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ANALYSIS OF THE FICTITIOUS DOMAIN METHOD WITH L^2-PENALTY FOR ELLIPTIC AND PARABOLIC PROBLEMS*

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Abstract. The fictitious domain method with L^2-penalty for elliptic and parabolic problems are considered, respectively. The regularity theorems and a priori estimates for L^2-penalty problems are given. We derive error estimates for penalization and finite element interpolation with $P1$-element. Numerical experiments are performed to confirm the theoretical results.

Key words. fictitious domain method, error estimate, penalization

AMS subject classifications. 65M85

1. Introduction. The purpose of this paper is to establish a mathematical study of the fictitious domain method for elliptic and parabolic problems. The fictitious domain method is well known to be based on a reformulation of the original problem in a larger spatial domain, called the fictitious domain, with a simple shape. Then, the fictitious domain can be discretized by a uniform mesh, independent of the original boundary. The advantage of this approach is that we can avoid the time-consuming construction of a boundary-fitted mesh. Furthermore, this approach will be useful to solve time-dependent moving-boundary problems. In our previous reports ([14, 15]), we developed a mathematical theory for the H^1-penalty fictitious domain method for elliptic and parabolic problems. The aim of this paper is to establish rigorous estimates of the errors induced by L^2 penalization and finite element interpolation. We examine the L^2 penalization by studying the H^2 regularity and estimates of the L^2-penalty problem, which is a different approach from [1], where the L^2 penalization for Navier-Stokes equation is considered without numerical analysis. Thanks to our regularity and estimate results, the finite element analysis becomes easy to treat. Our error estimates in the H^1 norm of L^2 penalization for elliptic and parabolic problems maintain the sharpness of those for Navier-Stokes problems in [1]; moreover, we show the error estimates of L^2 norm. The convergence of L^2 penalization for elliptic and parabolic problems has been proved in [7]; however, no error estimate has been found, neither the finite element analysis. Our analysis method presented here can also be applied to Stokes and Navier-Stokes problems with little difficulty.

The rest of this paper is arranged as follow. In Sect. 2, we consider the elliptic problem. We first show the error estimates for L^2 penalization, then we turn to the finite element approximation. And Sect. 3 is devoted to the parabolic problem, as the same way to the elliptic case. The numerical experiments to validate the theoretical results are presented in the last section.

2. The fictitious domain method with L^2-penalty for elliptic problem. Let Ω be a bounded connected domain in $\mathbb{R}^2$. Throughout this paper, we assume that the boundary $\partial\Omega = \Gamma$ is of class C^2. We consider the original elliptic problem (EQ):

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma, \end{cases} \tag{2.1}$$

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the weak form of which reads as:

$$\begin{cases} \text{Find } u \in H_0^1(\Omega), \text{ such that} \\ (\nabla u, \nabla v)_\Omega = (f, v)_\Omega, \ \forall v \in H_0^1(\Omega), \end{cases} \tag{2.2}$$

where $(\cdot,\cdot)_\Omega$ denotes the inner-product of $L^2(\Omega)$, f is a given function of $L^2(\Omega)$.

To implement the fictitious domain method with L^2 penalization, we assume there exists a rectangular domain $D \supset \Omega$ and denote $\Omega_1 = D \backslash \overline{\Omega}$. The L^2-penalty problem (EQ_ϵ) reads as

$$\begin{cases} \text{Find } u_\epsilon \in H_0^1(D), \text{ such that} \\ (\nabla u_\epsilon, \nabla v)_D + \dfrac{1}{\epsilon}(u_\epsilon, v)_{\Omega_1} = (\tilde{f}, v)_D, \ \forall v \in H_0^1(D), \end{cases} \tag{2.3}$$

where $0 < \epsilon \ll 1$, and $\tilde{f} \in L^2(D)$ is some extension of f onto D. Applying Green's formula, (EQ_ϵ) is equivalent to

$$\begin{cases} -\Delta u_\epsilon = f \text{ in } \Omega, \quad -\Delta u_\epsilon + \dfrac{1}{\epsilon} u_\epsilon = \tilde{f} \text{ in } \Omega_1, \\ u_\epsilon|_\Omega = u_\epsilon|_{\Omega_1} \text{ on } \Gamma, \quad \left.\dfrac{\partial u_\epsilon}{\partial n}\right|_\Omega = \left.\dfrac{\partial u_\epsilon}{\partial n}\right|_{\Omega_1} \text{ on } \Gamma, \ u_\epsilon = 0 \text{ on } \partial D, \end{cases} \tag{2.4}$$

where $v|_\Omega$ denotes the restriction of v on Ω, and n is the outer normal vector.

2.1. Error estimate for penalization. In the following, we denote by C some constant independent of ϵ, and $\|\cdot\|_{n,\Omega}$ is the norm of $H^n(\Omega)$. We show the main theorem for error estimate of penalization,

THEOREM 2.1. *There exist unique solutions $u \in H^1(\Omega)$ and $u_\epsilon \in H_0^1(D)$ for (2.2) and (2.3), respectively. Moreover, if $\tilde{f}$, the extension of f, satisfies $\|\tilde{f}\|_{0,\Omega_1} \leq C\|f\|_{0,\Omega}$, then we have $u_\epsilon|_\Omega \in H^2(\Omega)$, $u_\epsilon|_{\Omega_1} \in H^2(\Omega_1)$,*

$$\|u_\epsilon\|_{2,\Omega} \leq C\|f\|_{0,\Omega}, \ \|u_\epsilon\|_{2,\Omega_1} \leq C\epsilon^{-\frac{1}{4}}\|f\|_{0,\Omega}, \ \|u_\epsilon\|_{0,\Omega_1} \leq C\epsilon^{\frac{3}{4}}\|f\|_{0,\Omega}. \tag{2.5}$$

Furthermore, we have

$$\|u_\epsilon\|_{1,\Omega_1} \leq C\epsilon^{\frac{1}{4}}\|f\|_{0,\Omega}, \ \|u - u_\epsilon|_\Omega\|_{1,\Omega} \leq \epsilon^{\frac{1}{4}}\|f\|_{0,\Omega}, \ \|u - u_\epsilon|_\Omega\|_{0,\Omega} \leq \epsilon^{\frac{1}{2}}\|f\|_{0,\Omega}. \tag{2.6}$$

Before stating the proof, we give some lemmas.

LEMMA 2.2. *For $g \in H^{\frac{1}{2}}(\Gamma)$ and $\eta > 0$, there exists $v = v_\eta \in H^2(\Omega)$ such that,*

$$\frac{\partial v}{\partial n} = g, \ \|v\|_{0,\Omega} \leq C\eta^3\|g\|_{\frac{1}{2},\Gamma}, \ |v|_{2,\Omega} \leq C\eta^{-1}\|g\|_{\frac{1}{2},\Gamma},$$

where

$$|v|_{2,\Omega}^2 = \sum_{i,j=1}^{2} \|\frac{\partial^2 v}{\partial x_i \partial x_j}\|_{0,\Omega}^2,$$

is the semi-norm of $H^2(\Omega)$.

Proof. We only consider the case that $\Omega = \mathbb{R}_+^N$, since for general domain, transformations of domains between Ω and $\mathbb{R}_+^N$ can be applied(see [12]).

We set $\hat{v}(\xi)$ is the Fourier transform of $v(x)$. $\xi' = (\xi_1, \dots, \xi_{N-1})$. We add a slightly change of the extension formula in [9](Theorem 5.2, Chapter 2), and

$$\hat{v}(\xi', x_N) = x_N \exp\left(-(1+|\xi'|)\eta^{-2}, x_N\right)\hat{g}(\xi'). \tag{2.7}$$

Indeed, let $|\alpha| \le 2$, let us consider $w_\alpha = D^\alpha v$ in $\mathbb{R}^N_+$ and set $w_\alpha = 0$ for $x_N < 0$. Let us denote $\alpha = (\alpha_1, \dots, \alpha_N,)$, and $\alpha = (\alpha', \alpha_N)$. Hence $\hat{w_\alpha}(\xi)$ is a finite sum of expressions like

$$aI(\xi) = a\int_0^\infty e^{(-ix_N\xi_N)}(\xi')^{\alpha'}((1+|\xi'|)\eta^{-2})^{\alpha_N - j} x_N^{1-j}\exp\left(-(1+|\xi'|)\eta^{-2}, x_N\right)\hat{g}(\xi')dx_N,$$

where a is a constant, $j = 0, 1$. We have:

$$I(\xi) = \frac{(\xi')^{\alpha'}((1+|\xi'|)\eta^{-2})^{\alpha_N - j}\hat{g}(\xi')}{((1+|\xi'|)\eta^{-2} + i\xi_N)^{2-j}},$$

and so

$$\|I(\xi)\|^2_{0,\mathbb{R}^N} = C\int_{\mathbb{R}^{N-1}} (\xi')^{2\alpha'}((1+|\xi'|)\eta^{-2})^{2\alpha_N - 3}|\hat{g}(\xi')|^2 d\xi' \le \begin{cases} C\eta^{-2}\|g\|^2_{\frac{1}{2},\Gamma}, & \alpha_N = 2, \\ C\eta^{2}\|g\|^2_{\frac{1}{2},\Gamma}, & \alpha_N = 1, \\ C\eta^{6}\|g\|^2_{\frac{1}{2},\Gamma}, & \alpha_N = 0. \end{cases}$$

Thus, we show the results. □

Lemma 2.3. *For $f \in L^2(\Omega)$, there exists a unique solution $u \in H^2(\Omega) \cap H^1_0(\Omega)$ such that*

$$\begin{cases} -\Delta u + \dfrac{1}{\epsilon}u = f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma, \end{cases}$$

with estimates $\|u\|_{2,\Omega} \le C\|f\|_{0,\Omega}$ and $\|u\|_{0,\Omega} \le C\epsilon\|f\|_{0,\Omega}$.

Proof. The existence and uniqueness of $u \in H^1_0(\Omega)$ is obvious in view of the Lax-Milgram theory. Setting $v = u$ into the weak form

$$(\nabla u, \nabla v)_\Omega + \frac{1}{\epsilon}(u, v)_\Omega = (f, v)_\Omega,$$

we have $\|u\|^2_{0,\Omega} \le \epsilon(f, u)_\Omega \le \epsilon\|f\|_{0,\Omega}\|u\|_{0,\Omega}$. Hence, $\|u\|_{0,\Omega} \le C\epsilon\|f\|_{0,\Omega}$. Since $f - \frac{1}{\epsilon}u \in L^2(\Omega)$, we have $u \in H^2(\Omega)$, and $\|u\|_{2,\Omega} \le C(\|f\|_{0,\Omega} + \frac{1}{\epsilon}\|u\|_{0,\Omega})$ by the standard regularity theory of elliptic equations. □

Lemma 2.4. *Replacing the boundary condition of the problem in Lemma 2.3 with the Neumann boundary $\frac{\partial v}{\partial n} = g$, for $g \in H^{\frac{1}{2}}(\Gamma)$, we have $\|u\|_{0,\Omega} \le C(\epsilon\|f\|_{0,\Omega} + \epsilon^{\frac{3}{4}}\|g\|_{\frac{1}{2},\Gamma})$, $\|u\|_{2,\Omega} \le C(\|f\|_{0,\Omega} + \epsilon^{-\frac{1}{4}}\|g\|_{\frac{1}{2},\Gamma})$.*

Proof. By Lemma 2.2, there exists $v \in H^2(\Omega)$ such that $\|v\|_{0,\Omega} \le C\epsilon^{\frac{3}{4}}\|g\|_{\frac{1}{2},\Gamma}$ and $\|v\|_{2,\Omega} \le C\epsilon^{-\frac{1}{4}}\|g\|_{\frac{1}{2},\Gamma}$. Setting $w = u - v$, we have

$$-\Delta w + \frac{1}{\epsilon}w = f - \Delta v + \frac{1}{\epsilon}v \;\text{ in } \Omega, \quad \frac{\partial w}{\partial n} = 0 \;\text{ on } \Gamma.$$

With the same analogue of the proof of Lemma 2.3, we can obtain the results. □

Remark 1. *If we replace the boundary condition of the problem in Lemma 2.4 with the mixed boundary* $\frac{\partial v}{\partial n} = g$ *on* Γ_1*, for* $g \in H^{\frac{1}{2}}(\Gamma_1)$*, and* $u = 0$ *on* Γ_2*, where* $\overline{\Gamma_1} \cap \overline{\Gamma_2} = \emptyset, \Gamma_1 \cup \Gamma_2 = \Gamma$ *, we can obtain the same results of Lemma 2.4.*

It has been proved(see Theorem I-4 in [7]) for $\tilde{f}$ being the zero extension of f that

$$\|u_\epsilon - u\|_{1,\Omega} \to 0, \quad \frac{1}{\sqrt{\epsilon}}\|u_\epsilon\|_{0,\Omega_1} \to 0, \quad \text{as } \epsilon \to 0, \tag{2.8}$$

Before the proof of Theorem 2.1, we show a simple a priori estimate here. Substituting $v = u_\epsilon$ in (2.3), we have

$$\|u_\epsilon\|_{1,\Omega}^2 + \|u_\epsilon\|_{1,\Omega_1}^2 + \frac{1}{\epsilon}\|u_\epsilon\|_{0,\Omega_1}^2 \le \|f\|_{0,\Omega}\|u_\epsilon\|_{0,\Omega} + \frac{1}{2}\epsilon\|\tilde{f}\|_{0,\Omega_1}^2 + \frac{1}{2\epsilon}\|u_\epsilon\|_{0,\Omega_1}^2,$$

with assumption that $\|\tilde{f}\|_{0,\Omega_1} \le C\|f\|_{0,\Omega}$, we have

$$\|u_\epsilon\|_{1,D} \le C\|\tilde{f}\|_{0,\Omega}, \quad \frac{1}{\sqrt{\epsilon}}\|u_\epsilon\|_{0,\Omega_1} \le C\|\tilde{f}\|_{0,\Omega}. \tag{2.9}$$

Proof. [Proof of Theorem 2.1] For the time being, we admit $u_\epsilon|_\Omega \in H^2(\Omega)$ and

$$\|u_\epsilon\|_{2,\Omega} \le C\|f\|_{0,\Omega}. \tag{2.10}$$

In view of problem (2.4) and the trace theorem, we have

$$\left\|\frac{\partial u_\epsilon}{\partial n}\right\|_{\frac{1}{2},\Gamma} \le C\|u_\epsilon\|_{2,\Omega} \le C\|f\|_{0,\Omega}.$$

Then, by Remark 1, we can conclude that

$$\|u_\epsilon\|_{2,\Omega} \le C(\epsilon^{-\frac{1}{4}}\|f\|_{0,\Omega} + \|\tilde{f}\|_{0,\Omega_1}), \ \|u_\epsilon\|_{0,\Omega} \le C(\epsilon^{\frac{3}{4}}\|f\|_{0,\Omega} + \epsilon\|\tilde{f}\|_{0,\Omega_1}). \tag{2.11}$$

Since $|u_\epsilon|_{1,\Omega_1} \le C(\eta|u_\epsilon|_{2,\Omega_1} + \eta^{-1}\|u_\epsilon\|_{0,\Omega})$, $\forall \eta > 0$(see Theorem 7.27 in [3]), setting $\eta = \epsilon^{\frac{1}{2}}$, with the assumption that $\|\tilde{f}\|_{0,\Omega_1} \le C\|f\|_{0,\Omega}$, we have $\|u_\epsilon\|_{1,\Omega_1} \le C\epsilon^{\frac{1}{4}}\|f\|_{0,\Omega}$. Following from trace theorem,

$$\|u_\epsilon\|_{\frac{1}{2},\Gamma} \le C\|u_\epsilon\|_{1,\Omega_1} \le C\epsilon^{\frac{1}{4}}\|f\|_{0,\Omega}.$$

Setting $\phi = u_\epsilon|_\Omega - u$, ϕ satisfies, in the sense of distribution,

$$-\Delta\phi = 0 \text{ in } \Omega, \quad \phi = u_\epsilon \text{ on } \Gamma,$$

which gives the error estimates of penalization in H^1 norm in view of the isomorphism of operator Δ,

$$\|u_\epsilon|_\Omega - u\|_{1,\Omega} = \|\phi\|_{1,\Omega} \le C(\|-\Delta\phi\|_{-1,\Omega} + \|\phi\|_{\frac{1}{2},\Gamma}) \le C\|u_\epsilon\|_{\frac{1}{2},\Gamma} \le C\epsilon^{\frac{1}{4}}\|f\|_{0,\Omega}. \tag{2.12}$$

To obtain an error estimate in L^2 norm, we introduce the adjoint problems for (2.2) and (2.3), which read as,

$$\begin{cases} \text{Find } u_F \in H_0^1(\Omega), \text{ such that} \\ (\nabla u_F, \nabla v)_\Omega = (F, v)_\Omega, \ \forall v \in H_0^1(\Omega), \end{cases} \tag{2.13}$$

$$\begin{cases} \text{Find } u_{F\epsilon} \in H_0^1(D), \text{ such that} \\ (\nabla u_{F\epsilon}, \nabla v)_D + \dfrac{1}{\epsilon}(u_{F\epsilon}, v)_{\Omega_1} = (\tilde{F}, v)_D, \ \forall v \in H_0^1(D), \end{cases} \tag{2.14}$$

for any $F \in L^2(\Omega)$, and the extension of F, $\tilde{F} \in L^2(D)$, satisfying $\|\tilde{F}\|_{0,\Omega_1} \le C\|F\|_{0,\Omega}$.

Apparently, we can obtain the a priori estimates and H^1 norm penalization error estimate, like (2.11) and (2.12), for the adjoint problems (2.13) and (2.14), such that

$$\|u_{F\epsilon}\|_{2,\Omega} \le C(\epsilon^{-\frac{1}{4}}\|F\|_{0,\Omega} + \|\tilde{F}\|_{0,\Omega_1}), \ \|u_{F\epsilon}\|_{0,\Omega} \le C(\epsilon^{\frac{3}{4}}\|F\|_{0,\Omega} + \epsilon\|\tilde{F}\|_{0,\Omega_1}). \tag{2.15}$$

$$\|u_{F\epsilon}|_\Omega - u_F\|_{1,\Omega} \le C\epsilon^{\frac{1}{4}}\|F\|_{0,\Omega}. \tag{2.16}$$

Denoting by $\tilde{u}$ and $\tilde{u}_F$ the zero extension of u and u_F, respectively, one can show that

$$(\nabla u_\epsilon, \nabla \tilde{u}_F)_D = (\tilde{u}_F, \tilde{f})_D = (u_F, f)_\Omega = (\nabla u_F, \nabla u)_\Omega = (F, u)_\Omega = (\tilde{F}, \tilde{u})_D = (\nabla u_{F\epsilon}, \nabla \tilde{u})_D,$$

following from which, we have

$$(\nabla(u_{F\epsilon} - \tilde{u}_F), \nabla(u_\epsilon - \tilde{u}))_D = (\tilde{F}, u_\epsilon - \tilde{u})_D - \frac{1}{\epsilon}(u_{F\epsilon}, u_\epsilon)_{\Omega_1}.$$

Let $\tilde{F} = u_\epsilon - \tilde{u}$, we have

$$\|u_\epsilon - \tilde{u}\|_{0,\Omega}^2 + \|u_\epsilon\|_{0,\Omega_1}^2 = (\nabla(u_{F\epsilon} - \tilde{u}_F), \nabla(u_\epsilon - \tilde{u}))_D + \frac{1}{\epsilon}(u_{F\epsilon}, u_\epsilon)_{\Omega_1},$$

Following from (2.11), (2.12), (2.15) and (2.16), we have

$$\|u_\epsilon|_\Omega - u\|_{0,\Omega} \le C\epsilon^{\frac{1}{2}}\|f\|_{0,\Omega}. \tag{2.17}$$

Thus, we obtain the a priori estimates of u_ϵ, and the error estimates of penalization in H^1 norm and L^2 norm.

At this stage, we go back to the beginning of the proof, it iremains to show (2.10). For interface problem (2.4), $u_\epsilon|_\Omega \in H^2(\Omega)$ follows the standard regularity theory, but we need to show the norm is independent of ϵ. We use the well-known method of tangential differential quotients due to Nirenberg(see Theorem 2.2.2.3 in [4], Appendix in [10], or Theorem 3.1 in [14]).

Let U_j be an open subset in $\mathbb{R}^2$, and there exists C^2 diffeomorphism Φ_j, with $\Psi_j = \Phi_j^{-1}$, $j = 1, 2, \ldots, N$, such that

$$\overline{\Omega} \subset \cup_{j=1}^N \Phi_j(U_j) \subset D. \ U_{j0} := \Psi_j(\Phi_j(U_j) \cap \Omega) = \mathbb{R}_+^2 \cap U_j,$$

$$U_{j1} := \Psi_j(\Phi_j(U) \cap \Omega_1) = \mathbb{R}_-^2 \cap U_j, \quad j = 1, 2, \ldots, N.$$

And also, there exists $\theta_j \in C_0^\infty(\overline{\Omega})$ with *supp* $\theta_j \subset \Phi_j(U_j)$, $j = 1, 2, \ldots, N$, with

$$\sum_{j=1}^N \theta_j = 1, \text{ on } \overline{\Omega}.$$

Hence, $(\theta_j u_\epsilon) \circ \Phi_j \in H_0^1(U_j)$, $j = 1, 2, \ldots, N$. We omit the index j and write U, U_1, U_0, Φ, Ψ, θ instead. Setting $u_1 = \theta u_\epsilon$ and $u_2 := (\theta u_\epsilon) \circ \Phi$. If $\|u_1\|_{2,\Omega}$

or $\|u_2\|_{2,U_0}$ are bounded by $C\|f\|_{0,\Omega}$, then, taking a summation of $j=1,\ldots,N$, we get $\|u\|_{2,\Omega} \le C\|f\|_{0,\Omega}$. (1)The case $U_1=\emptyset$. It is apparently that $u_1 \in H^2(\Omega)$ and $\|u_1\|_{2,\Omega} \le C\|\tilde{f}\|_{0,D}$. (2)The case $U_0 \neq \emptyset$, and $U_1 \neq \emptyset$. Setting $D_i = \frac{\partial}{\partial x_i}$, $(i=1,2)$, $u_2 \in H_0^1(U)$ satisfies

$$\sum_{i,j=1}^{2} \int_U a_{ij} D_i u_2 D_j v dx + \frac{1}{\epsilon} \sum_{i,j=1}^{2} \int_{U_1} D_i u_2 D_j v |D\Phi| dx = (f_2, v),\ \forall v \in H_0^1(U), \quad (2.18)$$

where $f_2 = (\theta\tilde{f} + \nabla u_\epsilon \nabla\theta + \nabla\cdot(u_\epsilon\nabla\theta)) \circ \Phi|D\Phi|$.

$$a_{ij} = (\sum_{k=1}^{2} D_k\psi_i D_k\psi_j) \circ \Phi|D\Phi|,\ i,j=1,2,\ \Psi=(\psi_1,\psi_2).$$

Let $\tilde{u}_2$ be the zero extension of u_2 onto $\mathbb{R}^2$. Substituting $v = \frac{\tau_h - 1}{h}\frac{\tau_{-h}-1}{h}\tilde{u}_2$ into (2.18), where τ_h is the translation operator with $\tau_h\phi(x) = \phi(x_1+h, x_2)$, $\phi(x) \in L^2(\mathbb{R}^2)$, after some computation, we have

$$\begin{aligned} &\sum_{i=1}^{2} \left\| D_i \left(\frac{\tau_h - 1}{h}\tilde{u}_2 \right) \right\|_{0,U}^2 + \frac{1}{\epsilon} \sum_{i=1}^{2} \left\| \frac{\tau_h - 1}{h}\tilde{u}_2 \right\|_{0,U_1}^2 \\ \le& C \sum_{i=1}^{2} \left\| D_i \left(\frac{\tau_h - 1}{h}\tilde{u}_2 \right) \right\|_{0,U}^2 + C\frac{1}{\epsilon}\|\tilde{u}_2\|_{0,U_1}^2 + C\|f_2\|_{0,U}^2, \end{aligned}$$

applying (2.8) or (2.9), we have $\sum_{i=1}^{2} \left\| D_i\left(\frac{\tau_h-1}{h}\tilde{u}_2\right)\right\|_{U_0} \le C\|f\|_{0,\Omega}$. Let $h \to 0$, we conclude $D_iD_1u_2 \in L^2(U_0)$, and $\|D_iD_1u_2\|_{0,U_0} \le C\|\tilde{f}\|_{0,\Omega}$, for $i=1,2$(we use several lemmas of Theorem 2.2.2.3 in [4] here). Then, we see that,

$$D_2^2 u_2 = \frac{1}{a_{22}}(f_2 - \sum_{k+l\le 3} D_l(a_{kl}D_k u_2) - D_2 a_{22} D_2 u_2), \quad \text{in } U_0,$$

following from which, we obtain $\|u_2\|_{2,U_0} \le C\|\tilde{f}\|_{0,\Omega}$. Hence, we complete the proof. □

2.2. Error estimate for finite element approximation. We introduce a Cartesian mesh to the rectangular domain D to get a uniform triangulation $\mathcal{T}_h$, where h is the maximum diameter of the triangles of $\mathcal{T}_h$. $V_h(D) \subset H_0^1(D)$ is the subspace of all piecewise linear continuous functions subordinate to $\mathcal{T}_h$. The discrete problem for (2.3) reads as,

$$\begin{cases} \text{Find } u_{\epsilon h} \in V_h(D), \text{ such that} \\ (\nabla u_{\epsilon h}, \nabla v_h)_D + \frac{1}{\epsilon}(u_{\epsilon h}, v_h)_{\Omega_1} = (\tilde{f}, v_h)_D,\ \forall v_h \in V_h(D), \end{cases} \quad (2.19)$$

To consider the error estimates of $u_\epsilon - u_{\epsilon h}$, we give the following lemma.

Lemma 2.5. *u_ϵ and $u_{\epsilon h}$ are the solutions of (2.3) and (2.19), respectively, then*

$$\|\nabla(u_\epsilon - u_{\epsilon h})\|_{0,D} + \frac{1}{\sqrt{\epsilon}}\|u_\epsilon - u_{\epsilon h}\|_{0,\Omega_1} \le C \inf_{v_h \in V_h(D)} \left(\|\nabla(u_\epsilon - v_h)\|_{0,D} + \frac{1}{\sqrt{\epsilon}}\|u_\epsilon - v_h\|_{0,\Omega_1} \right). \quad (2.20)$$

Proof. It follows from $(\nabla(u_\epsilon - u_{\epsilon h}), \nabla v_h)_D + \frac{1}{\epsilon}(u_\epsilon - u_{\epsilon h}, v_h) = 0, \forall v_h \in V_h(D)$. □

THEOREM 2.6. *Suppose that u_ϵ and $u_{\epsilon h}$ are the solutions of (2.3) and (2.19), respectively, then*

$$\|\nabla(u_\epsilon - u_{\epsilon h})\|_{0,D} + \frac{1}{\sqrt{\epsilon}}\|u_\epsilon - u_{\epsilon h}\|_{0,\Omega_1} \leq C\|f\|_\Omega(h^{\frac{1}{2}} + \epsilon^{\frac{1}{4}}), \tag{2.21}$$

$$\|u_\epsilon - u_{\epsilon h}\|_{0,\Omega} \leq C\|f\|_{0,\Omega}(h^{\frac{1}{2}} + \epsilon^{\frac{1}{4}})^2. \tag{2.22}$$

Proof. We define some notations first. K is some closed triangle of $\mathcal{T}_h$, and we denote $\Lambda(K) = (\nu_1^K, \nu_2^K, \nu_3^K)$ as the set of all vertices of K. $T_\Gamma = \{K | K \cap \Gamma \neq \emptyset\}$, $T' = \{K \subset \Omega | K \cap T_\Gamma = \emptyset\}$. $I_h u_\epsilon$ is the linear interpolation of u_ϵ. We define v_h by setting,

$$v_h(\nu) = \begin{cases} 0 & \text{for } \nu \in \Lambda(K), K \subset T_\Gamma \cup \overline{\Omega_1}, \\ u_\epsilon(\nu) & \text{for all other vertices } \nu, \end{cases}$$

and substitute this v_h into (2.20), then, following from the a priori estimates in Theorem 2.1, we have

$$\|u_\epsilon - v_h\|_{0,\Omega_1} = \|u_\epsilon\|_{0,\Omega} \leq C\epsilon^{\frac{3}{4}}\|f\|_{0,\Omega},$$

$$\begin{aligned} &\|\nabla(u_\epsilon - v_h)\|_{0,D} = \|\nabla(u_\epsilon - v_h)\|_{0,\Omega} \\ \leq &C(\|\nabla(u_\epsilon - I_h u_\epsilon)\|_{0,T'} + \|\nabla u_\epsilon\|_{0,\Omega\setminus T'} + \|\nabla v_h\|_{0,\Omega\setminus T'}) \\ \leq &C\left(h + h^{\frac{1}{2}} + h^{\frac{1}{2}}\right)\|u_\epsilon\|_{2,\Omega} \leq C\left(h + h^{\frac{1}{2}}\right)\|f\|_{0,\Omega}, \end{aligned}$$

(see Theorem 4.4 in [14] for the detailed proof of this estimate), which gives (2.21). Then, setting $\tilde{F} = 1_\Omega(u_\epsilon - u_{\epsilon h})$ and $v = u_\epsilon - u_{\epsilon h}$ in the adjoint problem (2.14), where $1_\Omega = 1$ in Ω, and $1_\Omega = 0$ in others, applying (2.21) and the prior estimates in Theorem 2.1, we have

$$\begin{aligned} &\|F\|_{0,\Omega}^2 = \|u_\epsilon - u_{\epsilon h}\|_{0,\Omega}^2 = (\nabla u_{F\epsilon}, \nabla(u_\epsilon - u_{\epsilon h}))_D + \frac{1}{\epsilon}(u_{F\epsilon}, u_\epsilon - u_{\epsilon h})_{\Omega_1} \\ =&(\nabla u_{F\epsilon} - v_h, \nabla(u_\epsilon - u_{\epsilon h}))_D + \frac{1}{\epsilon}(u_{F\epsilon} - v_h, u_\epsilon - u_{\epsilon h})_{\Omega_1}, \quad \forall v_h \in V_h(D), \\ \leq &C(\epsilon^{\frac{1}{4}} + h^{\frac{1}{2}})\|F\|_{0,\Omega}(\epsilon^{\frac{1}{4}} + h^{\frac{1}{2}})\|f\|_{0,\Omega} + C\frac{1}{\epsilon}\epsilon^{\frac{1}{2}}(\epsilon^{\frac{1}{4}} + h^{\frac{1}{2}})\|F\|_{0,\Omega}\epsilon^{\frac{1}{2}}(\epsilon^{\frac{1}{4}} + h^{\frac{1}{2}})\|f\|_{0,\Omega}, \end{aligned}$$

which shows (2.22), and the proof is completed. □

3. The fictitious domain method with L^2-penalty for parabolic problem. Let us consider the original parabolic problem (PQ) in cylindrical domain $Q_T = \Omega \times [0,T]$, with $0 < T < \infty$, and $\Sigma_T = \partial\Omega \times (0,T]$, then (PQ) reads as,

$$\begin{cases} u_t - \Delta u = f(x,t) & \text{in } Q_T, \\ u = 0 & \text{on } \Sigma_T, \\ u(x,0) = u_0 & \text{in } \Omega, \end{cases} \tag{3.1}$$

where $f \in L^2(Q_T)$, $u_0 \in H_0^1(\Omega)$. The weak form of (3.1) reads as,

$$\begin{cases} \text{Find } u \in H_0^{1,0}(Q_T),\ u_t \in H_0^{-1,0}(Q_T),\ s.t. \\ \langle u_t, v\rangle_{Q_T} + (\nabla u, \nabla v)_{Q_T} = (f, v)_{Q_T},\ \forall v \in H_0^{1,0}(Q_T) \\ u(x,0) = u_0, \end{cases} \tag{3.2}$$

where $H_0^{1,0}(Q_T) = L^2(0,T;H_0^1(\Omega))$, $H_0^{-1,0}(Q_T) = L^2(0,T;H^{-1}(\Omega))$, and $\langle\cdot,\cdot\rangle_{Q_T}$ is the dual product of $H_0^{1,0}(Q_T)$ and $H_0^{-1,0}(Q_T)$.

To implement the fictitious domain method, we set $D_T = D \times [0,T]$, $Q_{T1} = \Omega_1 \times [0,T]$ and $\mathcal{S}_T = \partial D \times (0,T]$, then the L^2-penalty problem (PQ_ϵ) reads as,

$$\begin{cases} u_{\epsilon t} - \Delta u_\epsilon + \dfrac{1}{\epsilon} 1_{Q_{T1}} u_\epsilon = \tilde{f}(x,t) & \text{in } D_T, \\ u = 0 & \text{on } \mathcal{S}_T, \\ u(x,0) = \tilde{u}_0 & \text{in } D, \end{cases} \tag{3.3}$$

where $\tilde{u}_0 \in H_0^1(D)$ is the zero extension of u, and $\tilde{f} \in L^2(D_T)$ is some extension of f satisfying $\|\tilde{f}\|_{0,Q_{T1}} \le C\|f\|_{0,Q_T}$. The weak form of (3.3) reads as

$$\begin{cases} \text{Find } u_\epsilon \in H_0^{1,0}(D_T),\ u_{\epsilon t} \in H^{-1,0}(D_T),\ s.t. \\ \langle u_{\epsilon t}, v\rangle_{D_T} + (\nabla u_\epsilon, \nabla v)_{D_T} + \dfrac{1}{\epsilon}(u_\epsilon, v)_{Q_{T1}} = (\tilde{f}, v)_{D_T},\ \forall v \in H_0^{1,0}(D_T) \\ u_\epsilon(x,0) = \tilde{u}_0. \end{cases} \tag{3.4}$$

3.1. Error estimate for penalization. Before deriving the error estimate, we define some spaces and show a regularity result for the L^2-penalty problem.

$$D(A_\epsilon) = \left\{ u \in H_0^1(D) \mid u|_\Omega \in H^2(\Omega),\ u|_{\Omega_1} \in H^2(\Omega_1),\ \left.\frac{\partial u}{\partial n}\right|_{\Omega,\Gamma} = \left.\frac{\partial u}{\partial n}\right|_{\Omega_1,\Gamma} \right\},$$

with norm $\|u\|_{D(A_\epsilon)} = \|u\|_{2,\Omega} + \epsilon^{\frac14}\|u\|_{2,\Omega_1} + \epsilon^{-\frac34}\|u\|_{0,\Omega_1}$ $(0 < \epsilon \ll 1)$. We define an operator $A_\epsilon : D(A_\epsilon) \to L^2(D)$, $u \mapsto -\Delta u + \frac{1}{\epsilon}1_{\Omega_1}u$. Following from Theorem 2.1, we know A_ϵ is invertible, satisfying $\|u\|_{D(A_\epsilon)} \le C\|A_\epsilon u\|_{0,D}$, for $u \in D(A_\epsilon)$. Then we have the following lemma, which is an analogue to THeorem 5.1 of Chapter 4 in [6].

LEMMA 3.1. *For any $v \in D(A_\epsilon)$, there exists $\xi_0 \in \mathbb{R}$ and $C > 0$, such that $\forall p = \xi + i\eta$, $\xi > \xi_0$, $\eta \in \mathbb{R}$, we have*

$$\|(A_\epsilon + p)v\|_{0,D} \ge C(\|v\|_{D(A_\epsilon)} + |p|\|v\|_{0,D}).$$

Proof. We define $\Lambda_\epsilon = A_\epsilon - e^{i\theta}D_y^2$, and $w(x,y) = z(y)e^{i\mu y}v(x)$, where $\mu \in \mathbb{R}$, $z \in C_0^\infty(\mathbb{R})$, *supp* $z \in [-1,1]$, $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. By virtue of the ellipticity of Λ_ϵ, which follows from that of A_ϵ, we have

$$\|\Lambda_\epsilon w\|_{L^2(D\times\mathbb{R}_y)} + \|w\|_{L^2(D\times\mathbb{R}_y)} \ge C(\|w\|_{L^2(\mathbb{R}_y;D(A_\epsilon))} + \|w\|_{H^2(\mathbb{R}_y;L^2(D))}).$$

Since $\|w\|_{L^2(\mathbb{R}_y;D(A_\epsilon))} = C\|v\|_{D(A_\epsilon)}$,

$$\|w\|_{H^2(\mathbb{R}_y;L^2(D)))} \ge C|\mu|^2\|v\|_{L^2(D)} - C(1+|\mu|)\|v\|_{L^2(D)},$$

$$\|\Lambda_\epsilon w\|_{L^2(D\times\mathbb{R}_y)} \leq C\|(A_\epsilon + e^{i\theta}\mu^2)v\|_{L^2(D)} + C(1+|\mu|)\|v\|_{L^2(D)},$$

choosing $\mu \geq \xi_0$, with ξ_0 sufficiently large, we have obtained the result. □

With the help of Lemma 3.1, following from Theorem 4.2 in Chapter 4 of [6], we have the following regularity theorem.

THEOREM 3.2. *For $F \in L^2(D_T)$, there exists a unique solution $v \in L^2(0,T;D(A_\epsilon)) \cap H^1(0,T;L^2(D))$ satisfying*

$$\begin{cases} v_t + A_\epsilon v = F & \text{in } D_T, \\ v = 0 & \text{on } \mathcal{S}_T, \\ v(x,0) = 0 & \text{in } D, \end{cases} \tag{3.5}$$

with $\|v_t\|_{L^2(D_T)} + \|v\|_{L^2(0,T;D(A_\epsilon))} \leq C\|F\|_{L^2(D_T)}$.

REMARK 2. *For $u_0 \in H_0^1(\Omega)$, there exists $v \in H_0^{1,0}(Q_T) \cap L^2(0,T;H^2(\Omega))$ satisfies*

$$v_t - \Delta v = 0 \text{ in } Q_T, \quad v = u_0 \text{ in } \Omega,$$

with

$$\|v_t\|_{L^2(Q_T)} + \|v\|_{L^2(0,T;H^2(\Omega))} \leq C\|u_0\|_{H^1(\Omega)}.$$

Let $\tilde{v}$ be the zero extension of v onto D_T, recalling the assumption that $\tilde{u}_0$ is the zero extension of u_0, we have $w = u_\epsilon - \tilde{v}$ satisfies (3.5) with $F = \tilde{f}$, which gives the regularity result for (3.3),

$$\|u_{\epsilon t}\|_{L^2(D_T)} + \|u_\epsilon\|_{L^2(0,T;D(A_\epsilon))} \leq C(\|\tilde{f}\|_{L^2(D_T)} + \|u_0\|_{H^1(\Omega)}). \tag{3.6}$$

With this regularity result, we can derive the error estimate for L^2 penalization.

THEOREM 3.3. *Suppose that u and u_ϵ are the solutions for (3.1) and (3.3), respectively. Then, we have*

$$\|u_{\epsilon t} - u_t\|_{H^{-1,0}(Q_T)} + \|u_\epsilon - u\|_{H^{1,0}(Q_T)} \leq C\epsilon^{\frac{1}{4}}(\|f\|_{0,\Omega} + \|u_0\|_{1,\Omega}), \tag{3.7}$$

$$\|u_\epsilon - u\|_{L^2(Q_T)} \leq C\epsilon^{\frac{1}{2}}(\|f\|_{0,\Omega} + \|u_0\|_{1,\Omega}). \tag{3.8}$$

Proof. In view of Remark 2, we have

$$\epsilon^{\frac{1}{4}}\|u_\epsilon\|_{L^2(0,T;H^2(\Omega_1))} + \epsilon^{-\frac{3}{4}}\|u_\epsilon\|_{L^2(Q_{T1})} \leq C(\|f\|_{0,\Omega} + \|u_0\|_{1,\Omega}),$$

which gives,

$$\|u_\epsilon\|_{L^2(0,T;H^1(\Omega_1))} \leq C\epsilon^{\frac{1}{4}}(\|f\|_{0,\Omega} + \|u_0\|_{1,\Omega}),$$

$$\|u_\epsilon\|^2_{0,\Sigma_T} \leq \|u_\epsilon\|_{L^2(0,T;H^1(\Omega_1))}\|u_\epsilon\|_{L^2(Q_{T1})} \leq C\epsilon(\|f\|_{0,\Omega} + \|u_0\|_{1,\Omega})^2.$$

Then, (3.2) and (3.4) yield that,

$$(u_{\epsilon t} - u_t, v)_{Q_T} + (\nabla(u_\epsilon - u), \nabla v)_{Q_T} - \int_{\Sigma_T} \frac{\partial(u_\epsilon - u)}{\partial n} v ds = 0, \ \forall v \in H^{1,0}(Q_T),$$

and setting $v = u_\epsilon|_{Q_T} - u$, we have

$$\|u_\epsilon(T) - u(T)\|_{0,\Omega}^2 + \|\nabla(u_\epsilon - u)\|_{0,Q_T}^2 \le C \left\| \frac{\partial(u_\epsilon - u)}{\partial n} \right\|_{0,\Sigma_T} \|u_\epsilon\|_{0,\Sigma_T}.$$

Hence,

$$\|u_\epsilon(T) - u(T)\|_{0,\Omega} + \|\nabla(u_\epsilon - u)\|_{0,Q_T} \le C\|u_\epsilon\|_{0,\Sigma_T}^{\frac{1}{2}} \le C\epsilon^{\frac{1}{4}}(\|f\|_{0,\Omega} + \|u_0\|_{1,\Omega}). \quad (3.9)$$

And it is apparently that

$$\|u_{\epsilon t} - u_t\|_{H^{-1,0}(Q_T)} \le C\|\nabla(u_\epsilon - u)\|_{0,Q_T} \le C\epsilon^{\frac{1}{4}}(\|f\|_{0,\Omega} + \|u_0\|_{1,\Omega}).$$

To estimate $\|u_\epsilon - u\|_{0,Q_T}$ we introduce the adjoint problems for (3.2) and (3.4),

$$\begin{cases} \text{Find } w \in H_0^{1,0}(Q_T),\ w_t \in H_0^{-1,0}(Q_T),\ s.t. \\ \langle w_t, v\rangle_{Q_T} - (\nabla w, \nabla v)_{Q_T} = (H, v)_{Q_T},\ \forall v \in H_0^{1,0}(Q_T) \\ w(x,T) = w_0, \end{cases} \quad (3.10)$$

$$\begin{cases} \text{Find } w_\epsilon \in H_0^{1,0}(D_T),\ w_{\epsilon t} \in H^{-1,0}(D_T),\ s.t. \\ \langle w_{\epsilon t}, v\rangle_{D_T} - (\nabla w_\epsilon, \nabla v)_{D_T} - \dfrac{1}{\epsilon}(w_\epsilon, v)_{Q_{T1}} = (\tilde{H}, v)_{D_T},\ \forall v \in H_0^{1,0}(D_T) \\ w_\epsilon(x,T) = \tilde{w}_0, \end{cases} \quad (3.11)$$

where $w_0 \in H_0^1(\Omega)$, $H \in L^2(Q_T)$, $\tilde{w}_0$ is the zero extension of w_0, and $\tilde{H}$ is some extension of H satisfying $\|\tilde{H}\|_{L^2(Q_{T1})} \le C\|H\|_{L^2(Q_T)}$. Setting $H = u_\epsilon|_{Q_T} - u$, and $\tilde{H}$ the zero extension of H, we have

$$\|H\|_{0,Q_T}^2 = (w_{\epsilon t} - \tilde{w}_t, u_\epsilon - \tilde{u})_{D_T} - (\nabla(w_\epsilon - \tilde{w}), \nabla(u_\epsilon - \tilde{u}))_{D_T} - \frac{1}{\epsilon}(w_\epsilon, u_\epsilon)_{Q_{T1}},$$

then (3.8) follows from (3.9) and Remark 2, which completes the proof. ∎

3.2. Error estimate for finite element approximation. We employ the backward Euler approximation for a time-discretization. Setting the time step $k > 0$, $T = Nk$, where N is some integer, we consider the discrete problem $(PQ_{\epsilon h})$ defined as:

$$\begin{cases} \text{Find } \{U^n\}_{n=0}^N \subset V_h(D),\ s.t.\ n = 1, \dots, N, \\ (\partial U^n, v_h)_D + (\nabla U^n, \nabla v_h)_D + \dfrac{1}{\epsilon}(U^n, v_h)_{\Omega_1} = (\tilde{f}(t_n), v_h)_D,\ \forall v_h \in V_h(D), \\ U^0 = \tilde{u}_{0h}, \end{cases} \quad (3.12)$$

where $\partial U^n = (U^n - U^{n-1})/k$.

Before showing the error estimate, we define the *Ritz projection* operator $R_h : H_0^1(D) \to V_h(D)$ as

$$A_\epsilon(u - R_h u, v_h) = 0, \quad \forall v_h \in V_h(D),$$

where

$$A_\epsilon(u, v) \equiv (\nabla u, \nabla v)_D + \frac{1}{\epsilon}(u, v)_{\Omega_1}.$$

Then, as consequences of the previous section, we have, for $u \in D(A_\epsilon)$,

$$\|u - R_h u\|_{1,D} + \frac{1}{\sqrt{\epsilon}}\|u - R_h u\|_{0,\Omega_1} \le C(\epsilon^{\frac{1}{4}} + h^{\frac{1}{2}})\|u\|_{D(A_\epsilon)}, \tag{3.13}$$

$$\|u - R_h u\|_{0,\Omega} \le C(\epsilon^{\frac{1}{4}} + h^{\frac{1}{2}})^2\|u\|_{D(A_\epsilon)}. \tag{3.14}$$

We shall show the following theorem, which is an analogue to Theorem 1.5 in [11] with assumption that $\tilde{u}_{0h} = R_h\tilde{u}_0$. This error estimate is simple but not optimal, and one may apply the method from [15][8] to obtain some better results.

THEOREM 3.4. *With U^n and u_ϵ the solutions of (3.12) and (3.3), respectively, we have, for $t_n = nk$, $n > 0$,*

$$\|U^n - u_\epsilon(t_n)\|_{0,D} \le C(\epsilon^{\frac{1}{4}} + h^{\frac{1}{2}})^2(\|u_\epsilon\|_{D(A_\epsilon)} + \int_0^{t_n} \|u_{\epsilon t}\|_{D(A_\epsilon)} ds) + k\int_0^{t_n} \|u_{\epsilon tt}\|_{0,D} ds, \tag{3.15}$$

$$\|\nabla(U^n - u_\epsilon(t_n))\|_{0,D} \le C(\epsilon^{\frac{1}{4}} + h^{\frac{1}{2}})(\|u_{\epsilon t}\|_{L^2(0,t_n;D(A_\epsilon))} + Ck|u_{\epsilon tt}\|_{L^2(0,t_n;L^2(D))}. \tag{3.16}$$

REMARK 3. *Theorem 3.4 requires a higher regularity of (3.3), which can be obtained by the same method introduced in the previous sections for elliptic and parabolic problems, with smoother assumption of f, u_0, $\tilde{f}$, $\tilde{u}_0$.*

Proof. [Proof of Theorem 3.4] We write

$$U^n - u_\epsilon(t_n) = (U^n - R_h u_\epsilon(t_n)) + (R_h u_\epsilon(t_n) - u_\epsilon(t_n)) = \theta^n + \rho^n,$$

and ρ^n is bounded as claimed in (3.14) and (3.13). For θ^n, (3.12), (3.4), and the definition of R_h yields that

$$(\partial\theta^n, v_h)_D + (\nabla\theta^n, \nabla v_h)_D + \frac{1}{\epsilon}(\theta^n, v_h)_{\Omega_1} = -(w^n, v_h)_D,\ \forall v_h \in V_h(D),\ n \ge 1, \tag{3.17}$$

where

$$w^n = R_h(\partial u_\epsilon(t_n)) - u_{\epsilon t}(t_n) = (R_h - I)\partial u_\epsilon(t_n) + (\partial u_\epsilon(t_n) - u_{\epsilon t}(t_n)) = w_1^n + w_2^n.$$

choosing $v_h = \theta^n$, we have

$$(\partial\theta^n, \theta^n)_D + \|\nabla\theta^n\|_{0,D}^2 + \frac{1}{\epsilon}\|\theta^n\|_{0,\Omega_1}^2 \le \|w^n\|_{0,D}\|\theta^n\|_{0,D},$$

which yields

$$\|\theta^n\|_{0,D} + Ck\sum_{j=1}^{n}(\|\theta^j\|_{1,D} + \frac{1}{\sqrt{\epsilon}}\|\theta^j\|_{0,\Omega_1}) \le \|\theta^0\|_{0,D} + k\sum_{j=1}^{n}\|w_1^j\|_{0,D} + k\sum_{j=1}^{n}\|w_2^j\|_{0,D}.$$

Since we have assume that $\tilde{u}_{0h} = R_h\tilde{u}_0$, $\theta^0 = 0$. Further, we write

$$w_1^j = (R_h - I)k^{-1}\int_{t_{j-1}}^{t_j} u_{\epsilon t} ds = k^{-1}\int_{t_{j-1}}^{t_j} (R_h - I)u_{\epsilon t} ds,$$

$$kw_2^j = u_\epsilon(t_j) - u_\epsilon(t_{j-1}) - ku_{\epsilon t}(t_j) = -\int_{t_{j-1}}^{t_j} (s - t_{j-1}) u_{\epsilon tt}(s) ds.$$

Together with our estimates, we show the (3.15).

To obtain $\|\nabla(U^n - u_\epsilon(t_n))\|_{0,D}$, we choose $v_h = \partial\theta^n$ in (3.17) to obtain, with the assumption $\theta^0 = 0$,

$$\|\nabla\theta^n\|_{0,D}^2 + \frac{1}{\epsilon}\|\theta^n\|_{0,\Omega_1}^2 \leq k \sum_{j=1}^n \|w^j\|_{0,D},$$

which gives (3.16). □

4. Numerical experiments. We give two numerical experiments for elliptic and parabolic problems, respectively, to show that the L^2-error is bounded by $(\sqrt{\epsilon}+h)$ and the H^1-norm error is bounded by $(\epsilon^{\frac{1}{4}} + h^{\frac{1}{2}})$, which is according to our analysis on L^2 penalization and finite element error estimates. For elliptic problem, we consider the problem

$$-\Delta u = 1 \text{ in } \Omega, \quad u = 0 \text{ on } \Gamma,$$

where $\Omega = \{(x,y) \mid x^2 + y^2 < 1\}$. To implement the fictitious domain method, we set the domain $D = \{-1.2 < x, y < 1.2\}$. We solve the problem (2.19). First, fixing $h = 0.01$, we show the errors for different ϵ, see Figure 4.1; then, setting $\epsilon = 10^{-6}$, we observe the errors dependents on different h, see Figure 4.2.

For parabolic problem, we consider the problem,

$$u_t - \Delta u = f \text{ in } \Omega \times [0,T], \quad u = 0 \text{ on } \Sigma_T, \quad u(x,0) = u_0 \text{ in } \Omega,$$

where $f = -\frac{1}{2}\sin(\frac{t}{2})\sin(x^2+y^2-1) - 4\cos(x^2+y^2-1)\cos(\frac{t}{2}) + 4\sin(x^2+y^2-1)(x^2+y^2-1)\cos(\frac{t}{2})$, $u_0 = \sin(x^2+y^2-1)$. Then, setting $T = 0.4$, the time-step $k = 0.02$, we solve the problem (3.12). Fixing $h = 0.01$, we show the errors for different ϵ, see Figure 4.3; then, setting $\epsilon = 10^{-6}$, we observe the errors dependents on different h, see Figure 4.4.

In our numerical results(using Freefem++), the error estimate becomes bad for very small ϵ(see Figure 4.1,4.3), this may due to the interpolation of the function $\frac{1}{\epsilon}1_{\Omega_1}$ when doing computation. To find a simple and effective way to approximating $\frac{1}{\epsilon}1_{\Omega_1}$ is important for the fictitious domain method with L^2 or H^1 penalization. However, we skip this part and left it for the future work.

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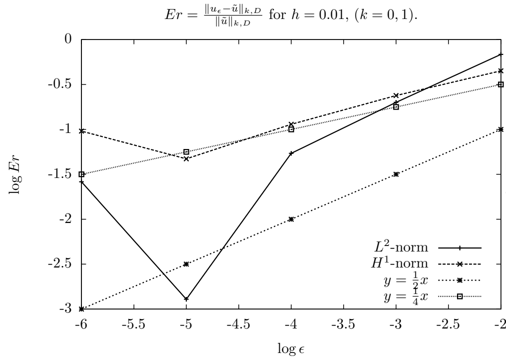

Fig. 4.1. $\frac{\|u_\epsilon - \tilde{u}\|_{k,D}}{\|\tilde{u}\|_{k,D}}$ *for* $h = 0.01$*,* $k = 0, 1$

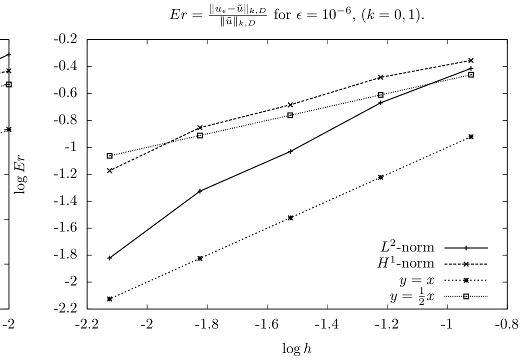

Fig. 4.2. $\frac{\|u_\epsilon - \tilde{u}\|_{k,D}}{\|\tilde{u}\|_{k,D}}$ *for* $\epsilon = 1e - 6$*,* $k = 0, 1$.

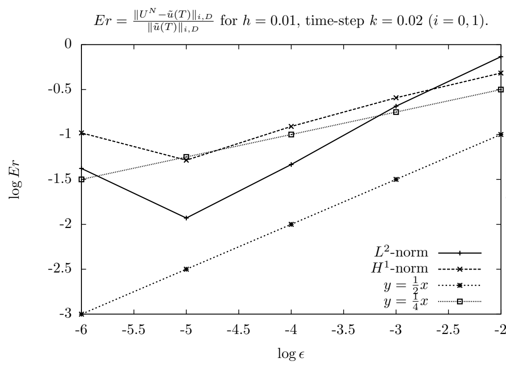

Fig. 4.3. $\frac{\|U^N - \tilde{u}(T)\|_{i,D}}{\|\tilde{u}(T)\|_{i,D}}$ *for* $h = 0.01$*,* $i = 0, 1$.

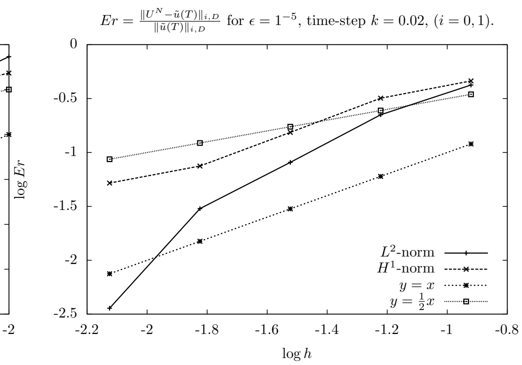

Fig. 4.4. $\frac{\|U^N - \tilde{u}(T)\|_{i,D}}{\|\tilde{u}(T)\|_{i,D}}$ *for* $\epsilon = 1e - 5$*,* $i = 0, 1$.

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