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Large deviations for simple random walk on percolations with long-range correlations

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Abstract

We show quenched large deviations for the simple random walk on percolation models with long-range correlations defined by Drewitz, Ráth and Sapozhnikov [3], which contain supercritical Bernoulli percolations, random interlacements, the vacant set of random interlacements and the level set of the Gaussian free field. Our result is an extension of Kubota's result [8] for supercritical Bernoulli percolations.

1 Introduction

In the research of percolation, it is important to understand geometric properties of clusters and behaviors of random walks on the clusters. In the case of supercritical Bernoulli percolation, Antal and Pisztora [1] gave large deviation estimates for the graph distance of two sites lying in the same cluster. Kubota [8] showed quenched large deviations for the simple random walk on supercritical Bernoulli percolation on $\mathbb{Z}^d$. The strategy of proof in [8] is similar to the one in Zerner [10], which showed large deviations for random walk in random environment. However, the configurations of percolation fluctuate and the random walk has non-elliptic transition probabilities. These obstructions are overcame by using [1] Theorem 1.1.

Drewitz, Ráth and Sapozhnikov [3] considered percolation models on $\mathbb{Z}^d$ with long range correlation satisfying some conditions (Assumption 1.1 in below). They obtained large deviation estimates for the graph distance, which are similar to [1] Theorem 1.1 and a shape theorem for balls in the graph distance. The percolation model they considered is a generalization of the

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supercritical Bernoulli site percolation on $\mathbb{Z}^d$. Moreover, the conditions are satisfied by random interlacements, the vacant set of random interlacements and the level set of the Gaussian free field.

In this paper, we show quenched large deviation principles for the simple random walk on percolation models $\{P_u\}_u$ considered by Drewitz, Ráth and Sapozhnikov. Our strategy of proof follows the one in [10] and [8]. In [8], the fact that P_p is a product measure on $\{0,1\}^{\mathbb{Z}^d}$ is essentially used in order to show that the Lyapunov exponent $\alpha_\lambda(\cdot)$ is subadditive. However, in the case under consideration, P_u is *not* necessarily a product measure. To get over this obstruction, we use some ergodic theoretical results for commutative transformations, specifically, Furstenberg and Katznelson's theorem [5] and Tao [9] Theorem 1.1.

Now we describe the setting. Let $d \geq 2$. We write $|x|_\infty = \max_{1\leq i\leq d} |x_i|$, and, $|x|_1 = \sum_{1\leq i\leq d} |x_i|$ for $x = (x_1, \dots, x_d) \in \mathbb{R}^d$. Let $B(x,r) = \{y \in \mathbb{Z}^d : |x-y|_\infty \leq \lfloor r \rfloor\}$, $x \in \mathbb{Z}^d$, $r \geq 0$.

Let us denote a configuration of $\{0,1\}^{\mathbb{Z}^d}$ by $\omega = (\omega(x))_{x\in\mathbb{Z}^d}$. Let $\mathcal{C} = \mathcal{C}(\omega) = \{x \in \mathbb{Z}^d : \omega(x) = 1\}$, $\omega \in \{0,1\}^{\mathbb{Z}^d}$. We regard $\mathcal{C}(\omega)$ as a subgraph of $\mathbb{Z}^d$ in which the set of edges is $\{\{x,y\} : x, y \in \mathcal{C}(\omega), |x-y|_1 = 1\}$. Let $C_x = C_x(\omega)$ be the connected component in $\mathcal{C}(\omega)$ containing x. Let $\mathcal{C}_r$, $r \in [0, +\infty]$, be the set of $x \in \mathbb{Z}^d$ such that l_1-diameter of C_x is larger than or equal to r. Let $D(x,y)$ be the graph distance in $\mathcal{C}$ between x and y. Let $D(x,y) = +\infty$ if x and y are in different connected components in $\mathcal{C}$.

Let θ_x, $x \in \mathbb{Z}^d$, be the canonical shifts on $\{0,1\}^{\mathbb{Z}^d}$, that is, $\theta_x(\omega)(\cdot) = \omega(x+\cdot)$, $\omega \in \{0,1\}^{\mathbb{Z}^d}$. Let $\Phi_y : \{0,1\}^{\mathbb{Z}^d} \to \{0,1\}$, $y \in \mathbb{Z}^d$, be the map defined by $\Phi_y(\omega) = \omega(y)$.

Let $0 \leq a < b$. Following [3], we assume that a family of probability measures $\{P_u\}_{a<u<b}$ on $\{0,1\}^{\mathbb{Z}^d}$ satisfies the following conditions.

Assumption 1.1. (P1) P_u is invariant and ergodic with respect to the lattice shifts θ_x, $x \in \mathbb{Z}^d \setminus \{0\}$, $u \in (a,b)$.
(P2) For any $u_1 < u_2$ and any increasing event G, $P_{u_1}(G) \leq P_{u_2}(G)$.
(P3) There exist constants $R_P, L_P < +\infty$, $\epsilon_P, \chi_P > 0$, and a real valued function f_P with $f_P(t) \geq \exp((\log t)^{\epsilon_P})$, $t \geq L_P$, such that
$P_{u_2}(A_1 \cap A_2) \leq P_{u_1}(A_1)P_{u_1}(A_2) + \exp(-f_P(L))$, and,
$P_{u_1}(B_1 \cap B_2) \leq P_{u_2}(B_1)P_{u_2}(B_2) + \exp(-f_P(L))$
for any pair $(R, L, u_1, u_2, x_1, x_2, A_1, A_2, B_1, B_2)$ with the following conditions,
(i) $R \geq R_P$ is an integer.
(ii) $L \geq 1$ is an integer.
(iii) u_1, u_2 are real numbers such that $a < u_1 < u_2 < b$ and $u_2 \geq (1+R^{-\chi_P})u_1$.
(iv) $x_1, x_2 \in \mathbb{Z}^d$ such that $|x_1 - x_2|_\infty \geq RL$.
(v) A_i (resp. B_i), $i = 1, 2$, are decreasing (resp. increasing) events such that

A_i (resp. B_i) $\in \sigma(\Phi_y : y \in B(x_i, 10L))$.
(S1) (connectivity) There exists $f_S : (a,b) \times \mathbb{Z}_+ \to \mathbb{R}$ such that for any $u \in (a,b)$, there exist $\Delta_S(u)$ and $R_S(u)$ such that $f_S(u,R) \geq (\log R)^{1+\Delta_S(u)}$ for $R \geq R_S(u)$. Moreover, for any $R \geq 1$,

$$P_u(\mathcal{C}_R \cap B(0,R) \neq \emptyset) \geq 1 - \exp(-f_S(u,R)), \text{ and,}$$

$$P_u\left(\bigcap_{x,y \in \mathcal{C}_{R/10} \cap B(0,R)} \{x \text{ and } y \text{ are connected in } \mathcal{C} \cap B(0,2R)\}\right) \geq 1 - \exp(-f_S(u,R)).$$

(S2) (density) $u \mapsto P_u(0 \in \mathcal{C}_\infty)$ is positive and continuous.

The family of supercritical Bernoulli site percolations on $\mathbb{Z}^d$ satisfies the assumptions **(P1)**-**(P3)** and **(S1)**-**(S2)**. **(P3)** is trivial because P_p is a product measure on $\{0,1\}^{\mathbb{Z}^d}$. We see **(S1)** by Grimmett's book [7] (7.89) and (8.98). and **(S2)** by [7] (8.8).

Fix $u \in (a,b)$. By **(S1)**, $\mathcal{C}_\infty$ is non-empty and connected, P_u-a.s. and hence $\mathcal{C}_\infty$ is a unique infinite cluster, P_u-a.s. Let $\Omega_0 = \{0 \in \mathcal{C}_\infty\}$. we define the probability measure $\mathbb{P}$ on $\{0,1\}^{\mathbb{Z}^d}$ by $\mathbb{P}(A) = P_u(A|\Omega_0)$.

Let us define the random walk on the infinite cluster by the Markov chain $((X_n)_{n\geq 0}, (P_\omega^x)_{x \in \mathcal{C}_\infty(\omega)})$ on $\mathcal{C}_\infty(\omega)$ whose transition probabilities are given by $P_\omega^x(X_0 = x) = 1$,

$$P_\omega^x(X_{n+1} = x + e | X_n = x) = \frac{1}{2d} 1_{\{\omega(e)=1\}} \circ \theta_x, \ |e|_1 = 1, \text{ and,}$$

$$P_\omega^x(X_{n+1} = x | X_n = x) = \frac{1}{2d} \sum_{e' : |e'|_1 = 1} 1_{\{\omega(e')=0\}} \circ \theta_x.$$

The following theorem is our main result.

Theorem 1.2. *The law of X_n/n obeys the following large deviation principles with rate function $I(x) = \sup_{\lambda \geq 0}(\alpha_\lambda(x) - \lambda)$, $x \in \mathbb{R}^d$, where $\alpha_\lambda(\cdot)$ is the function on $\mathbb{R}^d$ defined in Section 3.*
(1) *Upper bound : For any closed set A in $\mathbb{R}^d$, we have $\mathbb{P}$-a.s. ω,*

$$\limsup_{n\to\infty} \frac{\log P_\omega^0(X_n/n \in A)}{n} \leq -\inf_{x \in A} I(x). \tag{1.1}$$

(2) *Lower bound : For any open set B in $\mathbb{R}^d$, we have $\mathbb{P}$-a.s. ω,*

$$\liminf_{n\to\infty} \frac{\log P_\omega^0(X_n/n \in B)}{n} \geq -\inf_{x \in B} I(x). \tag{1.2}$$

2 Preliminaries

Let H_y be the first hitting time to y for the random walk $(X_n)_n$.

Let $\lambda \geq 0$. For $x, y, z \in \mathcal{C}_\infty$, we let

$$a_\lambda(x,y) = a_\lambda^\omega(x,y) = -\log E_\omega^x[\exp(-\lambda H_y)1_{\{H_y<+\infty\}}], \text{ and,}$$

$$d_\lambda(x,y) = \max\{a_\lambda(x,y), a_\lambda(y,x)\}.$$

By the strong Markov property of $(X_n)_n$,

$$a_\lambda(x,z) \leq a_\lambda(x,y) + a_\lambda(y,z),\ x,y,z \in \mathcal{C}_\infty. \tag{2.1}$$

By considering a path from x to y of length $D(x,y)$ in $\mathcal{C}_\infty$,

$$d_\lambda(x,y) \leq (\lambda + \log(2d))D(x,y),\ x,y \in \mathcal{C}_\infty. \tag{2.2}$$

Let $T_x : \Omega \to \mathbb{N} \cup \{+\infty\}$ be the map defined by $T_x(\omega) = \inf\{n \geq 1 : nx \in \mathcal{C}_\infty(\omega)\}$, $x \in \mathbb{Z}^d \setminus \{0\}$, where we let $\inf \emptyset = +\infty$. We define the maps $\Theta_x : \Omega_0 \to \Omega_0$ by $\Theta_x\omega = \theta_x^{T_x(\omega)}\omega$. By the Poincaré recurrence theorem, Θ_x is well-defined up to measure 0. By Lemma 3.3 in Berger and Biskup [2], Θ_x is invertible measure-preserving and ergodic with respect to $\mathbb{P}$. Let $T_x^{(n)} = \sum_{k=0}^{n-1} T_x \circ \Theta_x^k$. Then, by Birkhoff's ergodic theorem and Kac's theorem, we have that for $x \in \mathbb{Z}^d \setminus \{0\}$,

$$\lim_{n\to\infty} \frac{T_x^{(n)}}{n} = \mathbb{E}[T_x] = P_u(\Omega_0)^{-1},\ \mathbb{P}\text{ -a.s. and in } L^1(\mathbb{P}). \tag{2.3}$$

2.1 Some Lemmas

In this subsection, we describe some assertions derived from [3] Theorem 1.3.

By [3] Theorem 1.3, we have that for any $u \in (a,b)$, there exist $c_u > 0$ and $C_u < +\infty$ such that for any $x \in \mathbb{Z}^d$,

$$P_u\left(D(0,x) > C_u|x|_1, 0 \leftrightarrow x\right) \leq C_u \exp(-c_u(\log|x|_1)^{1+\Delta_S}). \tag{2.4}$$

Noting (2.4), we can show the following assertions by using the arguments in the proofs of Garet and Marchand [6] Lemma 2.2 and Lemma 2.4 respectively. We omit the proofs.

Lemma 2.1. *There exist $C_1, C_2 > 0$ such that for any $r \geq 1$ and for any y with $|y|_1 \leq r$,*

$$P_u\left(D(0,y) \leq (3r)^d, 0 \leftrightarrow y\right) \leq C_1 \exp(-C_2(\log r)^{1+\Delta_S}).$$

Lemma 2.2. *There exists $C_3 > 0$ such that $\mathbb{E}[D(0, T_x x)] \le C_3 |x|_1$, $x \in \mathbb{Z}^d$.*

Noting (2.4) and Lemma 2.1, we can show the following by using the arguments in the proof of [8], Lemma 3.1, or, in the proof of [10] Lemma 6. We omit the proof.

Lemma 2.3. *Let $\lambda \ge 0$. Then the following holds $\mathbb{P}$-a.s. : For any $\epsilon \in \mathbb{Q} \cap (0, +\infty)$, there exists a positive number N such that for any $x \in \mathcal{C}_\infty$ with $|x|_1 \ge N$,*

$$\sup\{d_\lambda(x, y) : y \in \mathcal{C}_\infty, |x - y|_1 \le \epsilon |x|_1\} \le (\lambda + \log(2d)) C_u \epsilon |x|_1.$$

3 Lyapunov exponents

Let the *Lyapunov exponents* $\alpha_\lambda(x) = P_u(\Omega_0) \inf_{n \ge 1} \mathbb{E}[a_\lambda(0, T_x^{(n)} x)]/n$, for $\lambda \ge 0$ and $x \in \mathbb{Z}^d$. They are obtained by Kingman's subadditive ergodic theorem as the following.

Proposition 3.1. *Let $\lambda \ge 0$ and $x \in \mathbb{Z}^d \setminus \{0\}$. Then,*

$$\lim_{n \to \infty} \frac{a_\lambda(0, T_x^{(n)} x)}{T_x^{(n)}} = \alpha_\lambda(x), \ \mathbb{P}\text{-}a.s.$$

Proof. Fix $\lambda \ge 0$ and $x \in \mathbb{Z}^d \setminus \{0\}$. Let $W_{m,n} = a_\lambda(T_x^{(m)} x, T_x^{(n)} x)$, $0 \le m < n$. Then, by using (2.1), (2.2) and Lemma 2.2, we see that $W_{m+1,n+1} = W_{m,n} \circ \Theta_x$, $W_{0,n} \le W_{0,m} + W_{m,n}$, and, $W_{m,n} \in L^1(\{0,1\}^{\mathbb{Z}^d}, \mathbb{P})$, $0 \le m < n$. Now we can apply Kingman's subadditive ergodic theorem to $\{W_{m,n}\}_{0 \le m < n}$ and obtain

$$\lim_{n \to \infty} \frac{a_\lambda(0, T_x^{(n)} x)}{n} = \inf_{n \ge 1} \frac{\mathbb{E}[a_\lambda(0, T_x^{(n)} x)]}{n}, \ \mathbb{P}\text{-a.s.}$$

By (2.3), we have that

$$\lim_{n \to \infty} \frac{a_\lambda(0, T_x^{(n)} x)}{T_x^{(n)}} = \alpha_\lambda(x), \ \mathbb{P}\text{-a.s.}$$

□

We need the following lemma to show the subadditivity of the Lyapunov exponents.

Lemma 3.2. *Let $z_1, z_2 \in \mathbb{Z}^d$. Then,*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} P_u(\Omega_0 \cap \theta_{z_1}^{-i}\Omega_0 \cap \theta_{z_2}^{-i}\Omega_0) \text{ exists and positive.}$$

We denote this limit by b_{z_1,z_2}.

Proof. By Tao [9] Theorem 1.1, there exists a function $g \in L^2(\{0,1\}^{\mathbb{Z}^d}, P_u)$ such that

$$\frac{1}{n}\sum_{i=1}^{n} 1_{\Omega_0} \circ \theta_0^i \cdot 1_{\Omega_0} \circ \theta_{z_1}^i \cdot 1_{\Omega_0} \circ \theta_{z_2}^i \to g,\ n \to \infty, \text{ in } L^2(P_u).$$

Since θ_0 is the identity map on $\{0,1\}^{\mathbb{Z}^d}$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} P_u(\Omega_0 \cap \theta_{z_1}^{-i}\Omega_0 \cap \theta_{z_2}^{-i}\Omega_0) = \int_{\{0,1\}^{\mathbb{Z}^d}} g dP_u.$$

Since $P_u(\Omega_0) > 0$, it follows from Furstenberg and Katznelson's theorem [5] that

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} P_u(\Omega_0 \cap \theta_{z_1}^{-i}\Omega_0 \cap \theta_{z_2}^{-i}\Omega_0) > 0.$$

These complete the proof. □

Proposition 3.3. *Let $x, y \in \mathbb{Z}^d$ and $q \in \mathbb{N}$. Then, we have that*
(i) $\alpha_\lambda(x + y) \le \alpha_\lambda(x) + \alpha_\lambda(y)$.
(ii) $\alpha_\lambda(qx) = q\alpha_\lambda(x)$.
(iii) $\lambda|x|_1 \le \alpha_\lambda(x) \le (\lambda + \log(2d))C_3 P_u(\Omega_0)|x|_1$, where C_3 is the constant in Lemma 2.2.

Proof. We can see the assertion (ii) by using the methods taken in the proof of [8], Corollary 2.4. By noting (2.2) and Lemma 2.2, we have $\mathbb{E}[a(0, T_x x)] \le (\lambda + \log(2d))C_3|x|_1$ and hence $\alpha_\lambda(x) \le (\lambda + \log(2d))C_3 P_u(\Omega_0)|x|_1$. $\lambda|x|_1 \le \alpha_\lambda(x)$ is shown by using the methods taken in the proof of [8], Lemma 2.2. Thus we have the assertion (iii).

Now we show the assertion (i). We can assume without loss of generality that $x, y, x + y \in \mathbb{Z}^d \setminus \{0\}$.

For $z_1, z_2 \in \mathbb{Z}^d$, let

$$A_{z_1,z_2} = \{z_1, z_2 \in \mathcal{C}_\infty,\ a_\lambda(z_1, z_2) \le C_u(\lambda + \log(2d))|z_1 - z_2|_1\},$$

where C_u is the constant in (2.4). Let

$$A_i = A_{0,ix} \cap A_{0,i(x+y)} \cap A_{ix,i(x+y)}.$$

By (2.1),

$$\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\left[\frac{a_\lambda(0,i(x+y))}{i},A_i\right]$$

$$\leq \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\left[\frac{a_\lambda(0,ix)}{i},A_i\right]+\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\left[\frac{a_\lambda(ix,i(x+y))}{i},A_i\right].$$

Now it is sufficient to show the following convergences.

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\left[\frac{a_\lambda(0,i(x+y))}{i},A_i\right]=\alpha_\lambda(x+y)\frac{b_{x,x+y}}{P_u(\Omega_0)}. \tag{3.1}$$

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\left[\frac{a_\lambda(0,ix)}{i},A_i\right]=\alpha_\lambda(x)\frac{b_{x,x+y}}{P_u(\Omega_0)}. \tag{3.2}$$

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\left[\frac{a_\lambda(ix,i(x+y))}{i},A_i\right]=\alpha_\lambda(y)\frac{b_{x,x+y}}{P_u(\Omega_0)}. \tag{3.3}$$

Here b denotes the constant defined in Lemma 3.2.

Now we prepare the following lemma.

Lemma 3.4.

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\mathbb{P}(A_i)=\frac{b_{x,x+y}}{P_u(\Omega_0)}.$$

Proof. By Lemma 3.2, it is sufficient to show that

$$\lim_{i\to\infty}P_u(A_i^c\cap\Omega_0\cap\theta_x^{-i}\Omega_0\cap\theta_{x+y}^{-i}\Omega_0)=0.$$

By (2.2) and (2.4),

$$P_u\left(A_i^c\cap\Omega_0\cap\theta_x^{-i}\Omega_0\cap\theta_{x+y}^{-i}\Omega_0\right)$$

$$\leq P_u\left(\Omega_0\cap\theta_x^{-i}\Omega_0\cap A_{0,ix}^c\right)+P_u\left(\Omega_0\cap\theta_{x+y}^{-i}\Omega_0\cap A_{0,i(x+y)}^c\right)+P_u\left(\theta_x^{-i}\Omega_0\cap\theta_{x+y}^{-i}\Omega_0\cap A_{ix,i(x+y)}^c\right)$$

$$\begin{aligned}\leq P_u\left(D(0,ix)>C_u i|x|_1, 0\leftrightarrow ix\right)&+P_u\left(D(0,i(x+y))>C_u i|x+y|_1, 0\leftrightarrow i(x+y)\right)\\ &+P_u\left(D(ix,i(x+y))>C_u i|y|_1, ix\leftrightarrow i(x+y)\right)\end{aligned}$$

$$\leq 3C_u\exp\left(-c_u(\log(i\min\{|x|_1,|x+y|_1,|y|_1\}))^{1+\Delta_S}\right).$$

Since $x,y,x+y\neq 0$, $\exp\left(-c_u(\log(i\min\{|x|_1,|x+y|_1,|y|_1\}))^{1+\Delta_S}\right)\to 0$, $i\to\infty$. This completes the proof of Lemma 3.4. □

We show (3.2). First, we have that

$$\mathbb{E}\left[\frac{a_\lambda(0,ix)}{i}, A_i\right] = \mathbb{E}\left[\frac{a_\lambda(0,ix)}{i} - \alpha_\lambda(x), A_i\right] + \alpha_\lambda(x)\mathbb{P}(A_i).$$

By Lemma 3.4, it is sufficient to show that

$$\mathbb{E}\left[\left|\frac{a_\lambda(0,ix)}{i} - \alpha_\lambda(x)\right| 1_{A_i}\right] \to 0, \ i \to \infty. \tag{3.4}$$

By Proposition 3.1, we have that

$$\left|\frac{a_\lambda(0,ix)}{i} - \alpha_\lambda(x)\right| 1_{A_i} \le \left|\frac{a_\lambda(0,ix)}{i} - \alpha_\lambda(x)\right| 1_{\{0,ix\in\mathcal{C}_\infty\}} \to 0, \ i \to \infty, \ \mathbb{P}\text{-a.s.}$$

By the definition of A_i,

$$\left|\frac{a_\lambda(0,ix)}{i} - \alpha_\lambda(x)\right| 1_{A_i} \le C_u(\lambda + \log(2d)) + \alpha_\lambda(x), \ i \ge 1.$$

By the Lebesgue convergence theorem, we obtain (3.4). Thus (3.2) is shown.

We can show (3.1) in the same manner.

Finally we show (3.3). By Lemma 3.4, it is sufficient to show that

$$E_u\left[\left|\frac{a_\lambda(ix, i(x+y))}{i} - \alpha_\lambda(y)\right| 1_{A_i}\right] \to 0, \ i \to \infty. \tag{3.5}$$

By the shift invariance of P_u, we have

$$E_u\left[\left|\frac{a_\lambda(ix, i(x+y))}{i} - \alpha_\lambda(y)\right| 1_{A_i}\right] = E_u\left[\left|\frac{a_\lambda(0,iy)}{i} - \alpha_\lambda(y)\right| 1_{\theta_x^i A_i}\right].$$

Now we have that $a_\lambda(0,iy) \le C_u(\lambda + \log(2d)) i |y|_1$ on $\theta_x^i A_i$. Hence,

$$\left|\frac{a_\lambda(0,iy)}{i} - \alpha_\lambda(y)\right| 1_{\theta_x^i A_i} \le C_u(\lambda + \log(2d))|y|_1 + \alpha_\lambda(y).$$

By Proposition 3.1,

$$\left|\frac{a_\lambda(0,iy)}{i} - \alpha_\lambda(y)\right| 1_{\theta_x^i A_i} \le \left|\frac{a_\lambda(0,iy)}{i} - \alpha_\lambda(y)\right| 1_{\{0,iy\in\mathcal{C}_\infty\}} \to 0, \ i \to \infty, P_u\text{-a.s.}$$

Thus we obtain (3.5) by using the Lebesgue convergence theorem and hence (3.3) is shown. These complete the proof of the assertion (i). $\square$

We can easily extend the Lyapunov exponent $\alpha_\lambda(\cdot)$ to the function on $\mathbb{R}^d$ and then we have the following. See [10] Proposition 3 for proof.

Proposition 3.5. *Let $\lambda \geq 0$. Then, there exists a function $\alpha_\lambda : \mathbb{R}^d \to [0, \infty)$ such that for any $x \in \mathbb{Z}^d \setminus \{0\}$, $\mathbb{P}$-a.s.,*

$$\lim_{n\to\infty} \frac{a_\lambda(0, T_x^{(n)} x)}{T_x^{(n)}} = \alpha_\lambda(x).$$

Moreover, for any $x, y \in \mathbb{R}^d$ and for any $q \in (0, +\infty)$, $\alpha_\lambda(qx) = q\alpha_\lambda(x)$, $\alpha_\lambda(x+y) \leq \alpha_\lambda(x) + \alpha_\lambda(y)$, and, $\lambda|x|_1 \leq \alpha_\lambda(x) \leq (\lambda + \log(2d))C_3 P_u(\Omega_0)|x|_1$.

We state some properties of the Lyapunov exponent. See [10] Proposition 3 for proof.

Lemma 3.6. *(i) $x \mapsto \alpha_\lambda(x)$ is convex on $\mathbb{R}^d$.*
(ii) $\lambda \mapsto \alpha_\lambda(x)$ is concave on $[0, +\infty)$.
(iii) $(\lambda, x) \mapsto \alpha_\lambda(x)$ is continuous on $(0, +\infty) \times \mathbb{R}^d$.

4 Shape theorem

First, we state the following lemma, which is essentially the same as [6] Lemma 5.5.

Lemma 4.1. *Let $z \in \mathbb{Z}^d \setminus \{0\}$. Let $\eta > 0$. Then, we have $\mathbb{P}$-a.s. that there exists a positive integer N such that for any $r \geq N$ there exists $k \in [(1-\eta)r, (1+\eta)r]$ such that $kz \in \mathcal{C}_\infty$.*

Let $\Omega_{1,\lambda,z}$ be the set with probability 1 such that the statement in Proposition 3.1 holds on the set for fixed λ, z. Let $\Omega_{2,z,\eta}$ be the set with probability 1 such that the statement in Lemma 4.1 holds on the set for fixed z, η. Let $\Omega_{3,\lambda}$ be the set with probability 1 such that the statement in Lemma 2.3 holds on the set for fixed λ. For $\lambda \geq 0$, we let

$$\Omega(\lambda) = \bigcap_{z\in\mathbb{Z}^d} \Omega_{1,\lambda,z} \cap \bigcap_{z\in\mathbb{Z}^d\setminus\{0\}, \eta\in\mathbb{Q}\cap(0,\infty)} \Omega_{2,z,\eta} \cap \Omega_{3,\lambda}.$$

We remark that $\mathbb{P}(\Omega(\lambda)) = 1$, $\lambda \geq 0$.

Proposition 4.2 (Shape theorem)**.** *We have $\mathbb{P}$-a.s. that for any $\lambda \geq 0$,*

$$\lim_{|x|_1\to\infty,\, x\in\mathcal{C}_\infty} \frac{a_\lambda(0, x) - \alpha_\lambda(x)}{|x|_1} = 0.$$

Proof. The following proof is essentially the same as the proof of [8] Theorem 1.2. By using Lemma 3.6 and the argument in the final part of [10] Theorem A, we see that it is sufficient to show that for any *fixed* $\lambda \geq 0$ and $\epsilon \in \mathbb{Q} \cap (0, 1)$, the following holds $\mathbb{P}$-a.s., there exists a positive integer N such that for any $x \in \mathcal{C}_\infty$ with $|x|_1 \geq N$, $|a_\lambda(0, x) - \alpha_\lambda(x)| \leq \epsilon |x|_1$.

Assume this statement fails. Then, there exist $\lambda_0 \geq 0$ and $\epsilon_0 > 0$ and an event A with positive probability such that on A, there exists a sequence $(x_n)_n \subset \mathcal{C}_\infty$ satisfying $|x_n|_1 \to \infty$, and $|a_{\lambda_0}(0, x_n) - \alpha_{\lambda_0}(x_n)| \geq \epsilon_0 |x_n|_1$, $n \geq 1$.

Take a configuration $\omega \in A \cap \Omega(\lambda_0)$ and a sequence $(x_n)_n$ in $\mathcal{C}_\infty(\omega)$ described as above. By taking a subsequence if necessary, we can assume that $x_n/|x_n|_1$ converges to a point $v \in \{z \in \mathbb{R}^d : |z|_1 \leq 1\}$.

Take $\eta \in \mathbb{Q} \cap (0, \infty)$, which is chosen small enough later. Let $v' \in S^{d-1} \cap \mathbb{Q}^d$ such that $|v - v'| < \eta$. Let $M \in \mathbb{N}_{\geq 1}$ such that $Mv' \in \mathbb{Z}^d$. Let $x'_n = \lfloor |x_n|_1/M \rfloor Mv'$, $n \geq 1$. By recalling Lemma 4.1 and $\omega \in \Omega(\lambda_0)$, we have that for any n, there exists $k_n = k_n(\eta, \omega)$ such that $(1 - \eta)\lfloor |x_n|_1/M \rfloor \leq k_n \leq \lfloor |x_n|_1/M \rfloor$, and, $k_n M v' \in \mathcal{C}_\infty(\omega)$. Let $x''_n = k_n M v'$. Then,

$$\begin{aligned}
|x_n - x''_n|_1 &\leq |x_n - x'_n|_1 + |x'_n - x''_n|_1 \\
&\leq |x_n - |x_n|_1 v'|_1 + ||x_n|_1 v' - x'_n|_1 + M(\lfloor |x_n|_1/M \rfloor - k_n) \\
&\leq |x_n|_1 \left| \frac{x_n}{|x_n|_1} - v' \right| + M + \eta |x_n|_1.
\end{aligned}$$

Hence $|x_n - x''_n| \leq 3\eta |x_n|_1$ for sufficiently large n.

Recalling $x''_n = k_n M v' \in \mathcal{C}_\infty$, Proposition 3.1 and $\omega \in \Omega(\lambda_0)$, we have that

$$\lim_{n\to\infty} \frac{a_{\lambda_0}(0, x''_n)}{k_n} = \lim_{n\to\infty} \frac{a_{\lambda_0}(0, k_n M v')}{k_n} = \alpha_{\lambda_0}(Mv') = \frac{\alpha_{\lambda_0}(x''_n)}{k_n}.$$

Since $k_n \leq |x_n|_1$,

$$\lim_{n\to\infty} \frac{a_{\lambda_0}(0, x''_n) - \alpha_{\lambda_0}(x''_n)}{|x_n|_1} = 0.$$

Hence,

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{|a_{\lambda_0}(0, x_n) - \alpha_{\lambda_0}(x_n)|}{|x_n|_1} \\
&\leq \limsup_{n\to\infty} \frac{|a_{\lambda_0}(0, x_n) - a_{\lambda_0}(0, x''_n)|}{|x_n|_1} + \limsup_{n\to\infty} \frac{|\alpha_{\lambda_0}(x_n) - \alpha_{\lambda_0}(x''_n)|}{|x_n|_1}.
\end{aligned}$$

By recalling $|x_n - x''_n| \leq 3\eta |x_n|_1$ for sufficiently large n, it follows from (2.1) and Lemma 2.3 and $\omega \in \Omega(\lambda_0)$ that

$$\limsup_{n\to\infty} \frac{|a_{\lambda_0}(0, x_n) - a_{\lambda_0}(0, x''_n)|}{|x_n|_1} \leq \limsup_{n\to\infty} \frac{d_{\lambda_0}(x_n, x''_n)}{|x_n|_1} \leq 3\eta(\lambda_0 + \log(2d))C_u.$$

By Proposition 3.5,

$$\limsup_{n\to\infty} \frac{\alpha_{\lambda_0}(x_n - x_n'') \vee \alpha_{\lambda_0}(x_n'' - x_n)}{|x_n|_1} \leq 3\eta(\lambda_0 + \log(2d))C_3.$$

Thus we have

$$\limsup_{n\to\infty} \frac{|a_{\lambda_0}(0, x_n) - \alpha_{\lambda_0}(x_n)|}{|x_n|_1} \leq 3(\lambda_0 + \log(2d))(C_3 + C_u)\eta.$$

By recalling the definition of $(x_n)_n$, we have that $\epsilon_0 \leq 3(\lambda_0 + \log(2d))(C_3 + C_u)\eta$. However we can take $\eta < \epsilon_0/(3(\lambda_0 + \log(2d))(C_3 + C_u))$. This is a contradiction. $\square$

5 Large deviations

In this section, we show Theorem 1.2 by using the strategies taken in the proof of [8] Theorem 1.3.

Let $I(z) = \sup_{\lambda \geq 0}(\alpha_\lambda(x) - \lambda)$, $z \in \mathbb{R}^d$. Let $\mathcal{D}_I = (I < +\infty)$.

5.1 Proof of the upper bound

Let A be a closed set in $\mathbb{R}^d$. Since $|X_n|_1 \leq n$ for any $n \geq 1$ under P_ω^0 and $I(z) = +\infty$ for $z \in \mathbb{R}^d$ with $|z|_1 > 1$, we can assume without loss of generality that A is contained in the closed l_1-ball centered at 0 with radius 1 in $\mathbb{R}^d$. If $0 \in A$, then $\inf_{z \in A} I(z) = 0$ and hence the assertion holds. Hereafter we assume that $0 \notin A$.

Let $I^\delta(z) = (I(z) - \delta) \wedge (1/\delta)$ and $A_\lambda(\delta) = \{z \in A : \alpha_\lambda(z) - \lambda > \inf_{x \in A} I^\delta(x) - \delta\}$, $\lambda \geq 0$, $\delta > 0$. Since A is compact, there exist $\lambda_1, \dots, \lambda_m$ such that $A = \cup_{i=1}^m A_{\lambda_i}(\delta)$. Hence we have

$$\limsup_{n\to\infty} \frac{\log P_\omega^0(X_n \in nA)}{n} \leq \max_{1 \leq i \leq m} \limsup_{n\to\infty} \frac{\log P_\omega^0(X_n \in nA_{\lambda_i}(\delta))}{n}. \tag{5.1}$$

We will show that for $\lambda \geq 0$ and $\delta > 0$, the following holds $\mathbb{P}$-a.s.ω :

$$\limsup_{n\to\infty} \frac{\log P_\omega^0(X_n \in nA_\lambda(\delta))}{n} \leq \delta - \inf_{z \in A} I^\delta(z). \tag{5.2}$$

We can assume without loss of generality that $nA_\lambda(\delta)\cap\mathcal{C}_\infty(\omega)\neq\emptyset$. Then,

$$\begin{aligned}
P^0_\omega(X_n\in nA_\lambda(\delta)) &= \sum_{y\in nA_\lambda(\delta)\cap\mathcal{C}_\infty(\omega)} P^0_\omega(X_n=y)\\
&\le \sum_{y\in nA_\lambda(\delta)\cap\mathcal{C}_\infty(\omega)} P^0_\omega(H_y\le n)\\
&\le \sum_{y\in nA_\lambda(\delta)\cap\mathcal{C}_\infty(\omega)} \exp(\lambda n - a^\omega_\lambda(0,y))\\
&\le \left|nA_\lambda(\delta)\cap\mathbb{Z}^d\right|\exp(\lambda n - a^\omega_\lambda(0,y_{n,\lambda})),
\end{aligned}$$

for some $y_{n,\lambda}\in nA_\lambda(\delta)\cap\mathcal{C}_\infty(\omega)$.

Since $A_\lambda(\delta)$ is bounded, we have

$$\begin{aligned}
\frac{\log P^0_\omega(X_n\in nA_\lambda(\delta))}{n} &\le o(1)+\lambda-\frac{a_\lambda(0,y_{n,\lambda})}{n}\\
&= o(1)+\lambda-\alpha_\lambda\left(\frac{y_{n,\lambda}}{n}\right)-\frac{a_\lambda(0,y_{n,\lambda})-\alpha_\lambda(y_{n,\lambda})}{|y_{n,\lambda}|_1}\frac{|y_{n,\lambda}|_1}{n}.
\end{aligned} \tag{5.3}$$

Since $A_\lambda(\delta)\subset A$, $0\notin A$ and A is compact, $\mathrm{dist}(0,A_\lambda(\delta))>0$. Hence $|y_{n,\lambda}|_1\to\infty$, $n\to\infty$. Then, by Proposition 4.2 and boundedness of $A_\lambda(\delta)$, we have $\mathbb{P}$-a.s. that

$$\frac{a_\lambda(0,y_{n,\lambda})-\alpha_\lambda(y_{n,\lambda})}{|y_{n,\lambda}|_1}\frac{|y_{n,\lambda}|_1}{n}\to 0, n\to\infty.$$

Recalling (5.3) and $y_{n,\lambda}/n\in A_\lambda(\delta)$, we have $\mathbb{P}$-a.s. that

$$\limsup_{n\to\infty}\frac{\log P^0_\omega(X_n\in nA_\lambda(\delta))}{n}\le\lambda-\inf_{z\in A_\lambda(\delta)}\alpha_\lambda(z)\le\delta-\inf_{z\in A}I^\delta(z).$$

Thus we see that (5.2) holds $\mathbb{P}$-a.s. for fixed $\lambda\ge 0$ and $\delta>0$. By (5.1), we see that for fixed $\delta>0$ the following holds $\mathbb{P}$-a.s. :

$$\limsup_{n\to\infty}\frac{\log P^0_\omega(X_n\in nA)}{n}\le\delta-\inf_{z\in A}I^\delta(z).$$

By letting $\delta\to 0$, we see that (1.1) holds $\mathbb{P}$-a.s.

5.2 Proof of the lower bound

For $\lambda\ge 0$, $\omega\in\Omega_0$, $x,y\in\mathcal{C}_\infty(\omega)$, let

$$Q^{x,y}_{\lambda,\omega}(dX.)=\frac{\exp(-\lambda H_y(X.))1_{\{H_y(X.)<+\infty\}}}{E^x_\omega[\exp(-\lambda H_y)1_{\{H_y<+\infty\}}]}P^x_\omega(dX.).$$

Then we have the following lemma, which is essentially the same as [8] Lemma 4.1 and Fukushima and Kubota [4] Lemma 4.1. See the references for proof.

Lemma 5.1. *Let $x \in \mathbb{Q}^d \setminus \{0\}$. Let $\beta \in [0,1)$. Denote $v = x/|x|_1$. Denote $M \in \mathbb{N}_{\geq 1}$ such that $Mv \in \mathbb{Z}^d$. Denote $y_n^{(1)} = T_{Mv}^{(\lfloor P_u(\Omega_0)\beta n|x|/M \rfloor)} Mv$ and $y_n^{(2)} = T_{Mv}^{(\lfloor P_u(\Omega_0) n|x|/M \rfloor)} Mv$. Then, the following holds $\mathbb{P}$-a.s. : for any $\lambda \geq 0$ and $\gamma_1, \gamma_2 \in \mathbb{R}$ with $0 \leq \gamma_1 < \alpha'_{\lambda+}(x) \leq \alpha'_{\lambda-}(x) < \gamma_2$,*

$$\lim_{n\to\infty} Q_{\lambda,\omega}^{y_n^{(1)}, y_n^{(2)}} \left(\frac{H_{y_n^{(2)}}}{(1-\beta)n} \in (\gamma_1, \gamma_2) \right) = 1.$$

Now we start the proof of the lower bound.

First, we show that it is sufficient to show that for any fixed $z \in \mathbb{Q}^d \setminus \{0\} \cap \mathcal{D}_I$ and $r \in (0,\infty) \cap \mathbb{Q}$, the following holds $\mathbb{P}$-a.s. :

$$\liminf_{n\to\infty} \frac{\log P_\omega^0(X_n \in nB(z,r))}{n} \geq -I(z). \tag{5.4}$$

Let $B \subset \mathbb{R}^d$ be open. If $B \cap \mathcal{D}_I = \emptyset$, $-\inf_{z\in B} I(z) = -\infty$ and hence the assertion holds. Assume $B \cap \mathcal{D}_I \neq \emptyset$. Since $\mathcal{D}_I$ is convex and B is open, we see $B \cap \mathrm{int}\mathcal{D}_I \neq \emptyset$ and for any $z \in B \cap \mathcal{D}_I$, there exists $u < 1$ such that $uz \in B \cap \mathrm{int}\mathcal{D}_I$. Therefore, $\inf_{z \in B\cap\mathcal{D}_I} I(z) = \inf_{z\in B\cap \mathrm{int}\mathcal{D}_I} I(z)$. By the continuity of I on $\mathrm{int}\mathcal{D}_I$, $\inf_{z\in B} I(z) = \inf_{z \in B\cap\mathrm{int}\mathcal{D}_I \cap \mathbb{Q}^d} I(z)$. Take a point $z \in B \cap \mathrm{int}\mathcal{D}_I \cap \mathbb{Q}^d$ and $r > 0$ with $B(z,r) \subset B$ arbitrarily. By applying (5.4) to $B(z,r)$, we see that (1.2) holds $\mathbb{P}$-a.s. for B.

Now we show (5.4). Hereafter we fix z and r. Let $\lambda_*(z) = \sup\{\lambda \geq 0 : \alpha'_\lambda(z) \text{ exists and } \geq 1\}$, where $\alpha'_\lambda(z)$ denotes the derivative of $\alpha_\lambda(z)$ with respect to λ if it exists. Let $v = z/|z|_1$ and M be the least integer such that $Mv \in \mathbb{Z}^d$. Let $\Omega_{4,x,\beta}$ be the set with probability 1 such that the assertion in Lemma 5.1 holds and also $y_n^{(2)}/n \to x$, $n \to \infty$ (Cf. (2.3)), on the set, for fixed x, β.

Case 1. $\lambda_*(z) = 0$. In this case, we use the methods described in the proof of [8] Theorem 1.3. Let $y_n = y_n^{(2)}$, where $y_n^{(2)}$ is defined in Lemma 5.1 for $x = z$ and $\beta = 0$. Then, $y_n/n \to z$ on $\Omega_{4,z,0}$. Let $R > 0$ be an even integer. Then, for all sufficiently large n, $B(y_n, R) \subset nB(z,r)$.

$$\begin{aligned} P^0(X_n \in nB(z,r)) &\geq P^0(H_{y_n} \leq n,\, X_{m+H_{y_n}} \in B(y_n,R), \forall m \in [0,n]) \\ &\geq P^0(H_{y_n} \leq n) P^{y_n}(X_m \in B(y_n,R), \forall m \in [0,n]) \\ &\geq E^0[\exp(-\lambda H_{y_n}), H_{y_n} \leq n] P^{y_n}(X_R = y_n)^{n/R}. \end{aligned} \tag{5.5}$$

Applying Lemma 5.1 to the case $x = z$, $\beta = 0$, $\gamma_1 = 0$, and $\gamma_2 = 1$, we have that on $\Omega_{4,z,0}$, for any $\lambda > 0$ such that $\alpha'_\lambda(z)$ exists,

$$E^0[\exp(-\lambda H_{y_n}), H_{y_n} \le n] \sim E^0[\exp(-\lambda H_{y_n})] = \exp(-a_\lambda(0, y_n)).$$

By Proposition 3.1 and (5.5), we see that for any $\lambda > 0$ such that $\alpha'_\lambda(z)$ exists, we have that on $\Omega_{4,z,0} \cap \Omega_{1,\lambda,Mv}$,

$$\begin{aligned}\liminf_{n\to\infty} \frac{\log P^0(X_n \in nB(z,r))}{n} &\ge \liminf_{n\to\infty} \frac{-a_\lambda(0,y_n)}{n} + \liminf_{n\to\infty} \frac{\log P^{y_n}(X_R = y_n)}{R} \\ &= -\alpha_\lambda(z) + \liminf_{n\to\infty} \frac{\log P^{y_n}(X_R = y_n)}{R}.\end{aligned}$$

Since $\mathcal{C}_\infty$ is a subgraph of $\mathbb{Z}^d$, $P^{y_n}(X_R = y_n) \ge c_d R^{-d}$ for *any* $n \ge 1$, where c_d is a positive constant depending only on d. Therefore, by letting $R \to \infty$, we see that for any $\lambda > 0$ such that $\alpha'_\lambda(z)$ exists, on $\Omega_{4,z,0} \cap \Omega_{1,\lambda,Mv}$,

$$\liminf_{n\to\infty} \frac{\log P^0(X_n \in nB(z,r))}{n} \ge -\alpha_\lambda(z).$$

Since $\lambda_*(z) = 0$, we have that $I(z) = \lim_{\lambda\downarrow 0} \alpha_\lambda(z)$ and the following holds $\mathbb{P}$-a.s. :

$$\liminf_{n\to\infty} \frac{\log P^0(X_n \in nB(z,r))}{n} \ge -I(z).$$

This completes the proof of Case 1.

Case 2. $\lambda_*(z) \in (0, +\infty)$. In this case, we follow the strategy of proof of [10] Theorem B. Let $\epsilon \in (0, \lambda_*(z) \wedge 1)$. Then, By noting Lemma 3.6 and the assumption $\lambda_*(z) \in (0, \infty)$, there are $\rho \in (0,1)$, $\eta > 0$, and, λ_0, λ_2 such that
(1) $\alpha'_{\lambda_0}(z)$ and $\alpha'_{\lambda_2}(z)$ exist.
(2) $\lambda_*(z) - \epsilon < \lambda_0 \le \lambda_*(z) \le \lambda_2$.
(3) $\alpha_{\lambda_2}(z) < \alpha_{\lambda_*(z)}(z) + \epsilon$.
(4) $\rho\alpha'_{\lambda_0}(z) + (1-\rho)\alpha'_{\lambda_2}(z) + [-\eta, +\eta] \subset (1 - \epsilon r/2, 1 + \epsilon r/2)$.

Let $y_n^{(1)}$, $y_n^{(2)}$ as defined in Lemma 5.1 for $x = z$ and $\beta = \rho$. Since $y_n^{(2)}/n \to z$ on $\Omega_{4,z,\rho}$, $B(y_n^{(2)}, nr/2) \subset nB(z,r)$ for sufficiently large n. Then, for sufficiently large n,

$$\begin{aligned}\frac{1}{n}\log P^0(X_n \in nB(z,r)) &\ge \frac{1}{n}\log P^0\left(H_{y_n^{(2)}}/n \in (1 - \epsilon r/2, 1 + \epsilon r/2)\right) \\ &\ge \lambda_*(z)\left(1 - \frac{\epsilon r}{2}\right) + \frac{1}{n}\log E^0\left[\exp(-\lambda_*(z) H_{y_n^{(2)}}), A_n\right],\end{aligned}$$

where we let

$$A_n = \left\{ H_{y_n^{(1)}} \in n\rho(\alpha'_{\lambda_0}(z) + [-\eta, +\eta]) \right\} \\ \cap \left\{ \exists m \in n(1-\rho)(\alpha'_{\lambda_2}(z) + [-\eta, +\eta]) \text{ such that } X_{m+H_{y_n^{(1)}}} = y_n^{(2)} \right\}.$$

By the strong Markov property of $(X_n)_n$,

$$E^0 \left[\exp(-\lambda_*(z) H_{y_n^{(2)}}), A_n \right] = E^0 \left[\exp\left(-\lambda_*(z) H_{y_n^{(1)}} \right), \frac{H_{y_n^{(1)}}}{n\rho} \in \alpha'_{\lambda_0}(z) + [-\eta, +\eta] \right] \\ \times E^{y_n^{(1)}} \left[\exp\left(-\lambda_*(z) H_{y_n^{(2)}} \right), \frac{H_{y_n^{(2)}}}{n(1-\rho)} \in \alpha'_{\lambda_2}(z) + [-\eta, +\eta] \right].$$

Since $-\lambda_*(z) H_{y_n^{(1)}} \geq -\lambda_0 H_{y_n^{(1)}} + (\lambda_0 - \lambda_*(z)) n\rho(\alpha'_{\lambda_2}(z) + \eta)$ on the set $\{H_{y_n^{(1)}} \in n\rho(\alpha'_{\lambda_0}(z) + [-\eta, +\eta])\}$, and, $\lambda_*(z) \leq \lambda_2$, we have that on $\Omega_{4,z,\rho}$,

$$\liminf_{n \to \infty} \frac{1}{n} \log E^0 \left[\exp(-\lambda_*(z) H_{y_n^{(1)}}), A_n \right] \\ \geq \lambda_*(z)(1 - \epsilon r/2) + (\lambda_0 - \lambda_*(z))\rho(\alpha'_{\lambda_2}(z) + \eta) + a_1 + a_2, \quad (5.6)$$

where we let

$$a_1 = \liminf_{n \to \infty} \frac{1}{n} \log E^0 \left[\exp(-\lambda_0 H_{y_n^{(1)}}), H_{y_n^{(1)}} \in n\rho(\alpha'_{\lambda_0}(z) + [-\eta, +\eta]) \right], \text{ and,}$$

$$a_2 = \liminf_{n \to \infty} \frac{1}{n} \log E^{y_n^{(1)}} \left[\exp(-\lambda_2 H_{y_n^{(2)}}), H_{y_n^{(2)}} \in n(1-\rho)(\alpha'_{\lambda_2}(z) + [-\eta, +\eta]) \right].$$

By using Lemma 5.1 for $x = z$ and $\beta = 0$ and for $x = z$ and $\beta = \rho$, and then by using Proposition 3.1,

$$a_1 = \lim_{n \to \infty} \frac{\log E^0[\exp(-\lambda_0 H_{y_n^{(1)}})]}{n} = -\rho\alpha_{\lambda_0}(z), \text{ on } \Omega_{4,z,0} \cap \Omega_{1,\lambda_0}, \text{ and,}$$

$$a_2 = \lim_{n \to \infty} \frac{\log E^{y_n^{(1)}}[\exp(-\lambda_2 H_{y_n^{(2)}})]}{n} = -(1-\rho)\alpha_{\lambda_2}(z), \text{ on } \Omega_{4,z,\rho} \cap \Omega_{1,\lambda_2}.$$

Therefore we have that on $\Omega_{4,z,\rho} \cap \Omega_{4,z,0} \cap \Omega_{1,\lambda_0} \cap \Omega_{1,\lambda_2}$, the right hand side of (5.6) is larger than or equal to

$$\lambda_*(z) \left(1 - \frac{\epsilon r}{2} \right) + (\lambda_0 - \lambda_*(z))\rho(\alpha'_{\lambda_2}(z) + \eta) - \rho\alpha_{\lambda_0}(z) - (1-\rho)\alpha_{\lambda_2}(z).$$

By the assumption $\lambda_*(z) \in (0, +\infty)$, we have that $I(z) = \alpha_{\lambda_*(z)}(z) - \lambda_*(z)$. Recalling the properties (1) - (4) which ρ, λ_0 and λ_2 satisfy, we see that on $\Omega_{4,z,\rho} \cap \Omega_{4,z,0} \cap \Omega_{1,\lambda_0} \cap \Omega_{1,\lambda_2}$,

$$\liminf_{n\to\infty} \frac{\log P^0(X_n \in nB(z,r))}{n} \geq -I(z) - \lambda_*(z)\epsilon r - \epsilon(2 + \epsilon r).$$

By letting $\epsilon \to 0$, we see that (5.4) holds $\mathbb{P}$-a.s.

Case 3. $\lambda_*(z) = +\infty$. In this case, we use the methods taken in the proof of [4] Theorem 1.4. Since $\lambda_*(z) = +\infty$, we have $\lim_{\lambda\to\infty} \alpha'_\lambda(z) \leq 1$. Then, for any $u \in \mathbb{Q} \cap (0,1)$, there exists $\lambda(u) < \infty$ such that for any $\lambda \geq \lambda(u)$, $\alpha'_\lambda(uz) < 1$, and hence, $\lambda_*(uz) \in [0, \infty)$. If $u \in (0 \vee (1 - r/|z|), 1)$, we can take $r(u) \in \mathbb{Q}$ with $B(uz, r(u)) \subset B(z,r)$. By using Case 1 or 2, we have $\mathbb{P}$-a.s. that

$$\liminf_{n\to\infty} \frac{\log P^0(X_n \in nB(z,r))}{n} \geq \liminf_{n\to\infty} \frac{\log P^0(X_n \in nB(uz,r(u)))}{n} \geq -I(uz).$$

Since $I(uz) \leq uI(z) \leq I(z)$, we see that (5.4) holds $\mathbb{P}$-a.s.

Thus the proof of the lower bound (1.2) is completed.

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