{d}$. If $\varepsilon > 0$ and

$$(C.1) \quad E + B_{\lambda\varepsilon}(0) \subset A$$

then

$$E \subset A^{\dagger;\varepsilon} + B_{\lambda\varepsilon}(0).$$

Proof. Let $\xi \in E$. Then, by (C.1), $\xi + B_{\lambda\varepsilon}(0) = B_{\lambda\varepsilon}(\xi) \subset A$. The choice of λ guarantees that there exists $\zeta \in B_{\lambda\varepsilon}(\xi) \cap \varepsilon\mathbb{Z}^d \neq \emptyset$. Clearly $\xi \in B_{\lambda\varepsilon}(\zeta) \subset A^{\dagger;\varepsilon} + B_{\lambda\varepsilon}(0)$ and the proof is finished. \square

Lemma C.4. Let $E \subset \mathbb{R}^d$, $\varepsilon > 0$ and $\lambda > \frac{1}{2}\sqrt{d}$. Define $K := \{x \in E : B_{\lambda\varepsilon}(x) \subset E\}$. If $\mu := \text{dist}(E, K) < \infty$ then

$$(E + x) \subset (E + x)^{\dagger;\varepsilon} + B_{\lambda\varepsilon + \mu}(0) \quad \text{for any } x \in \mathbb{R}^d.$$

Proof. Choose $\delta > 0$ so that $\lambda - \delta > \frac{1}{2}\sqrt{d}$. Since $K + x + B_{(\lambda - \delta)\varepsilon}(0) \subset E + x$ by definition of K , we have by definition of μ and Lemma C.3

$$\begin{aligned} E + x &\subset K + x + B_{\mu + \delta\varepsilon}(0) \subset (E + x)^{\dagger;\varepsilon} + B_{(\lambda - \delta)\varepsilon}(0) + B_{\mu + \delta\varepsilon}(0) \\ &= (E + x)^{\dagger} + B_{\lambda\varepsilon + \mu}(0), \end{aligned}$$

and that is exactly what we wanted to prove. \square

Remark C.5. Note that if $E = \text{Cone}_{\nu, \theta}(0)$ for some $\nu \in \mathbb{R}^n \setminus \{0\}$ and $\theta \in (0, \frac{\pi}{2})$ then $\mu = \frac{\lambda}{\sin \theta} \varepsilon$ in Lemma C.4.

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