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Asymptotically self-similar solutions to curvature flow equations with prescribed contact angle

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ASYMPTOTICALLY SELF-SIMILAR SOLUTIONS TO CURVATURE FLOW EQUATIONS WITH PRESCRIBED CONTACT ANGLE AND THEIR APPLICATIONS TO GROOVE PROFILES DUE TO EVAPORATION-CONDENSATION

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Abstract. We study the asymptotic behavior of solutions to fully nonlinear second order parabolic equations including a generalized curvature flow equation which was introduced by Mullins in 1957 as a model of evaporation-condensation. We prove that, in the multi-dimensional half space, solutions of the problem with prescribed contact angle asymptotically converge to a self-similar solution of the associated problem under a suitable rescaling. Several properties of the profile function of the self-similar solution are also investigated. We show that the profile function has a corner and that the angles are determined by points at which the equation is degenerate. We also study the depth of the groove, which is represented by the value of the profile function at the boundary. Among other results it turns out that, as the contact angle tends to zero, the depth of the groove is well approximated by the linearized problem.

1. Introduction

We are concerned with the asymptotic behavior of solutions to second order parabolic equations with the Neumann boundary condition of the form

$$\text{(NP)}\begin{cases} \partial_t u(x,t) = F(\nabla u(x,t), \nabla^2 u(x,t)) & \text{in } \Omega\times(0,\infty), & (1.1)\\ u(x,0)=u_0(x) & \text{on } \overline{\Omega}, & (1.2)\\ \partial_{x_1} u(x,t)|_{x_1=0} = \beta > 0 & \text{on } \partial\Omega\times(0,\infty), & (1.3)\end{cases}$$

which we also denote by (NP; F, u_0). Here $\Omega = \{(x_1,\dots,x_n)\in \mathbf{R}^n \mid x_1>0\}$ is the half space, ∇u and $\nabla^2 u$ denote, respectively, the gradient and Hessian matrix of u with respect to x, and the initial data u_0 is bounded and uniformly continuous, i.e., $u_0 \in BUC(\overline{\Omega})$. A given real-valued function F is continuous and degenerate elliptic. Our goal in this paper is to prove that (viscosity) solutions of (NP) asymptotically converge to a self-similar solution of the associated problem, and study properties of a profile function of the self-similar solution.

Our study is motivated by evaporation-condensation model which was first proposed by a material scientist Mullins in [43]. Consider the situation that there are two crystal grain regions (solid phases) on the plane which consist of the same matter and differ only in their relative crystalline orientation. Let the two region be $\{(x,y)\mid x\geqq 0,\ y\leqq u(x,t)\}$ and $\{(x,y)\mid x\leqq 0,\ y\leqq \tilde{u}(x,t)\}$ at time $t\geqq 0$, where we assume $u(0,t)=\tilde{u}(0,t)$ so that a triple junction appears at the point $(0,u(0,t))$;

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see Figure 1. Moreover, we assume the symmetry, i.e., $u(x,t) = \tilde{u}(-x,t)$ for $x > 0$. The rest part on the plane is filled by gas. The intersection between the two crystal regions, which is called a grain boundary, is assumed to be stable on the line $x = 0$. We suppose that due to evaporation and condensation crystal atoms move between solid phases and gas phase. This mechanism leads development of a surface groove at the grain boundary, which we call a thermal groove, as in Figure 1. In this setting we study evolution of interfaces between crystal grains and gas. By

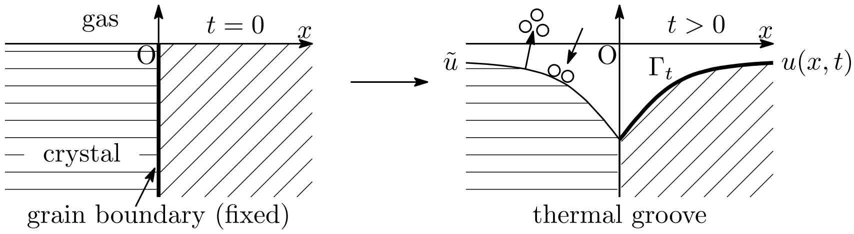

FIGURE 1. The thermal groove develops due to evaporation-condensation.

symmetry we consider the interface only in the right region, which we represent as $\Gamma_t := \{(x, u(x,t)) \in \mathbf{R}^2 \mid x \geqq 0\}$. According to Mullins' theory in [43] the evolution equation for Γ_t is given as

$$V_n = C_0 \left(1 - e^{-C_1 k}\right) \quad \text{on } \Gamma_t, \tag{1.4}$$

where V_n is the upward normal velocity of Γ_t, k is the upward (mean) curvature, and C_0, C_1 are positive constants. Thus, taking $C_0 = C_1 = 1$ for simplicity, we obtain the following partial differential equation for u:

$$\frac{u_t}{\sqrt{1+u_x^2}} = 1 - e^{-k} \tag{1.5}$$

in $\{x > 0\} \times \{t > 0\}$, where $(u_t, u_x, u_{xx}) = (\partial_t u, \partial_x u, \partial_{xx} u)$. Here we have invoked the formula $V_n = u_t/\sqrt{1+u_x^2}$, and also the curvature k is represented by $k = u_{xx}/\sqrt{1+u_x^2}^3$ ([22, Chapter 1.2, 1.4]). In this model a boundary condition on u at $x = 0$ is given as

$$u_x(0,t) \equiv \beta > 0, \tag{1.6}$$

which is the prescribed angle condition and results from equilibrium of tensions at the triple junction point $(0, u(0,t))$. Hence solving the Cauchy problem for (1.5) under the Neumann boundary condition (1.6) gives the surface profile due to evaporation-condensation. The problem (NP) is a generalized multidimensional case of this model.

In [43] Mullins approached the equation (1.5) via two approximations. He first applies the linear approximation of the exponential term, which is

$$1 - e^{-k} \approx k. \tag{1.7}$$

Then the original equation (1.5) simplifies to

$$v_t = \frac{v_{xx}}{1+v_x^2}, \tag{1.8}$$

which is the usual mean curvature flow equation for graphs. To solve (1.8) Mullins next applies the second approximation that

$$v_x \approx 0. \tag{1.9}$$

This condition comes from physical assumption that slopes on the surface are sufficiently small, which especially implies $\beta \ll 1$. Applying (1.9) to (1.8) finally yields

$$w_t = w_{xx}. \tag{1.10}$$

Since this is the simple heat equation, its classical solution w with the initial-boundary conditions $w(x,0) \equiv 0$ and $w_x(0,t) \equiv \beta > 0$ exists and has the explicit form; see Example 2.3. In this way Mullins concludes that the groove profile due to evaporation-condensation is given by the solution w. In particular, putting $x = 0$, Mullins computes the depth of the developing thermal groove at the origin, which is

$$-w(0,t) = 2\beta\sqrt{\frac{t}{\pi}} \approx 1.13\beta\sqrt{t}. \tag{1.11}$$

In this paper we aim at justifying these two approximations by Mullins. Namely, we rigorously discuss a relation among the three solutions u, v and w. The point in our study is that the solutions v of (1.8) and w of (1.10) are (forward) *self-similar*, i.e., they are of the form

$$v(x,t) = \sqrt{t}V\left(\frac{x}{\sqrt{t}}\right), \quad w(x,t) = \sqrt{t}W\left(\frac{x}{\sqrt{t}}\right).$$

The functions V and W are called *profile functions* of v and w, respectively. Then, as a justification for the first approximation, we prove

$$\frac{1}{\sqrt{t}}u(\sqrt{t}x,t) \to V(x) \quad \text{as } t \to \infty \tag{1.12}$$

in Theorem 3.4. This convergence result says that if we rescale the solution u of (1.5) in the above way, then it converges to the profile function V of the approximated equation. In other words, u itself is not necessarily self-similar, but it is asymptotically self-similar in the above sense.

We prove such an asymptotic result for more general problems of the form (NP) in Section 3. As a special structure of the equation (1.1) we direct our attention to homogeneity of F. Here we say F (or (1.1)) is *homogeneous* if F is positively homogeneous of degree 1 with respect to X, i.e., $F(p,X) = \lambda F(p, X/\lambda)$ for $\lambda > 0$. Evidently, the equation (1.8) is homogeneous. It also turns out that solutions of the homogeneous equations with the zero initial data are self-similar. Thus (1.8) can be generalized to homogeneous equations. In order to explain how we generalize (1.5) to the equation (1.1) with $G : \mathbf{R}^n \times \mathbf{S}^n \to \mathbf{R}^n$, we shall give an idea of the proof of (1.12). Let u and v be, respectively, a solution of (NP; G, u_0) and (NP; $F, 0$), where F is homogeneous. We prove the result (1.12) by showing that rescale functions of u converge to v; namely,

$$u_{(\lambda)}(x,t) := \frac{1}{\lambda}u(\lambda x, \lambda^2 t) \to v(x,t) \quad \text{as } \lambda \to \infty. \tag{1.13}$$

It is easy to see that this rescaled function $u_{(\lambda)}$ is a solution to the rescaled equation (NP; $G_\lambda, (u_0)_{(\lambda)}$) with $G_\lambda(p,X) = \lambda G(p, X/\lambda)$ and $(u_0)_{(\lambda)}(x) = u_0(\lambda x)/\lambda$. Since $(u_0)_{(\lambda)} \to 0$ as $\lambda \to \infty$, we can conclude that if G_λ converges to F, then the limit of $u_{(\lambda)}$ solves (NP; $F, 0$). By uniqueness the limit should be v, and hence we obtain

(1.13). Note that our convergence result (1.13) holds for a solution u of (NP; G, u_0) with an arbitrary initial data $u_0 \in BUC(\overline{\Omega})$.

In this way we are led to introduce a notion that G is *asymptotically homogeneous*, which roughly means that G approximates some homogeneous function in a suitable sense. To be more precise, we require that $G_\lambda(p, x) := \lambda G(p, X/\lambda)$ converge to some homogeneous F as $\lambda \to \infty$. By a simple calculation we see that (1.5) is asymptotically homogeneous with the limit (1.8). Accordingly the asymptotic homogeneity is a generalized notion containing (1.5), and the Mullins' first approximation is then generalized to

$$G \approx F.$$

To show the convergence of $u_{(\lambda)}$ to v rigorously we employ stability results of viscosity solutions. Due to comparison principle for (NP), we see that the upper and lower relaxed limit of $u_{(\lambda)}$, which are a sub- and supersolution respectively, should agree with v provided that the relaxed limits exist. Thus the remaining problem, which is our main difficulty, is to show the existence of the relaxed limits. This is achieved by constructing suitable barriers which are of order $O(\sqrt{t})$ as $t \to \infty$; see Lemma 3.5 and the proof of Theorem 3.4.

We turn to the second approximation by Mullins, to which we dedicate Section 5. Since the solution v of (1.8) and w of (1.10) are self-similar, we consider only their profile functions. Our main interest is to examine adequateness of Mullins' conclusion (1.11) concerning the depth of the thermal groove at the origin. For this purpose we compare the depths of two profile functions at the origin; one is the original depth $-V(0)(= -v(0, 1))$ which comes from (1.8) and the other is the approximated depth $-W(0)(= -w(0, 1))$ corresponding to (1.10). Recall that $-W(0)$ has the explicit form that $-W(0) = 2\beta/\sqrt{\pi}$ by (1.11). We prove among other results that, in Mullins' problem, $-W(0)$ is the third order approximation of $-V(0)$, i.e.,

$$-V(0) = -W(0) + O(\beta^3) \quad \text{as } \beta \to 0. \tag{1.14}$$

In this paper we discuss such comparison of the two depths for more general equations. From results for the general case we deduce (1.14). To discuss the general case let us consider (NP) with a homogeneous F. Since the problem (NP; $F, 0$) does not include the variables $x_2, \ldots, x_n$, its self-similar solution depends only on x_1 and t. Thus, in what follows we let the spatial dimension n be one so that the profile function V is defined on $\mathbf{R}$. Then it turns out that V satisfies the ordinary differential equation of the form

$$V(\xi) - \xi V'(\xi) = a(V'(\xi))V''(\xi) \quad \text{in } (0, \infty), \tag{1.15}$$

where a is given by $a(p) := -2F(p, -1)$. Note that $a(p) = 2/(1 + p^2)$ in Mullins' case since $F(p, X) = X/(1 + p^2)$ for (1.8). Let us recall the Mullins' second approximation which replaces the first derivative v_x by zero. As its analogue, for the general equation (1.15) we replace $a(V'(\xi))$ in the right hand side by $a(0)$, i.e., we apply

$$a(V'(\xi)) \approx a(0).$$

This is a generalized Mullins' second approximation. The resulting approximated equation is

$$W(\xi) - \xi W'(\xi) = a(0)W''(\xi) \quad \text{in } (0, \infty), \tag{1.16}$$

which represents the heat equation if we return (1.16) to the parabolic problem. Let V and W be, respectively, the unique viscosity solution of (1.15) and (1.16) with the boundary conditions that $V'(0) = \beta$ and $V(\infty) = 0$. A well-posedness of these equations in the viscosity sense is a consequence of that of parabolic equations (NP). We also remark that W has the explicit form. In this general setting we prove that the estimate

$$0 \leqq \frac{V(0) - W(0)}{\beta} \leqq C\left(a(0) - \min_{[0,\beta]} a\right) \tag{1.17}$$

holds for some positive constant C independent of β. This result implies $-V(0) = -W(0) + o(\beta)$ as $\beta \to 0$ for general equations and (1.14) for Mullins' case where $a(p) = 2/(1+p^2)$. The main tool for the proof of (1.17) is comparison principle. Namely, if we have a subsolution V_1 and a supersolution V_2, then we obtain an inequality $V_1 \leqq V_2$ and in particular $-V_1(0) \geqq -V_2(0)$. To this end we seek a suitable sub- or supersolution of the ordinary differential equation. We also deduce a couple of other estimates on the depth by the comparison method.

Our another interest is degenerate cases. We study (1.15) when $a(p)$ is allowed to be zero. Even in such degenerate cases the unique solution to (1.15) exists in the viscosity sense. As an instructive example, we now let $a(p) = 0$ for $p \in [q^-, q^+]$ and $a(p) > 0$ otherwise. Then a simple observation indicates that the unique solution

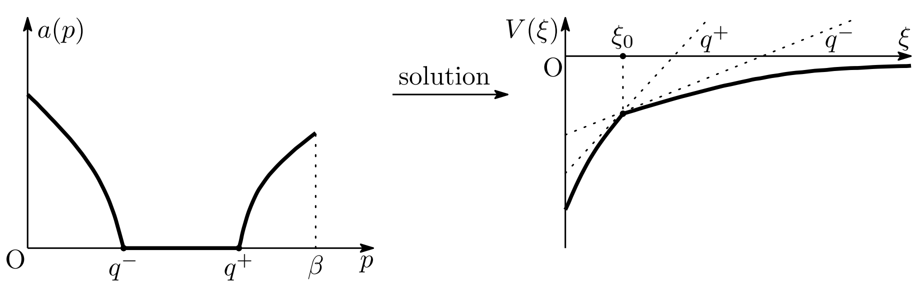

FIGURE 2. The profile function V has a corner when the equation is degenerate.

V has a corner whose angles are determined by q^- and q^+. Indeed, if we admit that V is negative and increasing (these properties are shown in Proposition 4.3), we notice by (1.15) that $0 > a(V'(\xi))V''(\xi)$. This implies $V'(\xi) \notin [q^-, q^+]$; in other words, the derivative of V jumps over the interval $[q^-, q^+]$. Rigorous statement and its proof on the corner of the viscosity solution V are given in Theorem 4.10, where we prove that there exists a unique $\xi_0 \in (0,\infty)$ such that the left and right derivatives of V at ξ_0 are, respectively, q^+ and q^-; see Figure 2.

Since the solution V of (1.15) is a profile function of the (forward) self-similar solution, it is natural to expect relation between V and the Wulff shape, which minimizes the total surface energy among all sets with the same volume. Although our interface Γ_t is now unbounded, we are able to relate the corner of the profile function V to that of the associated Wulff shape in the following way. For a given surface energy density $\gamma : S^{n-1} = \{x \in \mathbf{R}^n \mid |x| = 1\} \to (0,\infty)$ we define a Wulff shape associated with γ by

$$\mathrm{Wulff}(\gamma) = \bigcap_{|q|=1} \{x \in \mathbf{R}^n \mid \langle x, q\rangle \leqq \gamma(q)\},$$

where $\langle\cdot,\cdot\rangle$ is the standard inner product in $\mathbf{R}^n$. Let us consider the evolution equation of the form

$$V_n = M(\mathbf{n})k_\gamma \quad \text{on } \Gamma_t, \tag{1.18}$$

where $M : S^{n-1} \to (0,\infty)$ is the mobility, $\mathbf{n}$ is the oriented normal vector on Γ_t, and k_γ is the anisotropic curvature with respect to the surface energy density γ. See, e.g., [22, Chapter 1.3] for the definition of k_γ. We now let $n = 2$ and assume that Γ_t is represented by a graph, i.e., $\Gamma_t = \{(x, u(x,t)) \in \mathbf{R}^2\}$. Then, choosing $\mathbf{n}$ as the upward normal vector and using the formula

$$k_\gamma = (\tilde\gamma''(\arg\mathbf{n}) + \tilde\gamma(\arg\mathbf{n}))k,$$

where $\arg\mathbf{n}$ is the argument of $\mathbf{n}$ and $\tilde\gamma(\theta) := \gamma(\cos\theta, \sin\theta)$, we see that (1.18) is rewritten as

$$\frac{u_t}{\sqrt{1+u_x^2}} = M\left(\frac{(-u_x,1)}{\sqrt{1+u_x^2}}\right)(\tilde\gamma''(\arg(-u_x,1)) + \tilde\gamma(\arg(-u_x,1)))\,\frac{u_{xx}}{\sqrt{1+u_x^2}^3}.$$

The profile function of the self-similar solution of this equation satisfies the ordinary differential equation (1.15) with a of the form

$$a(p) = 2M\left(\frac{(-p,1)}{\sqrt{1+p^2}}\right)(\tilde\gamma''(\arg(-p,1)) + \tilde\gamma(\arg(-p,1)))\,\frac{1}{\sqrt{1+p^2}^3}.$$

Therefore we see that $a(p) = 0$ for all $p \in [q^-, q^+]$ if and only if $\tilde\gamma''(\theta)+\tilde\gamma(\theta) = 0$ for all $\theta \in [\arg(-q^-,1), \arg(-q^+,1)]$. The latter condition on γ leads the corner point of $\mathrm{Wulff}(\gamma)$ at which the slope of each tangent line is in $[q^-, q^+]$. This agrees with the corner of our profile function shown in Figure 2.

Let us explain why the equation (1.5) (or (1.4)) and the boundary condition (1.6) appear in Mullins' model. The exponential term in (1.4) comes from the Gibbs-Thompson formula in physics. This formula asserts that the vapor pressure p in equilibrium with the surface is given as

$$\log\left(\frac{p}{p_0}\right) = -C_1 k, \tag{1.19}$$

where p_0 is the atmospheric pressure and C_1 is a positive constant. Now, recall that the only mechanism operative in the transport of matter is evaporation-condensation. Thereby the normal velocity V_n is determined by the difference between the effect by condensation and that by evaporation. According to kinetic theory their effects are in proportion to pressures p_0 and p, respectively, and thus

$$V_n = C_2(p_0 - p) \tag{1.20}$$

with $C_2 > 0$. It is now clear that (1.19) and (1.20) lead the equation (1.4) by letting $C_0 = C_2 p_0$. The prescribed angle condition (1.6) is a consequence of equilibrium of tensions. More precisely, the resultant of the grain boundary tension $(0, -\gamma_b) \in \mathbf{R}^2$ and two surface tensions $(\pm\gamma_s\cos\theta, \gamma_s\sin\theta) \in \mathbf{R}^2$ is assumed to vanish at $(0, u(0,t))$, where $\gamma_b > 0$ and $\gamma_s > 0$ are, respectively, the boundary free energy and the surface free energy per unit area and θ is the slope angle of u at $x = 0$. Thus we have $2\gamma_s \sin\theta = \gamma_b$, which implies (1.6).

In [43] Mullins proposes another mechanism for the development of surface groove, which is surface diffusion. If we take the surface diffusion into account, the resulting equation describing the surface profile becomes a fourth order nonlinear parabolic equation. In this paper, however, we do not discuss such effect by surface diffusion so that only second order equations appear in our study. As a result, we are able to apply the viscosity solution theory ([16]) to study the problem. Mullins gives a criterion for judging which mechanism dominates the development of surface. According to [43] for magnesium under high pressure the profile is completely shaped by evaporation-condensation after a very short time while surface diffusion plays a dominant role for a very long time for gold under low pressure. See, e.g., [11, 32, 42, 58] for the studies of fourth order equations related to the surface diffusion.

We next state previous work related to our study. Many authors investigate asymptotic behaviors of solutions to curvature flow type equations. We first refer the reader to [26], where surfaces evolving by the mean curvature over a domain in $\mathbf{R}^n$ are studied under the zero Neumann boundary condition. It is shown that the solution converges to a constant function as $t \to \infty$. In [2] Altschuler and Wu study Cauchy problems for quasilinear equations of the form $u_t = (a(u_x))_x$ on $\{0 \leqq x \leqq d\} \times [0,\infty)$. They prove that solutions of the problem asymptotically converge to a solution which moves at a constant speed. The same authors obtain in [3] a similar convergence result for surfaces over a convex domain in $\mathbf{R}^2$, but they deal with only the curvature flow equation.

Asymptotic behaviors of graph solutions to free boundary problems are also studied in the literature. The paper [13] treats a quasilinear parabolic equation $u_t = (a(u_x))_x$ under a two point free boundary condition. (The same problem restricted to the equation (1.8) can be found in [15].) In [13] two half-lines are given radially from the origin and solutions are required to have intersections with them, which are the free boundary points, at prescribed contact angles. A global existence and uniqueness of solutions to the parabolic problem are established. A convergence result to a self-similar solution is deduced together with its convergence rate in the sense of the Hausdorff metric. The parabolic equation in [13] is not allowed to be degenerate, but our results concerning a well-posedness and the asymptotic behavior include degenerate cases. A similar setting to [13] is found in [39], where a one-point free boundary problem is considered. The paper [13] deals with expanding interfaces while the preserving case and the shrinking case for the same problem are discussed in [24].

For graphs defined on a whole space, their convergence results to a self-similar solution are obtained in [20, 31]. The paper [20] studies mean curvature evolutions written as graphs over $\mathbf{R}^n$. Under a suitable rescaling the convergence result is obtained for initial data satisfying a linear growth condition and further assumptions. Ishimura, the author of [31], considers the spatially one dimensional equation (1.8) in $\mathbf{R} \times (0,\infty)$ with prescribed opening angle conditions; that is, $v_x \to K_1$ as $x \to \infty$ and $v_x \to -K_2$ as $x \to -\infty$ for given constants $K_1, K_2 > 0$.

Curvature flow equations with constant driving force

$$\frac{v_t}{\sqrt{1+v_x^2}} = \frac{v_{xx}}{\sqrt{1+v_x^2}^3} + c \tag{1.21}$$

and asymptotic convergences to traveling fronts are studied in several works. In [17] the authors consider (1.21) for $(x,t) \in (0,\infty) \times (0,\infty)$ with the zero Neumann condition at $x = 0$ and the opening angle condition at $x = \infty$. It is shown that the solution v converges to a traveling wave solution as $t \to \infty$ when c is positive, while for a negative c convergence to a self-similar solution is proven in the sense that $t^{-1}|v(x,t) - tQ(x/t)| \to 0$ as $t \to \infty$, where $tQ(x/t)$ is a solution of (1.21). The explicit form of Q is also found in [17]. Note that the way of rescaling is different from ours. The papers [48, 45] studies asymptotics of solutions to (1.21) on $\mathbf{R} \times (0,\infty)$ when c is positive. Convergence results to a traveling V-shaped solution are obtained for spatially decaying and non-decaying initial perturbations in [48] and [45], respectively. For the explicit form of the V-shaped front, see [47]. The reader is also referred to [44] for convergence to a traveling line.

The paper [46] is related to the Mullins' second approximation (1.9) and asymptotic stability of constant solutions. There it is shown that

$$\sup_{x \in \mathbf{R}} |v(x,t) - w(x,t)| = O(1/\sqrt{t}) \quad \text{as } t \to \infty, \tag{1.22}$$

where v and w are, respectively, the solution of the Cauchy problem for (1.8) and (1.10) in $\mathbf{R} \times (0,\infty)$ with the same initial data. Moreover, using the results, the authors of [46] obtain a necessary and sufficient condition on initial data that ensures $u \to 0$ uniformly or pointwisely as $t \to \infty$. In our Neumann problem on the half space, however, a similar convergence result to (1.22) does not hold since

$$\sup_{x \in [0,\infty)} |v(x,t) - w(x,t)| = \sqrt{t} \sup_{\xi \in [0,\infty)} |V(\xi) - W(\xi)| \to \infty \quad \text{as } t \to \infty$$

for two different self-similar solutions $v(x,t) = \sqrt{t}V(x/\sqrt{t})$ and $w(x,t) = \sqrt{t}W(x/\sqrt{t})$.

Asymptotic shapes of expanding interfaces represented by a level set function are obtained in [29]. There the evolution equation $V_n = -\mathrm{tr}(E(\mathbf{n})D\mathbf{n}) + \nu(\mathbf{n})$ on Γ_t is considered, and it is shown that $\Gamma_t/t \to \partial\mathrm{Wulff}(\nu)$ as $t \to \infty$ in the Hausdorff metric. We remark that the limit is not the Wulff shape of the surface energy density in this work. To prove this large time asymptotics the authors study the limit of rescaled viscosity solutions of second order parabolic equations, and consider the corresponding stationary equations which the limit function satisfies. The result says that if u is a viscosity subsolution (resp. supersolution) of $\partial_t u + F_1(\nabla u, \nabla^2 u) + F_2(\nabla u) = 0$, then the (relaxed) limit of $u(tx,t)$ as $t \to \infty$ is a viscosity subsolution (resp. supersolution) of $-\langle x, \nabla u\rangle + F_2(\nabla u) = 0$. Note that this limit equation is first order while the second order equation (1.15), which V in (1.12) should satisfy, appears in our study.

Motion by curvature with triple junctions such as the point $(0, u(0,t))$ in Mullins' model is studied in [14]. There a planar domain surrounded by other phase domains is considered, and at each junction point three intersection angles are assumed to satisfy the Herring condition which is determined by interfacial energies. The authors of [14] give conditions for existence of self-similar stationary, expanding or shrinking solutions to the problem. Plane curves having the triple junction are also treated in [57], where the authors study evolving three curves by curvature forming 120 degree angles at their common start point. The authors of [57] derive several properties of solutions to (1.15) with $a(p) = 1/(1+p^2)$ and prove the unique existence of self-similar expanding solutions. As a study of expanding self-similar solutions we finally refer the reader to [21] for evolution by a crystalline curvature flow.

A generalized Mullins' model is proposed in [56, 49]. The author of [56] considers the model including a strain energy. In [49] Ogasawara studies evaporation-condensation model under a temperature gradient and proves an existence of stationary solutions to the resulting parabolic equation of the form $u_t = F(u, u_x, u_{xx})$. See also [50] for flattening properties of solutions to the generalized problem. Such flattening properties are also studied in [33, 37, 34, 38] for equations of the type (1.8) and in [35, 36] for those of the type (1.5).

Interestingly, an exact representation of the solution to (1.8) with $v_x(0,t) \equiv \beta$ and $v(x,0) \equiv 0$ is obtained by Broadbridge in [10]. However, we do not employ the formula in the present paper since generalization of the problem is one of our aims and the formula is rather complicated to handle. In [5] the authors obtain upper and lower bounds on the solution to (1.15) of the Mullins' case by solving two auxiliary problems which are relatively easily solvable and employing the comparison principle. They conclude accurate estimates of the depth when β is large, but an estimate allowing β to be small such as (1.17) is not stated in [5]. See Remark 5.3 for comparison with our results concerning the depth. The paper [52] gives exact solutions to wider classes of nonlinear equations, but solutions to (1.8) constructed there do not satisfy the prescribed angle condition (1.6). In [53, 12] exact solutions of the separated form $\phi(x) + \psi(t)$ are investigated. We also refer [1] for solvablity of the equation (1.8) on $I \times (0,\infty)$, where I is a bounded interval. Under the zero Dirichlet or Neumann boundary condition, the authors of [1] establish the existence of weak, strong and classical solutions and asymptotic behaviors of the classical solutions. The paper [40] shows the existence of classical solutions to more general degenerate parabolic equations.

A well-posedness of the problem (NP) is established in the sense of viscosity solutions in Section 2. We thus interpret the boundary condition (1.3) in the viscosity sense, that is, we require solutions to satisfy either (1.1) or (1.3) on the boundary. As a result, we observe that the unique solution may not satisfy (1.3) in the classical sense when the equation is degenerate (Proposition 4.6 (1)). Such generalized boundary conditions, which naturally appear when we take the limit in the vanishing viscosity method, was first introduced by Lions in [41]; see also [51]. The well-posedness is obtained in [41] for first order equations with Neumann or oblique conditions involving applications to optimal control, differential games and ergodic problems. After their works, uniqueness and existence results for oblique boundary problems in the viscosity sense were established in [9] for first order cases and in [28, 27, 6] for second order cases. In [18, 19] the authors approach oblique problems on domains involving corners. All of these studies treat continuous equations while equations with singularity in ∇u like the mean curvature flow equation for level sets are discussed in [23, 54] under the zero Neumann boundary condition. As relatively general results for second order singular equations with nonlinear boundary conditions, we refer the reader to [7, 30]. Compared with [30], the paper [7] deals with more general equations and boundary conditions, but domains are more restrictive.

Unfortunately, all the above results treat a bounded domain with respect to the space variables. As far as the author know, [55] is the only paper which proves a well-posedness of the Neumann type problems on an unbounded domain. In [55] Sato established comparison and existence results for second order singular equations under the capillary boundary condition:

$$\partial_{x_1} u = k|\nabla u| \quad \text{with } -1 < k < 1,$$

which does not cover our boundary condition (1.3). Although it might be possible to extend the previous results for bounded domains to our problem (NP) by modifying their proofs suitably, we give in the present paper complete proofs of comparison and existence theorem for (NP) to make the paper self-contained. Neumann problems in half-space type domains are also treated in [4, 8], where the authors studies ergodic problems and homogenization.

This paper is organized as follows. In Section 2 we establish comparison and existence results of viscosity solutions to (NP). Section 3 is devoted to the asymptotic profile. We prove (1.13), i.e., asymptotic self-similarity of the solution to the equation of the type (1.5). In Section 4 we consider the ordinary differential equation (1.15) and its solution. We show the solution has a corner if the equation is degenerate. Section 5 concerns the depth of the thermal groove at the origin. Several estimates for the depth including (1.17) are obtained.

2. A well-posedness of Neumann problems

2.1. **Definition of solutions.** Throughout this paper we set $\Omega := \{(x_1, \ldots, x_n) \in \mathbf{R}^n \mid x_1 > 0\}$. We first introduce a notion of viscosity solutions for (NP). The boundary condition (1.3) is interpreted in the (weak) viscosity sense. Our basic assumption on F is

(F0) $F : \mathbf{R}^n \times \mathbf{S}^n \to \mathbf{R}$ is continuous and degenerate elliptic.

Here $\mathbf{S}^n$ denotes the space of real $n \times n$ symmetric matrices with the usual ordering, i.e., $X \leqq Y$ if $\langle X\xi, \xi\rangle \leqq \langle Y\xi, \xi\rangle$ for all $\xi \in \mathbf{R}^n$. We say F is *degenerate elliptic* if $F(p, X) \leqq F(p, Y)$ for all $p \in \mathbf{R}^n$ and $X, Y \in \mathbf{S}^n$ with $X \leqq Y$.

Definition 2.1 (Viscosity solution)**.** We say $u : \overline{\Omega} \times [0, \infty) \to \mathbf{R}$ is a *viscosity subsolution* (resp. *supersolution*) of (NP) if u is bounded from above (resp. below) on $\overline{\Omega} \times [0, T)$ for every $T > 0$, $u^*(\cdot, 0) \leqq u_0$ (resp. $u_*(\cdot, 0) \geqq u_0$) on $\overline{\Omega}$ and

$$\begin{cases} \partial_t \phi(x,t) - F(\nabla\phi(x,t), \nabla^2\phi(x,t)) \leqq 0 \text{ (resp. } \geqq 0) & \text{if } x_1 > 0, \\ \partial_t \phi(x,t) - F(\nabla\phi(x,t), \nabla^2\phi(x,t)) \leqq 0 \text{ (resp. } \geqq 0) & \\ \text{or } \ \beta - \partial_{x_1}\phi(x,t) \leqq 0 \text{ (resp. } \geqq 0) & \text{if } x_1 = 0 \end{cases} \tag{2.1}$$

whenever $u^* - \phi$ (resp. $u_* - \phi$) attains its maximum (resp. minimum) at (x, t) for $\phi \in C^{2,1}(\overline{\Omega} \times [0, \infty))$. If u is both a viscosity sub- and supersolution, u is said to be a *viscosity solution*.

Here by a $C^{2,1}$ function we mean that derivatives $\partial_t \phi$, $\nabla\phi$ and $\nabla^2\phi$ are continuous. If $u^* < \infty$ (resp. $u_* > -\infty$) on $\overline{\Omega} \times [0, \infty)$ and u satisfies (2.1), u is said to be a viscosity subsolution (resp. supersolution) of (1.1) and (1.3). In the definition above, u^* and u_* stand for an *upper* and *lower semicontimuous envelope* of u respectively. Namely,

$$u^*(x,t) = \lim_{\delta \to 0} \sup\{u(y,s) \mid (y,s) \in \overline{\Omega} \times [0,\infty),\ |x-y| + |t-s| \leqq \delta\},$$

$$u_*(x,t) = \lim_{\delta \to 0} \inf\{u(y,s) \mid (y,s) \in \overline{\Omega} \times [0,\infty),\ |x-y| + |t-s| \leqq \delta\}.$$

As a boundary condition we consider not $\partial_{x_1}\phi(x,t)-\beta=0$ but $\beta-\partial_{x_1}\phi(x,t)=0$ so that consistency between a classical subsolution (resp. supersolution) and a viscosity subsolution (resp. supersolution) holds.

Proposition 2.2 (Consistency)**.** *Assume* (F0). *Let* $u \in C^{2,1}(\overline{\Omega}\times(0,\infty))$ *and assume that*

$$\begin{cases} \partial_t u(x,t) \leqq F(\nabla u(x,t), \nabla^2 u(x,t)) & \text{if } x_1 > 0, \qquad (2.2)\\ \beta - \partial_{x_1} u(x,t) \leqq 0 & \text{if } x_1 = 0. \qquad (2.3)\end{cases}$$

Then u *is a viscosity subsolution of* (1.1) *and* (1.3)*.*

Proof. Take any $\phi \in C^{2,1}(\overline{\Omega}\times(0,\infty))$ such that $u-\phi$ attains its maximum at $(x,t)\in\overline{\Omega}\times(0,\infty)$. In the case where $x_1>0$, the inequality $\partial_t\phi(x,t)\leqq F(\nabla\phi(x,t),\nabla^2\phi(x,t))$ follows from (2.2) and the degenerate ellipticity of F. If $x_1=0$, we see at once that $\partial_{x_1}\phi(x,t)\geqq\partial_{x_1}u(x,t)$, and consequently $\beta-\partial_{x_1}\phi(x,t)\leqq 0$ by (2.3). □

It is known that for a general boundary condition $B(x,u(x),\nabla u(x))=0$ the consistency holds if a map $\lambda\mapsto B(x,r,p-\lambda\nu(x))$ is nonincreasing on $[0,\infty)$, where $\nu(x)$ is the unit outward normal vector at a boundary point x. We refer the reader to [16, Proposition 7.2] or [22, Proposition 2.3.3] for more details.

Example 2.3. We consider the heat equation

$$\partial_t u(x,t) = A\Delta u(x,t), \tag{2.4}$$

i.e., $F(p,X)=A\cdot\operatorname{tr}(X)$ with $A>0$, where $\operatorname{tr}(X)$ denotes the trace of $X\in\mathbf{S}^n$. Then the unique solution of (NP; F, 0), which is also given by Mullins in [43], is

$$u(x,t) = h_{\beta,A}(x_1,t) := -2\beta\sqrt{At}\cdot\operatorname{ierfc}\left(\frac{x_1}{2\sqrt{At}}\right). \tag{2.5}$$

Here $\operatorname{ierfc}(x)$ is the integral error function

$$\operatorname{ierfc}(x) = \int_x^\infty \operatorname{erfc}(z)dz,$$

and $\operatorname{erfc}(x)$ is the error function

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-z^2}dz.$$

We now differentiate $h=h_{\beta,A}$ to obtain

$$\partial_t h(x_1,t) = -\beta\sqrt{\frac{A}{t}}\cdot\operatorname{ierfc}\left(\frac{x_1}{2\sqrt{At}}\right) - \frac{\beta x_1}{2t}\cdot\operatorname{erfc}\left(\frac{x_1}{2\sqrt{At}}\right),$$
$$\partial_{x_1}h(x_1,t) = \beta\cdot\operatorname{erfc}\left(\frac{x_1}{2\sqrt{At}}\right), \quad \partial_{x_1x_1}h(x_1,t) = \frac{-\beta}{\sqrt{\pi At}}e^{-x_1^2/(4At)}.$$

Employing the formula

$$\operatorname{ierfc}(\xi) + \xi\cdot\operatorname{erfc}(\xi) = \frac{1}{\sqrt{\pi}}e^{-\xi^2}$$

with $\xi = x_1/(2\sqrt{At})$, we observe that h indeed solves (2.4) in the classical sense. Thus h is also a viscosity solution of (NP; F, 0) by Proposition 2.2. By the formula (2.5) or the derivatives of h we notice that $h(\cdot,t)$ is negative, increasing and (strictly) concave on $[0,\infty)$. It turns out that these properties still hold for viscosity solutions of more general equations; see Proposition 4.3.

Example 2.4. We seek viscosity sub- and supersolutions of (NP; $F,0$) which have the form of (2.5). Assume (F0) and

(F1) $F(p,X)=\lambda F(p,X/\lambda)$ for all $(p,X)\in\mathbf{R}^n\times\mathbf{S}^n$ and $\lambda>0$.

We simply say F is *homogeneous* if F satisfies (F1). For $\gamma\geqq 0$ we set

$$m(\gamma)=\min_{0\leqq\theta\leqq 1}\{-F(\theta\gamma e_1,-I_{1,1})\},\quad M(\gamma)=\max_{0\leqq\theta\leqq 1}\{-F(\theta\gamma e_1,-I_{1,1})\},\tag{2.6}$$

where $e_1=(1,0,\dots,0)$ and $I_{1,1}$ denotes the matrix with 1 in the $(1,1)$ entry and 0 elsewhere. We then notice that $m(\gamma)\geqq 0$ since F is degenerate elliptic by (F0) and satisfies $F(p,O)=0$ for all $p\in\mathbf{R}^n$ by (F1), where O is the zero matrix. For the function $h=h_{\gamma,A}$ given as (2.5) we observe

$$\begin{aligned}F(\nabla h,\nabla^2 h)&=F(\partial_{x_1}h\cdot e_1,\partial_{x_1x_1}h\cdot I_{1,1})=-\partial_{x_1x_1}h\cdot F(\partial_{x_1}h\cdot e_1,-I_{1,1})\\&\begin{cases}\leqq m(\gamma)\cdot\partial_{x_1x_1}h=(m(\gamma)/A)\cdot\partial_t h,\\ \geqq M(\gamma)\cdot\partial_{x_1x_1}h=(M(\gamma)/A)\cdot\partial_t h.\end{cases}\end{aligned}$$

Taking account of the boundary condition (1.3), we conclude that $h_{\gamma,A}$ is a viscosity subsolution of (NP; $F,0$) if $\gamma\geqq\beta$ and $A\geqq M(\gamma)$ while $h_{\gamma,A}$ is a viscosity supersolution of (NP; $F,0$) if $0\leqq\gamma\leqq\beta$ and $0<A\leqq m(\gamma)$.

2.2. **Comparison principle.** We show uniqueness of viscosity solutions to (NP) via comparison principle. Define $U_T:=\overline{\Omega}\times\overline{\Omega}\times[0,T)$ for $T>0$.

Theorem 2.5 (Comparison principle)**.** *Assume* (F0). *Let u and v be, respectively, a viscosity subsolution and a viscosity supersolution of* (NP). *Then*

$$K=K[u,v]:=\lim_{\theta\to 0}\sup\{u^*(x,t)-v_*(y,t)\mid(x,y,t)\in U_T,\ |x-y|<\theta\}\leqq 0$$

for every $T>0$. In particular, $u^*\leqq v_*$ on $\overline{\Omega}\times[0,\infty)$.

In the proof of Theorem 2.5 we use an auxiliary function $\mathcal{F}:\overline{U_T}\to\mathbf{R}\cup\{-\infty\}$ of the form

$$\mathcal{F}(x,y,t)=u^*(x,t)-v_*(y,t)-\Psi(x,y,t)$$

with

$$\Psi(x,y,t)=\frac{|x-y|^2}{2\varepsilon}+\beta(x_1-y_1)+\delta\{\rho(x_1)+\rho(y_1)\}+\gamma(|x|^2+|y|^2)+\frac{\alpha}{T-t}.$$

Here $\alpha,\gamma,\delta,\varepsilon\in(0,1)$ are constants and ρ is given by $\rho(r)=(1+r)^{-1}$. Note that $\rho'(0)=-1$. It then follows from an elementary calculation that for all $(x,y,t)\in U_T$

$$\beta-\partial_{x_1}\Psi(x,y,t)\geqq\delta\quad\text{if } x_1=0,\tag{2.7}$$

$$\beta+\partial_{y_1}\Psi(x,y,t)\leqq-\delta\quad\text{if } y_1=0\tag{2.8}$$

and

$$\lim_{(\gamma,\delta)\to(0,0)}\nabla^2_{(x,y)}\Psi(x,y,t)=\frac{1}{\varepsilon}\begin{pmatrix}I&-I\\-I&I\end{pmatrix},\tag{2.9}$$

where I is the identity matrix with dimension n.

Lemma 2.6. *Assume the same hypotheses of Theorem 2.5. Let $T>0$ and suppose $K=K[u,v]>0$. Then,*

(1) *$\mathcal{F}$ attains a maximum on $\overline{U_T}$ at some $(\hat{x}, \hat{y}, \hat{t})$ with $\hat{t} < T$.*
(2) *There exists a constant $\eta \in (0, 1]$ such that*

$$\max_{\overline{U_T}} \mathcal{F} > K' \tag{2.10}$$

for all $\alpha, \gamma, \delta \in (0, \eta)$, where $K' := K/7$.
(3) $\sup_{\gamma,\delta,\varepsilon\in(0,\eta)} |\hat{x} - \hat{y}| < \infty$ *and* $\lim_{\varepsilon\to 0} \sup_{\gamma,\delta\in(0,\eta)} |\hat{x} - \hat{y}| = 0$.
(4) $\lim_{(\gamma,\delta)\to(0,0)}(\gamma\hat{x}, \gamma\hat{y}) = (0, 0)$ *for all $\varepsilon \in (0, \eta)$.*
(5) *There exists a constant $\eta_0 \in (0, \eta)$ such that $\hat{t} > 0$ for all $\gamma, \delta, \varepsilon \in (0, \eta_0)$.*

Proof. (1) This follows from an upper semicontinuity of $\mathcal{F}$ and the facts that $\mathcal{F}(x, y, T) = -\infty$ and $\mathcal{F} \to -\infty$ as $|x| \to \infty$ or $|y| \to \infty$.

(2) By the definition of K there exists some $\theta_0 > 0$ such that for all $\theta \in (0, \theta_0]$

$$u^*(x_\theta, t_\theta) - v_*(y_\theta, t_\theta) > 6K' \tag{2.11}$$

holds for some $(x_\theta, y_\theta, t_\theta) \in U_T$ with $|x_\theta - y_\theta| < \theta$. Take

$$\theta = \min\left\{\theta_0,\ \sqrt{2K'\varepsilon},\ K'/\beta\right\}.$$

By this choice we have

$$\frac{|x_\theta - y_\theta|^2}{2\varepsilon} \leqq K', \quad \beta(x_{\theta 1} - y_{\theta 1}) \leqq K'. \tag{2.12}$$

We next choose $\eta \in (0, 1]$ as

$$\eta = \min\left\{1,\ K'/2,\ K'(|x_\theta|^2 + |y_\theta|^2 + 1)^{-1},\ K'(T - t_\theta)\right\},$$

and then for $\alpha, \gamma, \delta \in (0, \eta)$

$$\delta\{\rho(x_{\theta 1}) + \rho(y_{\theta 2})\} \leqq K', \quad \gamma(|x_\theta|^2 + |y_\theta|^2) \leqq K', \quad \frac{\alpha}{T - t_\theta} \leqq K'. \tag{2.13}$$

Thus (2.11)–(2.13) yield (2.10).

(3) Take $M > 0$ so that $u^* - v_* \leqq M$ on $\overline{U_T}$. By (2.10) we have

$$K' < u^*(\hat{x}, \hat{t}) - v_*(\hat{y}, \hat{t}) - \Psi(\hat{x}, \hat{y}, \hat{t}) \leqq M - \frac{|\hat{x} - \hat{y}|^2}{2\varepsilon} + \beta|\hat{x} - \hat{y}|.$$

Thus by an elementary calculation

$$|\hat{x} - \hat{y}| \leqq \varepsilon\beta + \sqrt{\varepsilon^2\beta^2 + 2\varepsilon M},$$

which implies our assertions.

(4) By (2.10) again we see

$$K' \leqq M + \beta|\hat{x} - \hat{y}| - \gamma(|\hat{x}|^2 + |\hat{y}|^2).$$

Therefore $\sup_{\gamma,\delta\in(0,\eta)} \gamma(|\hat{x}|^2 + |\hat{y}|^2) < \infty$, and so

$$\gamma(|\hat{x}| + |\hat{y}|) \leqq \sqrt{2\gamma}\sqrt{\gamma(|\hat{x}|^2 + |\hat{y}|^2)} \to 0 \quad \text{as } (\gamma, \delta) \to (0, 0).$$

(5) Suppose by contradiction that there were some sequence $\{(\varepsilon_j, \delta_j, \gamma_j)\}_{j=1}^\infty$ which satisfies $\lim_{j\to\infty}(\varepsilon_j, \delta_j, \gamma_j) = (0, 0, 0)$ and $\hat{t} = \hat{t}(\varepsilon_j, \delta_j, \gamma_j) = 0$. Then

$$\mathcal{F}(\hat{x}, \hat{y}, \hat{t}) = u^*(\hat{x}, 0) - v_*(\hat{y}, 0) - \Psi(\hat{x}, \hat{y}, 0) \leqq u_0(\hat{x}) - u_0(\hat{y}) - \beta(\hat{x}_1 - \hat{y}_1),$$

and the right hand side converges to 0 as $j \to \infty$ by (3) and the uniform continuity of u_0. This is a contradiction to (2.10). □

Proof of Theorem 2.5. By virtue of (3) in Lemma 2.6 we may assume

$$\lim_{(\gamma,\delta)\to(0,0)}(\hat{x}-\hat{y})=\bar{p}$$

for some $\bar{p}\in\mathbf{R}^n$ by taking a subsequence if necessary. We now apply the Crandall-Ishii lemma ([16, Theorem 8.3]) to $\mathcal{F}$. Since (2.7) and (2.8) hold, there exists $(X,Y)\in\mathbf{S}^n\times\mathbf{S}^n$ such that

$$\partial_t\Psi(\hat{x},\hat{y},\hat{t})\leqq F(\nabla_x\Psi(\hat{x},\hat{y},\hat{t}),X)-F(-\nabla_y\Psi(\hat{x},\hat{y},\hat{t}),-Y) \tag{2.14}$$

and

$$-\left(\frac{1}{\varepsilon}+|A|\right)I\leqq\begin{pmatrix}X & O\\ O & Y\end{pmatrix}\leqq A+\varepsilon A^2. \tag{2.15}$$

Here $A:=\nabla^2_{(x,y)}\Psi(\hat{x},\hat{y},\hat{t})$ and $|A|:=\sup\{|\langle A\xi,\xi\rangle|\mid\xi\in\mathbf{R}^n,\ |\xi|=1\}$. Note that

$$\begin{aligned}\nabla_x\Psi(\hat{x},\hat{y},\hat{t})&=\frac{\hat{x}-\hat{y}}{\varepsilon}+\{\beta+\delta\rho'(\hat{x}_1)\}e_1+2\gamma\hat{x},\\ \nabla_y\Psi(\hat{x},\hat{y},\hat{t})&=-\frac{\hat{x}-\hat{y}}{\varepsilon}-\{\beta-\delta\rho'(\hat{y}_1)\}e_1+2\gamma\hat{y},\\ \partial_t\Psi(\hat{x},\hat{y},\hat{t})&=\frac{\alpha}{(T-\hat{t})^2}.\end{aligned}$$

In view of (2.9) and (2.15) we may assume that (X,Y) converges to some $(\bar{X},\bar{Y})\in\mathbf{S}^n\times\mathbf{S}^n$ as $(\gamma,\delta)\to(0,0)$. Then the limit $(\bar{X},\bar{Y})$ satisfies

$$\begin{pmatrix}\bar{X} & O\\ O & \bar{Y}\end{pmatrix}\leqq\frac{3}{\varepsilon}\begin{pmatrix}I & -I\\ -I & I\end{pmatrix},$$

and in particular $\bar{X}+\bar{Y}\leqq O$. Letting $(\gamma,\delta)\to(0,0)$ in (2.14), we have

$$\frac{\alpha}{T^2}\leqq F\left(\frac{\bar{p}}{\varepsilon}+\beta e_1,\bar{X}\right)-F\left(\frac{\bar{p}}{\varepsilon}+\beta e_1,-\bar{Y}\right).$$

This is a contradiction since F is degenerate elliptic. □

Corollary 2.7 (Uniqueness)**.** *Assume* (F0)*. Then* (NP) *admits at most one viscosity solution, and the solution is continuous on* $\overline{\Omega}\times[0,\infty)$*.*

Proof. If u and v are two viscosity solutions of (NP), we have $u^*\leqq v_*$ and $v^*\leqq u_*$ on $\overline{\Omega}\times[0,\infty)$ by Theorem 2.5. These inequalities imply our assertions. □

Corollary 2.8 (Contraction property)**.** *Assume* (F0)*. Let* $u_{01},u_{02}\in BUC(\overline{\Omega})$*. Let* u_1 *and* u_2 *be, respectively, a viscosity solution of* (NP; F,u_{01}) *and that of* (NP; F,u_{02})*. Then we have* $\sup_{\overline{\Omega}\times[0,\infty)}|u_1-u_2|\leqq\sup_{\overline{\Omega}}|u_{01}-u_{02}|$.

Proof. Let $d=\sup_{\overline{\Omega}}|u_{01}-u_{02}|$. Then it is easily seen that u_2+d is a viscosity solution of (NP; $F,u_{02}+d$). Since $u_{01}\leqq u_{02}+d$ on $\overline{\Omega}$, Theorem 2.5 gives $u_1\leqq u_2+d$ on $\overline{\Omega}\times[0,\infty)$. In the same manner we obtain $u_2\leqq u_1+d$ on $\overline{\Omega}\times[0,\infty)$. □

2.3. **Existence result.** We prove the existence of viscosity solutions by *Perron's method* ([16, Section 4]). An important step is to construct a *lower* and *upper barrier*, which are a viscosity sub- and supersolution of (NP) satisfying the given initial data. We first prepare stability results for viscosity solutions. For the proofs we refer the reader to [16, Lemma 4.2, Lemma 6.1] or [22, Lemma 2.4.1, Theorem 2.3.5].

Proposition 2.9 (Stability)**.** *Assume* (F0)*.*

(1) *Let $\mathcal{S}$ be a nonempty subset of*

$$\{v \mid v \text{ is a viscosity subsolution of } (1.1) \text{ and } (1.3)\}.$$

Let $u(x,t) := \sup_{v\in\mathcal{S}} v(x,t)$. If $u^* < \infty$ on $\overline{\Omega}\times[0,\infty)$, then u is a viscosity subsolution of (1.1) *and* (1.3)

(2) *Assume that F^ε satisfies* (F0)*, and let u^ε be a a viscosity subsolution of* (1.1) *with $F = F^\varepsilon$ and* (1.3) *for each $\varepsilon > 0$. If $F \geqq \limsup^*_{\varepsilon\to 0} F^\varepsilon$ on $\mathbf{R}^n\times\mathbf{S}^n$ and $\overline{u} := \limsup^*_{\varepsilon\to 0} u^\varepsilon < \infty$ on $\overline{\Omega}\times[0,\infty)$, then $\overline{u}$ is a viscosity subsolution of* (1.1) *and* (1.3).

To apply Perron's method we need only (1) while (2) plays an important role in Section 3, where we discuss a local uniform convergence of solutions. We recall a notion of relaxed limits appearing in (2). For a subset $L \subset \mathbf{R}^N$ and functions $h^\varepsilon : L \to \mathbf{R}$ with $\varepsilon > 0$ we define an *upper relaxed limit* $\overline{h} = \limsup^*_{\varepsilon\downarrow 0} h^\varepsilon$ (resp. *lower relaxed limit* $\underline{h} = \liminf_{*\varepsilon\downarrow 0} h^\varepsilon$) : $\overline{L} \to \mathbf{R}\cup\{\pm\infty\}$ as

$$\overline{h}(z) := \limsup_{(\varepsilon,y)\to(0,z)} h^\varepsilon(y) = \lim_{\delta\downarrow 0}\sup\{h^\varepsilon(y) \mid y\in L,\ |y-z|<\delta,\ 0<\varepsilon<\delta\}$$

$$(\text{resp. } \underline{h}(z) := \liminf_{(\varepsilon,y)\to(0,z)} h^\varepsilon(y) = \lim_{\delta\downarrow 0}\inf\{h^\varepsilon(y) \mid y\in L,\ |y-z|<\delta,\ 0<\varepsilon<\delta\}).$$

If $\overline{h} = \underline{h}$ in L, then h^ε converges to $h := \overline{h} = \underline{h}$ locally uniformly in L as $\varepsilon\to 0$.

Proposition 2.10 (Barriers)**.** *Assume* (F0)*. Then* (NP) *has a viscosity subsolution w^- and a viscosity supersolution w^+ such that $w^-(x,t) \leqq u_0(x) \leqq w^+(x,t)$ for all $(x,t)\in\overline{\Omega}\times[0,\infty)$ and $u_0(x) = w^\pm(x,0) = \lim_{(z,t)\to(x,0)} w^\pm(z,t)$ for all $x\in\overline{\Omega}$.*

Proof. We give the proof only for a subsolution since a similar argument applies for a supersolution.

1. Let $\omega(r) = \sup_{|x-y|\leqq r}|u_0(x)-u_0(y)|$ and $f(r) = r - \arctan r$. Then for each $\varepsilon > 0$ there exists $C_0(\varepsilon) > 0$ such that $\omega(r) \leqq \varepsilon + C_0(\varepsilon)f(r)$ for all $r \geqq 0$. Set $C(\varepsilon) := \max\{4C_0(\varepsilon),\ 4C_0(\varepsilon)/\beta,\ 1\} \geqq 1$. Since $f(r+s) \leqq 4\{f(r)+f(s)\}$ for $r,s\geqq 0$, we see that

$$\omega(|x-y|) \leqq \varepsilon + \beta C(\varepsilon)f(|x_1-y_1|) + C(\varepsilon)f(|x'-y'|) \tag{2.16}$$

for all $x = (x_1,x')\in\mathbf{R}^n$ and $y = (y_1,y')\in\mathbf{R}^n$. We also remark that $f\in C^2(\mathbf{R})$, $f(0) = f'(0) = f''(0) = 0$, $0\leqq f'\leqq 1$ and $0\leqq f''\leqq 1/2$ in $\mathbf{R}$.

2. For $\varepsilon\in(0,1)$ and $y\in\Omega$ we define

$$v_{\varepsilon,y}(x,t) := u_0(y) - \varepsilon - \frac{\beta C(\varepsilon)}{f'(y_1)}f(|x_1-y_1|) - C(\varepsilon)f(|x'-y'|) - Mt,$$

where $M = M(\varepsilon,y) > 0$ is a large constant. Then $v_{\varepsilon,y}\in C^{2,1}(\overline{\Omega}\times[0,\infty))$ and $v_{\varepsilon,y}(x,t)\leqq u_0(x)$ on $\overline{\Omega}\times[0,\infty)$ from (2.16). By the boundedness of f' and f'' we see that $|\nabla_x v_{\varepsilon,y}|$ and $|\nabla_x^2 v_{\varepsilon,y}|$ are also bounded on $\overline{\Omega}\times[0,\infty)$. We thus have

$$-M \leqq F(\nabla_x v_{\varepsilon,y}(x,t), \nabla_x^2 v_{\varepsilon,y}(x,t))$$

for sufficiently large M. We also compute

$$\partial_{x_1} v_{\varepsilon,y}(x,t)|_{x_1=0} = -\frac{\beta C(\varepsilon)}{f'(y_1)}\{-f'(y_1)\} \geqq \beta,$$

and therefore $v_{\varepsilon,y}$ is a viscosity subsolution of (1.1) and (1.3) by Proposition 2.2. Consequently Proposition 2.9 (2) ensures that the supremum of $v_{\varepsilon,y}$

$$w^-(x,t) = \sup\{v_{\varepsilon,y}(x,t) \mid \varepsilon \in (0,1),\ y \in \Omega\}$$

is also a viscosity subsolution of (1.1) and (1.3). By definition w^- is lower semicontinuous and satisfies $w^-(x,t) \leqq u_0(x)$ on $\overline{\Omega} \times [0,\infty)$. In particular w^- is bounded from above.

3. We next show $w^-(x,0) \geqq u_0(x)$ for all $x \in \overline{\Omega}$. We see $w^-(x,0) \geqq v_{\varepsilon,x}(x,0) = u_0(x) - \varepsilon$ if $x \in \Omega$, and so $w^-(x,0) \geqq u_0(x)$ holds. Let $x \in \partial\Omega$. Taking $y = (y_1, x')$, we then have

$$w^-(x,0) \geqq v_{\varepsilon,y}(x,0) \geqq u_0(x) - \varepsilon - \frac{\beta C(\varepsilon)}{f'(y_1)} f(y_1).$$

Letting $y_1 \to 0$ first and then $\varepsilon \to 0$, we obtain $w^-(x,0) \geqq u_0(x)$.

4. Since w^- is lower semicontinuous, for all $x \in \overline{\Omega}$

$$\begin{aligned} u_0(x) = w^-(x,0) &\leqq \liminf_{(z,t)\to(x,0)} w^-(z,t) \leqq \limsup_{(z,t)\to(x,0)} w^-(z,t) \\ &\leqq \limsup_{(z,t)\to(x,0)} u_0(z) = u_0(x). \end{aligned}$$

Hence $\lim_{(z,t)\to(x,0)} w^-(z,t) = u_0(x)$. We thus conclude that w^- satisfies the required properties. □

Remark 2.11. By the same way as in Step 3 we obtain a more general estimate that $w^-(x,t) \geqq u_0(x) - Mt$ for all $(x,t) \in \overline{\Omega} \times [0,\infty)$. This implies that w^- is bounded from below on $\overline{\Omega} \times [0,T)$ for every $T > 0$. Similarly, we are able to construct w^+ in Proposition 2.10 such that it is bounded from above on $\overline{\Omega} \times [0,T)$ for every $T > 0$.

Theorem 2.12 (Existence by Perron's method)**.** *Assume* (F0). *Then* (NP) *admits at least one viscosity solution.*

Proof. Let

$$\mathcal{S} = \left\{ v \;\middle|\; \begin{array}{c} v \text{ is a viscosity subsolution of (NP)} \\ \text{such that } w^- \leqq v \leqq w^+ \text{ on } \overline{\Omega} \times [0,\infty) \end{array} \right\},$$

where w^- and w^+ are functions in Proposition 2.10. Since $w^- \in \mathcal{S}$, the set $\mathcal{S}$ is nonempty. We demonstrate that $u(x) := \sup_{v\in\mathcal{S}} v(x)$ is a viscosity solution of (NP). By definition we have $w^- \leqq u \leqq w^+$ on $\overline{\Omega} \times [0,\infty)$. We then notice that $u^*(\cdot,0) = u_*(\cdot,0) = u_0$ on $\overline{\Omega}$ and that u is bounded on $\overline{\Omega} \times [0,T)$ for all $T > 0$ by Remark 2.11. Proposition 2.9 (1) ensures that u is a subsolution of (NP). We also see that u is a viscosity supersolution of (NP) since u is a maximal subsolution in the sense that $u(x_0,t_0) < v(x_0,t_0)$ for some $v \in \mathcal{S}$ and $(x_0,t_0) \in \overline{\Omega} \times (0,\infty)$ if u were not a supersolution. See [22, Lemma 2.4.2] for more details. □

3. Asymptotic behavior

To study the asymptotic behavior self-similar solutions of (NP) play an important role in our study.

Definition 3.1. Let $u : \overline{\Omega} \times [0,\infty) \to \mathbf{R}$.

(1) We define a *rescaled function* $u_{(\lambda)}$ of u as $u_{(\lambda)}(x,t) := u(\lambda x, \lambda^2 t)/\lambda$ for $\lambda > 0$.

(2) We say u is *self-similar* if $u = u_{(\lambda)}$ for all $\lambda > 0$, or equivalently $u(x,t) = \sqrt{t}U(x/\sqrt{t})$ for some $U : [0,\infty) \to \mathbf{R}$. We call U a *profile function* of u.

Note that, if u is self-similar, the profile function U of u is represented by $U(x) = u(x,1)$. We next introduce a notion of asymptotic homogeneity. We consider $G : \mathbf{R}^n \times \mathbf{S}^n \to \mathbf{R}$ such that G is not necessarily homogeneous but it approximates some homogeneous F in a suitable sense. To state the rigorous meaning of the approximation we define

$$G_\lambda(p,X) := \lambda G\left(p, \frac{X}{\lambda}\right)$$

for $\lambda > 0$. We say G is *asymptotically homogeneous* if G satisfies the following:

(F2) G_λ converges to some $F : \mathbf{R}^n \times \mathbf{S}^n \to \mathbf{R}$ as $\lambda \to \infty$ locally uniformly in $\mathbf{R}^n \times \mathbf{S}^n$.

We call F in (F2) the limit of G. We also remark that the limit F satisfies (F0) and (F1) whenever G satisfies (F0) and (F2). Thus the limit F is always homogeneous. The function $G(p,X) = \sqrt{1+p^2}(1-e^{-k})$ with $k = X/\sqrt{1+p^2}^3$, which represents (1.5) in Mullins' case, is indeed asymptotically homogeneous with the limit $F(p,X) = X/(1+p^2)$ corresponding to (1.8). This follows from the fact that $f_\lambda(z) := \lambda(1-e^{-z/\lambda}) - z$ converges to 0 as $\lambda \to \infty$ locally uniformly in $\mathbf{R}$.

Remark 3.2. If u is a viscosity solution of (NP; G, u_0), then the rescaled function $u_{(\lambda)}$ is a viscosity solution of (NP; $G_\lambda, (u_0)_{(\lambda)}$), where

$$(u_0)_{(\lambda)}(x) = \frac{1}{\lambda}u_0(\lambda x).$$

Indeed, noting that

$$\begin{aligned} &\partial_t u_{(\lambda)}(x,t) = \lambda \partial_t u(\lambda x, \lambda^2 t), \\ &\nabla u_{(\lambda)}(x,t) = \nabla u(\lambda x, \lambda^2 t), \quad \nabla^2 u_{(\lambda)}(x,t) = \lambda \nabla^2 u(\lambda x, \lambda^2 t), \end{aligned}$$

we compute

$$\partial_t u_{(\lambda)}(x,t) = \lambda G(\nabla u(\lambda x, \lambda^2 t), \nabla^2 u(\lambda x, \lambda^2 t)) = \lambda G\left(\nabla u_{(\lambda)}(x,t), \frac{1}{\lambda}\nabla^2 u_{(\lambda)}(x,t)\right)$$

and

$$\partial_{x_1} u_{(\lambda)}(x,t) = \partial_{x_1} u(\lambda x, \lambda^2 t) = \beta$$

if u is a classical solution. In the case where u is not smooth, taking elements of semijets, we see that $u_{(\lambda)}$ solves (NP; $G_\lambda, (u_0)_{(\lambda)}$) in the viscosity sense. We also remark that if G is homogeneous, then $u_{(\lambda)}$ solves (NP; $G, (u_0)_{(\lambda)}$).

We prove that the unique solution of the homogeneous equation with the zero initial data is always self-similar. Several properties of the self-similar solution are also discussed.

Proposition 3.3 (Self-similar solution)**.** *Assume* (F0) *and* (F1)*. Let u be the unique viscosity solution of* (NP; $F, 0$)*. Then*

(1) *u is self-similar.*
(2) *$u \leqq 0$ on $\overline{\Omega} \times [0,\infty)$. If $F(0, -I_{1,1}) < 0$, then $u < 0$ on $\overline{\Omega} \times [0,\infty)$.*
(3) *$u(x,t) = u(x_1, 0, \dots, 0, t)$ for all $(x,t) \in \overline{\Omega} \times [0,\infty)$.*
(4) *$\lim_{x_1 \to \infty} u(x,t) = 0$ for all $t \geqq 0$.*

Proof. By Remark 3.2 we see that $u_{(\lambda)}$ is a viscosity solution of (NP; $F, 0$) for every $\lambda > 0$. Applying Theorem 2.5, we obtain $u = u_{(\lambda)}$. This implies (1). Combining Example 2.4 with Theorem 2.5, we observe that $h_{\beta,M(\beta)} \leqq u \leqq h_{\beta,m(\beta)}$ on $\overline{\Omega} \times [0,\infty)$, where h is the function in (2.5) and $m(\beta)$, $M(\beta)$ are defined as (2.6). Thus the first assertion in (2) and (4) hold. If $F(0, -I_{1,1}) < 0$, then we have $m(\gamma) < 0$ for sufficiently small $\gamma \in (0, \beta]$. Then a supersolution $h_{\gamma,m(\gamma)}$ is negative on $\overline{\Omega} \times [0,\infty)$, so that u is also negative. We finally prove (3). For $a \in \mathbf{R}^{n-1}$ we set $w_a(x,t) := u(x_1, x' - a, t)$, where $x' = (x_2, \ldots, x_n)$. Then it is easy to see that w_a is also a viscosity solution of (NP; $F, 0$) since F and the initial-boundary conditions do not depend on x'. By the uniqueness we obtain $u = w_a$. In particular, for fixed $(x,t) \in \overline{\Omega} \times [0,\infty)$ we have $u(x,t) = w_{x'}(x,t) = u(x_1, 0, \ldots, 0, t)$. □

Our main result on asymptotic convergence is

Theorem 3.4 (Asymptotic behavior)**.** *Assume that G satisfies* (F0) *and* (F2) *with the limit F. Let u and v be, respectively, the unique viscosity solution of* (NP; G, u_0) *and that of* (NP; $F, 0$). *Then $u_{(\lambda)}$ converges to v as $\lambda \to \infty$ locally uniformly on $\overline{\Omega} \times [0,\infty)$.*

By Theorem 3.4 we see that $u_{(\sqrt{t})}(x,1)$ converges to $v(x,1)$ as $t \to \infty$ uniformly on every compact subset of $\overline{\Omega}$. This implies that (1.12) holds locally uniformly with respect to $x \in \overline{\Omega}$.

As is pointed out in Remark 3.2, the rescaled function $u_{(\lambda)}$ is a solution of (NP; $G_\lambda, (u_0)_{(\lambda)}$). Since the local uniform convergence of G_λ to F is assumed, in view of Proposition 2.9 (2) the relaxed limits $\overline{u}$ and $\underline{u}$ of $u_{(\lambda)}$ becomes a sub- and supersolution of (NP; $F, 0$), respectively, provided that the limits exist. To guarantee the existence of the relaxed limits we construct suitable barriers of (NP; G, u_0) whose rescaled families are locally uniformly bounded. Recalling Remark 2.11, we have rough estimates for u that $u_0(x) - Mt \leqq u(x,t) \leqq u_0(x) + Mt$. Then $u_0(\lambda x)/\lambda - M\lambda t \leqq u_{(\lambda)}(x,t) \leqq u_0(\lambda x)/\lambda + M\lambda t$, but this does not yields that $\overline{u}$ and $\underline{u}$ are real-valued. We construct the barriers so that they have the order $O(\sqrt{t})$ as $t \to \infty$.

Lemma 3.5. (1) *Assume that $g : [0,\infty) \to \mathbf{R}$ satisfies*

$$|g(t)| \leqq M(\sqrt{t} + 1) \quad \text{on } [0,\infty) \tag{3.1}$$

for some $M > 0$. Set $g_{(\lambda)}(t) := g(\lambda^2 t)/\lambda$, $\underline{g} := \liminf_{*\lambda\to\infty} g_{(\lambda)}$ and $\overline{g} := \limsup^*_{\lambda\to\infty} g_{(\lambda)}$. Then we have $-M\sqrt{t} \leqq \underline{g}(t) \leqq \overline{g}(t) \leqq M\sqrt{t}$ on $[0,\infty)$.

(2) *Assume that G satisfies* (F0) *and* (F2). *Then there exists $M_0 > 0$ such that*

$$\rho(t) := \sup_{|\theta|,|\sigma|\leqq 1} \left| G\left(\theta\beta e_1, \frac{\sigma I_{1,1}}{\sqrt{t}}\right)\right| \leqq \frac{M_0}{\sqrt{t}} \tag{3.2}$$

for all $t \geqq 1$. Moreover

$$g(t) := \begin{cases} 0 & (0 \leqq t \leqq 1), \\ \int_1^t \rho(s)ds & (t > 1) \end{cases} \tag{3.3}$$

satisfies (3.1) *with $M = 2M_0$.*

Obviously, the estimate (3.2) holds if G is homogeneous. For a general G, roughly speaking, (3.2) still holds since G is approximately homogeneous.

Proof. (1) Fix $t_0 \geqq 0$. Let $\delta > 0$ and take $t \geqq 0$, $\lambda > 0$ such that $|t - t_0| \leqq \delta$, $\lambda \geqq 1/\delta$. We then observe

$$|g_{(\lambda)}(t)| = \frac{1}{\lambda}|g(\lambda^2 t)| \leqq \frac{1}{\lambda}M(\sqrt{\lambda^2 t}+1) = M\left(\sqrt{t}+\frac{1}{\lambda}\right) \leqq M(\sqrt{t_0+\delta}+\delta).$$

Thus, sending $\delta \to 0$ gives $-M\sqrt{t_0} \leqq \underline{g}(t_0) \leqq \overline{g}(t_0) \leqq M\sqrt{t_0}$.

(2) The second assertion is obvious if (3.2) holds. For $t \geqq 1$ we observe

$$\sqrt{t}\rho(t) \leqq \sup_{|\theta|,|\sigma|\leqq 1}\left|\sqrt{t}G\left(\theta\beta e_1, \frac{\sigma I_{1,1}}{\sqrt{t}}\right) - F(\theta\beta e_1, \sigma I_{1,1})\right| + \sup_{|\theta|,|\sigma|\leqq 1}|F(\theta\beta e_1, \sigma I_{1,1})|.$$

The first term of the right hand side converges to 0 as $t \to \infty$ by assumption while the second term is a constant independent of t. Therefore (3.2) follows. □

Proof of Theorem 3.4. Let w^- and w^+ be barriers constructed in the proof of Proposition 2.10. Then there exists $C > 0$ such that $-C \leqq w^- \leqq w^+ \leqq C$ on $\overline{\Omega} \times [0,2]$ by Remark 2.11. Define $\Phi : \overline{\Omega} \times [0,\infty) \to \mathbf{R}$ as

$$\Phi(x,t) := -C + h(x_1,t) - g(t).$$

Here h and g are the functions given by (2.5) and (3.3), respectively. We choose $A = \beta^2/\pi$ in (2.5) so that $0 \leqq \partial_{x_1}h \leqq \beta$ and $-1/\sqrt{t} \leqq \partial_{x_1x_1}h \leqq 0$ in $\overline{\Omega} \times (0,\infty)$. By the definition of g, we then find that Φ and $-\Phi$ are, respectively, a viscosity subsolution and a viscosity supersolution of

$$\partial_t u(x,t) = G(\nabla u(x,t), \nabla^2 u(x,t)) \tag{3.4}$$

in $\Omega \times (1,\infty)$ and (1.3). Indeed, the boundary condition is easy to check, and for $(x,t) \in \Omega \times (1,\infty)$ we compute

$$\partial_t\Phi(x,t) \leqq -g'(t) = -\sup_{|\theta|,|\sigma|\leqq 1}\left|G\left(\theta\beta e_1, \frac{\sigma I_{1,1}}{\sqrt{t}}\right)\right| \leqq G(\nabla\Phi(x,t), \nabla^2\Phi(x,t)).$$

Here we have used the facts that $\partial_t h \leqq 0$, $0 \leqq \partial_{x_1}h \leqq \beta$ and $-1/\sqrt{t} \leqq \partial_{x_1x_1}h \leqq 0$. A similar argument yields that $-\Phi$ is a supersolution. Since $\Phi \leqq w^- \leqq w^+ \leqq -\Phi$ on $\overline{\Omega} \times [0,2]$, we see that $\tilde{w}^- := \max\{w^-, \Phi\}$ and $\tilde{w}^+ := \min\{w^+, -\Phi\}$ are, respectively, a viscosity subsolution and a viscosity supersolution of (3.4) in $\Omega \times (0,\infty)$ and (1.3). Noting that $(\tilde{w}^-)^*(x,0) = u_0(x) = (\tilde{w}^+)_*(x,0)$ on $\overline{\Omega}$, we see by Theorem 2.5 that $(\tilde{w}^-)^* \leqq u \leqq (\tilde{w}^+)_*$ in $\overline{\Omega} \times [0,\infty)$. In particular, we have $\Phi_{(\lambda)} \leqq u_{(\lambda)} \leqq -\Phi_{(\lambda)}$. Taking $\liminf_{*\lambda\to\infty}$ and $\limsup^*_{\lambda\to\infty}$, we obtain

$$h(x_1,t) - \overline{g}(t) \leqq \underline{u}(x,t) \leqq \overline{u}(x,t) \leqq -h(x_1,t) + \overline{g}(t) \quad \text{in } \overline{\Omega} \times [0,\infty),$$

where $\underline{u} := \liminf_{*\lambda\to\infty} u_{(\lambda)}$ and $(\overline{u}, \overline{g}) := \limsup^*_{\lambda\to\infty}(u_{(\lambda)}, g_{(\lambda)})$. Therefore Lemma 3.5 implies that $\underline{u}$ and $\overline{u}$ are bounded on $\overline{\Omega} \times [0,T)$ for every $T > 0$ and that $\underline{u}(x,0) = \overline{u}(x,0) = 0$ on $\overline{\Omega}$.

Now, since $u_{(\lambda)}$ is a viscosity solution of (NP; $G_\lambda, (u_0)_{(\lambda)}$) for every $\lambda > 0$, Proposition 2.9 (2) and (F2) imply that $\overline{u}$ and $\underline{u}$ are, respectively, a viscosity subsolution and a viscosity supersolution of (NP; $F, 0$). By Theorem 2.5 we have $\overline{u} \leqq \underline{u}$ in $\overline{\Omega} \times [0,\infty)$, and therefore $\overline{u} \equiv \underline{u} \equiv v$ since v is now the unique solution of (NP; $F, 0$). As a result, $u_{(\lambda)}$ converges to v locally uniformly in $\overline{\Omega} \times [0,\infty)$. □

Remark 3.6. If G is homogeneous in Theorem 3.4, i.e., $G \equiv F$, then $u_{(\lambda)}$ converges to v uniformly on $\overline{\Omega} \times [0, \infty)$. Indeed, since $u_{(\lambda)}$ solves (NP; $F, (u_0)_{(\lambda)}$), the contraction property (Corollary 2.8) ensures

$$\sup_{\overline{\Omega} \times [0,\infty)} |u_{(\lambda)} - v| \leqq \sup_{\overline{\Omega}} |(u_0)_{(\lambda)} - 0| = \frac{1}{\lambda} \sup_{\overline{\Omega}} |u_0|.$$

We thus obtain the uniform convergence of $u_{(\lambda)}$ together with its convergence rate.

Remark 3.7. We derive a sufficient condition for (F2). Let $G : \mathbf{R}^n \times \mathbf{S}^n \to \mathbf{R}$ and consider a linear approximation of G such as (1.7). Suppose that G is of the form $G(p, X) = H(p, f(p, X))$ with some continuous and homogeneous f. We expand H as $H(p, z) = z \cdot \partial_z H(p, 0) + z \cdot r(p, z)$, where we have assumed $H(p, 0) = 0$. Then

$$\lambda G\left(p, \frac{X}{\lambda}\right) = \lambda H\left(p, \frac{1}{\lambda} f(p, X)\right) = f(p, X) \cdot \partial_z H(p, 0) + f(p, X) \cdot r\left(p, \frac{1}{\lambda} f(p, X)\right).$$

Thus G satisfies (F2) with the limit $F(p, X) = f(p, X) \cdot \partial_z H(p, 0)$ if the reminder term $r(p, z/\lambda)$ converges to zero as $\lambda \to \infty$ locally uniformly with respect to (p, z). This setting includes Mullins' problem, which corresponds to the case where $H(p, X) = \sqrt{1 + p^2}(1 - e^{-z})$ and $f(p, X) = X/\sqrt{1 + p^2}^3$.

4. Profile functions

In this section we study the profile function of the unique self-similar solution to (NP; $F, 0$) with a homogeneous F. Our main interest is the configuration of its graph, especially the corner of the graph when $F(p, X)$ is allowed to be 0 even if $X \neq 0$.

We first derive the ordinary differential equation which the profile function should solve. Assume (F0) and (F1). Let v be a viscosity solution of (NP; $F, 0$). According to Proposition 3.3 (3), $v(x, t)$ is independent of $(x_2, \ldots, x_n)$. Thus we hereafter assume $n = 1$ so that u and F are, respectively, defined on $\mathbf{R} \times [0, \infty)$ and $\mathbf{R} \times \mathbf{R}$. We let $V : [0, \infty) \to \mathbf{R}$ be the profile function of v, i.e.,

$$V(x) = v(x, 1). \tag{4.1}$$

Since v is self-similar, we have

$$v(x, t) = \sqrt{t}\, v\left(\frac{x}{\sqrt{t}}, 1\right) = \sqrt{t}\, V\left(\frac{x}{\sqrt{t}}\right). \tag{4.2}$$

We now differentiate v to find

$$\partial_t v(x, t) = \frac{1}{2\sqrt{t}} V\left(\frac{x}{\sqrt{t}}\right) - \frac{x}{2t} V'\left(\frac{x}{\sqrt{t}}\right),$$
$$\partial_x v(x, t) = V'\left(\frac{x}{\sqrt{t}}\right), \quad \partial_{xx} v(x, t) = \frac{1}{\sqrt{t}} V''\left(\frac{x}{\sqrt{t}}\right)$$

provided that v is smooth. Substituting these derivatives for (1.1), we have

$$\frac{1}{2\sqrt{t}}\left\{V\left(\frac{x}{\sqrt{t}}\right) - \frac{x}{\sqrt{t}} V'\left(\frac{x}{\sqrt{t}}\right)\right\} = F\left(V'\left(\frac{x}{\sqrt{t}}\right), \frac{1}{\sqrt{t}} V''\left(\frac{x}{\sqrt{t}}\right)\right).$$

Multiplying the both sides by $2\sqrt{t}$ and letting $\xi = x/\sqrt{t}$, we are led to

$$V(\xi) - \xi V'(\xi) = 2F(V'(\xi), V''(\xi)).$$

Here we have used (F1). We consider this equation with the boundary condition at $\xi = 0$ and $\xi = \infty$:

$$\text{(FODE)} \begin{cases} V(\xi) - \xi V'(\xi) = 2F(V'(\xi), V''(\xi)) \quad \text{in } (0,\infty), & (4.3) \\ V'(0) = \beta > 0, & (4.4) \\ \lim_{\xi\to\infty} V(\xi) = 0. & (4.5) \end{cases}$$

To impose (4.5) is natural in terms of Proposition 3.3 (4). Since F is now defined on $\mathbf{R}\times\mathbf{R}$ and satisfies (F1), we notice that F is written as

$$F(p,X) = \begin{cases} F(p,1)X & \text{if } X \geqq 0, \\ -F(p,-1)X & \text{if } X \leqq 0. \end{cases}$$

Thus the right hand side of (4.3) is linear with respect to $V''(\xi)$ as long as the sign of $V''(\xi)$ does not change. By (F0) we also find that $F(p,1)$ and $-F(p,-1)$ are nonnegative continuous functions of p.

We say a function $V : [0,\infty) \to \mathbf{R}$ is a *classical solution* of (FODE) if $V \in C^2(0,\infty)\cap C^1[0,\infty)$ and V satisfies (4.3)–(4.5). Here we define $V'(0)$ as the right derivative:

$$V'(0) := \lim_{\xi\downarrow 0} \frac{V(\xi)-V(0)}{\xi}.$$

A *viscosity subsolution* of (FODE) is a function $V : [0,\infty) \to \mathbf{R}$ such that V is bounded from above on $[0,\infty)$, V^* satisfies (4.5) and

$$\begin{cases} V^*(\xi) - \xi p \leqq 2F(p,X) & \text{if } \xi > 0, \\ V^*(0) \leqq 2F(p,X) \ \text{ or } \ \beta - p \leqq 0 & \text{if } \xi = 0 \end{cases}$$

for all $(p,X) \in J^{2,+}V^*(\xi)$ with $\xi \geqq 0$. The definitions of a viscosity supersolution and a viscosity solution of (FODE) are similar so are omitted. The set of all second order superjets and subjets of V at ξ on $[0,\infty)$ are denoted by $J^{2,+}V(\xi)$ and $J^{2,-}V(\xi)$, respectively. Namely,

$$J^{2,+}V(\xi) = \{(\phi'(\xi),\phi''(\xi)) \mid \phi \in C^2[0,\infty) \text{ and } V-\phi \text{ attains its maximum at } \xi\},$$
$$J^{2,-}V(\xi) = \{(\phi'(\xi),\phi''(\xi)) \mid \phi \in C^2[0,\infty) \text{ and } V-\phi \text{ attains its minimum at } \xi\}.$$

Remark 4.1. Assume (F0) and (F1). Although (4.3) was derived under the assumption that v is smooth, the consistency between (NP; F, 0) and (FODE) holds in the viscosity sense as well.

- (Consistency) If V is a viscosity subsolution of (FODE), then v given as (4.2) is a viscosity subsolution of (NP; F, 0). Conversely, if v is a viscosity subsolution of (NP; F, 0), then V given as (4.1) is a viscosity subsolution of (FODE). Similar statements hold for supersolutions.

Due to this consistency we have the comparison and existence of viscosity solutions to (FODE). These assertions follow from the results for the time-dependent case in Section 2.

- (Comparison principle) If U and V are, respectively, a viscosity subsolution and supersolution of (FODE), then $U^* \leqq V_*$ on $[0,\infty)$.
- (Existence) There exists a continuous viscosity solution of (FODE).

Example 4.2. Let us consider the linearized equation

$$\text{(LODE)}\begin{cases} V(\xi) - \xi V'(\xi) = BV''(\xi) & \text{in } (0,\infty), \\ (4.4), (4.5) \end{cases} \tag{4.6}$$

with $B > 0$. This equation corresponds to the case that $2F(p,1) = -2F(p,-1) = B$ for all $p \in \mathbf{R}$ in (FODE). Choosing $A = B/2$ in (2.5), we see that the unique classical solution of (LODE) is

$$H_{\beta,B}(\xi) := -\beta\sqrt{2B} \cdot \operatorname{ierfc}\left(\frac{\xi}{\sqrt{2B}}\right). \tag{4.7}$$

Note that the derivatives of $H_{\beta,B}$ up to the second order are

$$H'_{\beta,B}(\xi) = \beta \cdot \operatorname{erfc}\left(\frac{\xi}{\sqrt{2B}}\right), \quad H''_{\beta,B}(\xi) = -\beta\sqrt{\frac{2}{\pi B}} \cdot e^{-\xi^2/2B}. \tag{4.8}$$

In the rest of this section we consider the problem of the form

$$\text{(ODE)} = \text{(ODE}; a, \beta)\begin{cases} V(\xi) - \xi V'(\xi) = a(V'(\xi))V''(\xi) & \text{in } (0,\infty), \\ (4.4), (4.5) \end{cases} \tag{4.9}$$

with nonnegative $a \in C(\mathbf{R})$. Although (ODE) is a special case of (FODE), it turns out that the both problems are equivalent; see Remark 4.8. Our basic assumption on a is

(A1) $a \in C(\mathbf{R})$, $a \geqq 0$ in $\mathbf{R}$ and $a(0) > 0$.

Recall that Mullins' equation (1.8) corresponds to (ODE) with $a(p) = 2/(1+p^2)$. We list fundamental properties of a viscosity solution to (ODE).

Proposition 4.3. *Assume* (A1). *Let V be the unique viscosity solution of* (ODE). *Let $(p, X) \in J^{2,-}V(\xi_0)$ with $\xi_0 > 0$. Then*

(1) *$V < 0$ on $[0,\infty)$.*
(2) *V is increasing on $[0,\infty)$, i.e, $V(\xi_1) < V(\xi_2)$ if $0 \leqq \xi_1 < \xi_2$.*
(3) *$p > 0$ and $X < 0$.*
(4) *V is strictly concave on $[0,\infty)$, i.e., $V((1-\lambda)\xi_1 + \lambda\xi_2) > (1-\lambda)V(\xi_1) + \lambda V(\xi_2)$ for all $\lambda \in (0,1)$ and $\xi_1, \xi_2 \in [0,\infty)$ with $\xi_1 < \xi_2$.*
(5) *$a(p) > 0$.*

Proof. (1) This is a consequence of the second assertion of Proposition 3.3 (2).

(2) We suppose that $0 > V(\xi_1) \geqq V(\xi_2)$ with $\xi_1 < \xi_2$. In view of (4.5) we then have $\min_{[\xi_1,\infty)} V = V(\eta) < 0$ for some $\eta \in (\xi_1, \infty)$. Thus $(0,0) \in J^{2,-}V(\eta)$, so that $V(\eta) - \eta \cdot 0 \geqq a(0) \cdot 0 = 0$ since V is a supersolution. However, this is contract to (1).

(3) (5) The monotonicity of V yields that $p \geqq 0$. We then notice that $a(p)$ must be positive and that X must be negative since $0 > V(\xi) - \xi p \geqq a(p)X$. We show that $p > 0$ after the proof of (4).

(4) We suppose that $V((1-\lambda)\xi_1 + \lambda\xi_2) \leqq (1-\lambda)V(\xi_1) + \lambda V(\xi_2)$ for some $\lambda \in (0,1)$ and $\xi_1, \xi_2 \in [0,\infty)$ with $\xi_1 < \xi_2$. We now take the parabola $\phi \in C^2(\mathbf{R})$ which passes through three points $(\xi_1, V(\xi_1))$,$((1-\lambda)\xi_1 + \lambda\xi_2, V((1-\lambda)\xi_1 + \lambda\xi_2))$ and $(\xi_2, V(\xi_2))$. Then ϕ'' is a nonnegative constant c and $\min_{[\xi_1,\xi_2]}(V - \phi) = (V - \phi)(\eta)$ for some $\eta \in (\xi_1, \xi_2)$. Thus $(\phi'(\eta), c) \in J^{2,-}V(\eta)$, which contradicts (3) since $c \geqq 0$.

(3) Let $\xi_1, \xi_2 > 0$ with $\xi_1 < \xi_0 < \xi_2$. Since V is concave and increasing, we observe that

$$\frac{V(\xi_0) - V(\xi_1)}{\xi_0 - \xi_1} \geqq \frac{V(\xi_2) - V(\xi_0)}{\xi_2 - \xi_0} > 0.$$

We next take $\phi \in C^2(0,\infty)$ such that $\min_{(0,\infty)}(V - \phi) = (V - \phi)(\xi_0)$ and $(p, X) = (\phi'(\xi_0), \phi''(\xi_0))$. Then

$$\frac{\phi(\xi_0) - \phi(\xi_1)}{\xi_0 - \xi_1} \geqq \frac{V(\xi_0) - V(\xi_1)}{\xi_0 - \xi_1}.$$

Combining the two inequalities above and letting $\xi_1 \uparrow \xi_0$, we obtain $p > 0$. □

Remark 4.4. Since V is concave on $[0,\infty)$, we see by Aleksandrov's theorem ([16, Theorem A.2]) that V is twice differentiable almost everywhere on $[0,\infty)$. Namely, $J^{2,+}V(\xi) \cap J^{2,-}V(\xi)$ is nonempty a.e. $\xi \in [0,\infty)$. Accordingly, V solves (4.9) almost everywhere in the classical sense.

Remark 4.5. Although the viscosity solution V in Proposition 4.3 may not be differentiable, we are able to deduce several properties of its one-side derivatives mainly from the strict concavity of V. We define the right derivative V_r' of V and the left derivative V_l' of V as follows:

$$V_r'(\xi_0) := \lim_{\xi \downarrow \xi_0} \frac{V(\xi) - V(\xi_0)}{\xi - \xi_0} \quad \text{for } \xi_0 \in [0,\infty),$$
$$V_l'(\xi_0) := \lim_{\xi \uparrow \xi_0} \frac{V(\xi) - V(\xi_0)}{\xi - \xi_0} \quad \text{for } \xi_0 \in (0,\infty).$$

Under the same hypotheses of Proposition 4.3 these limits indeed exist and enjoy the following properties.

(a) (One-side continuity) $V_r'(\xi_0) = \lim_{\xi\downarrow\xi_0} V_r'(\xi)$ for all $\xi_0 \in [0,\infty)$ and $V_l'(\xi_0) = \lim_{\xi\uparrow\xi_0} V_r'(\xi)$ for all $\xi_0 \in (0,\infty)$.
(b) (Monotonicity) $\beta \geqq V_r'(\xi_1) > V_l'(\xi_2) \geqq V_r'(\xi_2) > V_l'(\xi_3) > 0$ if $0 \leqq \xi_1 < \xi_2 < \xi_3$.
(c) (Limit as $\xi \to \infty$) $\lim_{\xi\to\infty} V_r'(\xi) = \lim_{\xi\to\infty} V_l'(\xi) = 0$. (If the limit were positive, $V(\xi)$ would not converge to zero as $\xi \to \infty$.)

If V is a classical solution, it is obvious that the range of V' on $[0,\infty)$ is $(0,\beta]$. In Corollary 4.12 we will determine the range of V_r' and V_l' when V is not necessarily a classical solution.

We discuss the angle $V'(0)$ at the origin for a viscosity solution V of (ODE).

Proposition 4.6 (Angle at the origin)**.** *Assume* (A1). *Let V be the unique viscosity solution of* (ODE).

(1) *We have*

$$V'(0) = q^- := \begin{cases} \beta & \textit{if } a(\beta) > 0, \\ \inf\{q \in (0,\beta] \mid a = 0 \textit{ on } [q,\beta]\} & \textit{if } a(\beta) = 0. \end{cases}$$

(2) *Let $\beta_1 > \beta$ and V_1 be the unique viscosity solution of* (ODE; a, β_1). *If $a = 0$ on $[\beta, \beta_1]$, then $V = V_1$ on $[0,\infty)$.*

Proof. (1) 1. We first prove that $V'(0)$ exists and $0 < V'(0) \leqq \beta$. Since V is strictly concave, we see that $(V(\xi) - V(0))/\xi$ is increasing as $\xi \downarrow 0$. Thus $V'(0)$ exists and $V'(0) \in (0, \infty]$ by the monotonicity of V. Suppose $V'(0) > \beta$. Then $(p, 0) \in J^{2,-}V(0)$ for every $p \in (\beta, V'(0))$; however, $V(0) - 0 \cdot p < a(p) \cdot 0$ and $\beta - p < 0$. This is a contradiction.

2. We next show that $V'(0) \geqq q^-$. Suppose $0 < V'(0) < q^-$. Then, by the definition of q^- there exists some $p \in (V'(0), \beta)$ such that $a(p) > 0$. We let $\phi(\xi) = -c\xi^2 + p\xi + V(0)$ for $c > 0$. Since $\phi(0) = V(0)$ and $\phi'(0) = p$, it follows that $(p, -2c) \in J^{2,+}V(0)$. We thus have $V(0) - 0 \cdot p \leqq a(p) \cdot (-2c)$, which is a contradiction for large $c > 0$.

3. If $q^- = \beta$, the proof has already been completed. Let $q^- < \beta$ and suppose that $q^- < V'(0) \leqq \beta$. Since $V_r'(\xi) \to V'(0)$ as $\xi \downarrow 0$, we see that $q^- < V_r' \leqq \beta$ on $[0, \varepsilon]$ for some small $\varepsilon > 0$. We now take $(p, X) \in J^{2,+}V(\xi_0) \cap J^{2,-}V(\xi_0)$ with $\xi_0 \in (0, \varepsilon)$; see Remark 4.4 for the existence of such ξ_0. Then $p = V_r'(\xi_0)$. However, we reach a contradiction that $0 > V(\xi_0) - \xi_0 \cdot p \geqq a(p)X = 0$ since $q^- < p \leqq \beta$.

(2) If we prove that V is a viscosity solution of (ODE; a, β_1), the conclusion follows. We only need to consider the boundary condition. Evidently, V is a supersolution of (ODE; a, β_1) since $\beta_1 - p \geqq \beta - p \geqq 0$ whenever $(p, X) \in J^{2,-}V(0)$; see Remark 4.7. We next take $(p, X) \in J^{2,+}V(0)$ and let $p < \beta_1$; otherwise $\beta_1 - p \leqq 0$ holds. In (1) we have shown $V'(0) = \inf\{q \in (0, \beta] \mid a = 0 \text{ on } [q, \beta]\}$. Since $V'(0) \leqq p < \beta_1$, we now have $a(p) = 0$ and therefore $V(0) - 0 \cdot p \leqq 0 = a(p)X$. □

Remark 4.7. Since $0 < V'(0) \leqq \beta$ by (1) above, we always have $\beta - p \geqq 0$ if $(p, X) \in J^{2,-}V(0)$ for a viscosity solution V. Indeed, if $V - \phi$ has its minimum at the origin, then $\phi'(0) \leqq V'(0) \leqq \beta$.

Remark 4.8. Let $F : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ satisfy (F0), (F1) and $F(0, -1) < 0$. It is not difficult to see that, if we replace $a(p)$ by $-2F(p, -1)$, the assertions in Proposition 4.3 and 4.6 still hold for a viscosity solution of the general problem (FODE). We thus find that (FODE) and (ODE) are equivalent in the following sense.

(i) If V is a viscosity solution of (FODE) with F satisfying (F0), (F1) and $F(0, -1) < 0$, then V is also a viscosity solution of (ODE) with $a(p) = -2F(p, -1)$.

(ii) If V is a viscosity solution of (ODE) with a satisfying (A1), then V is also a viscosity solution of (FODE) with $F(p, X) = a(p)X/2$ if $X \leqq 0$, and $F(p, X) = b(p)X$ for some nonnegative $b \in C(\mathbf{R})$ if $X \geqq 0$.

Indeed, when V is concave, we have $2F(p, X) = a(p)X$ for $(p, X) \in J^{2,-}V(\xi)$ with $\xi > 0$. We next let $(p, X) \in J^{2,+}V(\xi)$ with $\xi \geqq 0$. If $X \leqq 0$, then $2F(p, X) = a(p)X$. If $X > 0$, we see $(p, 0) \in J^{2,+}V(\xi)$ by concavity. Since $a(p) \cdot 0 \leqq a(p)X$ and $2F(p, 0) \leqq 2F(p, X)$, we finally conclude (i) and (ii). (It is easy to check the boundary condition by virtue of Remark 4.7.) Also, similar assertions to (i) and (ii) hold for classical solutions.

We next establish a unique existence result of classical solutions to (ODE). Recalling the property (5) in Proposition 4.3, we see that there is no classical solution of (ODE) if $a(\beta_0) = 0$ for some $\beta_0 \in (0, \beta)$. We thus need the positivity of a for the existence. Conversely, it turns out that a viscosity solution of (ODE), for which we have already known the unique existence, is actually a classical solution of (ODE) if a is positive.

Proposition 4.9 (C^2-regularity of viscosity solutions)**.** *Assume* (A1)*. Let V be the unique viscosity solution of* (ODE)*. If $a > 0$ on $[0, \beta]$, then V is a classical solution of* (ODE)*.*

Proof. 1. By virtue of Proposition 4.6 (1) the boundary condition (4.4) is now fulfilled. Since V_r is right continuous, the condition $V \in C^1[0, \infty)$ is satisfied if we prove $V \in C^1(0, \infty)$. In the rest of the proof we show $V \in C^2(0, l)$ for every $l > 0$.

2. Let

$$\psi(r) := \begin{cases} V(0) & \text{if } r \leqq V(0), \\ r & \text{if } V(0) \leqq r \leqq \beta, \\ \beta & \text{if } \beta \leqq r \end{cases}$$

and

$$b(p) := \max\{a(p),\ m_\beta\} \quad \text{with } m_\beta = \min_{q \in [0,\beta]} a(q).$$

Then we observe that V also satisfies

$$\psi(W(\xi)) - \xi\psi(W'(\xi)) = b(W'(\xi))W''(\xi) \quad \text{in } (0, l) \tag{4.10}$$

in the viscosity sense because $V(0) \leqq V \leqq 0$ and $0 \leqq p \leqq \beta$ for every $p \in J^{2,+}V(\xi) \cup J^{2,-}V(\xi)$ with $\xi \in (0, l)$; recall Proposition 4.3 (1), (2) and Remark 4.5 (b). We now solve the ordinary differential equation (4.10) with the boundary condition

$$W(0) = V(0) \quad \text{and} \quad W(l) = V(l). \tag{4.11}$$

According to [25, Theorem XII.4.2] there exists $U \in C^2(0, l) \cap C[0, l]$ which satisfies (4.10) and (4.11) in the classical sense. The reason why we introduced (4.10) is to guarantee that

$$f(\xi, r, p) := \frac{\psi(r) - \xi\psi(p)}{b(p)}$$

is continuous and bounded on $[0, l] \times (-\infty, \infty) \times (-\infty, \infty)$, which is assumed in [25, Theorem XII.4.2].

3. We assert that $V(0) \leqq U \leqq 0$ on $[0, l]$. If $U(\eta) > 0$ at a maximum point $\eta \in (0, l)$ of U, noting that $U'(\eta) = 0$ and $U''(\eta) \leqq 0$, we would reach a contradiction that

$$\psi(U(\eta)) - \eta\psi(U'(\eta)) > 0 \geq b(U'(\eta))U''(\eta).$$

Thus $U \leqq 0$. A similar argument yields $V(0) \leqq U$.

4. By Step 3 we find that U satisfies

$$W(\xi) - \xi\psi(W'(\xi)) = b(W'(\xi))W''(\xi) \quad \text{in } (0, \infty) \tag{4.12}$$

in the classical sense, and therefore in the viscosity sense. We now apply the comparison principle for a viscosity subsolution and a viscosity supersolution of (4.12). Such a comparison is ensured by [16, Theorem 3.3]; indeed, if we set $G(\xi, r, p, X) = r - \xi\psi(p) - b(p)X$, we have $G(\xi, r, p, X) - G(\xi, s, p, X) \geqq r - s$ for $r \geqq s$ and $G(\eta, r, \alpha(\xi - \eta), Y) - G(\xi, r, \alpha(\xi - \eta), X) \leqq \alpha|\xi - \eta|^2$ for $X \leqq Y$. We thus obtain $V = U$ on $[0, l]$, which implies $V \in C^2(0, l)$. □

Approximating a viscosity solution by classical solutions, we prove that its derivative takes the value p if $a(p) > 0$ and that the value of the derivative jumps over p if $a(p) = 0$. In other words, the solution has a corner when the equation is degenerate.

Theorem 4.10 (Corner of profile functions)**.** *Assume* (A1)*. Let V be the unique viscosity solution of* (ODE)*. Let $p \in (0,\beta)$.*

(1) *Assume that $a(p) > 0$. Then there exists a unique $\xi_p \in (0,\infty)$ such that $V \in C^2(I)$ and $V'(\xi_p) = p$ for some open interval I with $\xi_p \in I \subset (0,\infty)$.*
(2) *Assume that $a(p) = 0$. Let*

$$q^+ := \sup\{q \in [p,\beta] \mid a = 0 \text{ on } [p,q]\},$$
$$q^- := \inf\{q \in (0,p] \mid a = 0 \text{ on } [q,p]\}.$$

If $q^+ < \beta$, then there exists a unique $\xi_p \in (0,\infty)$ such that $V_l'(\xi_p) = q^+$ and $V_r'(\xi_p) = q^-$. If $q^+ = \beta$, then we have $V'(0) = q^-$.

Remark 4.11. If $q^- = q^+ = p$ in (2), then V is differentiable at ξ_p but not twice differentiable at ξ_p since $a(p) = 0$. See Proposition 4.3 (5).

Proof. The uniqueness assertions in (1) and (2) follow from the monotonicities of V_r' and V_l', which are ensured by Remark 4.5 (b). If $a > 0$ on $[0,\beta]$, the assertion in (1) is obvious since V' is bijection from $[0,\infty)$ to $(0,\beta]$; recall Remark 4.5 (b), (c) and Proposition 4.6 (1). Also, when $q^+ = \beta$ in (2), we have already proved $V'(0) = q^-$ in Proposition 4.6 (1).

(1) 1. Set $a_\delta(q) = \max\{a(q),\ \delta\}$ for $\delta \in (0, a(0)]$. Owing to the positivity of a_δ the unique solution V_δ of (ODE; a_δ,β) is smooth. Since a_δ converges to a uniformly, we see that V_δ converges to V as $\delta \to 0$ locally uniformly on $[0,\infty)$ by stability (Proposition 2.9 (2)).

2. Take $\varepsilon > 0$ small so that $[p-\varepsilon, p+\varepsilon] \subset (0,\beta)$ and $a > 0$ on $[p-\varepsilon, p+\varepsilon]$. Since V_δ is a classical solution of (ODE; a_δ,β) with a positive a_δ, there exist $\xi_\delta^-, \eta_\delta, \xi_\delta^+ \in (0,\infty)$ such that $\xi_\delta^- < \eta_\delta < \xi_\delta^+$ and $(V_\delta'(\xi_\delta^-), V_\delta'(\eta_\delta), V_\delta'(\xi_\delta^+)) = (p+\varepsilon, p, p-\varepsilon)$ for each $\delta > 0$. Then we observe

$$(p-\varepsilon)\xi_\delta^+ \leqq \int_0^{\xi_\delta^+} V_\delta'(\xi)d\xi = V_\delta(\xi_\delta^+) - V_\delta(0) \leqq -V_\delta(0).$$

Since $V_{a(0)}(0) \leqq V_\delta(0)$ by the comparison principle, we obtain $\xi_\delta^+ \leqq -V_{a(0)}(0)/(p-\varepsilon)$. Therefore we may assume that $(\xi_\delta^-, \eta_\delta, \xi_\delta^+) \to (\bar\xi^-, \bar\eta, \bar\xi^+)$ as $\delta \to 0$ by taking a subsequence if necessary.

3. We show that $-M \leqq V_\delta'' \leqq 0$ on $[\xi_\delta^-, \xi_\delta^+]$ for some $M > 0$ independent of δ. Take $c > 0$ such that $c \leqq a$ on $[p-\varepsilon, p+\varepsilon]$. Then, for $\xi \in [\xi_\delta^-, \xi_\delta^+]$ we have

$$V_\delta''(\xi) = \frac{V_\delta(\xi) - \xi V_\delta'(\xi)}{a(V_\delta'(\xi))} \geqq \frac{V_{a(0)}(0) - \xi_\delta^+(p+\varepsilon)}{c}.$$

Since $\{\xi_\delta^+\}_\delta$ is bounded by Step 2, we conclude that $V_\delta'' \geqq -M$ for some $M > 0$.

4. We next claim that $\bar\xi^- < \bar\eta < \bar\xi^+$. In fact, we compute

$$-\varepsilon = V_\delta'(\eta_\delta) - V_\delta'(\xi_\delta^-) = \int_{\xi_\delta^-}^{\eta_\delta} V_\delta''(\xi)d\xi \geqq -M(\eta_\delta^- - \xi_\delta^-),$$

which implies that $\bar\xi^- < \bar\eta$. The same argument yields that $\bar\eta < \bar\xi^+$.

5. Choose $\theta > 0$ small so that $J := [\bar\eta - \theta, \bar\eta + \theta] \subset (\bar\xi^-, \bar\xi^+)$. We then have $-M \leqq V_\delta'' \leqq 0$ on J for sufficiently small δ. Thus the Ascoli-Arzelà theorem ensures that V_δ' converges to some $U \in C(J)$ as $\delta \to 0$ uniformly on J by taking a subsequence. In particular, we have $U(\bar\eta) = \lim_{\delta\to 0} V_\delta'(\eta_\delta) = p$. Since V^δ converges

to V pointwise, we learn that $V \in C^1(\bar{\eta}-\theta, \bar{\eta}+\theta)$ and $V' = U$. Consequently $V'(\bar{\eta}) = p$.

6. We are able to show the C^2-regularity of V in the same way as in the proof of Proposition 4.9. Let $I = (a,b) := (\bar{\eta}-\theta/2, \bar{\eta}+\theta/2)$. Since $V \in C^1(\bar{\eta}-\theta, \bar{\eta}+\theta)$, for every $\xi \in I$ and $(p,X) \in J^{2,+}V(\xi) \cup J^{2,-}V(\xi)$ we have $V'(a) \geqq p \geqq V'(b)$ and $a(p) \geqq m$ for some $m > 0$. Thus V solves

$$\psi(W(\xi)) - \xi\psi(W'(\xi)) = b(W'(\xi))W''(\xi) \quad \text{in } I \tag{4.13}$$

in the viscosity sense. Here $\psi(r)$ and $b(p)$ are suitable modification of functions r and $a(p)$ respectively; see the proof of Proposition 4.9. Then V must agree with a classical solution of (4.13) with the boundary condition $W(a) = V(a)$ and $W(b) = V(b)$. Hence $V \in C^2(I)$.

(2) 1. Let $q^+ < \beta$. By the definitions of q^- and q^+ there exist sequences $\{q_n^-\}_n$ and $\{q_n^+\}_n$ such that $0 < q_n^- < q^- \leqq q^+ < q_n^+ < \beta$, $a(q_n^-) > 0$, $a(q_n^+) > 0$, $q_n^- \uparrow q^-$ as $n \to \infty$ and $q_n^+ \downarrow q^+$ as $n \to \infty$. Then we see by (1) that $(V'(\xi_n^-), V'(\xi_n^+)) = (q_n^-, q_n^+)$ for some $\xi_n^-, \xi_n^+ \in (0,\infty)$ such that $0 < \xi_n^+ \leqq \xi_{n+1}^+ \leqq \xi_{n+1}^- \leqq \xi_n^-$. By this monotonicity we let $\lim_{n\to\infty}(\xi_n^-, \xi_n^+) = (\bar{\xi}^-, \bar{\xi}^+)$, and then we have

$$V_l'(\bar{\xi}^+) = \lim_{\xi\uparrow\bar{\xi}^+} V_l'(\xi) = \lim_{n\to\infty} V_l'(\xi_n^+) = q^+,$$
$$V_r'(\bar{\xi}^-) = \lim_{\xi\downarrow\bar{\xi}^-} V_r'(\xi) = \lim_{n\to\infty} V_r'(\xi_n^-) = q^-.$$

2. It remains to prove that $\bar{\xi}^+ = \bar{\xi}^-$. Suppose that $\bar{\xi}^+ < \bar{\xi}^-$. We take $(p_0, X) \in J^{2,-}V(\eta_0)$ with $\eta_0 \in (\bar{\xi}^+, \bar{\xi}^-)$; recall Remark 4.4. We then have

$$p_0 \leqq V_r'(\eta_0) \leqq V_r'(\xi_n^+) = q_n^+ \quad \text{and} \quad p_0 \geqq V_l'(\eta_0) \geqq V_l'(\xi_n^-) = q_n^-.$$

Sending $n \to \infty$ yields that $q^- \leqq p_0 \leqq q^+$, and hence $a(p_0) = 0$. This is contrary to Proposition 4.3 (5). □

We are now in a position to determine the range of V_r' and V_l'. Define $R(V_r') := \{V_r'(\xi) \mid \xi \geqq 0\}$, $R(V_l') := \{V_l'(\xi) \mid \xi > 0\}$ and

$$\overline{\{a>0\}}^r := \left\{ p \in (0,\beta] \;\middle|\; \begin{array}{c} \text{there exists } \{q_n\}_{n=1}^\infty \subset (0,p] \text{ such that} \\ a(q_n) > 0 \text{ and } q_n \to p \text{ as } n \to \infty \end{array} \right\},$$
$$\overline{\{a>0\}}^l := \left\{ p \in (0,\beta) \;\middle|\; \begin{array}{c} \text{there exists } \{q_n\}_{n=1}^\infty \subset [p,\beta) \text{ such that} \\ a(q_n) > 0 \text{ and } q_n \to p \text{ as } n \to \infty \end{array} \right\}.$$

Corollary 4.12 (Range of derivatives)**.** *Assume* (A1). *Let V be the unique viscosity solution of* (ODE). *Then we have $R(V_r') = \overline{\{a>0\}}^r$ and $R(V_l') = \overline{\{a>0\}}^l$.*

Proof. The inclusion $R(V_r') \supset \overline{\{a>0\}}^r$ follows immediately from Theorem 4.10 (1) and (2). Let $p \in R(V_r')$, that is $p = V_r'(\xi)$ for some $\xi \geqq 0$. Evidently, we have $p \in \overline{\{a>0\}}^r$ if $a(p) > 0$. We let $a(p) = 0$. When $\beta = q^+ := \sup\{q \in [p,\beta] \mid a = 0 \text{ on } [p,q]\}$, Proposition 4.6 implies $V'(0) = q^- := \inf\{q \in (0,p] \mid a = 0 \text{ on } [q,p]\}$. By definition $q^- \leqq p$. Since we also have $q^- \geqq V_r'$ on $[0,\infty)$ by monotonicity, it follows that $p = q^- \in \overline{\{a>0\}}^r$. In the case where $\beta > q^+$, by Theorem 4.10 (2) we have $V_l'(\xi_p) = q^+$ and $V_r'(\xi_p) = q^-$ for some $\xi_p > 0$. Since $V_r'(\xi_p) = q^- \leqq p = V_r'(\xi)$, we see $\xi_p \geqq \xi$. If $\xi_p > \xi$, we would reach a contradiction that $V_r'(\xi) > V_l'(\xi_p) = q^+ \geqq p = V_r'(\xi)$. Thus $\xi = \xi_p$ and then $p = q^-$. This means $p \in \overline{\{a>0\}}^r$. We have thus proved $R(V_r') = \overline{\{a>0\}}^r$. A similar argument yields $R(V_l') = \overline{\{a>0\}}^l$. □

5. Depth of the thermal groove

We investigate the depth of the thermal groove. For a viscosity solution V of (ODE) we define

$$d(\beta) := -V(0). \tag{5.1}$$

This is the depth of the corresponding self-similar solution v in (4.2) at the origin when $t = 1$. Similarly, for the classical solution W of the linearized equation (LODE) with $B = a(0) > 0$ we define

$$L(\beta) := -W(0) = \beta\sqrt{\frac{2a(0)}{\pi}}, \tag{5.2}$$

where the second equality is due to (4.7) since $W = H_{\beta,a(0)}$.

Theorem 5.1 (Depth of the groove)**.** *Assume* (A1)*. Assume furthermore that* $a(p) \leqq a(0)$ *for all* $p > 0$*. Let* V *and* W *be, respectively, the unique viscosity solution of* (ODE) *and that of* (LODE) *with* $B = a(0)$*. Define* d *and* L *as in* (5.1) *and* (5.2)*. Then*

(1) $0 < d \leqq L$ *in* $(0, \infty)$.
(2) d *is nondecreasing in* $(0, \infty)$.
(3) $e_1(\beta) := \beta\sqrt{(2\min_{[0,\beta]} a)/\pi} \leqq d(\beta)$ *for all* $\beta > 0$.
(4)
$$0 \leqq \frac{L(\beta) - d(\beta)}{\beta} \leqq C\left(a(0) - \min_{[0,\beta]} a\right)$$
with $C = \sqrt{2/(\pi a(0))}$ *for all* $\beta > 0$*. In particular,* $\lim_{\beta\downarrow 0}(L(\beta)-d(\beta))/\beta = 0$.
(5) *If* a *is nonincreasing on* $[0, \infty)$*, then* $\lambda d(\beta) \leqq d(\lambda\beta)$ *for all* $\lambda \in [0, 1]$ *and* $\beta > 0$.
(6) $e_2(\beta) := \sqrt{\int_0^\beta a(p)p\,dp} \leqq d(\beta)$ *for all* $\beta > 0$.
(7) *If* $a(p) \geqq c/(1+p^2)$ *on* $[M, \infty)$ *for some* $c, M > 0$*, then* $\lim_{\beta\to\infty} d(\beta) = \infty$.

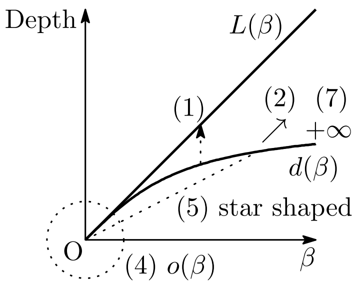

Figure 3. The assertions in Theorem 5.1 on the depth $d(\beta)$.

The estimate in (4) yields (1.14), which asserts that the depth of the linearized problem is the third order approximation in Mullins' case, i.e, $a(p) = 2/(1 + p^2)$. The main tool for the proof of (1)–(5) is the comparison principle while we calculate integrals to show (6) and (7).

Proof. (1) Fix $\beta > 0$. By Proposition 4.3 (1) the depth $d(\beta)$ is positive. We next observe that $W - \xi W' = a(0)W'' \leqq a(W')W''$ on $(0,\infty)$ since $W' \geqq 0$ and $W'' \leqq 0$. This inequality means that W is a subsolution of (ODE). We thus find by the comparison principle that $W \leqq V$ on $[0,\infty)$, and hence $d(\beta) \leqq L(\beta)$.

(2) Take $\beta_1, \beta_2 > 0$ with $\beta_1 < \beta_2$. Let V_1 and V_2 be, respectively, the unique viscosity solution of (ODE; a, β_1) and that of (ODE; a, β_2). It is then easily seen that V_1 is a supersolution of (ODE; a, β_2), and so $V_2 \leqq V_1$ on $[0,\infty)$ by the comparison principle. As a result we see that $d(\beta_1) \leqq d(\beta_2)$.

(3) Fix $\beta > 0$ and take $\beta_0 > 0$ such that $\min_{[0,\beta]} a = a(\beta_0)$. Clearly the claim holds if $a(\beta_0) = 0$. In the case where $a(\beta_0) > 0$ we consider the linearized equation (LODE) with $B = a(\beta_0)$. Then the unique classical solution is

$$U(\xi) := H_{\beta,a(\beta_0)}(\xi) = -\beta\sqrt{2a(\beta_0)} \cdot \operatorname{ierfc}\left(\frac{\xi}{\sqrt{2a(\beta_0)}}\right).$$

Since $0 \leqq U' \leqq \beta$ and $U'' \leqq 0$, we observe that $U - \xi U' = a(\beta_0)U'' \geqq a(U')U''$ on $(0,\infty)$. Thus U is a supersolution of (ODE; a, β). We now apply the comparison principle to obtain $V \leqq U$ on $[0,\infty)$. In particular, we have

$$d(\beta) \geqq -U(0) = \beta\sqrt{\frac{2a(\beta_0)}{\pi}} = e_1(\beta).$$

(4) It follows from (3) that

$$0 \leqq L(\beta) - d(\beta) \leqq \beta\sqrt{\frac{2a(0)}{\pi}} - \beta\sqrt{\frac{2\min_{[0,\beta]} a}{\pi}} \leqq C\beta\left(a(0) - \min_{[0,\beta]} a\right).$$

The second assertion in (4) is now obvious.

(5) Fix $\beta > 0$ and $\lambda \in (0,1)$. Let V_λ be the unique viscosity solution of (ODE; $a, \lambda\beta$). Set $\tilde{V} = \lambda V$. We now claim that $\tilde{V}$ is a supersolution of (ODE; $a, \lambda\beta$). Let $(p, X) \in J^{2,-}\tilde{V}(\xi)$, i.e., $(p/\lambda, X/\lambda) \in J^{2,-}V(\xi)$. If $\xi = 0$, we derive $\beta - (p/\lambda) \geqq 0$ from Remark 4.7. This means $\lambda\beta - p \geqq 0$. If $\xi > 0$, noting that $p \geqq 0$, $X \leqq 0$ and $V(\xi) - \xi \cdot (p/\lambda) \geqq a(p/\lambda)X/\lambda$, we have $\tilde{V}(\xi) - \xi p \geqq a(p/\lambda)X \geqq a(p)X$ since a is monotone. We thus conclude that $\tilde{V}$ is a supersolution of (ODE; $a, \lambda\beta$). Hence $V_\lambda \leqq \tilde{V}$ on $[0,\infty)$, and so $d(\lambda\beta) \geqq \lambda d(\beta)$.

(6) 1. We first let $a > 0$ on $[0,\infty)$. Then V is a classical solution of (ODE) by Proposition 4.9. We multiply the both sides of (4.9) by $V'(\xi)$ and integrate over $[0,\eta]$. We then have

$$\begin{aligned} I_1 &:= \int_0^\eta \{V(\xi) - \xi V'(\xi)\}V'(\xi)d\xi = \Big[\{V(\xi) - \xi V'(\xi)\}V(\xi)\Big]_0^\eta + \int_0^\eta \xi V''(\xi)V(\xi)d\xi \\ &= \{V(\eta)\}^2 - \eta V'(\eta)V(\eta) - \{V(0)\}^2 + \int_0^\eta \xi V''(\xi)V(\xi)d\xi \end{aligned}$$

from the left hand side while the right hand side becomes

$$I_2 := \int_0^\eta a(V'(\xi))V''(\xi)V'(\xi)d\xi = \int_\beta^{V'(\eta)} a(p)pdp = -\int_{V'(\eta)}^\beta a(p)pdp,$$

where we have used the change of variables that $p = V'(\xi)$. Since $V \leqq 0$, $V' \geqq 0$ and $V'' \leqq 0$, we see that $I_1 \geqq -\{V(0)\}^2 = -\{d(\beta)\}^2$. Thus

$$\{d(\beta)\}^2 \geqq -I_1 = -I_2 = \int_{V'(\eta)}^\beta a(p)pdp.$$

Letting $\eta \to \infty$ and recalling Remark 4.5 (c), we obtain the estimate in (6).

2. If a is not necessarily positive, we set $a_\delta(p) := \max\{a(p),\ \delta\}$ for $\delta > 0$. Then Step 1 yields $\int_0^\beta a_\delta(p)pdp \leqq \{V_\delta(0)\}^2$, where V_δ is the unique classical solution of (ODE; a_δ, β). Letting $\delta \to 0$ gives the desired conclusion since $V_\delta(0) \to V(0)$ by the stability; recall the argument in Step1 in the proof of Theorem 4.10 (1).

(7) For $\beta \geqq M$ we observe that

$$\{e_2(\beta)\}^2 = \int_0^\beta a(p)pdp \geqq \int_M^\beta \frac{cp}{1+p^2}dp = \frac{c}{2}\log\frac{1+\beta^2}{1+M^2}.$$

Thus (6) yields the claim. □

Remark 5.2. (1) We have actually derived several estimates not only at the origin but also on the whole $[0,\infty)$. In particular, by the proof of (1) and (3) we notice

$$\begin{aligned}&0 \leqq V(\xi) - W(\xi)\\ &\leqq \beta\left\{\sqrt{2a(0)}\cdot \mathrm{ierfc}\left(\frac{\xi}{\sqrt{2a(0)}}\right) - \sqrt{2a(\beta_0)}\cdot \mathrm{ierfc}\left(\frac{\xi}{\sqrt{2a(\beta_0)}}\right)\right\}\end{aligned}$$

for all $\xi \in [0,\infty)$, where $\beta_0 > 0$ is chosen so that $a(\beta_0) = \min_{[0,\beta]} a$.

(2) By virtue of Proposition 4.6 (2) we see that $\lim_{\beta\to\infty} d(\beta) \neq \infty$ if $a = 0$ on $[M,\infty)$ for some $M > 0$. Namely, the depth does not necessarily go to infinity.

Remark 5.3. In [5] the authors gives upper and lower bounds on the solution V of (ODE) with $a(p) = 1/2(1+p^2)$. There two auxiliary (ODE) with $a_1(p) = 1/(1+p)^2$ and $a_2(p) = 1/2(1+p)^2$ are considered, and the exact solution V_1 of (ODE; a_1, β) and V_2 of (ODE; a_2, β) are given in the implicit forms. Since $a_1 \geqq a \geqq a_2$, employing the comparison theorem, the authors conclude $V_1 \leqq V \leqq V_2$, and in particular they derive the estimate at the origin of the form

$$\sqrt{2\log\left(\frac{\beta}{\sqrt{\pi}}\right)} \geqq d(\beta) \geqq \sqrt{\log\left(\frac{\beta}{2\sqrt{\pi}}\right) + \frac{1}{4}} - \frac{1}{2} =: l_1(\beta).$$

The both sides of the above inequality are of order $O(\sqrt{\log\beta})$ as $\beta \to \infty$. Our result (6) also gives a lower bound on $d(\beta)$, which is

$$d(\beta) \geqq \sqrt{\int_0^\beta \frac{p}{2(1+p^2)}dp} = \sqrt{\frac{1}{4}\log(1+\beta^2)} =: l_2(\beta).$$

The right hand side $l_2(\beta)$ is of order $O(\sqrt{\log\beta})$, the same order as in [5]; however, by a direct calculation we see $\lim_{\beta\to\infty}(l_1(\beta) - l_2(\beta)) = \infty$. Thus our estimate (6) in Theorem 5.1 is rough in this sense, but it is shown more simply and directly by integration and is enough to prove $d(\beta) \to \infty$ as $\beta \to \infty$ in the Mullins' example.

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