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The critical values of exterior square L-functions on $GL(2)$

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0. Introduction

In number theory, the analytic properties of the critical values of automorphic L-functions are important in the conjectural framework of Deligne and Beilinson. For previous work of Harder [H], to compute the period integrals of the Eisenstein cohomology classes over the fundamental cycles formed by summing over a genus, we can get the special values of the associated L-function, together with some explicitly computed local factors. In this case, He considered the cohomology on the arithmetic quotients of the upper half plane of degree n. However, he has not come to consider its Eisenstein cohomology classes and has not prove the Deligne's conjecture. To understand the arithmetic of certain special values of L-functions based on the Deligne's conjecture, we should treat the Eisenstein cohomology of other manifold. In this article, we understand the analytic properties of such an Eisenstein cohomology classes to prove the Deligne's conjecture on the special values of the associated L-functions.

Let π be a cuspidal representation of GL(2). Our main object of this article is *the exterior square L-functions* of π defined by using the standard L-function

$$L(s,\pi)=\prod_{i=1}^{2}(1-\alpha_i p^{-s})^{-1}, \alpha_i \in \mathbb{C} \tag{0.1}$$

of π. We denote

$$L(s,\pi,\wedge^2)=\prod_{1\le i<j\le 2}(1-\alpha_i\alpha_j p^{-s})^{-1} \tag{0.2}$$

the exterior square L-functions on GL(2). To prove the Deligne's conjecture on the critical values of the exterior square L-functions, we find its period integral representation and confirm the analytic properties of suitable Eisenstein cohomology classes.

Since we will denote more precisely, our interesting situation confine G to $\mathrm{Sp}(2, \mathbb{Q})$. Let X be a symmetric spaces which is a quotient of $G(\mathbb{R})/K_\infty$. Here K_∞ is a maximal compact subgroup of $G(\mathbb{R})$. X is identified with the Siegel upper half space of degree 2. We take Γ be a torsion free arithmetic subgroup of G. It naturally acts on X for the identification. The cohomology classes of $H^*(\Gamma\backslash X)$ arise from classes on the boundary of the Borel-Serre compactification of $\Gamma\backslash X$. Since the compactification has the same homology type as $\Gamma\backslash X$, there is a restriction map from the cohomology of $\Gamma\backslash X$ to the cohomology of the boundary. The boundary components are parameterised by Γ-conjugacy classes of parabolic subgroups and are homotopic to quotients of X by a subgroup of Γ. The cohomology group $H^*(\Gamma\backslash \bar{X}, \mathbb{C})$ on the compactification decomposes the direct product of the cuspidal cohomology $H^*_{\mathrm{cusp}}(\Gamma\backslash \bar{X}, \mathbb{C})$ and so called Eisenstein cohomology $H^*_{\mathrm{Eis}}(\Gamma\backslash \bar{X}, \mathbb{C})$. It is known that the Eisenstein cohomology classes are represented by its suitable residue or the first term. Then our aim of this paper is to realize its classes using well-known arithmetic functions like Gamma function and zeta function and calculate its period integrals to describe the special values of automorphic L-functions.

Specially, if the degree of the cohomology group is 3, J. Schwermer showed in [Sc] that the cohomology classes of $H^3(\Gamma\backslash \bar{X}, \mathbb{C})$ are represented by the residue of a Eisenstein series $E(g, s)$ where its flat section is in the induced representation of the minimal parabolic subgroup of G (after, we will call it the minimal parabolic Eisenstein series) and the constant terms of the Eisenstein series of the maximal parabolic subgroups of G. Then first we will give the formula of the residue of the minimal parabolic Eisenstein series (Theorem 4). In order to give the explicit formula of the residue, it is necessary to describe Fourier expansion of the minimal parabolic Eisenstein series $E(g, s)$ along the minimal parabolic subgroup P of G and to carry out the differentiate for s (Theorem 3). However the Fourier expansion of the real analytic Siegel modular forms along P is not known, we will extend the results of [Na] for the holomorphic Siegel modular forms (Theorem 2).

Our main theorem of this paper is to compute $H = \mathrm{GL}(2) \times GL(2)$-period of the residue of the minimal parabolic Eisenstein series to bring out that it is the pure and simple critical value of the exterior square L-function $L(1, \pi, \wedge^2)$ (Theorem 5).

Theorem. *We define $\Omega_{\varphi^{(2)}}$ be the period integral of the flat section included in the minimal parabolic Eisenstein series $E(g, s)$. Then we have*

$$\Omega_{\varphi^{(2)}} \cdot \int_{H(\mathbb{Q})\backslash H(\mathbb{A})} \mathrm{Res}_{s=1} E^*(h, s) dh = L(1/2, \pi) L(1, \pi, \wedge^2), \tag{0.3}$$

where $L(1/2, \pi)$ be the special value of the standard L-function of automorphic cuspidal representation π.

The flow of its calculation is as follows. We use two theorems. One of it is the minimal parabolic Eisenstein series can be decomposed as the classical Eisenstein series on GL(2) and the Siegel-Eisenstein series associated to the maximal parabolic subgroup of G (Theorem 1), and the other one is the period becomes Bump and Friedberg's Rankin-Selberg integral referred in [BF]. Since the analyticity of it follows from the residue of the minimal parabolic Eisenstein series, then we hope to prove the Deligne's conjecture on the critical values of the exterior square L-functions to future application.

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Below, $\sigma(X)$ is set to the usual trace of the matrix X and $\delta(X)$ is the determinant of it. And $\mathrm{e}(\alpha)$ means $e^{2\pi\sqrt{-1}\alpha}$.

I. Minimal parabolic Eisenstein series for symplectic group

I-1. Group structure.

Let G be a symplectic group of degree 2 over the rationals $\mathbb{Q}$ which is defined by

$$G = \mathrm{Sp}(2,\mathbb{Q}) = \left\{ g = \mathrm{SL}(4,\mathbb{Q}) \mid {}^t g J_2 g = J_2 = \left(\begin{smallmatrix} 0_2 & -1_2 \\ 1_2 & 0_2 \end{smallmatrix}\right) \right\}. \tag{1.1}$$

Take its analytic subgroups N and A of G for

$$\begin{aligned} N &= N_0 \ltimes N_2 \\ &= \left\{ n = \left(\begin{array}{cc|cc} 1 & n_0 & & \\ & 1 & & \\ \hline & & 1 & \\ & & -n_0 & 1 \end{array} \right) \right\} \ltimes \left\{ \left(\begin{array}{cc|cc} 1 & & n_1 & n_2 \\ & 1 & n_2 & n_3 \\ \hline & & 1 & \\ & & & 1 \end{array} \right) \right\} \end{aligned} \tag{1.2}$$

and

$$A = \left\{ a(p) = \mathrm{diag}(a_1, a_2, a_1^{-1}, a_2^{-1}) \mid a_i > 0 \right\}, \tag{1.3}$$

where N is the maximal unipotent radical and A a maximal split torus of G. Then G has the Iwasawa decomposition $G = NAK$ for a fixed maximal compact subgroup $K = K_\infty \times \prod_{v<\infty} K_v$, where $K_\infty = G(\mathbb{R}) \cap \mathrm{O}(4) \cong \mathrm{U}(2)$ and $K_v = G(\mathbb{Z}_v)$ for $v < \infty$.

Let $M = Z_K(A)$ be the centraliser of A in K

$$M = \{ \mathrm{diag}(\varepsilon_1, \varepsilon_2, \varepsilon_1, \varepsilon_2) \mid \varepsilon_1, \varepsilon_2 \in \{\pm 1\} \}. \tag{1.4}$$

Then the minimal parabolic subgroup $P = NAM$ of G has the Langlands decomposition which is known by [L].

I-2. Principal series representation and Eisenstein series.

In this section, we recall the principal series representation of G for the minimal parabolic subgroup P.

Let $\mathfrak{a}$ be the Lie algebra of A. For $\lambda = (\lambda_1, \lambda_2) \in \mathfrak{a}_{\mathbb{C}}^* = \mathrm{Hom}_{\mathbb{R}}(\mathfrak{a}, \mathbb{C}) \cong \mathbb{C}^2$, we define a modulus quasi-character $e^\lambda : P \to \mathbb{R}_{>0}$ of P by

$$\mathrm{e}^\lambda(a(p)) = \exp(\lambda \log a(p)) \tag{1.5}$$

for the Langlands decomposition $p = na(p)m$, $n \in N$, $a(p) \in A$ and $m \in M$.

The irreducible unitary representation σ of M is given by the product of sign representations. It is specified by

$$\varepsilon_1 = \sigma(\mathrm{diag}(-1,1,-1,1)) \quad \text{and} \quad \varepsilon_2 = \sigma(\mathrm{diag}(1,-1,1,-1)). \tag{1.6}$$

For an irreducible cuspidal automorphic representation (π, V_π) of G, there exists a cuspidal data $(P, 1_N \otimes \mathrm{e}^{\lambda+\rho} \otimes \sigma)$, where ρ is the half-sum of the positive roots of P.

Definition 1. *Let σ be an irreducible unitary representation of M. For $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$, we define a principal series representation of G as an induced representation*

$$\begin{aligned}(1.7)\qquad &\mathrm{Ind}_P^G(1_N \otimes \mathrm{e}^{\lambda+\rho} \otimes \sigma)\\ &= \big\{\varphi : G \to V_\pi \mid \varphi(pg) = \mathrm{e}^{\lambda+\rho}(a(p))\sigma(m)\varphi(g),\ \forall (p,g) \in P \times G\big\}.\end{aligned}$$

We call function $\varphi : \mathbb{C} \times G \to \mathbb{C}$ a flat section of $\mathrm{Ind}_P^G(1_N \otimes \mathrm{e}^{\lambda+\rho} \otimes \sigma)$ when it satisfies the following conditions: For all $s \in \mathbb{C}$, $\varphi(s, \cdot) : G \to \mathbb{C}$ belongs to the space of the induced representation and its restricted in K is not depend on $s \in \mathbb{C}$.

Definition 2. *Let $\varphi_{\lambda+\rho} \in \mathrm{Ind}_P^G(1_N \otimes \mathrm{e}^{\lambda+\rho} \otimes \sigma)$ be a flat section. The minimal parabolic Eisenstein series for G is defined by*

$$(1.8)\qquad E(g,s) = \sum_{\gamma \in P \backslash G} \varphi_{\lambda+\rho}(s, \gamma g).$$

This series is absolutely convergent for $\mathrm{Re}\ s > 3/2$.

I-3. Relation among the Siegel-Eisenstein series.

In this subsection, we consider the relation between the minimal parabolic Eisenstein series and the Siegel-Eisenstein series of G. The Siegel-Eisenstein series is one of the most fascinating subject in number theory, for example, its analytic properties are very important. This relation among them is studied for some time, for instance in [Ba], [Sa] and [GMRV], however there is no evident paper to give an explicit formula of its Fourier expansion, prove its functional equation and give some information about poles using its relation. Then we bring out the relation among Siegel-Eisenstein series where its classical Siegel-Fourier expansion is extensively considered.

Now, let P_2 be the Siegel maximal parabolic subgroup of G. It has the Levi decomposition $P_2 = N_2 A_2 M_2$, where

$$(1.9)\quad N_2 = \left\{ n(x) = \begin{pmatrix} 1_2 & x \\ 0_2 & 1_2 \end{pmatrix} \,\middle|\, x = {}^t x \right\},\ A_2 = \{\mathrm{diag}(a, a, a^{-1}, a^{-1}) \mid a > 0\}$$

and

$$(1.10)\qquad M_2 = \left\{ m(a) = \begin{pmatrix} a & 0_2 \\ 0_2 & {}^t a^{-1} \end{pmatrix} \,\middle|\, a \in \mathrm{SL}^{\pm}(2) \right\}.$$

We can also define the Siegel-Eisenstein series for G using a flat section $\varphi^{(2)}_{\lambda_2} \in \mathrm{Ind}_{P_2}^G(1_{N_2} \otimes \mathrm{e}^{\lambda_2+\rho_2} \otimes \sigma_2)$ for $\rho_2 = (3/2, 3/2)$,

$$(1.11)\qquad E_2(g,s) = \sum_{\gamma \in P_2 \backslash G} \varphi^{(2)}_{\lambda_2+\rho_2}(s, \gamma g).$$

The first theorem of this paper is to show the relation between the minimal parabolic Eisenstein series and the Siegel-Eisenstein series.

Theorem 1. *Let $\varphi^{(2)}_{\lambda+\rho} \in \mathrm{Ind}_{P_2}^{G}(1_{N_2} \otimes \mathrm{e}^{\lambda+\rho} \otimes \sigma_2)$ be a flat section and f be a natural embedding from $\mathrm{GL}(2)$ to P_2 as $A \mapsto \begin{pmatrix} A & * \\ 0_2 & {}^tA^{-1} \end{pmatrix}$. For the embedding, we define $\bar{f}$ the isomorphism map from $B\backslash\mathrm{GL}(2)$ to $P\backslash P_2$. For all $g = p_2k \in P_2K = G$, we define the Eisenstein series on $\mathrm{GL}(2)$ by*

$$\varepsilon(g,s) = \sum_{\delta \in B\backslash GL(2)} \varphi^{(1)}_{\nu+\iota+\rho_2}(s, \delta\bar{f}^{-1}(p_2)), \tag{1.12}$$

where B is the standard Borel subgroup on $\mathrm{GL}(2)$ and $\varphi^{(1)}_{\nu+\iota+\rho_2}$ is a flat section in $\mathrm{Ind}_{B}^{\mathrm{GL}(2)}(1_{N_B} \otimes \mathrm{e}_B^{\nu+\iota+\rho_2} \otimes \sigma_B)$. Then the minimal parabolic Eisenstein series $E(g,s)$ is decomposed as follows.

$$E(g,s) = \sum_{\gamma \in P_2\backslash G} \varphi^{(2)}_{\lambda+\rho}(s, \gamma g)\varepsilon(\gamma g, s). \tag{1.13}$$

Proof. Since the Siegel maximal parabolic subgroup P_2 contains the minimal parabolic subgroup P which is a normal subgroup of P_2, then

$$E(g,s) = \sum_{\gamma_1 \in P\backslash P_2} \sum_{\gamma_2 \in P_2\backslash G} \varphi_{\lambda+\rho}(s, \gamma_1\gamma_2 g).$$

If we take an element of P, the inverse image of f is in the Borel subgroup B on $\mathrm{GL}(2)$. Then the first isomorphism theorem says that $P\backslash P_2 \cong B\backslash\mathrm{GL}(2)$ and we can replace the first summation with $B\backslash\mathrm{GL}(2)$.

From the character formula for the induced representation, if we consider the inclusion relations $P \subset P_0 \subset G$, then we have an isomorphism $\mathrm{Ind}_P^G(1_N \otimes \mathrm{e}^{\lambda+\rho} \otimes \sigma) \cong \mathrm{Ind}_{P_2}^G\left(\mathrm{Ind}_P^{P_2}(1_N \otimes \mathrm{e}^{\lambda+\rho} \otimes \sigma)\right)$ which is called induction in stages.

The Borel subgroup B has the Langlands decomposition $B = N_BA_BM_B$ such that

$$N_B = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \middle| \, b \in \mathbb{R} \right\}, A_B = \left\{ \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \middle| \, a_i \in \mathbb{R}_{>0} \text{ and } |a_1| \neq 1, |a_2| \neq 1 \right\}$$

and

$$M_B = \left\{ \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \middle| \, a_i \in \mathbb{R}_{>0} \text{ and } |a_1| = |a_2| = 1 \right\}.$$

We also define σ_B the irreducible unitary representation of B and e_B^{ν} the modulus quasi-character of B for $\nu \in \mathrm{Hom}_{\mathbb{R}}(\mathfrak{a}_B, \mathbb{C})$. Since the image of a representative α of $B\backslash\mathrm{GL}(2)$ by $\bar{f}$ becomes a representative $\bar{f}(\alpha)$ of $P\backslash P_2$, if we take φ' an element of $\mathrm{Ind}_P^{P_2}(1_N \otimes \mathrm{e}^{\lambda+\rho} \otimes \sigma)$ and define $\varphi^{(1)}(\alpha) = \varphi'(\bar{f}(\alpha))$, for $\alpha \in [A]$, then we have the isomorphism

$$\varphi^{(1)}_{\nu+\iota+\rho_2} \in \mathrm{Ind}_B^{\mathrm{GL}(2)}(1_{N_B} \otimes \mathrm{e}_B^{\nu+\iota+\rho_2} \otimes \sigma_B) \cong \mathrm{Ind}_P^{P_2}(1_N \otimes \mathrm{e}^{\lambda+\rho} \otimes \sigma) \ni \varphi'_{\lambda+\rho},$$

where $\iota = (1/2, -1/2)$ be the half-sum of positive roots for $\mathrm{GL}(2)$.

For all $g \in G$, we have

$$\begin{aligned} E(g,s) &= \sum_{\bar{f}^{-1}(\gamma_1) \in B\backslash\mathrm{GL}(2)} \sum_{\gamma_2 \in P_2\backslash G} \varphi^{(2)}_{\lambda+\rho}(s, \gamma_2 g)\varphi'_{\lambda+\rho}(s, \gamma_1(\gamma_2 g)|_{P_2}) \\ &= \sum_{\bar{f}^{-1}(\gamma_1) \in B\backslash\mathrm{GL}(2)} \sum_{\gamma_2 \in P_2\backslash G} \varphi^{(2)}_{\lambda+\rho}(s, \gamma_2 g)\varphi^{(1)}_{\nu+\iota+\rho_2}(s, \bar{f}^{-1}(\gamma_1)\bar{f}^{-1}((\gamma_2 g)|_{P_2})). \end{aligned}$$

The second summation is the Eisenstein series on $\mathrm{GL}(2)$. $\square$

II. Fourier expansion of the Eisenstein series along the minimal parabolic subgroup

However the classical Fourier expansion along the maximal parabolic subgroup are well-known, little is known concerning the Fourier expansion along the minimal parabolic subgroup P. One of the reason of it is that the unipotent radical N of P is non-abelian. We should extend the theory of Fourier analysis on non-abelian groups. According to the previous study of H. Narita discussed about an expansion of vector-valued holomorphic Siegel modular forms in [Na], the Fourier coefficients of the Fourier expansion along minimal parabolic subgroup are related to the maximal one. We should extend the results for the real analytic Eisenstein series. Also we prove the relation between the Fourier coefficients of the Fourier expansion along the minimal parabolic subgroup and the Siegel maximal parabolic subgroup as an expansion of the study of Narita and calculate the Fourier expansion of the real analytic Eisenstein series obtained in section 1, coming down to the maximal one.

II-1. Construction of the Fourier expansion along the minimal parabolic subgroup.

We define (π, H_π) be the spherical principal series representation of G and (τ, V_τ) be the irreducible finite dimensional representation of K. Let ι be the inclusion map from τ to π_K where π_K be the K-finite vectors in π. Then we can define a generalised Whittaker function as an image of the following map:

$$W_{k,T} \in \mathrm{Hom}_{(\mathfrak{g}_{\mathbb{C}},K)}(\pi_K, C^\infty_{\eta_T}(N\backslash G)_K) \to \mathrm{Hom}_K(\tau, C^\infty_{\eta_T}(N\backslash G)_K) \ni W_{k,T}\circ\iota, \tag{2.1}$$

where

$$\eta_T = L^2\text{-}\mathrm{Ind}_M^N \mathrm{e}(T\log) \tag{2.2}$$

and

$$C^\infty_{\eta_T}(N\backslash G)_K = \left\{ W_{k,T} : G \overset{C^\infty}{\to} H^\infty_{\eta_T} \;\middle|\; W_{k,T}(ng) = \eta_T(n) W_{k,T}(g),\ K\text{-finite} \right\} \tag{2.3}$$

for H_{η_T} the representation space of η_T and $H^\infty_{\eta_T}$ the space of C^∞-vectors in it. We remark that the space $\mathrm{Hom}_K(\tau, C^\infty_{\eta_T}(N\backslash G)_K)$ is equivalent to the following space such that

$$\left\{ W_{k,T} : G \overset{C^\infty}{\to} H^\infty_{\eta_T} \;\middle|\; W_{k,T}(ngk) = \eta_T(n)\tau(k) W_{k,T}(g) \right\}. \tag{2.4}$$

The explicit formula of the generalised Whittaker function at archimedean place was given by Niwa in [Ni], Theorem 1 and Proposition 2.

We take an arithmetic subgroup Γ of G which implies that the $\mathbb{Q}$-structure comes from a such of G and $N\cap\Gamma = N_\Gamma = (N_0\cap\Gamma)\ltimes(N_2\cap\Gamma)$. Since $N_\Gamma\backslash N$ is compact, its L^2-space is decomposed as the Hilbert space direct sum.

Proposition 1. *Let $\hat{N}$ is the unitary dual of N. We have*

$$L^2(N_\Gamma\backslash N) \cong \tilde{\bigoplus}_{(\eta, H_\eta)\in\hat{N}} \mathrm{Hom}_N(\eta, L^2(N_\Gamma\backslash N)) \otimes H_\eta. \tag{2.5}$$

PROOF. Its proof is described in [GGP], Chapter I. section 2.3. $\square$

As same as in [Na], if we choose a basis of $\mathrm{Hom}_N(\eta, L^2(N_\Gamma\backslash N))$ and decide the multiplicity of η in $L^2(N_\Gamma\backslash N)$, then we can describe the Fourier expansion of the real analytic Eisenstein series along the minimal parabolic subgroup P on G. Before stating our result, we prepare some notation.

Let $\mathcal{S}_2$ be a set of positive definite symmetric matrices of degree 2 with rational coefficients. We denote by $\bar{\mathcal{S}_2}$ the closure of $\mathcal{S}_2$ in V. Since $\mathrm{tr}(u, v)$ for $u, v \in V \cong N_2$ is a non-degenerate bilinear form on V, we define the dual lattice of $V_\Gamma = V \cap \Gamma = N_2 \cap \Gamma$ such that

$$\mathcal{S}_2^\vee = \left\{T \in \bar{\mathcal{S}_2} \mid \mathrm{tr}(T, S) \in \mathbb{Z},\ \forall S \in V_\Gamma\right\}. \tag{2.6}$$

Consider the natural action of the maximal unipotent radical $U_2(\mathbb{Q}) = \left\{\left(\begin{smallmatrix}1 & b\\ 0 & 1\end{smallmatrix}\right)\right\}$ of $\mathrm{GL}(2, \mathbb{Q})$ such that $(v, u) \mapsto {}^t uvu$ for all $u \in U_2(\mathbb{Q})$, we define

$$\mathcal{S}_{2,\sim}^\vee = \mathcal{S}_2/U_2(\mathbb{Q}). \tag{2.7}$$

If we consider the group structures of U_2 and N_0, the equivalences $U_2 \cap \Gamma \cong U_2(\mathbb{Z}) \cong N_0 \cap \Gamma$ hold, for $S \in \mathcal{S}_2^\vee$. Then we define

$$\mathcal{M}_{0,\sim}^\vee = \{T \in \mathcal{S}_2^\vee \mid {}^t uTu = S,\ \exists u \in U_2(\mathbb{Q})\}/U_2(\mathbb{Z}). \tag{2.8}$$

For $T \in \mathcal{M}_{0,\sim}^\vee$, let $\mathfrak{m}_T$ be a maximal subordinate subalgebra : a maximal left subalgebra containing the characteristic subalgebra and excluding the central generator, and $M_T = \exp(\mathfrak{m}_T)$.

PROPOSITION 2. *Let F be a real analytic Eisenstein series on G of weight k with respect to Γ. For any $n \in N$ and $g \in G$, the Fourier expansion of F along P is given by*

$$F(ng) = \sum_{S \in \mathcal{S}_{2,\sim}^\vee} \sum_{T \in \mathcal{M}_{0,\sim}^\vee(S)} F_{S,T}(g)\Theta_T(W_{k,T}(\cdot))(n), \tag{2.9}$$

where

$$\Theta_T(W_{k,T}(\cdot))(n) = \sum_{\gamma \in N_\Gamma \cap M_T\backslash N_\Gamma} W_{k,T}(\gamma n) \quad \text{and} \quad F_{S,T}(g) = W_{k,T}(1 \cdot g)^{-1}. \tag{2.10}$$

PROOF. This proof is almost the same as the paper of Narita ([Na], Theorem 5.8). Only different thing is that now F be a real analytic Eisenstein series. In this case, F generates the spherical principal series representation (π, H_π) at infinite places. Then for a basis $\{\Theta_T\}_{T \in \mathcal{M}_{0,\sim}^\vee}$ of $\mathrm{Hom}_N(\eta, L^2(N_\Gamma\backslash N))$, we can take the generalised Whittaker function $W_{k,T}(g) \in H_{\eta_T}^\infty$ on G of π. $\square$

We also discuss about the relation between this Fourier expansion and the Siegel Fourier expansion. There are three types of Fourier expansions on G by its parabolic subgroups. We call Siegel Fourier expansion as a Fourier expansion along Siegel maximal parabolic subgroup P_2. In the expansion, if we replace N with N_2 that is $N_0 \cong U_2$ is identity element, we get the following well-known expansion.

Theorem 2. *Let F be a real analytic Eisenstein series on G of weight k with respect to Γ. We take $T \in \mathcal{S}_2^\vee$ belonging to $\mathcal{M}_{0,\sim}^\vee(S)$ with some $S \in \mathcal{S}_2^\vee$. Define the Fourier expansion of F along the minimal parabolic subgroup P is*

$$F(g) = \sum_{S \in \mathcal{S}_{2,\sim}^\vee} \sum_{T \in \mathcal{M}_{0,\sim}^\vee(S)} F_{S,T}(g)\Theta_T(W_{k,T}(\cdot))(n) \tag{2.11}$$

and the maximal parabolic subgroup P_2 is

$$F(g) = \sum_{S \in \mathcal{S}_2^\vee} F_S(g)\mathrm{e}(\sigma(Sx)). \tag{2.12}$$

Then the relations of the both Fourier coefficients are given by

$$F_{S,T}(g) = F_S(g), \quad \text{and} \quad F_{S,T}(g) = F_{{}^tuSu}(g), \quad \text{for every } u \in U_2(\mathbb{Z}). \tag{2.13}$$

Proof. It is done to replace N with N_2 in the formula of the Fourier expansion of F along P in the previous proposition. Before all, in the part of $\Theta_T((W_{k,T})(\cdot))(n)$, it can be calculated as follows. For all $n(x) \in N_2 \subset P_2$,

$$\begin{aligned}\Theta_T\left((W_{k,T})(\cdot)\right)(n(x)) &= \chi_T(n(x))W_{k,T}(\cdot) = W_{k,T}(n(x)) \\ &= \eta_T(n(x)) = \mathrm{e}(\sigma(Tx)).\end{aligned}$$

Compare with the formulas of both Fourier expansions, we obtain the statement. □

II-2. Calculation of the Fourier expansion.

According to the previous subsection, we have to calculate the Fourier expansion as follows :

$$\begin{aligned} E(g,s) &= \sum_{S \in \mathcal{S}_{2,\sim}^\vee} \sum_{T \in \mathcal{M}_{0,\sim}^\vee(S)} E_{S,T}(g)\Theta_T\left(W_{k,T}(\cdot)\right)(n) \\ &= \sum_{S \in \mathcal{S}_2^\vee} E_S(g)\mathrm{e}(\sigma(Sx)), \end{aligned} \tag{2.14}$$

where the Fourier coefficients $E_{S,T}(g)$ is expressed as

$$\begin{aligned} E_{S,T}(g) = E_S(g) = \int_{N_2\backslash N_2(\mathbb{A})} &\sum_{\gamma \in P_2\backslash G} \varphi_{\lambda+\rho}^{(2)}(s, \gamma n(x) m\left(\left(\begin{smallmatrix} a_1 & a_2u \\ 0 & a_2 \end{smallmatrix}\right)\right)) \\ &\times \varepsilon(\gamma n(x) m\left(\left(\begin{smallmatrix} a_1 & a_2u \\ 0 & a_2 \end{smallmatrix}\right)\right), s)\mathrm{e}(-\sigma(Sx))dn(x), \end{aligned} \tag{2.15}$$

for a decomposition $g = n(x)m\left(\left(\begin{smallmatrix} a_1 & a_2u \\ 0 & a_2 \end{smallmatrix}\right)\right)k \in G$. Here we remark that the function $\varphi_{\lambda+\rho}^{(2)}(s,\cdot)$ and the Eisenstein series $\varepsilon(\cdot, s)$ are not depend on the maximal compact subgroup $k \in K$. Below, $E_S(g)$ are calculated concretely.

According to the structure of the Siegel maximal parabolic subgroup P_2, G has the Bruhat decomposition $G = \coprod_{i=0}^2 P_2 w_i P_2$, where

$$w_0 = 1_4, \quad w_1 = \begin{pmatrix} 1 & & & \\ & & & 1 \\ & & 1 & \\ & -1 & & \end{pmatrix} \quad \text{and} \quad w_2 = \begin{pmatrix} & & 1 & \\ & & & 1 \\ -1 & & & \\ & -1 & & \end{pmatrix} \tag{2.16}$$

be its Weyl group elements. It is well known that for any Weyl group element w_i, one has the decomposition for the unipotent subgroup of G such that $N_2 = N^{w_i} \cdot N_{w_i}$, where

$$N^{w_0} = N_2, \quad N^{w_1} = \left\{ n(x) \mid x = \begin{pmatrix} x_1 & x_2 \\ x_2 & 0 \end{pmatrix} \right\} \quad \text{and} \quad N^{w_2} = \{1_4\} \tag{2.17}$$

and

$$N_{w_0} = \{1_4\}, \quad N_{w_1} = \left\{ n(x) \mid x = \begin{pmatrix} 0 & 0 \\ 0 & x_3 \end{pmatrix} \right\} \quad \text{and} \quad N_{w_2} = N_2. \tag{2.18}$$

Then the Fourier coefficients $E_S(g)$ can be decomposed as the following.

Lemma 1. *Let $E_S(g)$ be the Fourier coefficients of the Fourier expansion of the minimal parabolic Eisenstein series formulated in* (2.15). *For all $S \in \mathcal{S}_2^\vee$, it is expressed as follows.*

$$\begin{aligned} E_S(g) = & \sum_{i=0}^{2} \sum_{\gamma \in Q_i \backslash M_2} \int_{N^{w_i} \backslash N^{w_i}(\mathbb{A})} \mathrm{e}(-\sigma(S\gamma^{-1}x\gamma)) dn(x) \\ & \times \int_{N_{w_i}(\mathbb{A})} \varphi^{(2)}_{\lambda+\rho}(s, w_i n(x) \gamma m(\left(\begin{smallmatrix} a_1 & a_2 u \\ 0 & a_2 \end{smallmatrix}\right))) \varepsilon\left(w_i n(x) \gamma m(\left(\begin{smallmatrix} a_1 & a_2 u \\ 0 & a_2 \end{smallmatrix}\right)), s\right) \\ & \times \mathrm{e}(-\sigma(S\gamma^{-1}x\gamma)) dn(x), \end{aligned} \tag{2.19}$$

where Q_i be the subgroup of M_2 defined by

$$Q_0 = M_2, \quad Q_1 = \left\{ m(a) \mid a = \begin{pmatrix} a_1 & b \\ 0 & a_2 \end{pmatrix} \in \mathrm{GL}(2) \right\} \quad \text{and} \quad Q_2 = M_2. \tag{2.20}$$

Proof. It is easy to check that $w_i^{-1} P_2 w_i \cap P_2 = N^{w_i} Q_i$ for all $i = 0$, 1 and 2. The coset $w_i^{-1} P_2 w_i \cap P_2 \backslash P_2$ is equivalent to the set $\{nm \mid n \in N_{w_i},\ m \in Q_i \backslash M\}$. This equivalence relation will be acquired by computing the residue class of the coset. Since the map of $P \to P w_i P$, $p \mapsto w_i p$ is surjective, then we have an isomorphism

$$w_i^{-1} P_2 w_i \cap P_2 \backslash P_2 \cong P_2 \backslash P_2 w_i P_2 \cong \{nm \mid n \in N_{w_i},\ m \in Q_i \backslash M\}.$$

The Fourier coefficients $E_S(g)$ is as follows when the sum running γ is rewritten using an isomorphism.

$$\begin{aligned} E_S(g) &= \sum_{i=0}^{2} \int_{N_2 \backslash N_2(\mathbb{A})} \sum_{\gamma \in P_2 \backslash P_2 w_i P_2} \varphi^{(2)}_{\lambda+\rho}(s, \gamma g) \varepsilon(\gamma g, s) \mathrm{e}(-\sigma(Sx)) dn(x) \\ &= \sum_{i=0}^{2} \sum_{\gamma \in Q_i \backslash M_2} \int_{N_2 \backslash N_2(\mathbb{A})} \sum_{\delta \in N_{w_i}} \varphi^{(2)}_{\lambda+\rho}(s, \delta\gamma g) \varepsilon(\delta\gamma g, s) \mathrm{e}(-\sigma(Sx)) dn(x). \end{aligned}$$

For the element in $N_2(\mathbb{A})$, transformation of the variable $n(x)$ to $\gamma^{-1}n(x)\gamma$ shows that

$$
\begin{aligned}
E_S(g) = & \sum_{i=0}^{2} \sum_{\gamma \in Q_i \backslash M_2} \int_{N^{w_i} \backslash N_2(\mathbb{A})} \varphi^{(2)}_{\lambda+\rho}(s, w_i n(x) \gamma m\left(\left(\begin{smallmatrix} a_1 & a_2 u \\ 0 & a_2 \end{smallmatrix}\right)\right)) \\
& \times \varepsilon(w_i n(x) \gamma m\left(\left(\begin{smallmatrix} a_1 & a_2 u \\ 0 & a_2 \end{smallmatrix}\right)\right), s) \mathrm{e}(-\sigma(S\gamma^{-1}x\gamma)) dn(x).
\end{aligned}
$$

Because of $N_2(\mathbb{A}) = N^{w_i}(\mathbb{A}) \cdot N_{w_i}(\mathbb{A})$, we obtain the statement. $\square$

The next opinion is obtained because we classify the Fourier coefficients $E_S(g)$ according to the rank of $S \in \mathcal{S}_2^{\vee}$ concretely using Lemma 1.

Proposition 3. *The assumptions and notations are same as in Lemma 1. The Fourier coefficients $E_S(g)$ can be described concretely as follows.*
(i) If $S = 0_2$, then we have

(2.21)
$$
\begin{aligned}
& \varphi^{(2)}_{\lambda+\rho}(s, m\left(\left(\begin{smallmatrix} a_1 & a_2 u \\ 0 & a_2 \end{smallmatrix}\right)\right)) \varepsilon(m\left(\left(\begin{smallmatrix} a_1 & a_2 u \\ 0 & a_2 \end{smallmatrix}\right)\right), s) \\
& + \int_{N_{w_1}(\mathbb{A})} \varphi^{(2)}_{\lambda+\rho}(s, w_1 n(x) m\left(\left(\begin{smallmatrix} a_1 & a_2 u \\ 0 & a_2 \end{smallmatrix}\right)\right)) \varepsilon(w_1 n(x) m\left(\left(\begin{smallmatrix} a_1 & a_2 u \\ 0 & a_2 \end{smallmatrix}\right)\right), s) dn(x) \\
& + \sum_{l \in \mathbb{Q}} \int_{N_{w_1}(\mathbb{A})} \varphi^{(2)}_{\lambda+\rho}(s, w_1 n(x) m\left(\left(\begin{smallmatrix} 0 & a_2 \\ a_1 & a_2(u+a) \end{smallmatrix}\right)\right)) \\
& \quad \times \varepsilon(w_1 n(x) m\left(\left(\begin{smallmatrix} 0 & a_2 \\ a_1 & a_2(u+a) \end{smallmatrix}\right)\right), s) dn(x) \\
& + \int_{N_2(\mathbb{A})} \varphi^{(2)}_{\lambda+\rho}(s, w_2 n(x) m\left(\left(\begin{smallmatrix} a_1 & a_2 u \\ 0 & a_2 \end{smallmatrix}\right)\right)) \varepsilon(w_2 n(x) m\left(\left(\begin{smallmatrix} a_1 & a_2 u \\ 0 & a_2 \end{smallmatrix}\right)\right), s) dn(x).
\end{aligned}
$$

(ii-1) If rank $S = 1$ *and* $S = \left(\begin{smallmatrix} s_1 & s_2 \\ s_2 & s_3 \end{smallmatrix}\right)$ *such that $s_2 = s_3 = 0$ or $s_1 s_3 = s_2^2$, then we have*

(2.22)
$$
\begin{aligned}
& \int_{N_2(\mathbb{A})} \varphi^{(2)}_{\lambda+\rho}(s, w_2 n(x) m\left(\left(\begin{smallmatrix} a_1 & a_2 u \\ 0 & a_2 \end{smallmatrix}\right)\right)) \varepsilon(w_2 n(x) m\left(\left(\begin{smallmatrix} a_1 & a_2 u \\ 0 & a_2 \end{smallmatrix}\right)\right), s) \mathrm{e}(-\sigma(Sx)) dn(x) \\
& + \int_{N_{w_1}(\mathbb{A})} \varphi^{(2)}_{\lambda+\rho}(s, w_1 n\left(\left(\begin{smallmatrix} 0 & 0 \\ 0 & x_3 \end{smallmatrix}\right)\right) m\left(\left(\begin{smallmatrix} 0 & a_2 \\ a_1 & a_2(u+s_2/s_1) \end{smallmatrix}\right)\right)) \\
& \times \varepsilon(w_1 n\left(\left(\begin{smallmatrix} 0 & 0 \\ 0 & x_3 \end{smallmatrix}\right)\right) m\left(\left(\begin{smallmatrix} 0 & a_2 \\ a_1 & a_2(u+s_2/s_1) \end{smallmatrix}\right)\right), s) \mathrm{e}(-s_1 x_3) dx_3.
\end{aligned}
$$

(ii-2) If rank $S = 1$ *and* $S = \left(\begin{smallmatrix} 0 & 0 \\ 0 & s_3 \end{smallmatrix}\right)$, *then we have*

(2.23)
$$
\begin{aligned}
& \int_{N_2(\mathbb{A})} \varphi^{(2)}_{\lambda+\rho}(s, w_2 n(x) m\left(\left(\begin{smallmatrix} a_1 & a_2 u \\ 0 & a_2 \end{smallmatrix}\right)\right)) \varepsilon(w_2 n(x) m\left(\left(\begin{smallmatrix} a_1 & a_2 u \\ 0 & a_2 \end{smallmatrix}\right)\right), s) \mathrm{e}(-\sigma(Sx)) dn(x) \\
& + \int_{N_{w_1}(\mathbb{A})} \varphi^{(2)}_{\lambda+\rho}(s, w_1 n\left(\left(\begin{smallmatrix} 0 & 0 \\ 0 & x_3 \end{smallmatrix}\right)\right) m\left(\left(\begin{smallmatrix} a_1 & a_2 u \\ 0 & a_2 \end{smallmatrix}\right)\right)) \\
& \times \varepsilon(w_1 n\left(\left(\begin{smallmatrix} 0 & 0 \\ 0 & x_3 \end{smallmatrix}\right)\right) m\left(\left(\begin{smallmatrix} a_1 & a_2 u \\ 0 & a_2 \end{smallmatrix}\right)\right), s) \mathrm{e}(-s_3 x_3) dx_3.
\end{aligned}
$$

(iii) If rank $S = 2$, *then we have*

(2.24)

$$\int_{N_2(\mathbb{A})} \varphi^{(2)}_{\lambda+\rho}(s, w_2 n(x) m\left(\left(\begin{smallmatrix} a_1 & a_2 u \\ 0 & a_2 \end{smallmatrix}\right)\right)) \varepsilon(w_2 n(x) m\left(\left(\begin{smallmatrix} a_1 & a_2 u \\ 0 & a_2 \end{smallmatrix}\right)\right), s) \mathrm{e}(-\sigma(Sx)) dn(x).$$

In the above proposition, what is necessary is to give the Iwasawa decomposition of $w_i n(x) m(a)$ for $a = \left(\begin{smallmatrix} a_1 & a_2 u \\ 0 & a_2 \end{smallmatrix}\right)$, $a = \left(\begin{smallmatrix} 0 & a_2 \\ a_1 & a_2(u+a) \end{smallmatrix}\right)$ or $a = \left(\begin{smallmatrix} 0 & a_2 \\ a_1 & a_2(u+s_2/s_1) \end{smallmatrix}\right)$ because the domain on the Eisenstein series $\varepsilon(g, s)$ is depend only on the diagonal block of $P_2 \subset G$ which was explain in Theorem 1. If we compute the Iwasawa decomposition, in all cases, $\bar{f}^{-1}(w_i n(x) m(a)) = \left(\begin{smallmatrix} a_1 & a_2 u \\ 0 & a_2 \end{smallmatrix}\right)$ for $\bar{f}$ given in Theorem 1.

Theorem 3. *Let $E(g, s)$ be the minimal parabolic Eisenstein series of weight $k \in 2\mathbb{Z}$. We put $g = m(a)$ for $a = \left(\begin{smallmatrix} a_1 & a_2 v \\ 0 & a_2 \end{smallmatrix}\right) \in \mathrm{GL}(2)$. The imaginary part of the action of $\sqrt{-1}$ multiple of the unit matrix of degree 2 on G is defined by y, that is $y = a^t a$. We also define the imaginary part of the action of a on $\sqrt{-1}$ as v and $\tau = u + \sqrt{-1}v$. Following we use the notations $\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$ and the Pochhammer symbol $(a)_i = \Gamma(a+i)/\Gamma(a)$. For all $n(x) \in N_2$, the Fourier expansion of the normalised minimal parabolic Eisenstein series*

(2.25)

$$\begin{aligned} E^*(n(x)g, s) &= \varepsilon^*(n(x)g, s) E^*_2(n(x)g, s) \\ &= \xi(s)\xi(2s-2)^2 (s/2)_{k/2}((s-1)/2)_{k/2}(s-1)_k \varepsilon(n(x)g, s-k) E(n(x)g, s-k) \end{aligned}$$

along the minimal parabolic subgroup P of G is given as following.

First, we show the Fourier expansion of the normalised Eisenstein series

$$\varepsilon^*(n(x)g, s) = \xi(2s-2)(s-1)_k \varepsilon(n(x)g, s-k) \tag{2.26}$$

such that

(2.27)

$$\begin{aligned} &\varepsilon^*(n(x)g, s) \\ &= \delta(y)^{\frac{1}{2}} v^{\frac{1}{2}-k} \Big\{ \delta(y)^{-1} v^s \xi(2s-2)(s-1)_k + v^{-s+1}\xi(2s-3)(s-k-1)_k \\ &+ \sum_{m=1}^{\infty} m^{-s+1} \sigma_{2s-3}(m) W_{-k, s-3/2}(4\pi m v)\, \mathrm{e}(m\tau) \\ &+ (s-k-1)_k (s-1)_k \sum_{m=1}^{\infty} m^{-s+1} \sigma_{2s-3}(m) W_{k, s-3/2}(4\pi m v)\, \mathrm{e}(-m\tau) \Big\}, \end{aligned}$$

where $\sigma_s(n)$ is a divisor sum defined by $\sigma_s(n) = \sum_{d|n} d^s$ and $W_{\nu,\mu}(z)$ is a W-Whittaker function which is given by the integral

$$W_{\nu,\mu}(z) = \frac{e^{-z/2} z^{\mu+1/2}}{\Gamma(-\nu+\mu+1/2)} \int_0^\infty e^{-tz} t^{-\nu+\mu-1/2}(1+t)^{\nu+\mu-1/2} dt, \tag{2.28}$$

for Re $(-\nu+\mu+1/2) > 0$ *and* Re $z > 0$.

Second, we show the Fourier expansion of the normalised Siegel-Eisenstein series

$$E_2^*(n(x)g,s)=\xi(s)\xi(2s-2)(s/2)_{k/2}((s-1)/2)_{k/2}E(n(x)g,s-k) \tag{2.29}$$

such that

(2.30)

$$\begin{aligned}
&E_2^*(g,s)=\varepsilon_2(g,s)+\delta(y)^{\frac{s-k}{2}}v^{\frac{1}{2}}2^{-k}(s-1)_k\xi(s)\xi(2s-2)\\
&+\delta(y)^{\frac{3-s-k}{2}}v^{\frac{1}{2}}2^{-k}(s-k-1)_k\xi(s-2)\xi(2s-3)\\
&+2v^{\frac{1}{2}}\sum_{\substack{S\in\mathcal{S}_2^\vee,\delta(S)=0,\\ S\neq 0_2,S>0}}\Biggl\{\delta(y)^{\frac{s-k}{2}}|\sigma(Sy)|^{-\frac{s}{2}}\xi(2s-2)F^{(1)}_{e(S)}(2-s)\\
&\quad\times((3-s-k)/2)_{k/2}\,W_{k/2,(s-1)/2}\,(4\pi|\sigma(Sy)|)\\
&+\delta(y)^{\frac{3-s-k}{2}-k}|\sigma(Sy)|^{\frac{s-3}{2}}\xi(2s-3)F^{(1)}_{e(S)}(s-1)\\
&\quad\times((s-k)/2)_{k/2}\,W_{k/2,s/2-1}\,(4\pi|\sigma(Sy)|)\Biggr\}\mathrm{e}(\sigma(Sx))\\
&+2^{1-2k}v^{\frac{1}{2}}\pi^{-k}\sum_{\substack{S\in\mathcal{S}_2^\vee,\delta(S)=0,\\ S\neq 0_2,S<0}}\Biggl\{\delta(y)^{\frac{s-k}{2}}|\sigma(Sy)|^{-\frac{s}{2}-k}\xi(2s-2)F^{(1)}_{e(S)}(2-s)\\
&\quad\times((3-s-k)/2)_{k/2}\,W_{k/2,(s-1)/2}\,(4\pi|\sigma(Sy)|)\\
&+\delta(y)^{\frac{3-s-k}{2}}|\sigma(Sy)|^{\frac{s-3}{2}-k}\xi(2s-3)F^{(1)}_{e(S)}(s-1)\\
&\quad\times((s-k)/2)_{k/2}\,W_{k/2,s/2-1}\,(4\pi|\sigma(Sy)|)\Biggr\}\mathrm{e}(\sigma(Sx))\\
&+2^3\pi^{-\frac{1}{2}}v^{\frac{1}{2}}\Biggl\{\sum_{\substack{S\in\mathcal{S}_2^\vee,\delta(S)>0,\\ S>0}}(2/\pi)^{-k}(\delta(2S))^{\frac{s+k-3}{2}}L^*(s-1,\chi)F^{(2)}_S(s)\\
&\quad\times(s-k-1)_{2k}\omega\,(2\pi y,S;(s+k)/2,(s-k)/2)\,\mathrm{e}(\sigma(Sy))\\
&+\sum_{\substack{S\in\mathcal{S}_2^\vee,\delta(S)>0,\\ S<0}}(2\pi)^{-k}(\delta(2S))^{\frac{s-k-3}{2}}(\delta(2y))^{-k}L^*(s-1,\chi)F^{(2)}_S(s)\\
&\quad\times(s-k-1)_{2k}\omega\,(2\pi y,S;(s+k)/2,(s-k)/2)\,\mathrm{e}(\sigma(Sy))\\
&+\sum_{\substack{S\in\mathcal{S}_2^\vee,\delta(S)<0,\\ S<0}}(\delta(y))^{-\frac{k}{2}}(-\delta(2S))^{\frac{s-3}{2}}(-\delta(2Sy))^{\frac{1}{4}}L^*(s-1,\chi)F^{(2)}_S(s)\\
&\quad\times((s-k)/2)_{k/2}\,((s-1)/2)_{k/2}\,\omega\,(2\pi y,S;(s+k)/2,(s-k)/2)\,\mathrm{e}(\sigma(Sy))\Biggr\},
\end{aligned}$$

where $\varepsilon_2(g,s)$ *is the Eisenstein series on* GL(2)*, its flat section is in* $\mathrm{Ind}(|\cdot|_{\mathbb{A}}^{s-1/2},|\cdot|_{\mathbb{A}}^{5/2-s})$ *and* $e(S)=gcd(S)$*. Here the definition of some functions remarked. For* $s\in\mathbb{C}$ *and* $S\in\mathcal{S}_2^\vee$*,*

$$F^{(1)}_{e(S)}(s)=\prod_{p|e(S)}F^{(1)}_p(e(S),s),\quad F^{(1)}_p(e(S),s)=\sum_{i=0}^{\mathrm{ord}_p e(S)}p^{-(s-1)i} \tag{2.31}$$

and

$$F_S^{(2)}(s) = \prod_{p|f} F_p^{(2)}(S,s), \tag{2.32}$$

$$F_p^{(2)}(S,s) = \sum_{i=0}^{\alpha_1} p^{i(2-s)} \left(\sum_{m=0}^{\alpha-i} p^{m(3-2s)} - \chi(p)p^{1-s} \sum_{j=0}^{\alpha-i-1} p^{j(3-2s)} \right).$$

Here we note that $-\delta(2S) = D(S)f^2$ *for the fundamental discriminant* $D(S)$ *and* $f \in \mathbb{Z}$, $\alpha_1 = \mathrm{ord}_p e(S)$, $\alpha = \mathrm{ord}_p f$ *and the Kronecker symbol* $\chi(\cdot) = \left(\frac{D(S)}{\cdot}\right)$. *We put* $L(s,\chi)$ *the Dirichlet's L-function of* χ *normalised by* $\pi^{-s/2}\Gamma(s/2)L(s,\chi) = L^*(s,\chi)$ *and* $\omega(y,S,\alpha,\beta)$ *the confluent hypergeometric function as the same notation in* [Sh].

The minimal parabolic Eisenstein series converges for Re $s > 3$*, but can be analytically continued on the whole of the complex plane as a meromorphic function of* s*. It satisfies a functional equation* s *to* $3-s$ *and has simple poles at* $s=1$ *and* 2.

Proof. In Proposition 3, if we calculate the Iwasawa A-part of the restriction for P_2 of $w_i n(x)m(a)$ for $i=1$ or 2 and for $a = \left(\begin{smallmatrix} a_1 & a_2 u \\ 0 & a_2 \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} 0 & a_2 \\ a_1 & a_2(u+b) \end{smallmatrix}\right)$, it is understand that all of it is not depend on $N_i(\mathbb{A})$ and explicitly given by $\left(\begin{smallmatrix} a_1 & a_2 u \\ 0 & a_2 \end{smallmatrix}\right)$. Then the local integrals (ii-1), (ii-2) and (iii) of Proposition 3 come down $\varepsilon(m(a),s)$ for $a = \left(\begin{smallmatrix} a_1 & a_2 u \\ 0 & a_2 \end{smallmatrix}\right)$ multiple of the integral of the flat section $\varphi^{(2)}_{\lambda+\rho}$ of the Siegel-Eisenstein series $E_2(g,s)$ and $\mathrm{e}(-\sigma(Sx))$ on $N_{w_i}(\mathbb{A})$. The Fourier expansion of $E_2(g,s)$ was considered by S. Mizumoto in [M] and Y. Hasegawa-T. Miyazaki in [HM]. Using these references and Shimura's explicit expression for the confluent hypergeometric functions in [Sh] and considering the shift of ρ, we get the formula (2.30). Especially, the local integrals of the second term and the third term of (2.21) in Proposition 3 are calculated by

$$\begin{aligned} &\left\{ \int_{N_{w_1(\mathbb{A})}} \varphi^{(2)}_{\lambda+\rho}\left(s, w_1 n(x) m\left(\left(\begin{smallmatrix} a1 & a_2 v \\ 0 & a_2 \end{smallmatrix}\right)\right)\right) \varepsilon\left(w_1 n(x) m\left(\left(\begin{smallmatrix} a_1 & a_2 v \\ 0 & a_2 \end{smallmatrix}\right)\right), s\right) dn(x) \right. \\ &\qquad + \sum_{l \in \mathbb{Q}} \int_{N_{w_1}(\mathbb{A})} \varphi^{(2)}_{\lambda+\rho}\left(s, w_1 n(x) m\left(\left(\begin{smallmatrix} 0 & a_2 \\ a_1 & a_2(u+b) \end{smallmatrix}\right)\right)\right) \\ &\qquad \left. \times \varepsilon\left(w_1 n(x) m\left(\left(\begin{smallmatrix} 0 & a_2 \\ a_1 & a_2(u+b) \end{smallmatrix}\right)\right), s\right) dn(x) \right\} \\ &= \sum_{\gamma \in B\backslash \mathrm{GL}(2)} \int_{N_{w_1}(\mathbb{A})} \varphi^{(2)}_{\lambda+\rho}\left(s, \gamma m\left(\left(\begin{smallmatrix} a_1 & a_2 v \\ 0 & a_2 \end{smallmatrix}\right)\right)\right) \varepsilon\left(\gamma m\left(\left(\begin{smallmatrix} a_1 & a_2 v \\ 0 & a_2 \end{smallmatrix}\right)\right), s\right) dn(x). \end{aligned}$$

Since the right hand side of the integral on $N_{w_1}(\mathbb{A})$ sets a flat section of the induced representation $\mathrm{Ind}_B^{\mathrm{GL}(2)}(|\cdot|_{\mathbb{A}}^{s-\frac{1}{2}} |\cdot|_{\mathbb{A}}^{\frac{5}{2}-s})$, so that if we take the summation of all $\gamma \in B\backslash\mathrm{GL}(2)$ is equal to the Eisenstein series on GL(2). In the well-known formula of the Fourier expansion of the Eisenstein series on GL(2), we add it to the shift of ρ.

The functional equation of the minimal parabolic Eisenstein series follows from the local functional equations such that $\xi(s) = \xi(1-s)$, $(-s)_n = (-1)^n(s-n+1)_n$, $F_b^{(1)}(s) = b^{1-s}F_b^{(1)}(1-s)$, $F_S^{(2)} = f^{3-2s}F_S^{(1)}(3-s)$, $\sigma_s(n) = |n|^s\sigma_{-s}(n)$, $L^*(s,\chi) =$

$| - \delta(2S)/f^2|^{\frac{1}{2}-s}L^*(1-s,\chi)$, $W_{\nu,\mu}(z) = W_{\nu,-\mu}(z)$ and $\omega(y,S;\alpha,\beta) = \omega(y,S;3-\alpha,3-\beta)$. If we exchange s to $3-s$ in (2.30), then the first term itself, the second term and the third term, the forth term and the fifth term, the sixth term and the seventh term, the eighth term and the ninth term respectively preserve the equality.

Since the function $\xi(s)$ has poles $-1/2$ at $s=0$ and 1 at $s=1$, then $E_2^*(g,s)$ has simple poles at $s=1$ and 2. $\square$

For this theorem, the Fourier expansion of the minimal parabolic Eisenstein series had written exactly, then the analytical properties of the Eisenstein series came to be found well.

3. Bump-Friedberg's zeta integrals and critical values of the exterior square L-functions

There is a fundamental problem of Langlands' theory of automorphic L-functions such that every general automorphic L-function initially defined as an Euler product in some half-plane, continues to a meromorphic function in all of $s \in \mathbb{C}$, with only finitely many poles, and a functional equation relating its values at s and $1-s$. It has been successfully attacked in general using two different methods. One of it is the explicit construction of zeta-integrals and the other one is the Langlands-Shahidi method using Eisenstein series and their Fourier coefficients.

According to the previous work of G. Harder in [H], he constructed the cohomology classes in the cohomology groups of arithmetic quotients and provided it with integral over suitable cycle. Then by summing it over the classes in the genus which is called the period integrals, the critical values of L-functions attached to algebraic Hecke characters were appeared.

In this section, to refer the way of [GRS], we calculate the $H = \mathrm{GL}(2) \times \mathrm{GL}(2)$-period integral of the residue of the Eisenstein series and it is shown clearly that the critical value of the exterior square L-function appear.

3-1. Eisenstein cohomology on arithmetic quotients of the Siegel upper half space of degree 2.

In the introduction of this paper, we had already reviewed the structures of cohomology groups of $\Gamma\backslash G/K = \Gamma\backslash X$ for an arithmetic torsion free subgroup $\Gamma \subset G(\mathbb{Z})$. Then we induct the result of J. Schwermer in [Sc], p. 254, about the Eisenstein cohomology classes on $\Gamma\backslash X$.

Proposition 4. *Let $E^*(g,s)$ be the minimal parabolic Eisenstein series of weight 6 which is defined in the previous section. The residue of $E^*(g,s)$ at $s=1$ is closed and harmonic and represents a non-trivial class in the Eisenstein cohomology of degree 3.*

Considering this proposition, we let calculate the period integrals of the residue of $E^*(g,s)$ at $s=1$ which is a cohomology class in $H^3(\Gamma\backslash\bar{X},\mathbb{C})$ and then obtain the critical values of the exterior square L-functions. Since the Fourier expansion of $E^*(g,s)$ was explicitly shown in Theorem 3 (II-2), then the residue of $E^*(g,s)$ at $s=1$ is fully-clarified by calculating the Laurent expansion around $s=1$ of zeta function, gamma function, the confluent hypergeometric functions and other special functions appearing in the Fourier expansion of the Eisenstein series.

Theorem 4. *The notations are same as in Theorem 3. For all $g = m\left(\left(\begin{smallmatrix} a_1 & a_2u \\ 0 & a_2 \end{smallmatrix}\right)\right) \in G$ and $s \in \mathbb{C}$, take $E^*(g,s)$ as the minimal parabolic Eisenstein series of weight*

$k \in 2\mathbb{Z}$. The residue of $E^*(g,s)$ at $s=1$ is explicitly given as following. For $k>0$,

(3.1)

$$\begin{aligned}
&\mathrm{Res}_{s=1}E^*(g,s) = \varepsilon(g,1)\cdot\delta(y)^{-\frac{k}{2}}v^{\frac{1}{2}}\Biggl\{2^{-k}\delta(y)\pi/6\\
&-2^{-k}\delta(y)^{\frac{1}{2}}\sum_{m=1}^{\infty}\sigma_{-1}(m)W_{-k,1/2}(4\pi mv)\mathrm{e}(m\tau)\\
&-2^2\pi^{-\frac{1}{2}}\sum_{\delta(S)=-\square<0}(-\delta(2S))^{-1}(-\delta(2Sy))^{\frac{1}{4}}F_S^{(2)}(1)\left((1-k)/2\right)_{k-1/2}\\
&\quad\times\omega\left(2\pi y,S;(1+k)/2,(1-k)/2\right)\mathrm{e}\left(\sigma(Sy)\right)\Biggr\}.
\end{aligned}$$

For $k=0$,

(3.2)

$$\begin{aligned}
&\mathrm{Res}_{s=1}E^*(g,s) = \varepsilon(g,1)(\delta(y)v)^{\frac{1}{2}}\Biggl\{\frac{1}{2}\left(\log(4\pi v/\delta(y))+\gamma\right)+2\log|\eta(\tau)|\\
&-2\sum_{\delta(S)=0,S\neq 0_2}\sigma_0(e(S))K_0\left(2\pi(|\sigma(Sy)|)\right)\mathrm{e}\left(\sigma(Sx)\right)\\
&-2\sum_{\delta(2S)=-\square<0}\sigma_0(e(S))K_0(2\pi\sqrt{\sigma(Sy)^2-\delta(2Sy)})\mathrm{e}\left(\sigma(Sx)\right)\Biggr\}.
\end{aligned}$$

Proof. Since for all $k\in 2\mathbb{Z}$, the Eisenstein series $\varepsilon^*(g,s)$ is entire at $s=1$, then it appears the value of $s=1$ in the part of the residue of the minimal parabolic Eisenstein series at $s=1$. On the other hand, the Siegel-Eisenstein series $E_2(g,s)$ has poles in the first term $\varepsilon_2(g,s)$, the second term and the last term for the Fourier expansion given in Theorem 3, (2.30) for $k>0$. In this case, the functions $\xi(s)$ and $\xi(2s-2)$ have a singularity. If the Dirichlet's character χ is trivial, i.e. $-\delta(2S)=\square$, $L^*(s-1,\chi)=(-\delta(2S))^{1/2}f^{-1}L^*(2-s,\chi)=\zeta(s)$ has a simple pole at $s=1$. We also count a zero of order 1 of functions $(s-1)_k$ or $((3-s-k)/2)_{k/2}$, (3.1) is obtained.

If $k=0$, the poles of $E_2(g,s)$ appear in $\varepsilon_2(g,s)$, the second term, the forth term, the fifth term and the last term. In this case, we remark that

$$W_{0,0}(4\pi|\sigma(Sy)|)=2|\sigma(Sy)|^{\frac{1}{2}}K_0(2\pi|\sigma(Sy)|)$$

and

$$\omega\left(2\pi y,S;1/2,1/2\right)=2^{-\frac{3}{2}}\pi^{\frac{1}{2}}(-\delta(Sy))^{\frac{1}{4}}K_0\left(2\pi\sqrt{\sigma(Sy)^2-\delta(2Sy)}\right).$$

where the relation between the hypergeometric functions, it says more correctly W-Whittaker function and the confluent hypergeometric function, and K-Bessel function. □

After this section, computing the period integrals for this residue, then we show that an exterior square L-function is included in the integrals.

3-2. Formulation of the period integrals.

Let H denotes the subgroup of G which composes fixed points of an involution θ of G defined over $\mathbb{Q}$. We called such an H a symmetric subgroup of G and the pair (G, H) a symmetric pair. In this article, we take H as $\mathrm{GL}(2) \times \mathrm{GL}(2)$ which embedded in G by

$$\left(\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}, \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}\right) \mapsto \begin{pmatrix} a_1 & & & a_2 \\ & b_1 & -b_2 & \\ & -b_3 & b_4 & \\ a_3 & & & a_4 \end{pmatrix}. \tag{3.3}$$

The sublattice $\Gamma^\vee = H_q(\Gamma\backslash\bar{X}, \mathbb{Z}) \subset H^{q-1}(\Gamma\backslash\bar{X}, \Omega^q(\Gamma\backslash\bar{X}))^\vee$ consists of linear forms on $H^{q-1}(\Gamma\backslash\bar{X}, \Omega^q(\Gamma\backslash\bar{X}))$ and these linear forms are natural inner product between q-cycles $H(\mathbb{Q})\backslash H(\mathbb{A}) \in \Omega_q(\Gamma\backslash\bar{X})$ and closed q-forms $f \in \Omega^q(\Gamma\backslash\bar{X})$ as follows.

$$\Big(H(\mathbb{Q})\backslash H(\mathbb{A}),\ f\Big) = \int_{H(\mathbb{Q})\backslash H(\mathbb{A})} f(h)dh. \tag{3.4}$$

The Stokes theorem says that the pairing is independent of the choice of representatives of the equivalence classes $H_q(\Gamma\backslash\bar{X}, \mathbb{C})$ and $H^q(\Gamma\backslash\bar{X}, \mathbb{C})$ and defines a pairing between them. We call the pairing a period integral. We want to give an explicit expression for that period integral.

3-3. Calculation of the period integrals of the Eisenstein cohomology classes.

If we refer to the calculation method of the article [GRS], applying the truncation operator to $E^*(g, s)$ and computing its integrals follow our main theorem.

Theorem 5. *Let (π, V_π) be a cuspidal representation of* $\mathrm{GL}(2)$. *We define $\Omega_{\varphi^{(2)}}$ the period attached to a flat section $\varphi^{(2)} \in V_\pi$ appearing in the minimal parabolic Eisenstein series $E^*(g, s)$. Then the integral over $H(\mathbb{Q})\backslash H(\mathbb{A})$ for $H =$* $\mathrm{GL}(2) \times \mathrm{GL}(2)$ *of the residue of the minimal parabolic Eisenstein series at $s = 1$ expressed as following.*

$$\Omega_{\varphi^{(2)}} \cdot \int_{H(\mathbb{Q})\backslash H(\mathbb{A})} \mathrm{Res}_{s=1} E^*(h, s)dh = L(1/2, \pi)L(1, \pi, \wedge^2). \tag{3.5}$$

Here $L(1/2, \pi)$ be the special value of the standard L-function of π.

Since we prove this main theorem, following result by D. Bump and S. Friedberg in [BF] such that the Rankin-Selberg integral which have been discovered involve Eisenstein series represents a product of two L-functions is useful.

Lemma 2. *Let $\varepsilon(g, s)$ be an Eisenstein series on* $\mathrm{GL}(2)$. *For all cusp form $\varphi \in V_\pi$ and $\alpha \in \mathbb{C}$, we have*

$$L(1/2, \pi)L(s, \pi, \wedge^2) = \int_{\mathrm{GL}(2,\mathbb{Q})\backslash \mathrm{GL}(2,\mathbb{A})} \varphi\left(\alpha, \begin{pmatrix} g & \\ & {}^t g^{-1} \end{pmatrix}\right) \varepsilon(g, s)dg. \tag{3.6}$$

We apply the truncation operator Λ^c for a real number $c > 1$ to the minimal parabolic Eisenstein series $E^*(g, s)$ such that

$$\begin{aligned} \Lambda^c E^*(g, s) &= E^*(g, s) - \sum_{\gamma \in P_2\backslash G} (\varphi(s, \gamma g) + M(s)\varphi(s, \gamma g))\chi_c(\gamma g) \\ &= \sum_{\gamma \in P_2\backslash G} \varphi(s, \gamma g)\chi^c(\gamma g) - \sum_{\gamma \in P_2\backslash G} M(s)\varphi(s, \gamma g)\chi_c(\gamma g), \end{aligned} \tag{3.7}$$

where the function in the summation is a constant term of $E^*(g,s)$, χ_c is the characteristic function on GL(2) such that for the all real number c, $\chi_c(g)$ satisfies

$$\chi_c(g) = \begin{cases} 1, & \delta_B(g) > c, \\ 0, & \delta_B(g) \leq c, \end{cases}$$

where δ_B is the modulus character of B. Naturally, $M(s)$ is the standard intertwining operators. Since its integrals are explicitly calculated by Ginzburg, Rallis and Soudry in [GRS] of Proposition 4 to describe the cosets appearing the unfolding of its integrals and summations respectively, taking the residue of the result and c maps to ∞, they obtain the following lemma.

Lemma 3. *If* Re s *is sufficiently large, then the following formulae are valid with a certain choice of measures. For* $\alpha \in \mathbb{C}$, *we have*

(3.8)
$$\int_{H(\mathbb{Q})\backslash H(\mathbb{A})} \mathrm{Res}_{s=1} E^*(h,s)dh = \int_{K_H} \int_{\mathrm{GL}(2,\mathbb{Q})\backslash \mathrm{GL}(2,\mathbb{A})} \varphi\left(\alpha, \begin{pmatrix} g & \\ & {}^tg^{-1} \end{pmatrix} k\right) dgdk,$$

where K_H *is the maximal compact subgroup of* H.

Using the above two lemmas, we prove our main theorem.

Proof. Since the minimal parabolic subgroup $E^*(g,s)$ is decomposed as in Theorem 1, then the integrand is decomposed by

$$\varphi\left(\alpha, \begin{pmatrix} g & \\ & {}^tg^{-1} \end{pmatrix} k\right) = \varphi^{(2)}\left(\alpha, \begin{pmatrix} g & \\ & {}^tg \end{pmatrix} k\right) \varepsilon\left(\begin{pmatrix} g & \\ & {}^tg \end{pmatrix} k, 1\right).$$

The property of $\varphi^{(2)}$ says that it can be separated by $\varphi^{(2)}\left(\alpha, \begin{pmatrix} g & \\ & {}^tg \end{pmatrix}\right) \varphi^{(2)}(\alpha,k)$ and the Eisenstein series on GL(2) is not depend on $k \in K$. Then the integral in Lemma 2 is equal to

$$\begin{aligned} &\int_{H(\mathbb{Q})\backslash H(\mathbb{A})} \mathrm{Res}_{s=1} E^*(h,s)dh \\ &= \Omega^{-1}_{\varphi^{(2)}} \cdot \int_{\mathrm{GL}(2,\mathbb{Q})\backslash \mathrm{GL}(2,\mathbb{A})} \varphi^{(2)}\left(\alpha, \left(\begin{smallmatrix} g & \\ & {}^tg \end{smallmatrix}\right)\right) \varepsilon\left(\left(\begin{smallmatrix} g & \\ & {}^tg \end{smallmatrix}\right) k, 1\right), \end{aligned}$$

where $\Omega^{-1}_{\varphi^{(2)}}$ means the period attached to $\varphi^{(2)}$. Since this integral is exactly the special value of the Rankin-Selberg integral at $s = 1$ which is found by Bump and Friedberg in Lemma 2 up to the multiple by $\Omega_{\varphi^{(2)}}$, we obtain our main theorem. □

Since the analytics of $L(1,\pi,\wedge^2)$ follows from that of $\mathrm{Res}_{s=1}E^*(g,s)$ referring to our main theorem, then it becomes now easily to prove the Deligne's conjecture on $L(1,\pi,\wedge^2)$.

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