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Abstract

In this paper, we propose a new hybridized discontinuous Galerkin method for the convection-diffusion-reaction problems with mixed boundary conditions. The coercivity of the convection-reaction part is achieved by adding an upwinding term. We give error estimates of optimal order in the piecewise H^1-seminorm. Furthermore, we show that the approximate solution of our scheme is close to that of the reduced problem when the diffusion coefficient is very small. Some numerical results are presented to verify the validity of our method.

1 Introduction

The discontinuous Galerkin method (DGM) is now widely applied to various problems in science and engineering because of its flexibility for the choice of approximate functions and of element shapes. An issue of DGM is, however, the size and band-widths of the resulting matrices could be much larger than those of the standard finite element method, since the DGM is formulated in terms of the usual node values defined in each elements together with those corresponding to inter-element discontinuities. In order to surmount this difficulty, it is worth-while trying to extend the idea of DGM by combining with the hybrid displacement method (see, for example, [8], [9], [10] and [11]). Thus, we introduce new unknown functions on inter-element edges. We can then obtain a formulation that the resulting discrete system contains inter-element unknowns only and, consequently, the size of the system becomes smaller. Recently, in [12], [13] and [14], the author and his

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colleagues proposed and analyzed a new class of DGM, the *hybridized DGM*, that is based on the hybrid displacement approach by stabilizing their old method ([10] and [11]). In [12], we examined our idea by using a linear elasticity problem as a model problem and offered several numerical examples to confirm the validity of our formulation. After that, we carried out theoretical analysis by using the Poisson equation as a model problem. In [14], stability and convergence of symmetric and nonsymmetric interior penalty methods of hybrid type were studied. The usefulness of the lifting operator in order to ensure a better stability was studied in [14]. Furthermore, B. Cockburn and his colleagues are actively contributing to the hybridized DGM for elliptic([18], [15] and [19]), Stokes and Navier-Stokes problems ([20], [21] and [22]).

The purpose of this paper is to propose a new hybridized DGM for stationary convection-diffusion-reaction problems with mixed boundary conditions. In [17], Cockburn et al. proposed hybridization for the diffusion-convection-reaction problems. The stability of their method is achieved by choosing stabilization parameters according to the convection. They reported a lot of numerical results and confirmed the validity of their schemes. However, error analysis seems to be not undertaken. The scheme we are going to propose is close to the original DGM for convection-diffusion problem ([3] and [4]) and is based on a certain upwinding technique. As is well-known, there are a lot of methods of upwinding. Our method, however, differs form any previous methods. For example, our upwinding technique does not need information on neighboring elements, whereas most of upwinding use information upwind elements. Instead, our upwinding method is introduced in terms of neighboring edges. To be more specific, we find a hybridized approximation to convection and reaction terms in the following form

$$\sum_{K\in\mathcal{T}_h}[(\boldsymbol{b}\cdot\nabla u_h+cu_h,\ v_h)_K+\langle u_h-\hat{u}_h,\alpha v_h-\beta\hat{v}_h\rangle_{\partial K}], \tag{1.1}$$

where coefficients α and β are decided to satisfy coercivity, as it will be shown later. Moreover, our proposed scheme is stable even when ε is sufficiently small and it can be applied to the case $\varepsilon = 0$. We furthermore give stability and optimal order error estimates.

Now let us formulate our continuous problem. Let Ω be a bounded polyhedral domain in R^n. In this paper, we propose a new hybridized discontinuous Galerkin method for the convection-diffusion-reaction problems with mixed boundary conditions:

$$-\varepsilon\Delta u+\boldsymbol{b}\cdot\nabla u+cu=f \text{ in } \Omega, \tag{1.2a}$$

$$u=g_D \text{ on } \Gamma_D, \tag{1.2b}$$

$$\varepsilon\nabla u\cdot\boldsymbol{n}=g_N \text{ on } \Gamma_N, \tag{1.2c}$$

where $\varepsilon > 0$ is the diffusion coefficient and $f \in L^2(\Omega)$, $\boldsymbol{b} \in W^{1,\infty}(\Omega)^n$, $c \in L^\infty(\Omega)$, $g_D \in H^{3/2}(\Omega)$, and $g_N \in H^{1/2}(\Omega)$ are given functions. We assume $\overline{\Gamma}_D \cup \overline{\Gamma}_N = \partial\Omega$, $\Gamma_D \cap \Gamma_N = \emptyset$, and that the inflow boundary is included by Γ_D, i.e.,

$$\Gamma^- := \{x \in \partial\Omega : \boldsymbol{b}(x) \cdot \boldsymbol{n}(x) < 0\} \subset \Gamma_D,$$

where $\boldsymbol{n}$ is the outward unit normal vector to $\partial\Omega$. Moreover, we assume that there exists a non-negative constant ρ_0 such that

$$\rho(x) := c(x) - \frac{1}{2}\mathrm{div}\boldsymbol{b}(x) \geq \rho_0 \geq 0, \quad \forall x \in \Omega. \tag{1.3}$$

Under these assumptions, there exists a unique weak solution $u \in H^1(\Omega)$ by the Lax-Milgram theory. We shall pose further regularity on u in the error analysis. This paper is organized as follows. In Section 2, we introduce finite element spaces to describe our method, and norms and projections to use in our error analysis. Section 3 is devoted to the formulation of our hybridized method, and mathematical analysis is given in Section 4. We explain why our proposed DGM is stable even when ε is close to 0 in Section 5. In Section 6, we report some results of numerical computations. Finally, we conclude this paper in Section 7.

2 Preliminaries

2.1 Notation

Function spaces and norms Let $\mathcal{T}_h = \{K_i\}_i$ be a triangulation of Ω in the sense of [13]. Thus, each $K \in \mathcal{T}_h$ is an m-polyhedral domain, where m denotes an integer $m \geq n+1$. The boundary ∂K of $K \in \mathcal{T}_h$ is composed of m-faces. We assume that m is bounded from above independently a family of triangulations $\{\mathcal{T}_h\}_h$, and ∂K does not intersect with itself. Furthermore, we set $h = \max_{K\in\mathcal{T}_h} h_K$, where h_K denotes the diameter of K. The *skeleton* of $\mathcal{T}_h$ is defined by

$$\Gamma_h := \bigcup_{K\in\mathcal{T}_h} \partial K.$$

We use the broken Sobolev space over $\mathcal{T}_h$ defined by

$$H^k(\mathcal{T}_h) = \{v \in L^2(\Omega) : v|_K \in H^k(K)\},$$

and L^2-spaces on Γ_h as follows

$$\begin{aligned} L^2_D(\Gamma_h) &= \{\hat{v} \in L^2(\Gamma_h) : \hat{v}|_{\Gamma_D} = g_D, \hat{v}|_{\Gamma_N} = 0\}, \\ L^2_0(\Gamma_h) &= \{\hat{v} \in L^2(\Gamma_h) : \hat{v}|_{\Gamma_D} = 0, \hat{v}|_{\Gamma_N} = 0\}. \end{aligned}$$

Then, we set $\boldsymbol{V} = H^2(\mathcal{T}_h) \times L^2_D(\Gamma_h)$ and $\boldsymbol{V}_0 = H^2(\mathcal{T}_h) \times L^2_0(\Gamma_h)$. The inner products are defined as follows

$$(u, v)_K = \int_K uv dx, \qquad \langle \hat{u}, \hat{v} \rangle_e = \int_e \hat{u}\hat{v} ds,$$

for an element K and an edge e, respectively. Let $\| \cdot \|_m$ and $| \cdot |_m$ be the usual Sobolev norms and seminorms. We introduce auxiliary seminorms:

$$|v|^2_{m,h} := \sum_{K \in \mathcal{T}_h} h_K^{2(m-1)} |v|^2_{m,K} \quad \text{for } v \in H^m(\mathcal{T}_h),$$

$$|\boldsymbol{v}|^2_{j,h} := \sum_{K \in \mathcal{T}_h} \sum_{e \subset \partial K} \left\| \sqrt{\frac{\eta_e}{h_e}} (v - \hat{v}) \right\|^2_{0,e} \quad \text{for } \boldsymbol{v} \in \boldsymbol{V},$$

where h_K is the diameter of K and h_e is the length of e. For error analysis, we define the *HDG norm* defined by

$$\begin{aligned}
|||\boldsymbol{v}|||^2 &:= |||\boldsymbol{v}|||^2_d + |||\boldsymbol{v}|||^2_{rc}, \\
|||\boldsymbol{v}|||^2_d &:= \varepsilon \left(|v|^2_{1,h} + |v|^2_{2,h} + |\boldsymbol{v}|^2_{j,h} \right), \\
|||\boldsymbol{v}|||^2_{rc} &:= \frac{1}{2} \sum_{K \in \mathcal{T}_h} \| |\boldsymbol{b} \cdot \boldsymbol{n}_K|^{1/2} (v - \hat{v}) \|^2_{0,\partial K} + \rho_0 \|v\|^2_{0,\Omega},
\end{aligned}$$

where $\boldsymbol{n}_K$ is the unit outward normal vector to ∂K and ρ_0 is the positive constant defined in (1.3).

Finite element spaces Let U_h and $\hat{U}_h$ be finite dimensional spaces of $H^2(\mathcal{T}_h)$ and of $L^2_D(\Gamma_h)$, respectively. Then we set $\boldsymbol{V}_h := U_h \times \hat{U}_h$, which is included by $\boldsymbol{V}$. Similarly, we define $\boldsymbol{V}_{0h} := U_h \times \hat{U}_{0h} \subset \boldsymbol{V}_0$. In this paper, we assume (H1) $\nabla v_h \in [U_h]^n$ for all $v_h \in U_h$ and that (H2) U_h includes the piecewise constant functions $\mathcal{P}^0(\mathcal{T}_h)$. For example, we can use polynomials of degree k as U_h or $\hat{U}_h$.

Projections Let P_h denote the L^2-projection from $H^2(\mathcal{T}_h)$ onto U_h, and let $\hat{P}_h$ denote the L^2-projection from $L^2_D(\Gamma_h)$ onto $\hat{U}_h$. We define $\boldsymbol{P}_h : \boldsymbol{V} \to \boldsymbol{V}_h$ by $\boldsymbol{P}_h \boldsymbol{v} := \{P_h v, \hat{P}_h \hat{v}\}$. We introduce the L^2-projection $\boldsymbol{P}^0_h : W^{1,\infty}(\Omega) \to \mathcal{P}^0(\mathcal{T}_h)^n$. We also use the projection $\tilde{\cdot}$ defined by $\tilde{\boldsymbol{v}} := \{v, \hat{v}|_{\Gamma_h \setminus \partial\Omega} + \hat{P}_h v|_{\partial\Omega}\}$, which affects only on $\partial\Omega$. In this paper we assume the approximate properties (H3): for all $v \in H^{k+1}(K)$, we have

$$|v - P_h v|_{i,K} \le C h^{k+1-i} |v|_{k+1,K} \quad \text{for } i = 0, 1, \tag{2.1}$$

$$\|v - \hat{P}_h(v|_e)\|_{0,e} \le C h^{k+1/2} |v|_{k+1,K}. \tag{2.2}$$

Remark Throughout this paper, a boldface lowercase letters except $\boldsymbol{b}$ and $\boldsymbol{n}$ denotes a function of $\boldsymbol{V}$, i.e., $\boldsymbol{v}$ indicates $\{v, \hat{v}\} \in \boldsymbol{V}$. Moreover, the symbol C denotes a generic constant.

2.2 Inequalities

Theorem 2.1. Let $K \in \mathcal{T}_h$ and e be an edge of K.

1. (Trace inequality) There exists a constant C independent of K and e such that

$$\|v\|_{0,e} \leq C h_e^{-1/2} \left(\|v\|_{0,K}^2 + h_K^2 |v|_{1,K}^2 \right)^{1/2} \qquad \forall v \in H^1(K). \tag{2.3}$$

2. (Inverse inequality) There exists a constant C independent of K such that

$$|v_h|_{1,K} \leq C h_K^{-1} \|v_h\|_{0,K} \qquad \forall v_h \in U_h. \tag{2.4}$$

Proof. See [5, p.745]. □

Lemma 2.2. Assume (H3). Let $v \in H^{k+1}$ and $\boldsymbol{v} = \{v, v|_{\Gamma_h}\}$. Then we have

$$\begin{aligned} |||\boldsymbol{v} - \boldsymbol{P}_h \boldsymbol{v}|||_d &\leq C \varepsilon^{1/2} h^k |v|_{k+1}, \\ |||\boldsymbol{v} - \boldsymbol{P}_h \boldsymbol{v}|||_{rc} &\leq C h^{k+1/2} |v|_{k+1}. \end{aligned}$$

Proof. This follows immediately from the definitions. □

3 A new hybridized DGM

We are able to state a new hybridized DGM, which we propose in this paper. We first state our formulation: Find $\boldsymbol{u}_h \in \boldsymbol{V}_h$ such that

$$B_h(\boldsymbol{u}_h, \boldsymbol{v}_h) = (f, v_h)_\Omega + \langle g_N, v_h \rangle_{\Gamma_N} \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{3.1}$$

where

$$B_h(\boldsymbol{u}_h, \boldsymbol{v}_h) := B_h^d(\boldsymbol{u}_h, \boldsymbol{v}_h) + B_h^{rc}(\boldsymbol{u}_h, \boldsymbol{v}_h), \tag{3.2}$$

$$B_h^d(\boldsymbol{u}_h, \boldsymbol{v}_h) = \varepsilon \sum_{K\in\mathcal{T}_h} \Bigg[(\nabla u_h, \nabla v_h)_K - \left\langle \frac{\partial u_h}{\partial n}, v_h - \hat{v}_h \right\rangle_{\partial K} - \left\langle \frac{\partial v_h}{\partial n}, u_h - \hat{u}_h \right\rangle_{\partial K} + \sum_{e\subset\partial K} \frac{\eta_e}{h_e} \langle u_h - \hat{u}_h, v_h - \hat{v}_h \rangle_e \Bigg], \tag{3.3}$$

$$B_h^{rc}(\boldsymbol{u}_h, \boldsymbol{v}_h) = \sum_{K\in\mathcal{T}_h} \Big[(\boldsymbol{b}\cdot\nabla u_h + c u_h, v_h)_K + \langle u_h - \hat{u}_h, [\boldsymbol{b}\cdot\boldsymbol{n}]_- v_h - [\boldsymbol{b}\cdot\boldsymbol{n}]_+ \hat{v}_h \rangle_{\partial K} \Big], \tag{3.4}$$

$$(f, v_h)_\Omega = \int_\Omega f v_h dx, \tag{3.5}$$

$$\langle g_N, v_h \rangle_{\Gamma_N} = \int_{\Gamma_N} g_N v_h ds. \tag{3.6}$$

Here η_e is a penalty parameter with $\eta_e \geq \eta_{min} > 0$, h_e is the length of an edge e, and the functions $[\,\cdot\,]_+$ and $[\,\cdot\,]_-$ are defined by

$$[x]_+ = \max(0, x), \quad [x]_- = \max(0, -x). \tag{3.7}$$

Note that $[x]_+ + [x]_- = |x|$ and $[x]_+ - [x]_- = x$.

Before proceeding to the analysis of (3.1), we state the derivation of it. Multiplying the both sides of (1.2) by a test function $v_h \in \boldsymbol{V}_h$ and integrating the both sides over Ω, we have, by integration by parts,

$$\sum_{K\in\mathcal{T}_h} \left[(\nabla u_h, \nabla v_h)_K - \left\langle \frac{\partial u_h}{\partial n}, v_h \right\rangle_{\partial K} + (\boldsymbol{b}\cdot\nabla u_h + c u_h, v_h)_K \right] = (f, v_h)_\Omega + \langle g_N, v_h \rangle_{\Gamma_N} \qquad \boldsymbol{v}_h \in \boldsymbol{V}_h. \tag{3.8}$$

We denote the diffusion part and convection part in (3.8) by D and C, respectively, i.e.,

$$D(u_h, v_h) := \sum_{K\in\mathcal{T}_h} \left[(\nabla u_h, \nabla v_h)_K - \left\langle \frac{\partial u_h}{\partial n}, v_h \right\rangle_{\partial K} \right], \tag{3.9}$$

$$C(u_h, v_h) := \sum_{K\in\mathcal{T}_h} (\boldsymbol{b}\cdot\nabla u_h + c u_h, v_h)_K. \tag{3.10}$$

We first derive our formulation of the diffusion part. From the continuity of the flux, we have

$$\sum_{K\in\mathcal{T}_h}\left\langle \frac{\partial u_h}{\partial n}, \hat{v}_h \right\rangle_{\partial K} = 0. \tag{3.11}$$

Adding (3.11) to (3.9) yields

$$D(u_h, v_h) = \sum_{K\in\mathcal{T}_h}\left[(\nabla u_h, \nabla v_h)_K - \left\langle \frac{\partial u_h}{\partial n}, v_h - \hat{v}_h \right\rangle_{\partial K}\right]. \tag{3.12}$$

Symmetrizing (3.12) and adding a penalty term

$$\sum_{K\in\mathcal{T}_h}\sum_{e\subset\partial K}\left\langle \frac{\eta_e}{h_e}(u_h - \hat{u}_h), v_h - \hat{v}_h \right\rangle_{\partial K}, \tag{3.13}$$

we obtain (3.3).

Next, we derive the formulation of the convection part. Let α and β be coefficients to be determined later, and consider the following form:

$$C_h(u_h, v_h) := \sum_{K\in\mathcal{T}_h}\left[(\boldsymbol{b}\cdot\nabla u_h + c u_h, v_h)_K + \langle u_h - \hat{u}_h, \alpha v_h - \beta \hat{v}_h\rangle_{\partial K}\right]. \tag{3.14}$$

The coefficients α and β are chosen so that

$$\begin{aligned} C_h(v_h, v_h) \;=\; \sum_{K\in\mathcal{T}_h}\Big[&(\rho v_h, v_h)_K + \langle((\boldsymbol{b}\cdot\boldsymbol{n})/2 + \alpha)\, v_h, v_h\rangle_{\partial K} \\ &- \langle(\alpha+\beta) v_h, \hat{v}_h\rangle_{\partial K} + \langle\beta \hat{v}_h, \hat{v}_h\rangle_{\partial K}\Big] \geq |||\boldsymbol{v}|||_{rc} \end{aligned} \tag{3.15}$$

for any $\boldsymbol{v}_h \in \boldsymbol{V}_h$. We can find the following sufficient conditions

$$\alpha + \beta = 2((\boldsymbol{b}\cdot\boldsymbol{n})/2 + \alpha) = |\boldsymbol{b}\cdot\boldsymbol{n}|,$$

from which it follows that

$$\alpha = [\boldsymbol{b}\cdot\boldsymbol{n}]_-, \quad \beta = [\boldsymbol{b}\cdot\boldsymbol{n}]_+.$$

Thus we obtain our formulation (3.1).

4 Error analysis

In this section, we shall establish an error estimates for (3.1).

Lemma 4.1. The following hold.

1. (Boundedness) There exists a constant $C_b^d > 0$ such that
$$|B_h^d(\boldsymbol{w}, \boldsymbol{v})| \leq C_b^d \, |||\boldsymbol{w}|||_d \, |||\boldsymbol{v}|||_d \quad \forall \boldsymbol{w}, \boldsymbol{v} \in \boldsymbol{V}. \tag{4.1}$$

2. (Coercivity) There exists a constant $C_c^d > 0$ such that
$$B_h^d(\boldsymbol{v}_h, \boldsymbol{v}_h) \geq C_c^d \, |||\boldsymbol{v}_h|||_d^2 \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h. \tag{4.2}$$

Proof. We first prove the boundedness. Applying the Schwarz inequality for each term of (3.3), we have

$$\begin{aligned} |B_h^d(\boldsymbol{w}, \boldsymbol{v})| \leq \; & \varepsilon \sum_{K \in \mathcal{T}_h} \Bigg[\|\nabla w\|_{0,K} \|\nabla v\|_{0,K} \\ & + \sum_{e \subset \partial K} \Bigg(\left\| \frac{\partial w}{\partial n} \right\|_{0,e} \|v - \hat{v}\|_{0,e} + \left\| \frac{\partial v}{\partial n} \right\|_{0,e} \|w - \hat{w}\|_{0,e} \\ & + \left\| \sqrt{\frac{\eta_e}{h_e}} (w - \hat{w}) \right\|_{0,e} \left\| \sqrt{\frac{\eta_e}{h_e}} (v - \hat{v}) \right\|_{0,e} \Bigg) \Bigg]. \end{aligned} \tag{4.3}$$

By the trace theorem, we have

$$\left\| \frac{\partial w}{\partial n} \right\|_{0,e} \leq C h_e^{-1/2} (|w|_{1,K}^2 + h_K^2 |w|_{2,K}^2)^{1/2}. \tag{4.4}$$

From (4.3), (4.4), and the Cauchy-Schwarz inequality, it follows that

$$|B_h^d(\boldsymbol{w}, \boldsymbol{v})| \leq \max(1 + C\eta_{min}^{-1/2}, 2) \, |||\boldsymbol{w}|||_d \, |||\boldsymbol{v}|||_d \, .$$

Next, we prove the coercivity. By definition,

$$B_h^d(\boldsymbol{v}_h, \boldsymbol{v}_h) \geq |v_h|_{1,h}^2 - 2 \sum_{K \in \mathcal{T}_h} \sum_{e \subset \partial K} \left\| \frac{\partial v_h}{\partial n} \right\|_{0,e} \|v_h - \hat{v}_h\|_{0,e} + |\boldsymbol{v}_h|_{j,h}^2 \tag{4.5}$$

By the trace theorem, the inverse inequality and the Young inequality, we have for $\delta \in (0,1)$,

$$
\begin{aligned}
2\left\|\frac{\partial v_h}{\partial n}\right\|_{0,e}\|v_h-\hat{v}_h\|_{0,e} &\leq \frac{2C}{h_e}|v_h|_{1,K}\|v_h-\hat{v}_h\|_{0,e} \qquad (4.6)\\
&\leq \frac{C}{\delta\eta_e}|v_h|_{1,K}^2+\delta\left\|\sqrt{\frac{\eta_e}{h_e}}(v_h-\hat{v}_h)\right\|_{0,e}^2 \quad \forall v_h \in U_h.
\end{aligned}
$$

From (4.5) and (4.6), we obtain

$$
B_h^d(\boldsymbol{v}_h,\boldsymbol{v}_h) \geq \left(1-\frac{C}{\delta\eta_{min}}\right)|v_h|_{1,h}^2+(1-\delta)\,|\boldsymbol{v}_h|_{j,h}^2, \tag{4.7}
$$

If $\eta_{min} > 4C$, then we can take $\delta = \sqrt{C/\eta_{min}} < 1/2$, which implies that

$$
1-\frac{C}{\delta\eta_{min}} > 1/2, \quad 1-\delta > 1/2.
$$

Hence we have

$$
B_h^d(\boldsymbol{v}_h,\boldsymbol{v}_h) \geq \frac{1}{2}(|v_h|_{1,h}^2+|\boldsymbol{v}_h|_{j,h}^2). =: \frac{1}{2}\,|||\boldsymbol{v}_h|||_{d,h}^2 .
$$

Since the norms $|||\cdot|||_d$ and $|||\cdot|||_{d,h}$ are equivalent each other over $\boldsymbol{V}_h$, we obtain the coercivity (4.2) . □

Lemma 4.2. The following hold.

1. There exists a constant $C_b^{rc} > 0$ such that for all $\boldsymbol{v} \in \boldsymbol{V}$, $\boldsymbol{w}_h \in \boldsymbol{V}_h$,
$$|B_h^{rc}(\tilde{\boldsymbol{v}} - \boldsymbol{P}_h\boldsymbol{v}, \boldsymbol{w}_h)| \leq C_b^{rc} \left|\!\left|\!\left| \tilde{\boldsymbol{v}} - \boldsymbol{P}_h\boldsymbol{v} \right|\!\right|\!\right|_{rc} \left|\!\left|\!\left| \boldsymbol{w}_h \right|\!\right|\!\right|_{rc}$$

2. (Coercivity) There exists a positive constant $C_c^{rc} > 0$ such that
$$B_h^{rc}(\boldsymbol{v}_h, \boldsymbol{v}_h) \geq C_c^{rc} \left|\!\left|\!\left| \boldsymbol{v}_h \right|\!\right|\!\right|_{rc}^2 \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_{0h}.$$

Proof. For the proof of (1), we first show the following equality:

$$\begin{aligned} B_h^{rc}(\boldsymbol{v}_h, \boldsymbol{w}_h) &= \sum_{K\in\mathcal{T}_h} \Big(-(v_h, \boldsymbol{b}\cdot\nabla w_h)_K + ((c - \mathrm{div}\boldsymbol{b})v_h, w_h)_K \\ &\quad + \langle [\boldsymbol{b}\cdot\boldsymbol{n}]_+ v_n - [\boldsymbol{b}\cdot\boldsymbol{n}]_- \hat{v}_h, w_h - \hat{w}_h\rangle_{\partial K} \Big). \end{aligned} \tag{4.8}$$

By Green's formula,

$$\begin{aligned} B_h^{rc}(\boldsymbol{v}_h, \boldsymbol{w}_h) &= \sum_{K\in\mathcal{T}_h} (v_h, -\boldsymbol{b}\cdot\nabla w_h)_K + ((c - \mathrm{div}\boldsymbol{b})v_h, w_h)_K \\ &\quad + \langle (\boldsymbol{b}\cdot\boldsymbol{n})v_h, w_h\rangle_{\partial K} + \langle v_n - \hat{v}_h, [\boldsymbol{b}\cdot\boldsymbol{n}]_- w_h - [\boldsymbol{b}\cdot\boldsymbol{n}]_+ \hat{w}_h\rangle_{\partial K} \\ &=: \sum_{K\in\mathcal{T}_h} (I_K + II_K + III_{\partial K}). \end{aligned}$$

Rewrite $III_{\partial K}$ as follows:

$$\begin{aligned} III_{\partial K} &= \langle ([\boldsymbol{b}\cdot\boldsymbol{n}]_+ - [\boldsymbol{b}\cdot\boldsymbol{n}]_-)v_h, w_h\rangle_{\partial K} + \langle v_h, [\boldsymbol{b}\cdot\boldsymbol{n}]_- w_h\rangle_{\partial K} \\ &\quad - \langle \hat{v}_h, [\boldsymbol{b}\cdot\boldsymbol{n}]_- w_h\rangle_{\partial K} - \langle v_h - \hat{v}_h, [\boldsymbol{b}\cdot\boldsymbol{n}]_+ \hat{w}_h\rangle_{\partial K} \\ &= \langle ([\boldsymbol{b}\cdot\boldsymbol{n}]_+ v_h - [\boldsymbol{b}\cdot\boldsymbol{n}]_- \hat{v}_h, w_h\rangle_{\partial K} - \langle v_h - \hat{v}_h, [\boldsymbol{b}\cdot\boldsymbol{n}]_+ \hat{w}_h\rangle_{\partial K}. \end{aligned}$$

Since

$$\sum_{K\in\mathcal{T}_h} \langle \hat{v}_h, ([\boldsymbol{b}\cdot\boldsymbol{n}]_+ - [\boldsymbol{b}\cdot\boldsymbol{n}]_-)\hat{w}_h\rangle_{\partial K} = 0,$$

we have

$$\sum_{K\in\mathcal{T}_h} III_{\partial K} = \sum_{K\in\mathcal{T}_h} \langle [\boldsymbol{b}\cdot\boldsymbol{n}]_+ v_h - [\boldsymbol{b}\cdot\boldsymbol{n}]_- \hat{v}_h, w_h - \hat{w}_h\rangle_{\partial K}.$$

Thus we obtain (4.8). Next, we will estimate I_K. Let us denote $\boldsymbol{\eta} = \{\eta, \hat{\eta}\} := \tilde{\boldsymbol{u}} - \boldsymbol{P}_h\boldsymbol{u}$, then

$$\begin{aligned} I_K &= (\boldsymbol{\eta}, -\boldsymbol{b}\cdot\nabla w_h)_K \\ &= (\boldsymbol{\eta}, (\boldsymbol{P}_h^0\boldsymbol{b} - \boldsymbol{b})\cdot\nabla w_h)_K - (\boldsymbol{\eta}, (\boldsymbol{P}_h^0\boldsymbol{b})\cdot\nabla w_h)_K. \end{aligned}$$

By the property of the projection $\boldsymbol{P}_h^0$ and $\boldsymbol{P}_h\boldsymbol{u} = \boldsymbol{P}_h\tilde{\boldsymbol{u}}$, we have

$$(\boldsymbol{\eta}, (\boldsymbol{P}_h^0\boldsymbol{b})\cdot\nabla w_h)_K = 0.$$

Using the inverse inequality, we see that

$$|I_K| = |(\boldsymbol{\eta}, (\boldsymbol{P}_h^0\boldsymbol{b} - \boldsymbol{b})\cdot\nabla w_h)_K| \le C|\boldsymbol{b}|_{1,\infty}\|\eta\|_{0,K}\|w_h\|_{0,K}. \tag{4.9}$$

By using the Schwarz inequality, we have

$$|II_K| \le C(|c|_{0,\infty} + |\boldsymbol{b}|_{1,K})\|v_h\|_{0,K}\|w_h\|_{0,K}, \tag{4.10}$$

and

$$\begin{aligned} \sum_{K\in\mathcal{T}_h} |III_{\partial K}| &\le \sum_{K\in\mathcal{T}_h} \langle |\boldsymbol{b}\cdot\boldsymbol{n}|(v_h - \hat{v}_h), w_h - \hat{w}_h\rangle_{\partial K} \qquad (4.11) \\ &\le \sum_{K\in\mathcal{T}_h} \||\boldsymbol{b}\cdot\boldsymbol{n}|^{1/2}(v_h - \hat{v}_h)\|_{0,\partial K} \cdot \||\boldsymbol{b}\cdot\boldsymbol{n}|^{1/2}(v_h - \hat{v}_h)\|_{0,\partial K} \end{aligned}$$

From (4.9), (4.10), and (4.11), we conclude that (4.8) holds.

We now turn to the proof of (2). By Green's formula, we have

$$\begin{aligned} B_h^{rc}(\boldsymbol{v}_h, \boldsymbol{v}_h) &= \sum_{K\in\mathcal{T}_h} \Big(\int_K (c - \mathrm{div}\boldsymbol{b}/2)v_h^2 dx + \frac{1}{2}\int_{\partial K} (\boldsymbol{b}\cdot\boldsymbol{n})v_h^2 dx \\ &\qquad + \langle v_h - \hat{v}_h, [\boldsymbol{b}\cdot\boldsymbol{n}]_- v_h - [\boldsymbol{b}\cdot\boldsymbol{n}]_+ \hat{v}_h\rangle_{\partial K} \Big) \\ &=: \sum_{K\in\mathcal{T}_h} (I_K + II_{\partial K} + III_{\partial K}). \end{aligned}$$

By the assumption (1.3), we have

$$I_K \ge \rho_0\|v_h\|_{0,K}^2.$$

Since $\boldsymbol{b}\cdot\boldsymbol{n} = [\boldsymbol{b}\cdot\boldsymbol{n}]_+ - [\boldsymbol{b}\cdot\boldsymbol{n}]_-$, we have

$$\begin{aligned} II_{\partial K} + III_{\partial K} &= \frac{1}{2}\left\langle v_h, [\boldsymbol{b}\cdot\boldsymbol{n}]_- v_h\right\rangle_{\partial K} - \left\langle \hat{v}_h, [\boldsymbol{b}\cdot\boldsymbol{n}]_- v_h\right\rangle_{\partial K} \\ &\qquad + \frac{1}{2}\left\langle v_h, [\boldsymbol{b}\cdot\boldsymbol{n}]_+ v_h\right\rangle_{\partial K} - \left\langle v_h, [\boldsymbol{b}\cdot\boldsymbol{n}]_+ \hat{v}_h\right\rangle_{\partial K} \\ &= \frac{1}{2}\left\langle ([\boldsymbol{b}\cdot\boldsymbol{n}]_- + [\boldsymbol{b}\cdot\boldsymbol{n}]_-)(v_h - \hat{v}_h), v_h - \hat{v}_h\right\rangle_{\partial K} \\ &\qquad + \frac{1}{2}\left\langle ([\boldsymbol{b}\cdot\boldsymbol{n}]_+ - [\boldsymbol{b}\cdot\boldsymbol{n}]_-)\hat{v}_h, \hat{v}_h\right\rangle_{\partial K} \\ &= \frac{1}{2}\left\langle |\boldsymbol{b}\cdot\boldsymbol{n}|(v_h - \hat{v}_h), v_h - \hat{v}_h\right\rangle_{\partial K} + \frac{1}{2}\left\langle (\boldsymbol{b}\cdot\boldsymbol{n})\hat{v}_h, \hat{v}_h\right\rangle_{\partial K}. \end{aligned} \tag{4.12}$$

Since

$$\sum_{K\in\mathcal{T}_h} \left\langle (\boldsymbol{b}\cdot\boldsymbol{n})\hat{v}_h, \hat{v}_h\right\rangle_{\partial K} = 0,$$

summing (4.12) over all elements $K \in \mathcal{T}_h$ gives us

$$\sum_{K\in\mathcal{T}_h} (II_{\partial K} + III_{\partial K}) = \frac{1}{2}\sum_{K\in\mathcal{T}_h} \||\boldsymbol{b}\cdot\boldsymbol{n}|^{1/2}(v_h - \hat{v}_h)\|_{0,\partial K}^2.$$

Thus we obtain the coercivity (4.8). □

From Lemma 4.1 and Lemma 4.2, we get the following lemma

Lemma 4.3. We have the following three properties.

1. (Galerkin orthogonality) Let u be the exact solution of (1.2), and let $\boldsymbol{u} = \{u, u|_{\Gamma_h}\}$. Let $\boldsymbol{u}_h$ be the approximate solution by (3.1). Then we have

$$B_h(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{v}_h) = 0 \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h.$$

2. There exists a constant C_b independent of h and ε such that

$$|B_h(\boldsymbol{v} - \boldsymbol{P}_h\boldsymbol{v}, \boldsymbol{w}_h)| \leq C_b \left|\!\left|\!\left| \boldsymbol{v} - \boldsymbol{P}_h\boldsymbol{v} \right|\!\right|\!\right| \left|\!\left|\!\left| \boldsymbol{w}_h \right|\!\right|\!\right| \qquad \boldsymbol{v} \in \boldsymbol{V}, \boldsymbol{w}_h \in \boldsymbol{V}_h.$$

3. (Coercivity) There exists a constant C_c independent of h and ε such that

$$B_h(\boldsymbol{v}_h, \boldsymbol{v}_h) \geq C_c \left|\!\left|\!\left| \boldsymbol{v}_h \right|\!\right|\!\right|^2 \qquad \boldsymbol{v}_h \in \boldsymbol{V}_{0h}. \tag{4.13}$$

Theorem 4.4. Let u be the exact solution of (1.2), and let $\boldsymbol{u} = \{u, u|_{\Gamma_h}\}$. Let $\boldsymbol{u}_h$ be the approximate solution by (3.1). Recall that we are assuming (H1), (H2), and (H3). If $u \in H^{k+1}(\Omega)$ then we have the following error estimate:

$$\left|\!\left|\!\left| \boldsymbol{u} - \boldsymbol{u}_h \right|\!\right|\!\right| \leq C(\varepsilon^{1/2} + h^{1/2})h^k |u|_{k+1}, \tag{4.14}$$

where C denotes a positive constant independent of h and ε.

Proof. By using the three properties in Lemma 4.3, we deduce that

$$\begin{aligned} C_c \left|\!\left|\!\left| \boldsymbol{u}_h - \boldsymbol{P}_h\boldsymbol{u} \right|\!\right|\!\right|^2 &\leq B_h(\boldsymbol{u}_h - \boldsymbol{P}_h\boldsymbol{u}, \boldsymbol{u}_h - \boldsymbol{P}_h\boldsymbol{u}) \\ &= B_h(\boldsymbol{u} - \boldsymbol{P}_h\boldsymbol{u}, \boldsymbol{u}_h - \boldsymbol{P}_h\boldsymbol{u}) \\ &= B_h(\boldsymbol{u} - \tilde{\boldsymbol{u}}, \boldsymbol{u}_h - \boldsymbol{P}_h\boldsymbol{u}) + B_h(\tilde{\boldsymbol{u}} - \boldsymbol{P}_h\boldsymbol{u}, \boldsymbol{u}_h - \boldsymbol{P}_h\boldsymbol{u}) \\ &\leq C_b(\left|\!\left|\!\left| \boldsymbol{u} - \tilde{\boldsymbol{u}} \right|\!\right|\!\right| + \left|\!\left|\!\left| \tilde{\boldsymbol{u}} - \boldsymbol{P}_h\boldsymbol{u} \right|\!\right|\!\right|) \left|\!\left|\!\left| \boldsymbol{u}_h - \boldsymbol{P}_h\boldsymbol{u} \right|\!\right|\!\right|. \end{aligned}$$

Hence we have

$$\left|\!\left|\!\left| \boldsymbol{u}_h - \boldsymbol{P}_h\boldsymbol{u} \right|\!\right|\!\right| \leq C(\left|\!\left|\!\left| \boldsymbol{u} - \tilde{\boldsymbol{u}} \right|\!\right|\!\right| + \left|\!\left|\!\left| \boldsymbol{u} - \boldsymbol{P}_h\boldsymbol{u} \right|\!\right|\!\right|).$$

By the triangle inequality and Lemma 2.2, we have

$$\begin{aligned} \left|\!\left|\!\left| \boldsymbol{u} - \boldsymbol{u}_h \right|\!\right|\!\right| &\leq \left|\!\left|\!\left| \boldsymbol{u}_h - \boldsymbol{P}_h\boldsymbol{u} \right|\!\right|\!\right| + \left|\!\left|\!\left| \boldsymbol{u} - \boldsymbol{P}_h\boldsymbol{u} \right|\!\right|\!\right| \\ &\leq (C+1) \left|\!\left|\!\left| \boldsymbol{u} - \boldsymbol{P}_h\boldsymbol{u} \right|\!\right|\!\right| + C \left|\!\left|\!\left| \boldsymbol{u} - \tilde{\boldsymbol{u}} \right|\!\right|\!\right| \\ &\leq C(\varepsilon^{1/2} + h^{1/2})h^k |u|_{k+1}. \end{aligned}$$

Thus, the proof is completed. □

5 The relation between $\boldsymbol{u}_h$ and the solution of the reduced problem

Let u_0 be the solution of the reduced problem of (1.2) :

$$\begin{aligned} \boldsymbol{b} \cdot \nabla u_0 + c u_0 &= f \text{ in } \Omega, &\text{(5.1a)}\\ u_0 &= g_D \text{ on } \Gamma_D. &\text{(5.1b)} \end{aligned}$$

Here we assume that $\Gamma_D = \Gamma_-$ and $g_N \equiv 0$, and suppose that the unique existence of a solution $u_0 \in H^2(\Omega)$ to (5.1). Let $\boldsymbol{u}_0 = \{u_0, u_0|_{\Gamma_h}\}$. The aim of this section is to show the approximate solution $\boldsymbol{u}_h$ is also close to $\boldsymbol{u}_0$ when ε is very small. This suggests that our hybridized DG method (3.1) is stable even when ε is sufficiently small.

Theorem 5.1. Let $\boldsymbol{u}_h$ be the approximate solution defined by (3.1), and let $\tilde{\boldsymbol{u}}_0$ be defined as above. Then we have the following inequality:

$$|||\boldsymbol{u}_h - \tilde{\boldsymbol{u}}_0||| \quad \leq C \left(|||\tilde{\boldsymbol{u}}_0|||_d + |||\tilde{\boldsymbol{u}}_0 - \boldsymbol{P}_h \tilde{\boldsymbol{u}}_0||| \right), \tag{5.2}$$

where C is a constant independent of ε and h.

Proof. By the consistency of $B_h^{rc}(\cdot,\cdot)$, we have

$$B_h^{rc}(\boldsymbol{u}_0, \boldsymbol{v}_h) = (f, v_h), \tag{5.3}$$

from which it follows that

$$B_h(\tilde{\boldsymbol{u}}_0, \boldsymbol{v}_h) = (f, v_h) + B_h^d(\boldsymbol{u}_0, \boldsymbol{v}_h) + B_h(\tilde{\boldsymbol{u}}_0 - \boldsymbol{u}_0, \boldsymbol{v}_h). \tag{5.4}$$

Subtracting (5.4) from (3.1) gives us

$$B_h(\tilde{\boldsymbol{u}}_0 - \boldsymbol{u}_h, \boldsymbol{v}_h) = B_h^d(\boldsymbol{u}_h^0, \boldsymbol{v}_h) + B_h(\tilde{\boldsymbol{u}}_0 - \boldsymbol{u}_0, \boldsymbol{v}_h) \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h. \tag{5.5}$$

Here we claim that

$$B_h(\tilde{\boldsymbol{u}}_0 - \boldsymbol{u}_0, \boldsymbol{v}_h) = B_h^d(\tilde{\boldsymbol{u}}_0 - \boldsymbol{u}_0, \boldsymbol{v}_h). \tag{5.6}$$

In fact, we have

$$\begin{aligned} B_h(\tilde{\boldsymbol{u}}_0 - \boldsymbol{u}_0, \boldsymbol{v}_h) &= B_h^d(\tilde{\boldsymbol{u}}_0 - \boldsymbol{u}_0, \boldsymbol{v}_h) + B_h^{rc}(\tilde{\boldsymbol{u}}_0 - \boldsymbol{u}_0, \boldsymbol{v}_h) \\ &= B_h^d(\tilde{\boldsymbol{u}}_0 - \boldsymbol{u}_0, \boldsymbol{v}_h) \\ &\quad + \left\langle \hat{P}_h(u_0|_{\partial\Omega}) - u_0|_{\partial\Omega}, [\boldsymbol{b}\cdot\boldsymbol{n}]_- v_h - [\boldsymbol{b}\cdot\boldsymbol{n}]_+ \hat{v}_h \right\rangle_{\partial\Omega\setminus\Gamma_-} \end{aligned}$$

Since $[\boldsymbol{b}\cdot\boldsymbol{n}]_-$ and $\hat{v}_h$ vanish on $\partial\Omega\setminus\Gamma_-$, we have (5.6). Thus (5.5) becomes

$$\begin{aligned} B_h(\tilde{\boldsymbol{u}}_0-\boldsymbol{u}_h,\boldsymbol{v}_h) &= B_h^d(\boldsymbol{u}_0,\boldsymbol{v}_h)+B_h^d(\tilde{\boldsymbol{u}}_0-\boldsymbol{u}_0,\boldsymbol{v}_h) \\ &= B_h^d(\tilde{\boldsymbol{u}}_h^0,\boldsymbol{v}_h) \quad \forall \boldsymbol{v}_h\in\boldsymbol{V}_h. \end{aligned} \tag{5.7}$$

Choosing $\boldsymbol{v}_h=\boldsymbol{u}_h-\boldsymbol{P}_h\tilde{\boldsymbol{u}}_0\in\boldsymbol{V}_{0h}$ in (4.13), we have

$$\begin{aligned} C_c\,|||\boldsymbol{u}_h-\boldsymbol{P}_h\tilde{\boldsymbol{u}}_0|||^2 &\leq B_h(\boldsymbol{u}_h-\boldsymbol{P}_h\tilde{\boldsymbol{u}}_0,\boldsymbol{u}_h-\boldsymbol{P}_h\tilde{\boldsymbol{u}}_0) \\ &= B_h(\boldsymbol{u}_h-\tilde{\boldsymbol{u}}_0,\boldsymbol{u}_h-\boldsymbol{P}_h\tilde{\boldsymbol{u}}_0)+B_h(\tilde{\boldsymbol{u}}_0-\boldsymbol{P}_h\tilde{\boldsymbol{u}}_0,\boldsymbol{u}_h-\boldsymbol{P}_h\tilde{\boldsymbol{u}}_0) \\ &\leq |B_h^d(\tilde{\boldsymbol{u}}_0,\boldsymbol{u}_h-\boldsymbol{P}_h\tilde{\boldsymbol{u}}_0)|+|B_h(\tilde{\boldsymbol{u}}_0-\boldsymbol{P}_h\tilde{\boldsymbol{u}}_0,\boldsymbol{u}_h-\boldsymbol{P}_h\tilde{\boldsymbol{u}}_0)| \\ &\leq C_b^d\,|||\tilde{\boldsymbol{u}}_0|||_d\,|||\boldsymbol{u}_h-\boldsymbol{P}_h\tilde{\boldsymbol{u}}_0|||_d+C_b\,|||\tilde{\boldsymbol{u}}_0-\boldsymbol{P}_h\tilde{\boldsymbol{u}}_0|||\,|||\boldsymbol{u}_h-\boldsymbol{P}_h\tilde{\boldsymbol{u}}_0||| \end{aligned}$$

Then we have

$$C_c\,|||\boldsymbol{u}_h-\boldsymbol{P}_h\tilde{\boldsymbol{u}}_0|||\leq C_b^d\,|||\tilde{\boldsymbol{u}}_0|||_d+C_b\,|||\tilde{\boldsymbol{u}}_0-\boldsymbol{P}_h\tilde{\boldsymbol{u}}_0|||\,.$$

By the triangle inequality,

$$\begin{aligned} |||\boldsymbol{u}_h-\tilde{\boldsymbol{u}}_0||| &\leq |||\boldsymbol{u}_h-\boldsymbol{P}_h\tilde{\boldsymbol{u}}_0|||+|||\boldsymbol{P}_h\tilde{\boldsymbol{u}}_0-\tilde{\boldsymbol{u}}_0||| \\ &\leq \frac{C_b^d}{C_c}\,|||\tilde{\boldsymbol{u}}_0|||_d+\left(1+\frac{C_b}{C_c}\right)|||\tilde{\boldsymbol{u}}_0-\boldsymbol{P}_h\tilde{\boldsymbol{u}}_0|||\,, \end{aligned}$$

where C_b^d, C_c, and C_b are independent of ε. Thus we obtain the inequality (5.2). □

6 Numerical results

6.1 Convection-dominated case

We consider the case that the diffusion coefficient is very small, $\varepsilon = 10^{-9}$, so that the exact solution has a boundary layer. Let Ω be the unit square domain, $\boldsymbol{b} = (1,1)^T$, and $c \equiv 0$. The example problem is as follows:

$$-\varepsilon \Delta u + \boldsymbol{b} \cdot \nabla u = f \text{ in } \Omega, \tag{6.1a}$$

$$u = 0 \text{ on } \Gamma_D = \partial\Omega, \tag{6.1b}$$

where f is given so that the exact solution is

$$u(x,y) = \sin(\pi x/2)\sin(\pi y/2)\left(1 - e^{(x-1)/\varepsilon}\right)\left(1 - e^{(y-1)/\varepsilon}\right).$$

This solution has a boundary layer near $x = 1$ or $y = 1$. The meshes we use are the rectangular meshes with the length of $h = 1/N$. We computed the approximate solutions for $h = 1/10, 1/20, 1/40$, and $1/80$ with linear elements. In Figure 3, we display the graph of the approximate solution for $h = 1/10$. We can see that no oscillation appears unlike the classical finite element method. Figure 1 shows that the convergence diagram in the L^2 norm and $H^1(\mathcal{T}_h)$ seminorm on $\Omega_{0.9} := (0, 0.9)^2$. Here we restrict the domain to $\Omega_{0.9}$ in order to remove the boundary layers. We observe that the convergence rates of the L^2-error and the $H^1(\mathcal{T}_h)$-error are optimal,i.e., h^2 and h, respectively. We also computed for $\varepsilon = 10^{-1}$ to compare with the convection-dominated case, see Figure 2. In this case, it can be observed that the convergence rates of the error on the entire Ω are h^2 and h in the L^2-norm and $H^1(\mathcal{T}_h)$-seminorm, respectively.

6.2 Rotating flow problem

Next, we consider the example where $\boldsymbol{b}$ is not constant. Let Ω be the unit square domain with a slit, i.e., $\Omega = (0,1)^2 \setminus \{(1/2, y) : 0 \le y \le 1/2\}$. We consider the same equation (6.1) for different coefficients: $\varepsilon = 10^{-9}$, $\boldsymbol{b} = (1/2 - y, x - 1/2)^T$, and $f \equiv 0$. The non-homogeneous Dirichlet boundary condition, $g_D(x,y) = \sin^2(2\pi y)$, is imposed on the inflow-side slit, and $g_D = 0$ otherwise, see Figure 4. We used the same meshes and finite element spaces as the previous example. In Figure 5, we display the graphs of the approximate solution u_h and $\hat{u}_h$ with $h = 1/20$. Figure 6 shows the cross section of u_h at $x = 1/2$, which confirms us that our method works well and is stable.

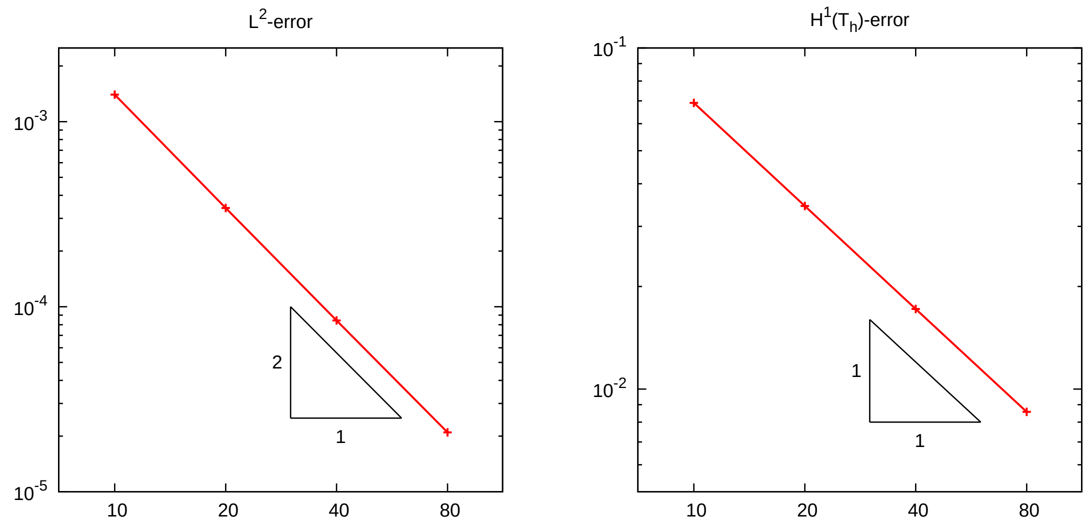

Figure. 1: L^2-error (left) and $H^1(\mathcal{T}_h)$-error(right) on $\Omega_{0.9}$ for $\varepsilon = 10^{-9}$.

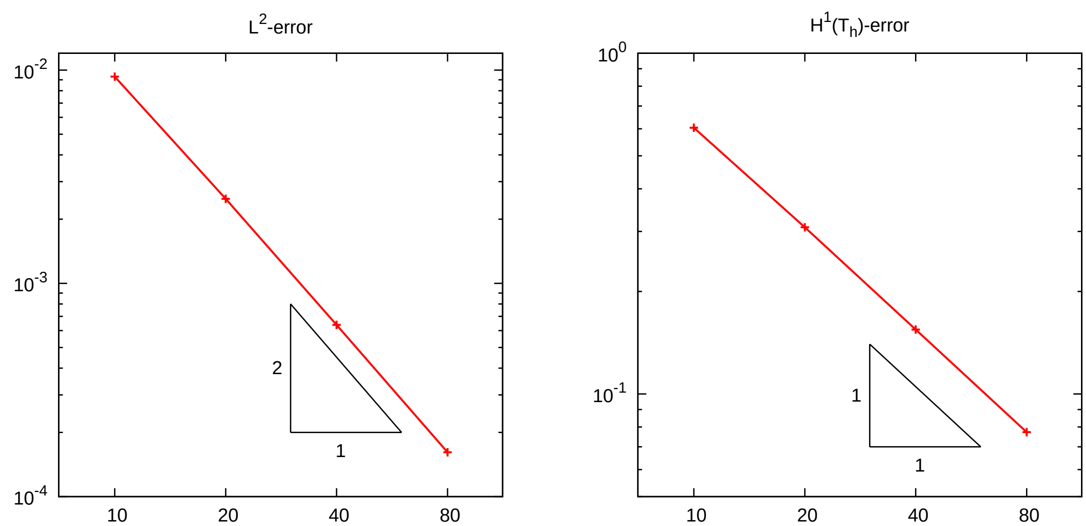

Figure. 2: L^2-error(left) and $H^1(\mathcal{T}_h)$-error(right) on Ω for $\varepsilon = 10^{-1}$.

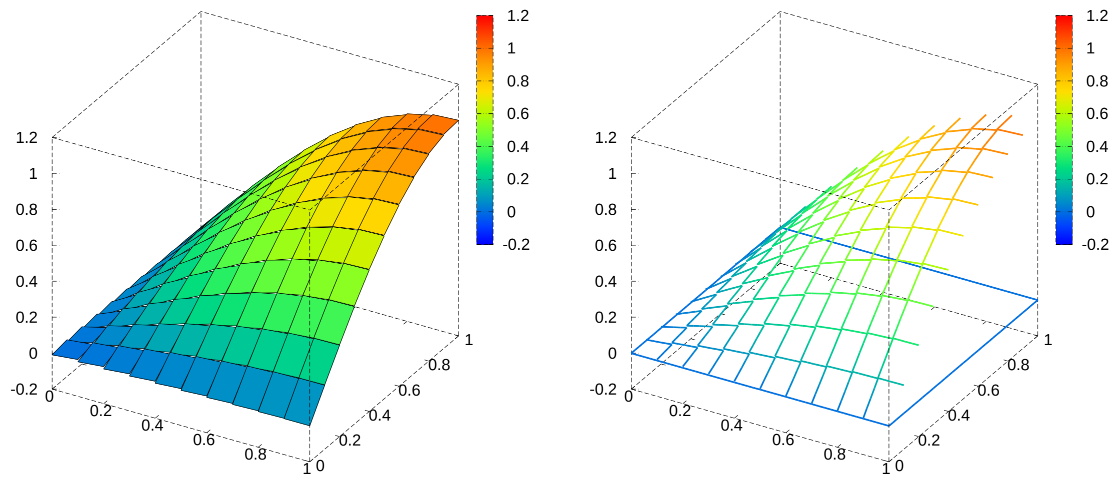

Figure. 3: Approximate solutions u_h(left) and $\hat{u}_h$(right) for $h = 1/10$ and $\varepsilon = 10^{-9}$.

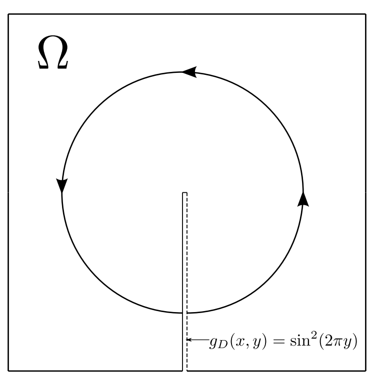

Figure. 4: Rotating flow problem.

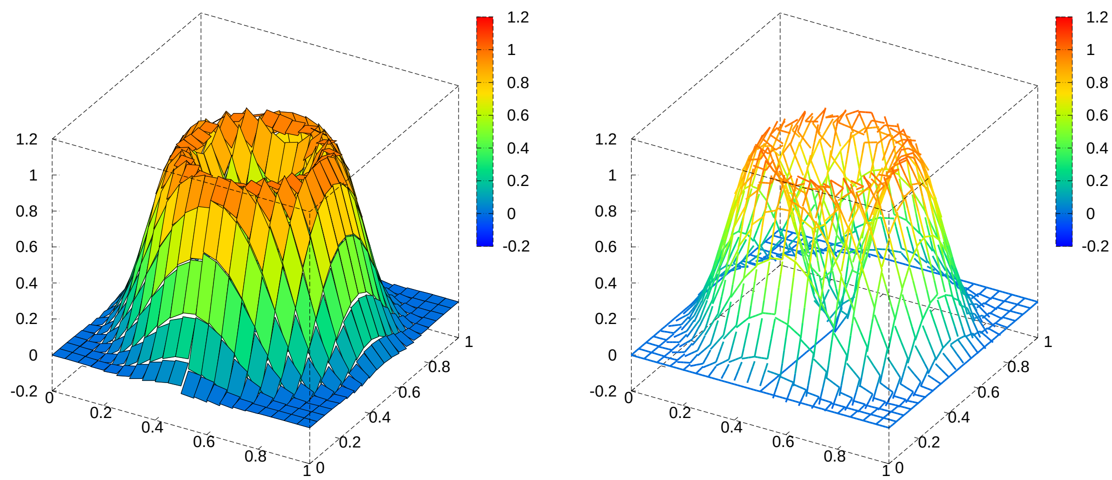

Figure. 5: Approximate solutions u_h(left) and $\hat{u}_h$(right) of the rotating flow problem for $h = 1/20$.

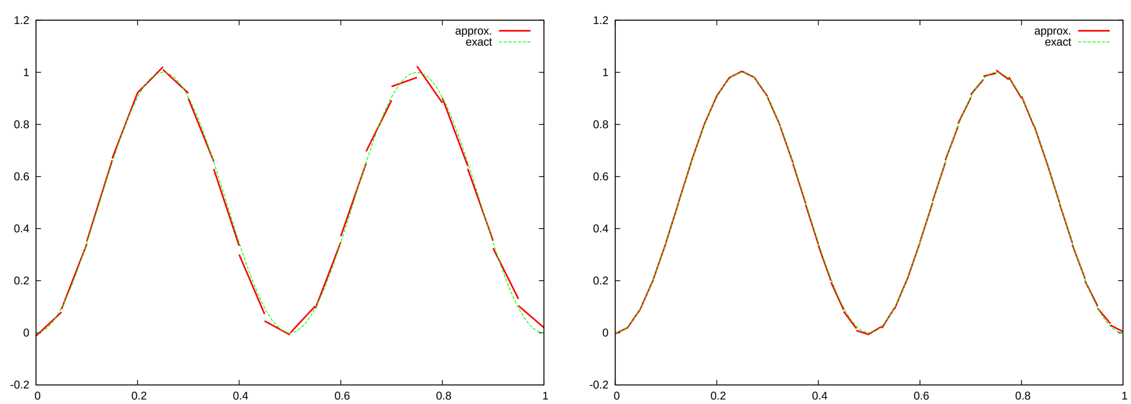

Figure. 6: Approximate solution u_h at $x = 1/2$ of the rotating flow problem for $h = 1/20$ (left) and $h = 1/40$ (right).

7 Conclusions

We have presented a new hybridized scheme for the convection-diffusion-reaction problems. In our formulation, a unwinding term is added to stabilize the convection-reaction part. As a result, our scheme is stable even when $\varepsilon \downarrow 0$. Indeed, numerical results show that no oscillation appears in our approximate solutions. We have proved the error estimates of optimal order in the HDG norm, and discussed the relation between our approximate solution and the solution of the reduced problem.

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