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Some regularity results for a certain class of de Rham's functional equations and stationary measures

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Abstract

We consider a certain class of de Rham's functional equations. We consider Hausdorff dimension of the measure whose distribution function is the solution. We give a necessary and sufficient condition for singularity. We also show that they have a relationship with stationary measures.

1 Introduction

de Rham [2] considered the following functional equation.

$$f(x) = \begin{cases} F_0(f(2x)) & 0 \le x \le 1/2 \\ F_1(f(2x-1)) & 1/2 \le x \le 1. \end{cases} \tag{1.1}$$

He showed that there exists a unique, continuous and strictly increasing solution f of (1.1), if F_0 and F_1 are strictly increasing contractions on $[0,1]$ such that $0 = F_0(0) < F_0(1) = F_1(0) < F_1(1) = 1$.

Let us denote $\Phi(A; z) = \dfrac{az+b}{cz+d}$ for a 2×2 real matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $z \in \mathbb{R}$.

Throughout this paper, we only consider the equation (1.1) for $F_i(x) = \Phi(A_i; x)$, $x \in [0,1]$, $i = 0,1$, such that 2×2 real matrices $A_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}$, $i = 0,1$, satisfy the following conditions (A1) - (A3).

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(A1) $0 = b_0 < \dfrac{a_0 + b_0}{c_0 + d_0} = \dfrac{b_1}{d_1} < \dfrac{a_1 + b_1}{c_1 + d_1} = 1$.
(A2) $a_i d_i - b_i c_i > 0$, $i = 0, 1$.
(A3) $(a_i d_i - b_i c_i)^{1/2} < \min\{d_i, c_i + d_i\}$, $i = 0, 1$.

The conditions (A1) - (A3) guarantee that $F_i := \Phi(A_i; \cdot)$, $i = 0, 1$, satisfy de Rham's conditions. Let us denote μ_f be the probability measure such that f is the distribution function of μ_f.

Let $\alpha = \min\{0, c_0/(d_0 - a_0), c_1/b_1\}$, $\beta = \max\{0, c_0/(d_0 - a_0), c_1/b_1\}$ and $\gamma = 1/\Phi(A_0; 1) = d_0/(a_0 + c_0) = d_1/b_1 > 1$. Let $p_0(x) = (x+1)/(x+\gamma)$ and $p_1(x) = 1 - p_0(x)$ for $x > -\gamma$. Let $s(p) = -p \log p - (1-p)\log(1-p)$ for $p \in [0, 1]$. We denote the s-dimensional Hausdorff measure, $s \in (0, 1]$, of $E \subset \mathbb{R}$ by $H_s(E)$ and the Hausdorff dimension of E by $\dim_H(E)$.

The followings are main results in this paper.

Theorem 1.1. (1) *There exists a Borel set K_0 such that $\mu_f(K_0) = 1$ and* $\dim_H(K_0) \leq \max\{s(p_0(y)); y \in [\alpha, \beta]\}/\log 2$.
(2) *We have that $\mu_f(K) = 0$ for any Borel set K with* $\dim_H(K) < \min\{s(p_0(y)); y \in [\alpha, \beta]\}/\log 2$.

Theorem 1.2. (1) *If both (i) $(c_0 + d_0 - 2a_0)(d_0 - a_0) = a_0 c_0$, and (ii) $(a_1 - 2c_1)(d_1 - 2b_1) = b_1 c_1$ are satisfied, then $\mu_f(dx) = (1 + 2c_0)/(-2c_0 x + 1 + 2c_0)^2 dx$. In particular, μ_f is absolutely continuous.*
(2) *If either (i) or (ii) fails, then there exists a Borel set K_1 such that $\mu_f(K_1) = 1$ and $\dim_H(K_1) < 1$. In particular, μ_f is singular.*

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2 Some Lemmas

First, we introduce some notation.

Let $X_n : [0, 1) \to \{0, 1\}$, $n \geq 1$ be given by $X_n(x) = [2^n x] - 2[2^{n-1}x]$, $x \in [0, 1)$. Let $\rho_n(i_1, \dots, i_n) = \mu_f(\{X_j = i_j, 1 \leq j \leq n\})$ for $n \geq 1$, $i_1, \dots, i_n \in \{0, 1\}$ and $R_n(x) = \rho_n(X_1(x), \dots, X_n(x))$ for $n \geq 1$ and $x \in [0, 1)$. Let
$I_n(x) = [\sum_{i=1}^n 2^{-j} X_j(x), \sum_{i=1}^n 2^{-j} X_j(x) + 2^{-n}) = [2^{-n}[2^n x], 2^{-n}([2^n x] + 1))$. Then, $x \in I_n(x)$, $x \in [0, 1)$, and, $X_n(y) = X_n(x)$ and $I_n(y) = I_n(x)$ for $y \in I_n(x)$. We have that $R_n(x) = \mu_f(\{X_j = X_j(x), 1 \leq j \leq n\}) = \mu_f(I_n(x))$. Let

$$\begin{pmatrix} p_n(x) & q_n(x) \\ r_n(x) & s_n(x) \end{pmatrix} = A_{X_1(x)} \cdots A_{X_n(x)}, x \in [0, 1).$$

Lemma 2.1. *Let $n \geq 1$ and $i_1, \dots, i_n \in \{0,1\}$. Then we have the followings.*
(1) $f\left(\sum_{i=1}^{n} 2^{-j} i_j\right) = \Phi(A_{i_1} \cdots A_{i_n}; 0)$ *and* $f\left(\sum_{i=1}^{n} 2^{-j} i_j + 2^{-n}\right) = \Phi(A_{i_1} \cdots A_{i_n}; 1)$.
(2) $R_{n+1}(x)/R_n(x) = p_{X_{n+1}(x)}(r_n(x)/s_n(x))$.

Proof. (1) We show the assertion by induction in n. If $n = 1$, it is a direct consequence of the equation (1.1). Assume that the assertion is valid for $n = m$.

Suppose that $i_1 = 0$. Since $f(y) = \Phi(A_0; f(2y))$ for $y \in [0, 1/2]$, we have that $f\left(\sum_{j=1}^{m+1} 2^{-j} i_j\right) = f\left(\sum_{j=2}^{m+1} 2^{-j} i_j\right) = \Phi\left(A_0; f\left(\sum_{j=1}^{m} 2^{-j} i_{j+1}\right)\right)$. By the assumption, we have that $f\left(\sum_{j=1}^{m} 2^{-j} i_{j+1}\right) = \Phi\left(A_{i_2} \cdots A_{i_{m+1}}; 0\right)$, and then
$\Phi\left(A_0; f\left(\sum_{j=1}^{m} 2^{-j} i_{j+1}\right)\right) = \Phi\left(A_0; \Phi\left(A_{i_2} \cdots A_{i_{m+1}}; 0\right)\right) = \Phi\left(A_{i_1} \cdots A_{i_{m+1}}; 0\right)$.
Similarly we have that $f\left(\sum_{j=1}^{m+1} 2^{-j} i_j + 2^{-m-1}\right) = \Phi\left(A_{i_1} \cdots A_{i_{m+1}}; 1\right)$.

Suppose that $i_1 = 1$. Since $f(y) = \Phi(A_1; f(2y-1))$ for $y \in [1/2, 1]$, we have that $f\left(\sum_{j=1}^{m+1} 2^{-j} i_j\right) = f\left(1/2 + \sum_{j=2}^{m+1} 2^{-j} i_j\right) = \Phi\left(A_1; f\left(\sum_{j=1}^{m} 2^{-j} i_{j+1}\right)\right)$.
By the assumption, we have that
$f\left(\sum_{j=1}^{m} 2^{-j} i_{j+1}\right) = \Phi\left(A_{i_2} \cdots A_{i_{m+1}}; 0\right)$ and then
$\Phi\left(A_1; f\left(\sum_{j=1}^{m} 2^{-j} i_{j+1}\right)\right) = \Phi\left(A_1; \Phi\left(A_{i_2} \cdots A_{i_{m+1}}; 0\right)\right) = \Phi\left(A_{i_1} \cdots A_{i_{m+1}}; 0\right)$.
Similarly we have that $f\left(\sum_{j=1}^{m+1} 2^{-j} i_j + 2^{-m-1}\right) = \Phi\left(A_{i_1} \cdots A_{i_{m+1}}; 1\right)$.

So the assertion is valid for $n = m+1$. Thus we obtain the assertion (1).

(2) By the assertion (1), we have that

$$\begin{aligned} R_k(x) = \mu_f(I_k(x)) &= \mu_f\left(\left[\sum_{i=1}^{k} \frac{X_j(x)}{2^j}, \sum_{i=1}^{k} \frac{X_j(x)}{2^j} + \frac{1}{2^k}\right)\right) \\ &= \Phi(A_{X_1(x)} \cdots A_{X_k(x)}; 1) - \Phi(A_{X_1(x)} \cdots A_{X_k(x)}; 0) \\ &= \frac{p_k(x)s_k(x) - q_k(x)r_k(x)}{s_k(x)(r_k(x) + s_k(x))}. \end{aligned}$$

We have that

$$\frac{R_{n+1}(x)}{R_n(x)} = \det A_{X_{n+1}(x)} \times \frac{s_n(x)\left(r_n(x) + s_n(x)\right)}{s_{n+1}(x)\left(r_{n+1}(x) + s_{n+1}(x)\right)}$$

$$= \frac{(\det A_{X_{n+1}(x)}) s_n(x)}{b_{X_{n+1}(x)} r_n(x) + d_{X_{n+1}(x)} s_n(x)} \times \frac{r_n(x) + s_n(x)}{\left(a_{X_{n+1}(x)} + b_{X_{n+1}(x)}\right) r_n(x) + \left(c_{X_{n+1}(x)} + d_{X_{n+1}(x)}\right) s_n(x)}.$$

If $X_{n+1}(x) = 0$, then, by noting that $b_0 = 0$, we have that

$$\frac{R_{n+1}(x)}{R_n(x)} = a_0 d_0 \times \frac{1}{d_0} \times \frac{r_n(x) + s_n(x)}{a_0\left(r_n(x) + \gamma s_n(x)\right)} = p_0\left(\frac{r_n(x)}{s_n(x)}\right).$$

If $X_{n+1}(x) = 1$, then, by noting that $a_1 + b_1 = c_1 + d_1$, we have that

$$\begin{aligned}\frac{R_{n+1}(x)}{R_n(x)} &= (a_1 d_1 - b_1 c_1) \times \frac{s_n(x)}{b_1 \left(r_n(x) + \gamma s_n(x)\right)} \times \frac{1}{a_1 + b_1} \\ &= (d_1 - b_1) \frac{s_n(x)}{b_1(r_n(x) + \gamma s_n(x))} = p_1 \left(\frac{r_n(x)}{s_n(x)} \right).\end{aligned}$$

Thus we obtain the assertion (2). □

Now we state some properties of $\Phi({}^t A_i; \cdot)$, $i = 0, 1$.

We remark that $\Phi({}^t A_0; \cdot)$ (resp. $\Phi({}^t A_1; \cdot)$) is well-defined and continuous on $\mathbb{R}$ (resp. $(-\gamma, \infty)$).

Lemma 2.2. (1) *$d_0 > a_0 > 0$, $b_1 + c_1 > 0$ and $\alpha > -1$.*
(2) *$\Phi({}^t A_0; z) = z$ if and only if $z = c_0/(d_0 - a_0)$.*
(3) *$\Phi({}^t A_1; z) = z$ if and only if $z = -1$ or $z = c_1/b_1$.*

Proof. (1) By (A2) and (A3), we have that $d_0 > 0$, and then $a_0 > 0$. By (A3) and (A1), we have that $0 < (a_0 d_0)^{1/2} = (a_0 d_0 - b_0 c_0)^{1/2} < d_0$ and then $0 < a_0 < d_0$.

By (A1), we have that $a_1 + b_1 = c_1 + d_1$ and then $a_1 d_1 - b_1 c_1 = (c_1 + d_1)(d_1 - b_1)$. By (A2) and (A3), we have that $c_1 + d_1 > 0$, and then $d_1 - b_1 > 0$. By (A3), we have that $0 < (c_1 + d_1)^{1/2}(d_1 - b_1)^{1/2} < c_1 + d_1$. Hence we have that $d_1 - b_1 < c_1 + d_1$, and then $b_1 + c_1 > 0$.

By (A2) and (A3), we have that $d_1 > 0$. By (A1), we have that $b_1 > 0$. Since $b_1 + c_1 > 0$, we see that $c_1/b_1 > -1$. Then, we have that $c_0/(d_0 - a_0) > -1$ by noting (A1) and $a_0 < d_0$. Now we have that $a = \min\{0, c_0/(d_0 - a_0), c_1/b_1\} > -1$.

(2) Since $b_0 = 0$, we have that $\Phi({}^t A_0; z) - z = -(d_0 - a_0)z/d_0 + c_0/d_0$. Since $d_0 > a_0$, we see that $\Phi({}^t A_0; z) = z$ if and only if $z = c_0/(d_0 - a_0)$.

(3) Since

$$\begin{aligned}\Phi({}^t A_1; z) - z &= \frac{-b_1 z^2 - (d_1 - a_1)z + c_1}{b_1 z + d_1} \\ &= \frac{(-b_1 z + c_1)(z + 1)}{b_1 z + d_1} = -\frac{(z + 1)(z - c_1/b_1)}{z + \gamma},\end{aligned}$$

we see that $\Phi({}^t A_1; z) = z$ if and only if $z = -1$ or $z = c_1/b_1$. □

Let $\mathcal{F}_n = \sigma(X_1, \dots, X_n)$, $n \ge 1$. Let $L_n = \sum_{i=1}^n E^{\mu_f}[-\log(R_i/R_{i-1})|\mathcal{F}_{i-1}]$ and $M_n = -\log R_n - L_n$, $n \ge 1$. Then we have the following.

Lemma 2.3. (1) *$L_{n+1}(x) - L_n(x) = s(p_0(r_n(x)/s_n(x)))$ for μ_f-a.s.$x \in [0, 1)$.*
(2) *$M_n/n \to 0$, $(n \to \infty)$ for μ_f-a.s.*

Proof. (1) It is sufficient to show that for any $x \in [0,1)$,

$$\int_{I_n(x)} s\left(p_0\left(\frac{r_n(y)}{s_n(y)}\right)\right)\mu_f(dy) = \int_{I_n(x)} -\log\left(\frac{R_{n+1}(y)}{R_n(y)}\right)\mu_f(dy).$$

Since $r_n(y)/s_n(y) = r_n(x)/s_n(x)$ for $y \in I_n(x)$, we see that

$$\int_{I_n(x)} s\left(p_0\left(\frac{r_n(y)}{s_n(y)}\right)\right)\mu_f(dy) = \mu_f(I_n(x))s\left(p_0\left(\frac{r_n(x)}{s_n(x)}\right)\right).$$

By Lemma 2.1(2), we see that $-\log\left(\mu_f(I_{n+1}(y))/\mu_f(I_n(y))\right) = -\log\left(R_{n+1}(y)/R_n(y)\right) = -\log p_{X_{n+1}(y)}\left(r_n(y)/s_n(y)\right)$ and

$$\int_{I_n(x)} -\log\left(\frac{R_{n+1}(y)}{R_n(y)}\right)\mu_f(dy) = \int_{I_n(x)} -\log\left(p_{X_{n+1}(y)}\left(\frac{r_n(y)}{s_n(y)}\right)\right)\mu_f(dy)$$

$$= -\mu_f\left(I_n(x)\cap\{X_{n+1}=0\}\right)\log p_0\left(\frac{r_n(x)}{s_n(x)}\right) - \mu_f\left(I_n(x)\cap\{X_{n+1}=1\}\right)\log p_1\left(\frac{r_n(x)}{s_n(x)}\right)$$

$$= \mu_f\left(I_n(x)\right)s\left(p_0(r_n(x)/s_n(x))\right),$$

which implies our assertion.

(2) By noting Jensen's inequality, we have that

$$\begin{aligned} E^{\mu_f}\left[(M_k - M_{k-1})^2\right] &\le 2\left(E^{\mu_f}\left[(-\log R_k + \log R_{k-1})^2\right] + E^{\mu_f}\left[(L_k - L_{k-1})^2\right]\right) \\ &\le 4E^{\mu_f}\left[(-\log R_k + \log R_{k-1})^2\right]. \end{aligned}$$

Let $C_0 = \sup\{x(\log x)^2 + (1-x)(\log(1-x))^2 : x \in [0,1]\} < +\infty$. We will show that $E^{\mu_f}\left[(\log(R_{n+1}/R_n))^2\right] \le C_0$ for any $n \ge 1$.

Let $\tau(p) = p(\log p)^2 + (1-p)(\log(1-p))^2$ for $p \in [0,1]$. We remark that $\tau(p) = \tau(1-p)$. Then we have that

$$E^{\mu_f}\left[(-\log R_n + \log R_{n-1})^2\right] = \sum_{k=0}^{2^n-1}\mu_f\left(I_n\left(\frac{k}{2^n}\right)\right)\left(\log\frac{R_n(k/2^n)}{R_{n-1}(k/2^n)}\right)^2$$

$$= \sum_{k=0}^{2^{n-1}-1}\left\{\mu_f\left(I_n\left(\frac{2k}{2^n}\right)\right)\left(\log\frac{R_n(2k/2^n)}{R_{n-1}(2k/2^n)}\right)^2 + \mu_f\left(I_n\left(\frac{2k+1}{2^n}\right)\right)\left(\log\frac{R_n(2k+1/2^n)}{R_{n-1}(2k+1/2^n)}\right)^2\right\}.$$

By noting that $R_{n-1}(2k/2^n) = R_{n-1}(2k+1/2^n) = R_{n-1}(k/2^{n-1})$, $\mu_f\left(I_n(2k/2^n)\right) = R_n(2k/2^n)$ and $\mu_f\left(I_n(2k+1/2^n)\right) = R_n(2k+1/2^n)$, we have that

$$E^{\mu_f}\left[\left(\log\frac{R_n}{R_{n-1}}\right)^2\right] = \sum_{k=0}^{2^{n-1}-1} R_{n-1}\left(\frac{k}{2^{n-1}}\right)\tau\left(\frac{R_n(k/2^{n-1})}{R_{n-1}(k/2^{n-1})}\right) \le \sum_{k=0}^{2^{n-1}} R_{n-1}\left(\frac{k}{2^{n-1}}\right)C_0 \le C_0.$$

Thus we have that $\sup_{k\geq 1} E^{\mu_f}[(M_k - M_{k-1})^2] \leq 4C_0 < +\infty$. Since $\{M_n\}$ is an $\{\mathcal{F}_n\}$-martingale, $\{M_n^2\}$ is an $\{\mathcal{F}_n\}$-submartingale. Noting that $M_0 = 0$, we have that $E^{\mu_f}[M_n^2] = \sum_{k=1}^n E^{\mu_f}[(M_k - M_{k-1})^2]$.

By Doob's submartingale inequality, we have that

$$\mu_f\left(\max_{1\leq k\leq 2^l} M_k^2 \geq \epsilon 4^l\right) \leq \frac{E^{\mu_f}[M_{2^l}^2]}{\epsilon 4^l} \leq \frac{4C_0}{\epsilon 2^l},\ l \geq 1,\ \epsilon > 0.$$

Now we have that for μ_f-a.s.x, there exists $m = m(x) \in \mathbb{N}$ such that $\max_{1\leq k\leq 2^l}(M_k(x)/2^l)^2 \leq \epsilon$, $l \geq m$, and then, $(M_n(x)/n)^2 \leq 4\epsilon$, $n \geq 2^m$. Then we see that $\limsup_{n\to\infty}(M_n/n)^2 \leq \epsilon$, μ_f-a.s., which implies our assertion. $\square$

Lemma 2.4. (1) *Suppose that* $\limsup_{n\to+\infty}(-\log R_n)/n \leq \theta_1$ *for a constant* θ_1*, then there exists a Borel set* K_0 *such that* $\mu_f(K_0) = 1$ *and* $\dim_H(K_0) \leq \theta_1/\log 2$*.*
(2) *Suppose that* $\liminf_{n\to+\infty}(-\log R_n)/n \geq \theta_2$ *for a constant* θ_2*, then we have that* $\mu_f(K) = 0$ *for any Borel set* K *with* $\dim_H(K) < \theta_2/\log 2$*.*

Proof. We denote the diameter of a set $G \subset \mathbb{R}$ by $\mathrm{diam}(G)$.

(1) Let $Y_{\epsilon,n} = \bigcap_{k\geq n}\{(-\log R_k)/k \leq \theta_1 + \epsilon\}$. Then we have that $\mu_f\left(\bigcup_{n\geq 1} Y_{\epsilon,n}\right) = 1$. Let $\mathcal{A}_{\epsilon,k}$ be the set of $I_k(x)$, $x \in [0,1)$, such that $R_k(x) \geq \exp(-k(\theta_1+\epsilon))$. Then, for any $k \geq n$, $\{I_k(x) \in \mathcal{A}_{\epsilon,k} : x \in Y_{\epsilon,n}\}$ is a 2^{-k}-covering of $Y_{\epsilon,n}$.

Since $\mu_f([0,1)) = 1$, we see that $\sharp(\mathcal{A}_{\epsilon,k})\exp(-k(\theta_1+\epsilon)) \leq 1$. Then

$$\sum_{I\in\mathcal{A}_{\epsilon,k}} \mathrm{diam}(I)^{(\theta_1+2\epsilon)/\log 2} = \sharp(\mathcal{A}_{\epsilon,k})\exp\left(-k(\theta_1+2\epsilon)\right) \leq \exp(-k\epsilon).$$

By letting $k \to +\infty$, we see $H_{(\theta_1+2\epsilon)/\log 2}(Y_{\epsilon,n}) = 0$.

Let $K_0 = \bigcap_{k\geq 1}\bigcup_{n\geq 1} Y_{1/k,n}$. Then, we have that $\mu_f(K_0) = 1$ and $H_{(\theta_1+2\epsilon)/\log 2}(K_0) = 0$ for any $\epsilon > 0$. Hence $\dim_H(K_0) \leq \theta_1/\log 2$.

(2) Let K be a Borel set such that $\dim_H(K) < \theta_2/\log 2$. Then, there exists $\epsilon > 0$ such that $H_{(\theta_2-\epsilon)/\log 2}(K) = 0$. Then, for any $n \geq 1$ and $\delta > 0$, there exist intervals $\{U(n,l)\}_{l=1}^\infty$ on $[0,1)$ such that $K \subset \bigcup_{l\geq 1} U(n,l)$ and $\mathrm{diam}(U(n,l)) < 2^{-n}$ for $l \geq 1$ and $\sum_{l\geq 1}\mathrm{diam}(U(n,l))^{(\theta_2-\epsilon)/\log 2} \leq \delta$. For each $l \geq 1$, let $k(n,l) > n$ be the integer such that $2^{-k(n,l)} \leq \mathrm{diam}(U(n,l)) < 2^{-(k(n,l)-1)}$.

Let $Z_{\epsilon,n} = \bigcap_{k\geq n}\{(-\log R_k)/k \geq \theta_2 - \epsilon\}$. Then we have that $\lim_{n\to\infty}\mu_f(Z_{\epsilon,n}) = \mu_f\left(\bigcup_{n\geq 1} Z_{\epsilon,n}\right) = 1$, and,

$$\mu_f\left(I_{k(n,l)}(y)\right) = R_{k(n,l)}(y) \leq \exp\left(-k(n,l)(\theta_2-\epsilon)\right) \leq \mathrm{diam}(U(n,l))^{(\theta_2-\epsilon)/\log 2},$$

for $y \in Z_{\epsilon,n}$ and $l \geq 1$.

Since $\mathrm{diam}(I_{k(n,l)}(x)) = 2^{-k(n,l)}$ and $\mathrm{diam}(U(n,l)) < 2^{-(k(n,l)-1)}$, we see that $\sharp \left\{ I_{k(n,l)}(x); I_{k(n,l)}(x) \cap U(n,l) \neq \emptyset \right\} \leq 3$ and that $\mu_f \left(K \cap Z_{\epsilon,n} \cap U(n,l) \right) \leq 3\mathrm{diam}(U(n,l))^{(\theta_2-\epsilon)/\log 2}$.

Noting that $K \subset \bigcup_{l \geq 1} U(n,l)$, we see that

$$\mu_f(K \cap Z_{\epsilon,n}) \leq \sum_{l\geq 1} \mu_f(K \cap Z_{\epsilon,n} \cap U(n,l)) \leq 3 \sum_{l \geq 1} \mathrm{diam}(U(n,l))^{(\theta_2-\epsilon)/\log 2} \leq 3\delta.$$

Since δ is taken arbitrarily, we see that $\mu_f(K \cap Z_{\epsilon,n}) = 0$. Recalling $\mu_f \left(\bigcup_{n\geq 1} Z_{\epsilon,n} \right) = 1$, we see that $\mu_f(K) = 0$. $\square$

3 Proofs of main Theorems

Lemma 3.1. *Let $n \geq 1$ and $i_1, \ldots, i_n \in \{0,1\}$. Then,* $\alpha \leq \Phi({}^tA_{i_n} \cdots {}^tA_{i_1}; \alpha) \leq \Phi({}^tA_{i_n} \cdots {}^tA_{i_1}; \beta) \leq \beta$.
In particular, $r_n(x)/s_n(x) \in [\alpha, \beta]$ for $n \geq 1$ and $x \in [0,1)$.

Proof. By noting Lemma 2.2, we have that $\Phi({}^tA_0; z) - z = -(d_0 - a_0)z/d_0 + c_0/d_0$ and $\Phi({}^tA_1; z) - z = -(z+1)(z - c_1/b_1)/(z+\gamma)$. We remark that $\alpha > -1 > -\gamma$. Since $\alpha \leq c_0/(d_0 - a_0), c_1/b_1 \leq \beta$, we see that $\alpha \leq \Phi({}^tA_i; \alpha) \leq \Phi({}^tA_i; \beta) \leq \beta$ for $i = 0, 1$.

Since $\Phi({}^tA_0; \cdot)$ and $\Phi({}^tA_1; \cdot)$ are increasing, we obtain the assertion by induction in n.

We have that $\alpha \leq 0 \leq \beta$ by the definition of α and β. Since $r_n(x)/s_n(x) = \Phi({}^tA_{X_n(x)} \cdots {}^tA_{X_1(x)}; 0)$, we see that $r_n(x)/s_n(x) \in [\alpha, \beta]$. $\square$

Now we show Theorem 1.1.

By noting Lemma 2.3 and Lemma 3.1, we see that for μ_f-a.s.,

$$\limsup_{n\to+\infty} \frac{-\log R_n}{n} = \limsup_{n\to\infty} \frac{L_n}{n} = \limsup_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} s\left(p_0\left(\frac{r_n(x)}{s_n(x)}\right)\right) \leq \max\left\{ s(p_0(y)); y \in [\alpha,\beta] \right\}$$

, and,

$$\liminf_{n\to+\infty} \frac{-\log R_n}{n} = \liminf_{n\to\infty} \frac{L_n}{n} = \liminf_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} s\left(p_0\left(\frac{r_n(x)}{s_n(x)}\right)\right) \geq \min\left\{ s(p_0(y)); y \in [\alpha,\beta] \right\}.$$

Let $\theta_1 = \max\{s(p_0(y)); y \in [\alpha,\beta]\}$ and $\theta_2 = \min\{s(p_0(y)); y \in [\alpha,\beta]\}$. Then, by Lemma 2.4(1) (resp. (2)), we obtain the assertion (1) (resp. (2)).

These complete the proof of Theorem 1.1.

Lemma 3.2. *Let $\mathbb{N}_i(x) = \{n \in \mathbb{N} : X_n(x) = i\}$ for $x \in [0,1)$, $i = 0,1$. Then,*

$$\liminf_{N\to\infty} \frac{|\mathbb{N}_0(x) \cap \{1,\dots,N\}|}{N} \geq p_0(\alpha) > 0,\ \mu_f\text{-a.s.}x.$$

Proof. Let $\zeta_N(x) = |\mathbb{N}_0(x) \cap \{1,\dots,N\}|$. Then, $\zeta_N(x) = \sum_{n=1}^N 1_{\{0\}}(X_n(x))$. Let $M_n = \sum_{i=1}^n \left(1_{\{0\}}(X_n) - p_0(\alpha)\right)$. Then, $\{M_n\}$ is an $\{\mathcal{F}_n\}$-submartingale because

$$E^{\mu_f}[M_{n+1} - M_n|\mathcal{F}_n](x) = E^{\mu_f}[1_{\{0\}}(X_{n+1}) - p_0(\alpha)|\mathcal{F}_n](x) = p_0\left(\frac{r_n(x)}{s_n(x)}\right) - p_0(\alpha) \geq 0.$$

We remark that $|M_{n+1} - M_n| = |1_{\{0\}}(X_{n+1}) - p_0(\alpha)| \leq 1 + p_0(\alpha)$ for μ_f-a.s.. By Azuma's inequality [1], we see that for $N \in \mathbb{N}$ and $0 < c < 1$,

$$\mu_f(\zeta_N < Ncp_0(\alpha)) = \mu_f(M_N < -N(1-c)p_0(\alpha)) \leq \exp\left(-\frac{N(1-c)^2 p_0(\alpha)^2}{2(1+p_0(\alpha))^2}\right).$$

Hence, for any $0 < c < 1$, $\liminf_{N\to\infty} \zeta_N/N \geq cp_0(\alpha)$ for μ_f-a.s.. Thus we obtain the assertion. $\square$

Lemma 3.3. *We assume that the condition (i) in Theorem 1.2 fails. Then,*
(1) *There exists $\epsilon_0 \in (0, 2(\gamma-1))$ such that for any $z \in \mathbb{R}$ with $|z - (\gamma-2)| \leq \epsilon_0$,*
$|\Phi({}^tA_0; z) - (\gamma-2)| > \epsilon_0$.

Let $A(x) = \{n \in \mathbb{N} : |r_n(x)/s_n(x) - (\gamma-2)| \leq \epsilon_0\}$, $B(x) = \mathbb{N} \setminus A(x)$, $C(x) = \{n \in A(x) : n-1 \in B(x)\}$ and $D(x) = B(x) \cup C(x)$. Then we have the followings.
(2) *$\mathbb{N}_0(x) \subset D(x)$ for $x \in [0,1)$.*
(3) *$\liminf_{N\to\infty} |B(x) \cap \{1,\dots,N\}|/N \geq p_0(\alpha)/2$, μ_f-a.s.x.*
(4) *Let $e_0 = s(p_0(\gamma - 2 + \epsilon_0)) < \log 2$. Then,*

$$\limsup_{N\to\infty} \frac{1}{N} \sum_{n=1}^N s\left(p_0\left(\frac{r_n(x)}{s_n(x)}\right)\right) \leq \log 2 - \frac{(\log 2 - e_0)p_0(\alpha)}{2},\ \mu_f\text{-a.s.}x.$$

Proof. (1) This is a direct consequence of the assumption that the condition (i) in Theorem 1.2 fails, that is, $\Phi({}^tA_0; \gamma-2) \neq \gamma-2$.

(2) It is sufficient to show that $\mathbb{N} \setminus D(x) \subset \mathbb{N}_1(x)$. We see that $\mathbb{N} \setminus D(x) = A(x) \cap (\mathbb{N} \setminus C(x)) = \{n \in A(x) : n-1 \in A(x)\}$. We assume that there exists $n \in \mathbb{N} \setminus D(x)$ such that $n \in \mathbb{N}_0(x)$. Since $n-1 \in A(x)$, we have that $|r_{n-1}(x)/s_{n-1}(x) - (\gamma-2)| \leq \epsilon_0$. Since $n \in \mathbb{N}_0(x)$, $r_n(x)/s_n(x) = \Phi({}^tA_0; r_{n-1}(x)/s_{n-1}(x))$. By the assertion (1), we see that $|r_n(x)/s_n(x) - (\gamma-2)| > \epsilon_0$. But this is contradict to $n \in A(x)$.

(3) By the assertion (2), we see that $|\mathbb{N}_0(x) \cap \{1,\dots,N\}| \le |D(x) \cap \{1,\dots,N\}|$. We have that $|C(x) \cap \{1,\dots,N\}| \le |B(x) \cap \{1,\dots,N\}|$ for any $N \ge 1$, by the injectivity of the map $h : C(x) \to B(x)$ given by $h(n) = n-1$. Then we see that $|D(x) \cap \{1,\dots,N\}| \le 2|B(x) \cap \{1,\dots,N\}|$, and then, $|\mathbb{N}_0(x) \cap \{1,\dots,N\}| \le 2|B(x) \cap \{1,\dots,N\}|$, for any $N \ge 1$.

By Lemma 3.2,

$$\liminf_{N\to\infty} \frac{|B(x) \cap \{1,\dots,N\}|}{N} \ge \frac{p_0(\alpha)}{2}, \mu_f\text{-}a.s.x.$$

Thus we obtain the assertion (3).

(4) By noting the definition of $B(x)$, we see that $s(p_0(r_n(x)/s_n(x))) < \max\{s(p_0(\gamma-2-\epsilon_0)), s(p_0(\gamma-2+\epsilon_0))\} = e_0$ for any $x \in [0,1)$ and $n \in B(x)$.

Now we have that

$$\frac{1}{N}\sum_{n=1}^{N} s\left(p_0\left(\frac{r_n(x)}{s_n(x)}\right)\right) = \frac{1}{N}\left(\sum_{n\in A(x), n\le N} + \sum_{n\in B(x), n\le N}\right) s\left(p_0\left(\frac{r_n(x)}{s_n(x)}\right)\right).$$

Let $\xi_N(x) = |B(x) \cap \{1,\dots,N\}|/N$. Then, by noting that $s(p_0(r_n(x)/s_n(x))) \le \log 2$, we see that

$$\frac{1}{N}\sum_{n\in A(x), n\le N} s\left(p_0\left(\frac{r_n(x)}{s_n(x)}\right)\right) \le \frac{|A(x) \cap \{1,\dots,N\}|}{N}\log 2 = (1-\xi_N(x))\log 2.$$

Now we have that

$$\frac{1}{N}\sum_{n\in B(x), n\le N} s\left(p_0\left(\frac{r_n(x)}{s_n(x)}\right)\right) \le \xi_N(x)e_0.$$

By noting that $e_0 < \log 2$, we see that

$$\limsup_{N\to\infty}\left((1-\xi_N(x))\log 2 + \xi_N(x)e_0\right) \le \log 2 - (\log 2 - e_0)\liminf_{N\to\infty}\xi_N(x).$$

By the assertion (3), we see that $\liminf_{N\to\infty}\xi_N(x) \ge p_0(\alpha)/2 > 0$ for μ_f-a.s.x. Thus we obtain the assertion (4). $\square$

Now we show Theorem 1.2 (1). We remark that $\Phi(cA; z) = \Phi(A; z)$ for any constant $c > 0$ and the conditions (A1) - (A3) remain valid for (cA_0, cA_1). Then, we can assume that $d_0 = 1$ and $b_1 = 1$.

By computation, we see that

$$A_0 = \begin{pmatrix} 1/2 & 0 \\ c_0 & 1 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 4c_0+1 & 1 \\ 2c_0 & 2(1+c_0) \end{pmatrix},$$

and $f(x) = \dfrac{x}{-2c_0x + 1 + 2c_0}$ satisfies the equation (1.1). This completes the proof of Theorem 1.2 (1).

Now we show Theorem 1.2 (2). We assume that the condition (i) fails. Then, by Lemma 2.3, we have that for μ_f-a.s.x,

$$\limsup_{N\to+\infty} \frac{-\log R_N(x)}{N} = \limsup_{N\to\infty} \frac{L_N(x)}{N} = \limsup_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} s\left(p_0\left(\frac{r_n(x)}{s_n(x)}\right)\right).$$

Then, by noting Lemma 3.3(4) and Lemma 2.4(1), we obtain the desired result.

We can show the assertion in the same manner if the condition (ii) fails. These complete the proof of Theorem 1.2(2).

4 A relationship with stationary measures

In this section, we state a relationship between a certain class of de Rham's functional equations and stationary measures.

We state a general setting. Let G be a semigroup and μ be a probability measure on G. Let M be a topological space. We assume that G acts on M measurably, that is, there is a map from $(g, x) \in G \times M$ to $g \cdot x \in M$ satisfying the following conditions :
(1) $(g_1 g_2) \cdot x = g_1 \cdot (g_2 \cdot x)$ for any $g_1, g_2 \in G$ and $x \in M$.
(2) $x \mapsto g \cdot x$ is measurable map on M for any $g \in G$.

We say that a probability measure ν on M is a *μ-stationary measure* if

$$\nu(B) = \int_G \nu(h^{-1}B)\mu(dh), \tag{4.1}$$

for any $B \in \mathcal{B}(M)$. Furstenberg [3] Lemma 1.2 showed that if M is a compact metric space, then there exists a μ-stationary measure.

Let

$$G = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M(2;\mathbb{R}) : ad > bc, b \ge 0, d > 0, 0 < a + b \le c + d \right\},$$

and, $M = [0, 1]$. Then G is a semigroup. We define a continuous action of G to M by $A \cdot z = \Phi(A; z)$. For (A_0, A_1) satisfying (A1)-(A3), we see that $A_0, A_1 \in G$. Let μ be a probability measure on G such that $\mu(A_0) = \mu(A_1) = 1/2$. Then we have the following.

Lemma 4.1. (1) *For* $k \geq 1$,

$$\begin{cases} A_0^{-1}(f(I_k(x))) = f(I_{k-1}(2x)),\ A_1^{-1}(f(I_k(x))) = \emptyset & x \in [0,1/2) \\ A_0^{-1}(f(I_k(x))) = \emptyset,\ A_1^{-1}(f(I_k(x))) = f(I_{k-1}(2x-1)) & x \in [1/2,1). \end{cases}$$

(2) *For any* μ*-stationary measure* ν *and* $k \geq 1$,

$$\nu(f(I_k(x))) = \begin{cases} \nu(f(I_{k-1}(2x)))/2 & x \in [0,1/2) \\ \nu(f(I_{k-1}(2x-1)))/2 & x \in [1/2,1). \end{cases}$$

(3) *There exists exactly one* μ*-stationary measure* ν.

Proof. (1) By Lemma 2.1(1), we see that
$f(I_k(x)) = \Phi(A_{X_1(x)} \cdots A_{X_k(x)}; [0,1)) = \Phi(A_{X_1(x)}; \Phi(A_{X_2(x)} \cdots A_{X_k(x)}; [0,1)))$.
We see that $f(I_{k-1}(2x)) = \Phi(A_{X_2(x)} \cdots A_{X_k(x)}; [0,1)) = A_0^{-1}(f(I_k(x)))$, $x \in [0,1/2)$, and, $f(I_{k-1}(2x-1)) = \Phi(A_{X_2(x)} \cdots A_{X_k(x)}; [0,1)) = A_1^{-1}(f(I_k(x)))$, $x \in [1/2,1)$. Since $\Phi(A_0; [0,1)) \cap \Phi(A_1; [0,1)) = \emptyset$, $A_1^{-1}(f(I_k(x))) = \emptyset$, $x \in [0,1/2)$, and, $A_0^{-1}(f(I_k(x))) = \emptyset$, $x \in [1/2,1)$. Thus we have the assertion (1).

(2) By noting the assertion (1) and (4.1), we obtain the desired result.

(3) Let ν_i, $i = 0,1$, be two μ-stationary measures. By the assertion (2), we see that $\nu_0(f(I_k(x))) = \nu_1(f(I_k(x)))$ for $k \geq 1$, $x \in [0,1)$. Let
$\mathcal{C} = \left\{ f(\sum_{i=1}^k 2^{-j} X_j(x)) : k \geq 1, x \in [0,1) \right\} = \left\{ f(l/2^k) : 0 \leq l \leq 2^{k-1}, k \geq 1 \right\}$.
Then, we have that $\nu_0([a,b)) = \nu_1([a,b))$ for $a, b \in \mathcal{C}$. Since f is continuous on $[0,1]$, $\mathcal{C}$ is dense in $[0,1]$. Thus we see that $\nu_0 = \nu_1$. □

Lemma 4.2. *Let* $g : [0,1] \to [0,1]$ *be the inverse function of the solution* f *of* (1.1). *Then,*
(1) g *is continuous and strictly increasing. Hence,* μ_g *is well-defined.*
(2) μ_f *is singular if and only if* μ_g *is so.*

Proof. (1) Noting that f is continuous and strictly increasing on $[0,1]$, $f(0) = 0$ and $f(1) = 1$, we obtain the desired result.

(2) Since $l([a,b)) = \mu_f\left(f^{-1}([a,b))\right) = \mu_g\left(g^{-1}([a,b))\right)$ for $0 \leq a \leq b \leq 1$, we see that $l(B) = \mu_f\left(f^{-1}(B)\right) = \mu_g\left(g^{-1}(B)\right)$ for any Borel set B.

We assume that μ_f is singular. Then, there exists a Borel set B_0 such that $\mu_f(B_0) = 0$ and $l(B_0) = 1$. Then, $\mu_g\left(g^{-1}(B_0)\right) = 1$ and $l\left(g^{-1}(B_0)\right) = \mu_f\left(f^{-1}(g^{-1}(B_0))\right) = \mu_f(B_0) = 0$. Thus we see that μ_g is singular.

We assume that μ_g is singular. Then, we see that μ_f is singular in the same manner as in the above argument. □

The following theorem gives a necessary and sufficient condition for the regularity of the stationary measure in this setting.

Theorem 4.3. *Let the conditions* (i) *and* (ii) *as in Theorem 1.2 and* ν *be a unique* μ*-stationary measure. Then, we have*
(1) ν *is absolutely continuous if and only if both* (i) *and* (ii) *hold.*
(2) ν *is singular if and only if either* (i) *or* (ii) *fails.*

Proof. It is sufficient to show "if" parts.

(1) By noting Theorem 1.2(1), we have that $f(x) = x/(-2c_0x + 2c_0 + 1)$ and then $g(y) = (2c_0 + 1)y/(2c_0y + 1)$. By Lemma 4.2(2), we have that μ_g is absolutely continuous and obtain the assertion (1).

(2) We see that $\mu_g(f(I_k(x))) = \mu_g(g^{-1}(I_k(x))) = 2^{-k}$, $x \in [0, 1)$, $k \geq 1$. By Lemma 4.1(1),
$\mu_g(f(I_k(x))) = \frac{1}{2}\left(\mu_g\left(A_0^{-1}(f(I_k(x)))\right) + \mu_g\left(A_1^{-1}(f(I_k(x)))\right)\right)$, $x \in [0, 1)$, $k \geq 1$. Then we see that (4.1) holds for $[a, b)$, $a, b \in \mathcal{C}$ and that μ_g is a μ-stationary measure. By noting Theorem 1.2(2), we have that μ_f is singular. By Lemma 4.2(2), we have that μ_g is singular and obtain the assertion (2). $\square$

5 Examples and remarks

The following example concerns the Lebesgue singular functions.

Example 5.1. Let us define 2×2 real matrices $A_{p,0}, A_{p,1}$, $p \in (0, 1)$, by

$$A_{p,0} = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}, \quad A_{p,1} = \begin{pmatrix} 1-p & p \\ 0 & 1 \end{pmatrix}.$$

Then, $(A_0, A_1) = (A_{p,0}, A_{p,1})$ satisfies the conditions (A1)-(A3).

Let f_p be the solution of (1.1) for $(A_0, A_1) = (A_{p,0}, A_{p,1})$. Then, as immediate consequences of main theorems, we have the followings.
(1) μ_{f_p} is absolutely continuous if $p = 1/2$, and μ_{f_p} is singular if $p \neq 1/2$.
(2) There exists a Borel set K_p such that $\mu_{f_p}(K_p) = 1$ and $\dim_H(K_p) \leq s(p)/\log 2$.
(3) $\mu_{f_p}(K) = 0$ for any Borel set K with $\dim_H(K) < s(p)/\log 2$.

The following example concerns the range of self-interacting walks on an interval in the author [5].

Example 5.2. Let $x_u = 2/(1 + \sqrt{1 + 8u^2})$, $u \geq 0$. Let $\tilde{A}_{u,i}$, $i = 0, 1$, be two 2×2 real matrices given by

$$\tilde{A}_{u,0} = \begin{pmatrix} x_u & 0 \\ -u^2x_u^2 & 1 \end{pmatrix}, \quad \tilde{A}_{u,1} = \begin{pmatrix} 0 & x_u \\ -u^2x_u^2 & 1 - u^2x_u^2 \end{pmatrix}, \quad u \geq 0.$$

Let $0 < u < \sqrt{3}$. Then $(A_0, A_1) = (\tilde{A}_{u,0}, \tilde{A}_{u,1})$ satisfies the conditions (A1)-(A3). Let g_u be the solution of (1.1) for $(A_0, A_1) = (\tilde{A}_{u,0}, \tilde{A}_{u,1})$. We remark that $\gamma = (1 - u^2 x_u^2)/x_u = (1 + x_u)/2x_u$. By the definition of x_u, we see that each of the conditions in Theorem 1.2 is equivalent to $x_u \neq 1/2$, that is, $u \neq 1$. Then, by Theorem 1.2, we have that μ_{g_u} is singular for $0 < u < \sqrt{3}$ and $u \neq 1$, and absolutely continuous for $u = 1$.

Let $0 < u < 1$. Then we have that $x_u > 1/2$, $\alpha = \min\{0, -1/2, -u^2 x_u\} = -1/2$, $\beta = 0$ and $\gamma < 3/2$. Hence we see that $\gamma - 2 < \alpha$, in particular, $\gamma - 2 \notin [\alpha, \beta]$. By Theorem 1.1, we see that there exists a Borel set $\tilde{K}_u$ such that $\dim_H(\tilde{K}_u) \leq s(p_0(\alpha))/\log 2 = s(x_u)/\log 2$ and $\mu_{g_u}(\tilde{K}_u) = 1$ and that $\mu_{g_u}(K) = 0$ for any Borel set K with $\dim_H(K) < s(p_0(\beta))/\log 2 = s(2x_u/(1 + x_u))/\log 2$.

Remark 5.3. (1) Pincus [6], [7] obtained results similar to Theorem 4.3. Hata [4] Corollary 7.4 showed the singularity of the solution of (1.1) under the assumptions similar to [7] Theorem 2.1.
(2) Let $T : [0, 1) \to [0, 1)$ be given by $T(x) = 2x \mod 1$. Then, by computation,

$$\mu_f\left(T^{-1}(A)\right) = \int_A \left(\frac{d\Phi(A_0; \cdot)}{dz}(f(y)) + \frac{d\Phi(A_1; \cdot)}{dz}(f(y)) \right) \mu_f(dy),\ A \in \mathcal{B}([0, 1)).$$

We see that T is a non-singular transformation on $[0, 1)$ with respect to μ_f, that is, $\mu_f \circ T^{-1} \ll \mu_f$ and $\mu_f \ll \mu_f \circ T^{-1}$.

Now it is natural to ask whether the non-singular dynamical system $([0, 1), \mu_f, T)$ is ergodic. The dynamical system $([0, 1], \mu_{f_p}, T)$ in Example 5.1 is invariant and ergodic. Also, we would like to see whether there exists a T-invariant measure λ on $[0, 1)$ which satisfies $\lambda \ll \mu_f$ and $\mu_f \ll \lambda$.

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