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1 Introduction and result

Let $d \geqq 1$, $T > 0$, $W_0 = \{w \in C([0,T]; \mathbf{R}^d);\ w(0) = 0\}$, and $\mathcal{B}(W_0^d)$ be a Borel algebra over W_0^d. Let μ be the Wiener measure on $(W_0^d, \mathcal{B}(W_0^d))$. Now let $a_i^k : [0,T] \to \mathbf{R}$, $b_i^k : [0,T] \to \mathbf{R}$, $i = 1, \ldots, N$, $k = 1, \ldots, d$, be continuous functions. Let $X : W_0^d \to \mathbf{R}$ be a random variable given by

$$X(w) = \sum_{i=1}^{N} \sum_{k,\ell=1}^{d} \int_0^T (\int_0^t b_i^\ell(s) dw^\ell(s)) a_i^k(t) dw^k(t).$$

Here stochastic integrals are Ito integrals. We assume that $E^\mu[\exp(X)] < \infty$.

Our concern is to compute the following.

$$E^\mu[\exp(\lambda \sum_{k=1}^{d} \int_0^T h^k(t) dw^k(t) + X)]$$

for $h \in L^2([0,T]; \mathbf{R}^d, dt)$ and $\lambda \in \mathbf{C}$.

Such a problem was considered by Ikeda-Kusuoka-Manabe [1] and [2] in special cases, and they gave explicit formulae. In the present paper, we consider general case and show that we can reduce this problem to a problem of a linear ordinary differential equation by using ideas in [1] and [2].

Let $\alpha^k : [0,T] \to \mathbf{R}^{2N}$, $k = 1, \ldots, d$, be given by

$$\alpha_j^k(t) = \begin{cases} a_j^k(t), & j = 1, \ldots, N, \\ b_{j-N}^k(t), & j = N+1, \ldots, 2N. \end{cases}$$

Let $J : \mathbf{R}^{2N} \to \mathbf{R}^{2N}$ be a linear operator given by

$$J((z_i)_{i=1}^{2N})_j = \begin{cases} -z_{j+N}, & j = 1, \ldots, N, \\ z_{j-N}, & j = N+1, \ldots, 2N. \end{cases}$$

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Let $\beta^k : [0,T] \to \mathbf{R}^{2N}$, $k = 1, \dots, d$, be given by $\beta^k(t) = J\alpha^k(t)$. Then we have

$$\beta^k_j(t) = \begin{cases} -b^k_j(t), & j = 1, \dots, N, \\ a^k_{j-N}(t), & j = N+1, \dots, 2N. \end{cases}$$

Let $c_{i,j} : [0,T] \to \mathbf{R}$, $i, j = 1, \dots, 2N$, be given by

$$c_{i,j}(t) = \sum_{k=1}^{d} \alpha^k_i(t) \beta^k_j(t)$$

Also, let $e_{i_1,i_2} : [0,T] \to \mathbf{R}$, $i_1, i_2 = 1, \dots, 2N$, be the solution to the following ODE

$$\frac{d}{dt} e_{i_1,i_2}(t) = \sum_{j=1}^{2N} c_{i_1,j}(t) e_{j,i_2}(t) \tag{1}$$

$$e_{i_1,i_2}(0) = \delta_{i_1,i_2}, \qquad i_1, i_2 = 1, \dots, 2N.$$

Let e be a $2N \times 2N$-matrix valued function defined in $[0,T]$ given by $e(t) = (e_{i,j}(t))_{i,j=1,\dots,2N}$, and let $\tilde{e}$ be a $N \times N$-matrix valued function defined in $[0,T]$ given by $\tilde{e}(t) = (e_{i,j}(t))_{i,j=1,\dots,N}$.

Let $\gamma^k : [0,T] \to \mathbf{R}^{2N}$, $k = 1, \dots, d$, be continuous functions given by

$$\gamma^k_i(t) = -\sum_{j=1}^{2N} e_{ji}(t) \beta^k_j(t), \qquad i = 1, \dots, 2N, \ t \in [0,T].$$

Now let $\Psi : L^2([0,T]; \mathbf{R}^d, dt) \to C([0,T]; \mathbf{R}^{2N})$ be bounded linear operators given by

$$(\Psi h)_i(t) = \sum_{k=1}^{d} \int_0^t \gamma^k_i(s) h^k(s) ds, \qquad t \in [0,T].$$

for $h \in L^2([0,T]; \mathbf{R}^d, dt)$.

The following is our main result.

Theorem 1 (1) *The $N \times N$-matrix $\tilde{e}(T)$ is invertible.*
(2) *Let $\tilde{e}(T)^{-1} = (\tilde{e}^{-1}_{i,j}(T))_{i,j=1,\dots,N}$ be the inverse matrix of $\tilde{e}(T)$. Let $d_{ij} \in \mathbf{R}$, $i, j = 1, \dots, N$, be given by*

$$d_{ij} = \sum_{r=1}^{N} \tilde{e}^{-1}_{i,r}(T) e_{r,N+j}(T).$$

Then $d_{i,j} = d_{j,i}$, $i, j = 1, \dots, N$.
(3) *For any $h \in L^2([0,T]; \mathbf{R}^d, dt)$ and $\lambda \in \mathbf{C}$,*

$$E^\mu[\exp(\lambda \sum_{k=1}^{d} \int_0^T h^k(t) dw^k(t) + X)]$$

$$= \det(\tilde{e}(T))^{-1/2} \exp(-\frac{1}{2} \sum_{i=1}^{n} \sum_{k=1}^{d} \int_0^T a^k_i(t) b^k_i(t) dt + \frac{\lambda^2}{2} \mathcal{A}(h,h)),$$

where

$$\mathcal{A}(h,h)$$
$$= ||h||_H^2 - \int_0^T (\frac{d}{dt}(\Psi h)(t), J(\Psi h)(t))_{\mathbf{R}^{2N}} dt$$
$$+ \sum_{i=1}^N J(\Psi h)(T)^i (\Psi h)(T)^i + \sum_{i,j=1}^N d_{ij} (\Psi h)(T)^i (\Psi h)(T)^j.$$

2 Preliminary Facts

Let $\tilde{H}$ be the Cameron Martin space of the Wiener space (W_0, μ), i.e.,

$$\tilde{H} = \{k \in W_0;\ k(t) \text{ is absolutely continuous in } t,\ \int_0^T |\frac{dk}{dt}(t)|^2 dt < \infty\},$$

$$(k_1, k_2)_{\tilde{H}} = \int_0^T \frac{dk_1}{dt}(t) \cdot \frac{dk_2}{dt}(t) dt.$$

Let $H = L^2([0,T]; \mathbf{R}^d, dt)$. Then the map $\Psi : \tilde{H} \to H$ corresponding k to $\dfrac{dk}{dt}$ is an isomorphism.

Let $\mathcal{E} : H \times H \to \mathbf{R}$ be a symmetric bilinear form given by

$$\mathcal{E}(h_1, h_2)$$
$$= \sum_{i=1}^N \sum_{k,\ell=1}^d \int_0^T \int_0^T 1_{\{t>s\}} a_i^k(t) b_i^\ell(s) h_1^k(t) h_2^\ell(s) dt ds$$
$$+ \sum_{i=1}^N \sum_{k,\ell=1}^d \int_0^T \int_0^T 1_{\{t>s\}} a_i^k(t) b_i^\ell(s) h_1^\ell(s) h_2^k(t) dt ds$$

for $h_1, h_2 \in H$.

The associated symmetric bounded linear operator $E : H \to H$ to $\mathcal{E}$ is given by

$$(Eh)^k(t)$$
$$= \sum_{i=1}^N \sum_{\ell=1}^d \int_0^T 1_{\{t>s\}} a_i^k(t) b_i^\ell(s) h^\ell(s) ds + \sum_{i=1}^N \sum_{\ell=1}^d \int_0^T 1_{\{s>t\}} a_i^\ell(s) b_i^k(t) h^\ell(s) ds$$
$$= \sum_{i=1}^N \sum_{\ell=1}^d a_i^k(t) \int_0^t b_i^\ell(s) h^\ell(s) ds - \sum_{i=1}^N \sum_{\ell=1}^d b_i^k(t) \int_0^t a_i^\ell(s) h^\ell(s) ds$$
$$+ \sum_{i=1}^N \sum_{\ell=1}^d b_i^k(t) \int_0^T a_i^\ell(s) h^\ell(s) ds, \qquad t \in [0,T],\ k = 1, \dots, d$$

for $h \in H$.

Since $E^\mu[\exp(X)] < \infty$, we see that $I_H - E : H \to H$ is invertible and positive-definite and we have

$$E^\mu[\exp(\lambda \sum_{k=1}^d \int_0^T h^k(t) dw^k(t) + X)]$$

$$= \det{}_2(I_H - E)^{-1/2} \exp(\frac{\lambda^2}{2}(h, (I_H - E)^{-1} h)_H), \qquad h \in H. \tag{2}$$

Let $V_{01} : H \to H$, and $A_r : H \to \mathbf{R}^N$, $r = 0, 1$, be given by

$$(V_{01} h)^k(t)$$

$$= \sum_{i=1}^N \sum_{\ell=1}^d a_i^k(t) \int_0^t b_i^\ell(s) h^\ell(s) ds - \sum_{i=1}^N \sum_{\ell=1}^d b_i^k(t) \int_0^t a_i^\ell(s) h^\ell(s) ds,$$

$$= \sum_{i=1}^{2N} \sum_{\ell=1}^d \beta_i^k(t) \int_0^t \alpha_i^\ell(s) h^\ell(s) ds, \qquad k = 1, \ldots, d,$$

$$(A_0 h)_i = \sum_{\ell=1}^d \int_0^T a_i^\ell(s) h^\ell(s) ds, \qquad h \in H,\ i = 1, \ldots, N,$$

and

$$(A_1 h)_i = \sum_{\ell=1}^d \int_0^T b_i^\ell(s) h^\ell(s) ds, \qquad h \in H,\ i = 1, \ldots, N.$$

Then we see that

$$E = V_{01} + A_1^* A_0.$$

Note that V_{01} is a Voltera type operator.

Proposition 2 (1) $\det_2(I_H - E)$

$$= \det(I_N - A_0 (I_H - V_{01})^{-1} A_1^*) \exp(\mathrm{trace} A_0 A_1^*),$$

where I_N is the identity map in $\mathbf{R}^N$.
In particular, the $N \times N$-matrix $I_N - A_0(I_H - V_{01})^{-1} A_1^*$ is invertible .
(2) $\qquad (I_H - E)^{-1}$

$$= (I_H - V_{01})^{-1} + (I_H - V_{01})^{-1} A_1^* (I_N - A_0 (I_H - V_{01})^{-1} A_1^*)^{-1} A_0 (I_H - V_{01})^{-1}.$$

Proof. Let $z \in \mathbf{C}$ for which $|z|$ is sufficiently small. Then we have

$$\det{}_2(I_H - zE)$$

$$= \det{}_2((I_H - zV_{01})(I_H - z(I_H - zV_{01})^{-1} A_1^* A_0)).$$

Since V_{01} is a Voltera type operator, we have $\det_2(I_H - zV_{01}) = 1$. So we have

$$\det{}_2(I_H - zE)$$

$$= \det{}_2(I_{\tilde H} - zV_{01}) \det{}_2(I_{\tilde H} - z(I_H - zV_{01})^{-1} A_1^* A_0) \exp(-\mathrm{trace}(z^2 V_{01} (I_H - zV_{01})^{-1} A_1^* A_0))$$

$$= \exp(-\sum_{k=2}^{\infty}\frac{1}{k}\mathrm{trace}((z(I_H - zV_{01})^{-1}A_1^*A_0)^k))\exp(-\mathrm{trace}(zA_0((I_H - zV_{01})^{-1} - I_H)A_1^*))$$

$$= \exp(-\sum_{k=1}^{\infty}\frac{1}{k}\mathrm{trace}((zA_0(I_H - zV_{01})^{-1}A_1^*)^k))\exp(\mathrm{trace}(zA_0A_1^*))$$

$$= \det(I_N - zA_0(I_H - zV_{01})^{-1}A_1^*)\exp(z\ \mathrm{trace}(A_0A_1^*)).$$

Also, we have

$$(I_H - zE)^{-1} = \{(I_H - zV_{01})(I_H - z(I_H - zV_{01})^{-1}A_1^*A_0)\}^{-1}$$

$$= (\sum_{k=0}^{\infty}(z(I_H - zV_{01})^{-1}A_1^*A_0)^k)(I_H - zV_{01})^{-1}$$

$$= \{I_H + \sum_{k=0}^{\infty} z(I_H - zV_{01})^{-1}A_1^*(zA_0(I_H - zV_{01})^{-1}A_1^*)^k A_0\}(I_H - zV_{01})^{-1}$$

$$= (I_H - zV_{01})^{-1} + z(I_H - zV_{01})^{-1}A_1^*(I_N - zA_0(I_H - zV_{01})^{-1}A_1^*)^{-1}A_0(I_H - zV_{01})^{-1}.$$

Since $\det_2(I_H - zE)$ and $(I_H - zE)^{-1}$ are holomorphic in $\mathbf{C}$ except a countable set which has no cluster point and are holomorphic around $z = 1$, we have our assertion.

3 Basic Computation

Let c be a $2N \times 2N$ matrix-valued continuous function defined in $[0, T]$ given by $c(t) = (c_{i,j}(t))_{i,j=1,\ldots,2N}$, and $e^{-1}(t)$ is the inverse matrix of $e(t)$, $t \in [0, T]$. Then we have

$$\frac{d}{dt}e(t) = c(t)e(t), \qquad \frac{d}{dt}e^{-1}(t) = -e^{-1}(t)c(t).$$

Note that

$$(Jc(t))_{i,j} = \sum_{k=1}^{d}\beta_i^k(t)\beta_j^k(t), \qquad i, j = 1, \ldots, N,$$

and so

$$c(t)^* = -(JJc(t))^* = Jc(t)J.$$

Therefore we have

$$\frac{d}{dt}(Je(t)^*J) = J(c(t)e(t))^*J = -Je(t)^*Jc(t).$$

Since $-Je(0)J = I_{2N}$, we have by the uniqueness of a solution to ODE

$$e^{-1}(t) = -Je(t)^*J.$$

So we see that

$$J\gamma^k(t) = -Je(t)^*J\alpha^k(t) = e^{-1}(t)\alpha^k(t) \qquad k = 1, \ldots, d.$$

Then we see that

$$J(\Psi h)(t) = \sum_{k=1}^{d}\int_0^t e(s)^{-1}\alpha^k(s)h^k(s)ds, \tag{3}$$

In this section, we prove the following.

Proposition 3 (1) *$I_N - A_0(I_H - V_{01})^{-1}A_1^* = \tilde{e}(T)$, and*

$$\det(I_N - A_0(I_H - V_{01})^{-1}A_1^*) = \det(\tilde{e}(T)).$$

(2) *For $h \in H$,*

$$(A_0(I_H - V_{01})^{-1}h)_i = \sum_{j=1}^{2N} e_{i,j}(T)(J(\Psi h)(T))_j, \qquad i = 1, \ldots, N.$$

(3) *For $h \in H$ and $v \in \mathbf{R}^N$,*

$$(h, (I_H - V_{01})^{-1}A_1^* v)_H = \sum_{i=1}^{N} v_i(\Psi h)_i(T), \qquad k = 1, \ldots, d, \ t \in [0, T].$$

(4) *For any $h \in H$,*

$$((I_H - V_{01})^{-1}h)^k(t) = h^k(t) - \sum_{j=1}^{2N} \gamma_j^k(t)(J(\Psi h)(t))_j, \qquad k = 1, \ldots, d, \ t \in [0, T].$$

In particular,

$$(h_1, (I_H - V_{01})^{-1}h_2)_H = (h_1, h_2)_H - \int_0^T (\frac{d}{dt}(\Psi h_1)(t), J(\Psi h_2)(t))_{\mathbf{R}^{2N}} dt$$

for $h_1, h_2 \in H$.

Proof. Let $f \in C_0^\infty((0, T); \mathbf{R}^d) \subset H$ and let $\xi = (I_H - V_{01})^{-1}f$. Then we have

$$\xi = f + V_{01}\xi$$

Let

$$\eta_i(t) = \sum_{k=1}^{d} \int_0^t \alpha_i^k(s)\xi^k(s)ds, \quad i = 1, \ldots, 2N.$$

Then we have

$$(A_0(I_H - V_{01})^{-1}f)_i = \eta_i(T), \qquad i = 1, \ldots, N. \tag{4}$$

Also we have

$$\xi^k(t) = f^k(t) + \sum_{i=1}^{2N} \beta_i^k(t)\eta_i(t), \qquad k = 1, \ldots, d,$$

and so we have

$$\frac{d}{dt}\eta_i(t) = \sum_{k=1}^{d} \alpha_i^k(t)\xi^k(t)$$

$$= \sum_{k=1}^{d} \alpha_i^k(t)f^k(t) + \sum_{j=1}^{2N} c_{ij}(t)\eta_j(t), \qquad i = 1, \ldots, 2N.$$

Note that $\eta_i(0)=0$, $i=1,\ldots,2N$. So we see that

$$\eta_i(t)=\sum_{j_1,j_2=1}^{2N}\sum_{\ell=1}^{d}e_{i,j_1}(t)\int_0^t e^{-1}_{j_1,j_2}(s)\alpha^{\ell}_{j_2}(s)f^{\ell}(s)ds$$

$$=\sum_{j=1}^{2N}e_{i,j}(t)(J(\Psi f)(t))_j,\qquad i=1,\ldots,2N,\ t\in[0,T].$$

This and Equation (4) imply the assertion (2), since $C_0^{\infty}((0,T);\mathbf{R}^d)$ is dense in H.

Also, we see that

$$\xi^k(t)=f^k(t)+\sum_{j_1,j_2=1}^{2N}\beta^k_{j_1}(t)e_{j_1,j_2}(t)(J(\Psi f)(t))_{j_2},$$

for $k=1,\ldots,d$, $t\in[0,T]$. This implies the assertion (4).

Let $v\in\mathrm{R}^N$. Then we have

$$(J(\Psi A_1^*v)(t))_i=\sum_{r=1}^{N}\sum_{j=1}^{2N}\sum_{\ell=1}^{d}v_r\int_0^t e^{-1}_{i,j}(s)\alpha^{\ell}_j(s)b^{\ell}_r(s)ds$$

$$=-\sum_{r=1}^{N}\sum_{j=1}^{2N}\sum_{\ell=1}^{d}v_r\int_0^t e^{-1}_{i,j}(s)c_{j,r}(s)ds=\sum_{r=1}^{N}v_r\int_0^t\frac{d}{ds}e^{-1}_{i,r}(s)ds=\sum_{r=1}^{N}v_re^{-1}_{i,r}(t)-v_i$$

Therefore

$$(h,(I_H-V_{01})^{-1}A_1^*v)$$

$$=(h,A_1^*v)+\sum_{j_1,j_2=1}^{2N}\sum_{i=1}^{N}v_i\int_0^T\beta^k_{j_1}(t)e_{j_1,j_2}(t)e^{-1}_{j_2,i}(t)h^k(t)dt$$

$$-\sum_{j=1}^{2N}\sum_{i=1}^{N}v_i\int_0^T\beta^k_j(t)e_{j,i}(t)h^k(t)dt$$

$$=\sum_{i=1}^{N}v_i\int_0^T\gamma^k_i(t)h^k(t)dt=\sum_{i=1}^{N}v_i(\Psi h)_i(T).$$

This implies the assertion (3).

So we have

$$((I_N-A_0(I_H-V_{01})^{-1}A_1^*)v)_i$$

$$=v_i+\sum_{k=1}^{d}\sum_{j_0=1}^{2N}\sum_{j_1=1}^{N}v_{j_1}\int_0^T a^k_i(t)\beta^k_{j_0}(t)e_{j_0,j_1}(t)dt$$

$$=v_i+\sum_{j_0=1}^{2N}\sum_{j_1=1}^{N}v_{j_1}\int_0^T c_{i,j_0}(t)e_{j_0,j_1}(t)dt$$

$$=v_i+\sum_{j=1}^{N}v_j(e_{i,j}(T)-\delta_{ij})=\sum_{j=1}^{N}e_{i,j}(T)v_j.$$

This implies the assertion (1).

4 Proof of Theorem

Proposition 4 (1) $d_{i,j} = d_{j,i}$, *for all* $i, j = 1, \ldots N$.
(2) *For any* $h_1, h_2 \in H$,

$$(h_1, (I-E)^{-1}h_2)_H$$

$$= (h_1, h_2)_H + \sum_{i=1}^{N} J(\Psi h_2)(T)^i (\Psi h_1)(T)^i - \int_0^T (\frac{d}{dt}(\Psi h_1)(t), J(\Psi h_2)(t))_{\mathbf{R}^{2N}} dt$$

$$+ \sum_{i,j=1}^{N} d_{ij}(\Psi h_2)(T))^i (\Psi h_1)(T)^j.$$

Proof. Note that $e(t)Je(t)^* = J$. This implies that for $i, j = 1, \ldots, N$, and

$$0 = (e(t)Je(t)^*)_{i,j} = -\sum_{r=1}^{N} e_{i,r}(t)e_{j,N+r}(t) + \sum_{r=1}^{N} e_{i,N+r}(t)e_{j,r}(t)$$

Let $f_{i,j} : [0, T] \to \mathbf{R}$, $i, j = 1, \ldots, N$, be given by

$$f_{i,j}(t) = \sum_{r=1}^{N} e_{i,r}(t)e_{j,N+r}(t),$$

and let $F(t)$ be an $N \times N$-matrix given by $F(t) = (f_{i,j}(t))_{i,j=1,\ldots,N}$. Then we have $F(t)^*$ $= F(t)$. Since we have

$$(d_{i,j})_{i,j=1,\ldots,N} = \tilde{e}^{-1}(T)(\tilde{e}^{-1}(T)F(T))^* = \tilde{e}^{-1}(T)F(T)(\tilde{e}^{-1}(T))^*,$$

we have the assertion (1).

By Propositions 2 and 3, we have for $h_1, h_2 \in H$,

$$(h_1, ((I_H - E)^{-1}h_2)_H$$

$$= (h_1, (I_H - V_{01})^{-1}h_2)_H + (h_1, (I_H - V_{01})^{-1}A_1^*(I_N - A_0(I_H - V_{01})^{-1}A_1^*)^{-1}A_0(I_H - V_{01})^{-1}h_2)_H$$

$$= (h_1, h_2)_H - \int_0^T (\frac{d}{dt}(\Psi h_1)(t), J(\Psi h_2)(t))_{\mathbf{R}^{2N}} dt$$

$$+ \sum_{i,j=1}^{N} \sum_{\ell=1}^{2N} \tilde{e}_{ij}^{-1}(T)e_{j,\ell}(T)J(\Psi h_2)(T)^\ell (\Psi h_1)(T)^i$$

$$= (h_1, h_2)_H - \int_0^T (\frac{d}{dt}(\Psi h_1)(t), J(\Psi h_2)(t))_{\mathbf{R}^{2N}} dt + \sum_{i=1}^{N} J(\Psi h_2)(T)^i (\Psi h_1)(T)^i$$

$$+ \sum_{i,j=1}^{N} d_{ij}(\Psi h_2)(T)^i (\Psi h_1)(T)^j$$

So we have the assertion (2).

Now Theorem 1 is an easy consequence of Propositions 2, 3, 4 and Equation (2).

5 A Remark

Let us define $N \times d$-matrix valued functions by $a(t) = (a_j^k(t))_{j=1,\dots,N\ k=1,\dots,d}$, $b(t) = (b_j^k(t))_{j=1,\dots,N\ k=1,\dots,d}$, $t \in [0,T]$. Also, let us define $2N \times d$-matrix valued functions $\alpha(t) = (\alpha_j^k(t))_{j=1,\dots,2N\ k=1,\dots,d}$, $\beta(t) = (\beta_j^k(t))_{j=1,\dots,2N\ k=1,\dots,d}$, $t \in [0,T]$.

Then we see that

$$c(t) = \alpha(t)\beta(t)^*, \qquad t \in [0,T].$$

So we have

$$e(t) = I_{2N} + \int_0^t c(s)e(s)ds$$

$$= I_{2N} + \sum_{k=1}^{\infty} \int_0^t ds_1 \int_0^{s_1} ds_2 \cdots \int_0^{s_{k-1}} ds_k\ c(s_1)\cdots c(s_k)$$

$$= I_{2N} + \sum_{k=1}^{\infty} \int_{0<s_k<\cdots<s_1<t} \alpha(s_1)\beta(s_1)^*\alpha(s_2)\beta(s_2)^*\cdots\alpha(s_k)\beta(s_k)^* ds_1 ds_2\cdots ds_k$$

$$= I_{2N} + \int_0^t \alpha(s)\beta(s)^* ds + \sum_{k=1}^{\infty} \int_{0<s_k<\cdots<s_1<t} \alpha(s_1)K(s_1,s_2)\cdots K(s_{k-1},s_k)\beta(s_k)^* ds_1 ds_2\cdots ds_k,$$

where $K(t,s)$ is a $d \times d$-matrix valued function given by

$$K(t,s) = \beta(t)^*\alpha(s) = a(t)^*b(s) - b(t)^*a(s), \quad 0 \leqq s \leqq t \leqq T.$$

So

$$K_{ij}(t,s) = \sum_{k=1}^{N} (a_k^i(t)b_k^j(s) - b_k^i(t)a_k^j(s)), \qquad i.j = 1,\dots,d,\ 0 \leqq s \leqq t \leqq T.$$

Let

$$E(t,r)$$

$$= K(t,r) + \sum_{k=1}^{\infty} \int_{r<s_k<\cdots<s_1<t} K(t,s_1)K(s_1,s_2)\cdots K(s_{k-1},s_k)K(s_k,r) ds_1 ds_2\cdots ds_k,$$

$0 \leqq r \leqq t \leqq T$. Then we see that

$$E(t,r) = K(t,r) + \int_0^t K(t,s)E(s,r)ds \qquad 0 \leqq r \leqq t \leqq T,$$

$$e(t) = I_{2N} + \int_0^t \alpha(s)\beta(s)^* ds + \int_{0<r<s<t} \alpha(s)E(s,r)\beta(r)^* dsdr,$$

and

$$\hat{e}(t) = I_N - \int_0^t a(s)b(s)^* ds - \int_{0<r<s<t} a(s)E(s,r)b(r)^* dsdr,$$

$$\gamma(t)^* = \beta(t)^*e(t) = \beta(t)^* + \int_0^t E(t,s)\beta(s)^* ds.$$

6 Special Case

Let K be a $2N \times 2N$ matrix and $\tilde{\alpha}^k \in \mathbf{R}^{2N}$, $k = 1, \ldots, d$. We assume that the matrix K satisfies

$$JKJ = K^*.$$

Note that the matrix K satisfies this condition, if and only if

$$K_{22} = -K_{11}^*,\ K_{12}^* = K_{12} \text{ and } K_{21}^* = K_{21},$$

where K_{ij}, $i, j = 1, 2$ are $N \times N$ matrix such that

$$K = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix}.$$

Now let $\alpha^k : [0, T] \to \mathbf{R}^N$, $k = 1, \ldots, d$ be given by

$$\alpha^k(t) = \exp(tK)\tilde{\alpha}^k, \qquad t \in [0, T],\ k = 1, \ldots, d.$$

Let $a_i^k : [0, T] \to \mathbf{R}$ and $b_i^k : [0, T] \to \mathbf{R}$, $i = 1, \ldots, N$, $k = 1, \ldots, d$, be given by

$$a_i^k(t) = \alpha_i^k(t), \quad b_i^k(t) = \alpha_{N+i}^k(t), \qquad t \in [0, T].$$

Let

$$X(w) = \sum_{i=1}^N \sum_{k,\ell=1}^d \int_0^T (\int_0^t b_i^\ell(s) dw^\ell(s)) a_i^k(t) dw^k(t).$$

We assume that if $E^\mu[\exp(X)] < \infty$. Then, for any $h \in L^2([0, T]; \mathbf{R}^d, dt)$ and $\lambda \in \mathbf{C}$,

$$E^\mu[\exp(\lambda \sum_{k=1}^d \int_0^T h^k(t) dw^k(t) + X)]$$

is computable by Theorem 1. In this section, we show that we can reduce all computation to linear ordinary differential equations with constant coefficients.

Note that

$$J \exp(tK) = \sum_{n=0}^\infty \frac{(-t)^n}{n!} J(KJ^2)^n = \sum_{n=0}^\infty \frac{(-t)^n}{n!} (JKJ)^n J = \exp(-tK^*)J,$$

and so we have

$$\beta^k(t) = J\alpha^k(t) = J \exp(tK)\tilde{\alpha}^k = \exp(-tK^*)J\tilde{\alpha}^k.$$

Then we see that

$$c(t) = \sum_{k=1}^d \alpha^k(t)\beta^k(t)^* = \exp(tK)L\exp(-tK),$$

where

$$L = \sum_{k=1}^d \tilde{\alpha}^k (J\tilde{\alpha}^k)^* = -\sum_{k=1}^d \tilde{\alpha}^k \tilde{\alpha}^{k*} J.$$

Note that the matrix L also satisfies $JLJ = L^*$. Since we have

$$\frac{d}{dt}(\exp(tK)\exp(t(L-K)) = \exp(tK)L\exp(t(L-K))$$
$$= c(t)\exp(tK)\exp(t(L-K)),$$

the uniqueness of the solution to the ordinary equation (1) implies that

$$e(t) = \exp(tK)\exp(t(L-K)), \qquad t \in [0,T].$$

Then we have

$$\gamma^k(t) = -e(t)^*\beta^k(t) = -\exp(t(L-K)^*))J\tilde{\alpha}^k, \qquad t \in [0,T],\ k = 1,\dots,d,$$

and

$$(\Psi h)(t) = \sum_{k=1}^{d}\int_0^t \gamma^k(s)h^k(s)ds$$
$$= -\int_0^t \exp(s(L-K)^*)(\sum_{k=1}^{d} J\tilde{\alpha}^k h^k(s))ds, \qquad t \in [0,T],$$

for $h \in L^2([0,T];\mathbf{R}^d, dt)$.

7 Examples 1

In this section, we shall see some examples that Theorem 1 is applicable. First example is known as harmonic oscillator.

Proposition 5 *For $\kappa > 0$ and $\lambda \in \mathbf{C}$, we have*

$$E\left[\exp\left(\lambda w(T) - \kappa\int_0^T w(t)^2dt\right)\right] = \frac{1}{\sqrt{\cosh(\sqrt{2\kappa}T)}}\exp\left(\frac{\lambda^2}{2}\frac{\tanh(\sqrt{2\kappa}T)}{\sqrt{2\kappa}}\right).$$

Proof. We define the quadratic Wiener functional given by

$$\tilde{X}(w) = -\kappa\int_0^T w(t)^2dt.$$

Then we see that $\tilde{X} = -\frac{\kappa T^2}{2} + X$, where

$$X(w) = \int_0^T 2\kappa(t-T)\int_0^t dw(s)dw(t).$$

We apply the computation of previous section for $N = 1$, $d = 1$,

$$K = \begin{pmatrix} 0 & 2\kappa \\ 0 & 0 \end{pmatrix}, \text{ and } \tilde{\alpha} = \begin{pmatrix} -2\kappa T \\ 1 \end{pmatrix}.$$

Then we see that $a_1(t) = 2\kappa(t-T)$ and $b_1(t) = 1$. Also, we have

$$J\tilde{\alpha} = \begin{pmatrix} -1 \\ -2\kappa T \end{pmatrix} \qquad L = J\tilde{\alpha}\tilde{\alpha}^* = \begin{pmatrix} 2\kappa T & -1 \\ 4\kappa^2 T^2 & -2\kappa T \end{pmatrix}.$$

Then we have

$$e(t) =$$

$$\begin{pmatrix} \cosh(\sqrt{2\kappa}t) - \sqrt{2\kappa}(t-T)\sinh(\sqrt{2\kappa}t) & 2\kappa t\cosh(\sqrt{2\kappa}t) - \sqrt{2\kappa}\big(1 + 2\kappa T(t-T)\big)\sinh(\sqrt{2\kappa}t) \\ -\frac{1}{\sqrt{2\kappa}}\sinh(\sqrt{2\kappa}t) & \cosh(\sqrt{2\kappa}t) - \sqrt{2\kappa}T\sinh(\sqrt{2\kappa}t) \end{pmatrix}.$$

Then γ_1 and γ_2 are given as

$$\gamma_1(t) = \cosh(\sqrt{2\kappa}t), \quad \gamma_2(t) = 2\kappa T\cosh(\sqrt{2\kappa}t) - \sqrt{2\kappa}\sinh(\sqrt{2\kappa}t).$$

Now we have

$$\begin{aligned} \tilde{e}(t) &= \cosh(\sqrt{2\kappa}t) - \sqrt{2\kappa}(t-T)\sinh(\sqrt{2\kappa}t), \\ d_{11} &= 2\kappa T - \sqrt{2\kappa}\tanh(\sqrt{2\kappa}T), \end{aligned}$$

and

$$\mathcal{A}(h,h) = \frac{1}{\sqrt{\kappa}}\tanh(\sqrt{\kappa}T).$$

for $h = 1$. From Theorem 1, we can easily show Proposition 5.

Corollary 6 *For $\kappa > 0$ and $x \in \mathbf{R}$, we have*

$$E\left[\exp(-\kappa\int_0^T w(t)^2 dt)\delta_x(w(T))\right] = \frac{1}{\sqrt{2\pi T}}\sqrt{\frac{\sqrt{2\kappa}T}{\sinh(\sqrt{2\kappa}T)}}\exp\Big(-\frac{1}{2}\frac{\sqrt{2\kappa}}{\tanh(\sqrt{2\kappa}T)}x^2\Big),$$

where $\delta_x(w(T))$ is the pull-back of the Dirac delta function at $x \in \mathbf{R}$ by the Wiener functional $w(T)$.

Proof. Applying Fourier transform for the Dirac delta function, we have the following;

$$\begin{aligned} & E\left[\exp(-\kappa\int_0^T w(t)^2 dt)\delta_x(w(T))\right] \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} E\left[\exp(-\kappa\int_0^T w(t)^2 dt)\exp(-i\xi(w(T)-x))\right] d\xi \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{\sqrt{\cosh(\sqrt{2\kappa}T)}}\exp\Big(-\frac{\xi^2}{2}\frac{\tanh(\sqrt{2\kappa}T)}{\sqrt{2\kappa}} + i\xi x\Big) d\xi \\ &= \frac{1}{\sqrt{2\pi T}}\sqrt{\frac{\sqrt{2\kappa}T}{\sinh(\sqrt{2\kappa}T)}}\exp\Big(-\frac{1}{2}\frac{\sqrt{2\kappa}}{\tanh(\sqrt{2\kappa}T)}x^2\Big). \end{aligned}$$

We applied Proposition 5 in the middle.

We shall see the next example. This is known as Levy's stochastic area.

Proposition 7 *For $\kappa > 0$ and $\lambda_1, \lambda_2 \in \mathbf{C}$, we have*

$$E\left[\exp\left(\lambda_1 w_1(T) + \lambda_2 w_2(T) + \kappa\big(\int_0^T w_2(t)dw_1(t) - \int_0^T w_1(t)dw_2(t)\big)\right)\right]$$
$$= \frac{1}{\cos(\kappa T)}\exp\big(\frac{\lambda_1^2+\lambda_2^2}{2\kappa}\tan(\kappa T)\big).$$

Proof. It is enough to show the formula for $\kappa = 1$. We define the quadratic Wiener functional given by

$$S(w) = \int_0^T w_2(t)dw_1(t) - \int_0^T w_1(t)dw_2(t).$$

Then we can apply Theorem 1 for $N = 2$, $d = 2$ and

$$a(t) = \begin{pmatrix} a_1^1 & a_1^2 \\ a_2^1 & a_2^2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad b(t) = \begin{pmatrix} b_1^1 & b_1^2 \\ b_2^1 & b_2^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then $\alpha^k : [0,T] \to \mathbb{R}^4$, $\beta^k : [0,T] \to \mathbb{R}^4, k = 1,2$ and $C \in M_4(\mathbf{R})$ are given by

$$\alpha^1(t) = \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \alpha^2(t) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \beta^1(t) = \begin{pmatrix} -1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \beta^2(t) = \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}, C = \begin{pmatrix} 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ -1 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 \end{pmatrix}.$$

Here $e : [0,T] \to M_4(\mathbf{R})$ satisfies the ODE

$$\frac{d}{dt}e(t) = Ce(t), \quad e(0) = I_4,$$

and the solution is given as

$$e(t) = \begin{pmatrix} E_{11}(t) & E_{12}(t) \\ -E_{12}(t) & E_{11}(t) \end{pmatrix},$$

where

$$E_{11}(t) = \frac{1}{2}(1+\cos(2t))I_2 + \frac{1}{2}\sin(2t)J_2, \quad E_{12}(t) = \frac{1}{2}(1-\cos(2t))J_2 + \frac{1}{2}\sin(2t)I_2.$$

Therefore we have

$$\tilde{e}(t) = E_{11}(t), \ \det(\tilde{e}(t)) = \frac{1}{2} + \frac{1}{2}\cos(2t),$$
$$\begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix} = E_{11}^{-1}(T)E_{12}(T) = \begin{pmatrix} \tan t & 0 \\ 0 & \tan t \end{pmatrix}.$$

Also we can easily show that

$$\mathcal{A}(h,h) = (\lambda_1^2 + \lambda_2^2)\tan(T),$$

where $h_1 = \lambda_1, h_2 = \lambda_2$. Now Proposition 7 is an easy consequence of Theorem 1.

Corollary 8 *For $\kappa > 0$ and $x \in \mathbf{R}$, we have*

$$E\left[\exp\big(\kappa\big(\int_0^T w_2(t)dw_1(t) - \int_0^T w_1(t)dw_2(t)\big)\big)\delta_x(w(t))\right]$$
$$= \frac{1}{2\pi T}\frac{\kappa T}{\sin(\kappa T)}\exp\left(-\frac{|x|^2}{2T}\frac{\kappa T}{\tan(\kappa T)}\right).$$

Proof. We can prove this formula as in the same way as Corollary 6 using Proposition 7.

8 Examples 2

We think of some examples using results in Section 5. Let $N \geqq 1$, $d = 2$, and $K_{k\ell}$, $k, \ell = 1, 2$, be $N \times N$-matrices given by

$$(K_{12})_{ij} = \begin{cases} -1, & \text{if } i = j+1,\ j = 1, \dots, N-1, \\ 0, & \text{otherwise,} \end{cases}$$

$(K_{21})_{ij} = -(K_{12})_{ji}$, $i, j = 1, \dots, N$, and $K_{12} = K_{21} = 0$. Let K be $2N \times 2N$-matrices given by

$$K = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix}.$$

Also, let $\tilde{\alpha}^k \in \mathbf{R}^{2N}$, $k = 1, 2$, be given by

$$\tilde{\alpha}^1_i = \begin{cases} 1, & i = 1, \\ 0, & i \neq 1, \end{cases}$$

and

$$\tilde{\alpha}^2_i = \begin{cases} 1, & i = 2N, \\ 0, & i \neq 2N. \end{cases}$$

Then we see that

$$a^1_i(t) = \frac{(-t)^{i-1}}{(i-1)!}, \qquad b^2_i(t) = \frac{t^{N-i}}{(N-i)!}, \qquad i = 1, \dots, N,$$

and $a^2(t) = b^1(t) = 0$. So

$$\begin{aligned} X &= \sum_{i=1}^N \int_0^T (\int_0^t \frac{s^{N-i}}{(N-i)!} dw^2(s)) \frac{(-t)^{i-1}}{(i-1)!} dw^1(t) \\ &= \frac{(-1)^{N-1}}{N!} \int_0^T (\int_0^t (t-s)^{N-1} dw^2(s)) dw^1(t). \end{aligned}$$

We also have

$$(J\tilde{\alpha}^1)_i = \begin{cases} -1, & i = N+1, \\ 0, & i \neq N+1, \end{cases}$$

and

$$(J\tilde{\alpha}^2)_i = \begin{cases} 1, & i = 2N, \\ 0, & i \neq 2N. \end{cases}$$

Let $\tau : \{1, 2, \dots, 2N\} \to \{1, 2, \dots, 2N\}$ be a permutation given by

$$\tau(1) = N+1,\ \tau(i) = i-1,\ i = 2, \dots, N,\ \ \tau(i) = i+1,\ i = N+1, \dots, 2N-1,\ \tau(2N) = N.$$

Then it is easy to see that

$$(L - K)_{ij} = \begin{cases} 1, & \text{if } i = 1, \dots, N, \text{ and } j = \tau(i), \\ -1, & \text{if } i = N+1, \dots, 2N, \text{ and } j = \tau(i), \\ 0, & \text{otherwise.} \end{cases}$$

Note that $\tau^n(i) \neq i$, $n = 1, \dots, 2N-1$, and $\tau^{2N}(i) = i$, for any $i = 1, \dots, 2N$. So it is easy to see that $(L-K)^{2N} = (-1)^N I_{2N}$.

For $M \geqq 1$, let

$$\varphi_M^{(k)}(z) = \sum_{n=0}^{\infty} \frac{z^{nM+k}}{(nM+k)!}, \qquad z \in \mathbf{C},\ k = 0, \dots, M-1.$$

Then we see that if N is even,

$$\exp(t(L-K)) = \sum_{k=0}^{2N-1} \varphi_{2N}^{(k)}(t)(L-K)^k,$$

and if N is odd,

$$\exp(t(L-K)) = \sum_{k=0}^{4N-1} \varphi_{4N}^{(k)}(t)(L-K)^k = \sum_{k=0}^{2N-1} (\varphi_{4N}^{(k)}(t) - \varphi_{4N}^{(2N+k)}(t))(L-K)^k.$$

Proposition 9 *For any $M \geqq 2$,*

$$\varphi_M^{(k)}(z) = \frac{1}{M} \sum_{j=0}^{M-1} \omega_M^{j(M-k)} \exp(\omega_M^j z), \qquad z \in \mathbf{C},\ k = 0, \dots, M-1.$$

Here

$$\omega_M = \exp(\frac{2\pi\sqrt{-1}}{M}).$$

Proof. Note that

$$\sum_{j=0}^{M-1} \exp(\omega_M^j z) = \sum_{k=0}^{\infty} (\frac{z^k}{k!} \sum_{j=0}^{M-1} \omega_M^{kj})$$

It is easy to see that $\sum_{j=0}^{M-1} \omega_M^{kj} = 0$, if k is not divisible by M, and that $\sum_{j=0}^{M-1} \omega_M^{kj} = M$, if k is divisible by M. So we see that

$$\sum_{j=0}^{M-1} \exp(\omega_M^j z) = M \sum_{n=0}^{\infty} \frac{z^{nM}}{(nM)!} = M\varphi_M^{(0)}(z).$$

Also, it is easy to see that

$$\varphi_M^{(k)}(z) = \frac{d^{M-k}}{dz^{M-k}} \varphi_M(z), \qquad k = 1, \dots, M-1.$$

So we have our assertion. ∎

It is easy to see that for $i, j = 1, \dots, N$, and $n = 0, 1, \dots, 2N-1$,

$$((L-K)^n)_{ij} = \begin{cases} 1, & \text{if } j = i - n, \\ (-1)^N, & \text{if } j = i - n + 2N, \\ 0, & \text{otherwise.} \end{cases}$$

Now we give a concrete computation in the case that N is even. In this case we have

$$-\gamma^1(t)^* = (J\tilde{\alpha}^1)^* \exp(t(L-K))$$

$$= (\varphi_{2N}^{(2N-1)}(t), \varphi_{2N}^{(2N-2)}(t), \ldots, \varphi_{2N}^{(N)}(t), \varphi_{2N}^{(0)}(t), -\varphi_{2N}^{(1)}(t), \varphi_{2N}^{(2)}(t), -\varphi_{2N}^{(3)}(t), \ldots, \varphi_{2N}^{(N-2)}(t), -\varphi_{2N}^{(N-1)}(t)),$$

and

$$\gamma^2(t)^* = -(J\tilde{\alpha}^1)^* \exp(t(L-K))$$

$$= (\varphi_{2N}^{(N-1)}(t), \varphi_{2N}^{(N-2)}(t), \ldots, \varphi_{2N}^{(0)}(t),$$
$$\varphi_{2N}^{(N)}(t), -\varphi_{2N}^{(N+1)}(t), \varphi_{2N}^{(N+2)}(t), -\varphi_{2N}^{(N+3)}(t), \ldots, \varphi_{2N}^{(2N-2)}(t), -\varphi_{2N}^{(2N-1)}(t)).$$

Also we have

$$(e_ij(t))_{i=1,\ldots,N\ j=1,\ldots,2N} =$$

$$\begin{pmatrix} \varphi_{2N}^{(0)}(t) & \varphi_{2N}^{(2N-1)}(t) & \varphi_{2N}^{(2N-2)}(t) & \cdots & \varphi_{2N}^{(N+1)}(t) & \varphi_{2N}^{(1)}(t) & \varphi_{2N}^{(2)}(t) & \cdots & \varphi_{2N}^{(N)}(t) \\ \varphi_{2N}^{(1)}(t) & \varphi_{2N}^{(0)}(t) & \varphi_{2N}^{(2N-1)}(t) & \cdots & \varphi_{2N}^{(N+2)}(t) & \varphi_{2N}^{(2)}(t) & \varphi_{2N}^{(3)}(t) & \cdots & \varphi_{2N}^{(N-1)}(t) \\ \varphi_{2N}^{(2)}(t) & \varphi_{2N}^{(1)}(t) & \varphi_{2N}^{(0)}(t) & \cdots & \varphi_{2N}^{(N+3)}(t) & \varphi_{2N}^{(3)}(t) & \varphi_{2N}^{(4)}(t) & \cdots & \varphi_{2N}^{(N-2)}(t) \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \varphi_{2N}^{(N-1)}(t) & \varphi_{2N}^{(N-2)}(t) & \varphi_{2N}^{(N-3)}(t) & \cdots & \varphi_{2N}^{(0)}(t) & \varphi_{2N}^{(N)}(t) & \varphi_{2N}^{(N+1)}(t) & \cdots & \varphi_{2N}^{(2N-1)}(t) \end{pmatrix}.$$

Since $\det\ \exp(tK_{11}) = 0$, we have

$$\det\ \tilde{e}(t) = \det \begin{pmatrix} \varphi_{2N}^{(0)}(t) & \varphi_{2N}^{(2N-1)}(t) & \varphi_{2N}^{(2N-2)}(t) & \cdots & \varphi_{2N}^{(N+1)}(t) \\ \varphi_{2N}^{(1)}(t) & \varphi_{2N}^{(0)}(t) & \varphi_{2N}^{(2N-1)}(t) & \cdots & \varphi_{2N}^{(N+2)}(t) \\ \varphi_{2N}^{(2)}(t) & \varphi_{2N}^{(1)}(t) & \varphi_{2N}^{(0)}(t) & \cdots & \varphi_{2N}^{(N+3)}(t) \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \varphi_{2N}^{(N-1)}(t) & \varphi_{2N}^{(N-2)}(t) & \varphi_{2N}^{(N-3)}(t) & \cdots & \varphi_{2N}^{(0)}(t) \end{pmatrix}.$$

Then we can compute

$$E^\mu[\exp(\lambda \sum_{k=1}^{2} \int_0^T h^k(t)dw^k(t) + \frac{(-1)^{N-1}}{N!} \int_0^T (\int_0^t (t-s)^{N-1} dw^2(s)) dw^1(t)))].$$

We can also handle the case that N is odd, though formulae become much complicated.

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