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Geodesic flows on spheres and the local Riemann-Roch numbers

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GEODESIC FLOWS ON SPHERES AND THE LOCAL RIEMANN-ROCH NUMBERS

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Abstract. We calculate the local Riemann-Roch numbers of the zero sections of T^*S^n and $T^*\mathbb{R}P^n$, where the local Riemann-Roch numbers are defined by using the S^1-bundle structure on their complements associated to the geodesic flows.

1. Introduction

In our previous papers [1], [2], and [3] we gave a formulation of index for Dirac-type operators on open manifolds when some additional structures are given on the ends of the base manifolds. A typical example is the *local Riemann-Roch number* for an open neighborhood of a singular fiber of (not necessarily completely) integrable system. In our previous papers we mainly considered the cases of global torus actions.

The explicit examples we calculated in [1], [2], and [3] were 2-dimensional cases and their products, which are integrable systems having tori as singular fibers. In this paper we determine the local Riemann-Roch number of T^*S^n when the additional structure is given by the geodesic flow for the standard metric. The two new aspects of this example is as follows: (1) The case T^*S^n is a Hamiltonian system having the zero section S^n as singularities. (2) On the complement of S^n we have the S^1-action induced from the geodesic flow, while the S^1-action cannot be extended to the whole spaces.

The examples we consider in this paper may appear as parts of the completely integrable systems on the moduli spaces of $SU(2)$ or $SO(3)$ flat connections on Riemann surfaces constructed in [4] [6] . A possible application of our framework is to give a direct proof of the Verlinde formula for this case based on the properties of our local Riemann-Roch numbers developed in [2]. We expect the calculation in this paper will be a key statement in this application.

The case of T^*S^n is the double covering of the case of $T^*\mathbb{R}P^n$. However their local Riemann-Roch numbers are not related by the multiplication by 2. This observation implies that the local Riemann-Roch number is a global invariant which is not expressed as integral of some local invariant.

We use the excision formula and the product formula to calculate the local Riemann-Roch number of T^*S^n. We first construct a compactification with a compatible almost complex structure. The Riemann-Roch number of the compactification is directly calculated by the usual Riemann-Roch formula. What we need is to identify the contribution

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of the zero section S^n. The other contributions to the whole Riemann-Roch number is calculated by the product formula. We obtain the required number by the excision formula and a simple subtraction.

The organization of the present paper is as follows. In Section 2 we state a formulation of local (or relative) index in [1] which will be convenient to use in this paper. In Section 3 we state our main theorem. In Section 4 we construct the three compactifications of T^*S^n with almost complex structures. In Section 4.4 we compare these three and show that we can use any of them to calculate the required local Riemann-Roch numbers. In Section 5 we show general properties of local models, in particular, a product formula for the product of discs or cylinders and closed manifolds, which is applied to the case of symplectic cuts. In Section 6 we prove our main theorem.

1.1. **Notations.** Let $E \to X$ be a vector bundle with Euclidean metric. For a positive number r we denote by $D_r(E)$, $D_r^o(E)$ and $S_r(E)$ the closed disc bundle, the open disc bundle and the circle bundle of radius r, respectively.

2. Circle actions and local Riemann-Roch number

In this section we recall our setting in [1], which allows us to define the local Riemann-Roch number and to have a localization formula of the Riemann-Roch number. For our convenience we explain in the category of almost Hermitian manifolds with S^1-actions. See [1] for the general version.

Let (M, J, g) be an almost Hermitian manifold, i.e., M is a smooth manifold, J is an almost complex structure on M and g is a Riemannian metric of M which is preserved by J. Let $(E, \nabla) \to M$ be a Hermitian vector bundle with connection. Let W_E be a $\mathbb{Z}/2$-graded Hermitian vector bundle $\wedge^\bullet TM \otimes E$, which has a structure of Clifford module bundle over TM. Suppose that there exists an open subset V of M with an action of the circle group S^1 which satisfies the following conditions.

(1) The complement $M \smallsetminus V$ is compact.
(2) The S^1-action preserves the almost Hermitian structure (J, g) on V.
(3) There are no fixed points, i.e., $V^{S^1} = \emptyset$.
(4) The restriction of (E, ∇) to any S^1-orbit does not have non-zero parallel sections.

In [1] we defined the *relative (or local) index* $\mathrm{ind}(M, V, W_E) = \mathrm{ind}(M, V)$ which satisfies the following.

Theorem 2.1. *The relative index* $\mathrm{ind}(M, V)$ *satisfies the following properties.*

(1) *If M is closed, then* $\mathrm{ind}(M, V)$ *coincides with the index of the spin^c Dirac operator acting on the space of sections of W_E.*
(2) $\mathrm{ind}(M, V)$ *is invariant under continuous deformations of the data, i.e., (J, g, E, ∇) and the S^1-action on V.*
(3) $\mathrm{ind}(M, V)$ *satisfies an excision formula.*
(4) $\mathrm{ind}(M, V)$ *satisfies a product formula.*

The relative index does not depend on the choice of the neighborhood of the complement $K := M \smallsetminus V$. Namely when M' is a neighborhood of K, if we put $V' := M' \smallsetminus K$, we have $\mathrm{ind}(M, V) = \mathrm{ind}(M', V')$ from the excision formula (3). Although the relative index depends on the data on a neighborhood of K we write $\mathrm{ind}_{\mathrm{loc}}(K)$ for this index if there are no confusion. If M is a symplectic manifold with a compatible almost complex structure and (E, ∇) is a prequantizing line bundle, then we denote it by $\mathrm{ind}(M, V) = RR(M, V)$ or $\mathrm{ind}_{\mathrm{loc}}(K) = RR_{\mathrm{loc}}(K)$ and call it the *local (or relative) Riemann-Roch number.*

3. Main theorem

Let S^n be the unit sphere centered at the origin in the $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$. We denote by V_0 the total space of the cotangent bundle of S^n and by V_{01} the complement of the zero section in V_0. We identify the cotangent bundle V_0 with the tangent bundle TS^n via the Riemannian metric. Hence V_0 is identified with the set of pairs of vectors (x, v) in $\mathbb{R}^{n+1}$ with $||x|| = 1$ and $x \cdot v = 0$, where $x \cdot v$ is the standard Euclidean inner product and $|| \cdot ||$ is the induced norm.

Consider the Liouville 1-form α and the standard symplectic structure $d\alpha$ of V_0. The Hamiltonian action for the Hamiltonian function

$$h_{01} : V_{01} \to \mathbb{R}, \ h_{01}(x, v) := ||v||$$

is periodic with the period 2π, and hence, induces a Hamiltonian (free) S^1-action on V_{01}. For $(x, v) \in V_{01}$ let $\mathbb{R}\langle x, v\rangle$ be the oriented 2-plane generated by $x, v \in \mathbb{R}^{n+1}$. The S^1-action on V_{01} is given by the standard rotation action of $S^1 = SO(2)$ on $\mathbb{R}\langle x, v\rangle$, which we call the normalized geodesic flow[1].

Consider the trivial bundle $V_0 \times \mathbb{C}$ with the connection 1-form α, which gives a prequantizing line bundle over V_0. Using the above S^1-action we can define the relative Riemann-Roch number $RR_{\mathrm{loc}}(S^n)$ of the zero section $S^n \subset V_0$, where we use the S^1-invariant $d\alpha$-compatible almost complex structure J_0, which is defined by the Levi-Civita connection of S^n as in Subsection 4.4. The following is the main theorem in the present paper.

Theorem 3.1.

$$RR_{\mathrm{loc}}(S^n) = \begin{cases} 2 & (n = 0) \\ 1 & (n \geq 1). \end{cases}$$

Since the integral of α along the S^1-orbit of $(x, v) \in V_{01}$ is given by $2\pi||v||$, the holonomy along the orbit is equal to $\exp(2\pi\sqrt{-1}||v||)$. In particular the following holds.

Lemma 3.2. *The holonomy along the S^1-orbit of $(x, v) \in V_{01}$ is trivial if and only if the norm of v is an integer.*

Let l be a positive integer. By the above lemma for any small neighborhood M_l of $h_{01}^{-1}(l)$ we can define the relative Riemann-Roch number $RR(M_l, M_l \smallsetminus h_{01}^{-1}(l)) = RR_{\mathrm{loc}}(h_{01}^{-1}(l))$. We will also show the following.

Theorem 3.3.

$$RR_{\mathrm{loc}}(h_{01}^{-1}(l)) = \begin{pmatrix} n+l-1 \\ n-1 \end{pmatrix} + \begin{pmatrix} n+l-2 \\ n-1 \end{pmatrix}.$$

Remark 3.4. (1) The isometric action of $O(n+1)$ on S^n extends to all the constructions in the subsequent sections. Using the corresponding equivariant version of the localization, we can show that the above identification of the local Riemann-Roch number in Theorem 3.1 still holds as virtual $O(n+1)$ representation, i.e., the equivariant local

[1] It is well-known that the Hamiltonian vector field for the Hamiltonian function $h_0(x, v) := \frac{1}{2}||v||^2$ gives the geodesic flow on V_0. Though this flow is periodic, it does not give an S^1-action in general. We need an S^1-action, so we use h_{01} insead of h_0.

Riemann-Roch character is equal to 1 for $n > 0$. It implies that for any finite subgroup G in $O(n+1)$ acting freely on S^n, the local Riemann-Roch number of S^n/G in $T(S^n/G)$ is equal to 1 for $n > 0$.
(2) In the setting of (1), for any nontrivial representation $\rho\colon G \to \{\pm 1\}$ we can twist the prequantizing line bundle by ρ to obtain another prequantizing line bundle on the quotient space $T(S^n/G)$. The local Riemann-Roch number S^n/G in $T(S^n/G)$ for this twisted prequantizing line bundle is similarly calculated and turns out to be 0 for $n > 0$. In particular, by taking $\{\pm 1\}$ as G and taking the identity map as ρ we can show that the local Riemann-Roch number of $\mathbb{R}P^n = S^n/G$ in $T\mathbb{R}P^n = TS^n/G$ with the twisted prequantizing line bundle is equal to 0 for $n > 0$.
(3) The above two examples imply that the local Riemann-Roch number is not always multiplicative with respect to finite covering.

4. Compactification of the cotangent bundle

4.1. **Symplectic cut.** For the prequantized symplectic manifold $(V_0, d\alpha, V_0 \times \mathbb{C}, \alpha)$ and the Hamiltonian S^1-action on V_{01} we can apply the construction of *symplectic cut* due to E. Lerman[5]. Namely for $r > 0$ let $\hat{h}_r : V_{01} \times \mathbb{C} \to \mathbb{R}$ be the function defined by

$$\hat{h}_r(x, v, w) := ||v|| + \frac{1}{2}|w|^2 - r.$$

$\hat{h}_r$ is a Hamiltonian for the diagonal S^1-action on $V_{01} \times \mathbb{C}$ and 0 is a regular value of $\hat{h}_r$. We define the symplectomorphism $\varphi\colon D_r^o(T^*S^n) \cap V_{01} \to \{(x, v, w) \in \hat{h}_r^{-1}(0)\colon w \neq 0\}/S^1$ by

$$\varphi(x, v) = \left[x, v, \sqrt{2(r - \|v\|)}\right].$$

Then the symplectic cut X_r is defined by gluing $D_r^o(T^*S^n)$ and $\hat{h}_r^{-1}(0)/S^1$ by φ, i.e.,

$$X_r := D_r^o(T^*S^n) \cup_\varphi \hat{h}_r^{-1}(0)/S^1.$$

By definition X_r can be identified with a union $D_r^o(T^*S^n) \cup h_{01}^{-1}(r)/S^1$ as a set. Note that X_r is a smooth symplectic manifold, and if r is a positive integer, then there exists the induced prequantizing line bundle L_r over X_r. We put $M_{(r)} := h_{01}^{-1}(r)/S^1$, which is a symplectic submanifold of X_r. The normal bundle $\nu_{(r)}$ of $M_{(r)}$ in X_r is isomorphic to $h_{01}^{-1}(r) \times_{S^1} \mathbb{C}_1$, where $\mathbb{C}_1$ is the one-dimensional representation of S^1 of weight 1.

Lemma 4.1. *The following gives an embedding of $\nu_{(r)}$ into X_r as a tubular neighborhood of $M_{(r)}$:*

$$\nu_{(r)} = h_{01}^{-1}(r) \times_{S^1} \mathbb{C}_1 \ni [x, v, z] \mapsto \left[x, \frac{v}{1+|z|^2}, \frac{\sqrt{2r}z}{\sqrt{1+|z|^2}}\right] \in \hat{h}_r^{-1}(0)/S^1 \subset X_r.$$

Remark 4.2. Though the smooth manifolds X_r for different r are all diffeomorphic to each other, (X_r, L_r) are not mutually isomorphic as prequantized symplectic manifolds.

4.2. **Grassmannian.** For three nonnegative integers a_0, a_1 and N with $a_0 + a_1 + 1 = N$, two oriented Grassmannians $\mathrm{Gr}^+_{a_0+1}(\mathbb{R}^{N+1})$ and $\mathrm{Gr}^+_{a_1+1}(\mathbb{R}^{N+1})$ can be identified by taking orthogonal complements. We put $X_{a_0,a_1,N} = \mathrm{Gr}^+_{a_0+1}(\mathbb{R}^{N+1}) = \mathrm{Gr}^+_{a_1+1}(\mathbb{R}^{N+1})$. For $i = 0, 1$, let γ_{a_i} be the total space of the tautological $\mathbb{R}^{a_i}$-bundle over $\mathrm{Gr}^+_{a_i}(\mathbb{R}^N)$ and $\gamma_{a_i}^\perp$ the orthogonal complement bundle of γ_{a_i} in $\mathrm{Gr}^+_{a_i}(\mathbb{R}^N) \times \mathbb{R}^N$.

Lemma 4.3. *$X_{a_0,a_1,N}$ is diffeomorphic to the manifold $\gamma_{a_0}^{\perp} \sqcup \gamma_{a_1}^{\perp}/\sim$, where $(P_0,v_0) \sim (P_1,v_1)$ for $(P_i,v_i) \in \gamma_{a_i}^{\perp}$ if $v_0, v_1 \neq 0$ and the following three relations are satisfied:*

$$P_0 \perp P_1, \quad \frac{v_0}{||v_0||} + \frac{v_1}{||v_1||} = 0, \quad ||v_0|| \cdot ||v_1|| = 1.$$

Proof. For $(P_i,v_i) \in \gamma_{a_i}^{\perp}$ let P be the subspace $(P_i \oplus \{0\}) \oplus \mathbb{R}(v_i \oplus 1)$ of $\mathbb{R}^{N+1}$. The map $\gamma_{a_i}^{\perp} \to X_{a_0,a_1,N}$, $(P_i,v_i) \mapsto P$ is compatible with the identification $\sim$ and induces the map $\gamma_{a_0}^{\perp} \sqcup \gamma_{a_1}^{\perp}/\sim \to X_{a_0,a_1,N}$.

We next construct the inverse map $X_{a_0,a_1,N} \to \gamma_{a_0}^{\perp} \sqcup \gamma_{a_1}^{\perp}/\sim$ as follows. Suppose that $P \in X_{a_0,a_1,N} = \mathrm{Gr}_{a_0+1}^{+}(\mathbb{R}^{N+1})$ is not contained in the subspace $\mathbb{R}^N \oplus \{0\} \subset \mathbb{R}^{N+1}$. We put $P_0 := P \cap (\mathbb{R}^N \oplus \{0\})$ and let $P_0^{\perp}$ be the orthogonal complement of P_0 in $\mathbb{R}^{N+1}$. Since P is not contained in $\mathbb{R}^N$, the orthogonal projection $\mathbb{R}^{N+1} \to \{0\} \oplus \mathbb{R} \subset \mathbb{R}^{N+1}$ gives an isomorphism between $P \cap P_0^{\perp}$ and $\{0\} \oplus \mathbb{R}$. Define $v_0 \in P_0^{\perp} \cap \mathbb{R}^N$ so that $v_0 \oplus 1$ gives an oriented basis of the line $P \cap P_0^{\perp}$. By definition (P_0,v_0) is an element in $\gamma_{a_0}^{\perp}$. If $P^{\perp}$ is not contained in $\mathbb{R}^N$, then we can define $(P_1,v_1) \in \gamma_{a_1}^{\perp}$ as in the above way for $P^{\perp}$. The map $P \mapsto (P_i,v_i)$ is compatible with the identification $\sim$ and induces the inverse map. □

Hereafter we put $a_0 = 1$, $a_1 = n-1$, $N = n+1$ and $X_{1,n-1,n+1} = X$. In this case we have $\mathrm{Gr}_{a_0}^{+}(\mathbb{R}^N) = S^n$, $\gamma_{a_0}^{\perp} = TS^n = V_0$ and $\mathrm{Gr}_{a_1}^{+}(\mathbb{R}^N) = \mathrm{Gr}_{n-1}^{+}(\mathbb{R}^{n+1}) = \mathrm{Gr}_2^{+}(\mathbb{R}^{n+1})$. Let γ_2 be the tautological plane bundle over $\mathrm{Gr}_2^{+}(\mathbb{R}^{n+1})$. In this setting the identification $\sim$ can be described as follows: Let r be a positive real number. For $(x,v) \in S_r(TS^n)$ we define $(P_1,v_1) \in S_{1/r}(\gamma_2)$ by $P_1^{\perp} := \mathbb{R}\langle x,v\rangle$ and $v_1 := -v/r^2$. For $(P_1,v_1) \in S_{1/r}(\gamma_2)$ we put $v := r^2 v_1$ and define $x \in S^n$ so that x is a tangent vector of the circle $P_1^{\perp} \cap S^n$ at $\frac{v}{||v||}$ and $\left\{x, \frac{v}{||v||}\right\}$ is an oriented orthonormal basis of $P_1^{\perp}$. These two maps give a diffeomorphism between $S_r(TS^n)$ and $S_{1/r}(\gamma_2)$ which induces the identification

$$X = TS^n \cup \gamma_2/\sim .$$

Under the above diffeomorphism $S_r(TS^n) \cong S_{1/r}(\gamma_2)$ the circle of radius r in $\mathbb{R}\langle x,v\rangle$ corresponds to the circle of radius $1/r$ in $P_1^{\perp} \in \mathrm{Gr}_2^{+}(\mathbb{R}^{n+1})$, and the S^1-action induced by the normalized geodesic flow is the principal S^1-action on $S_{1/r}(\gamma_2)$. It implies $\mathrm{Gr}_2^{+}(\mathbb{R}^{n+1}) = h_{01}^{-1}(r)/S^1$ for all $r > 0$. Moreover we have the following.

Lemma 4.4. *X_r is diffeomorphic to X for any $r > 0$.*

Proof. We first take and fix a positive number $\varepsilon > 0$ small enough so that $\varepsilon < \frac{r}{2}$ and $(r-\varepsilon)\varepsilon < 1$. Let $\rho : [0,r) \to \mathbb{R}$ be a smooth function such that $\rho(s) = 1$ $(0 \le s \le r - 2\varepsilon)$, $\rho(s) = \frac{1}{\sqrt{s(r-s)}}$ $(r - \varepsilon \le s < r)$ and ρ is strictly increasing on $[r-2\varepsilon, r)$. We define a diffeomorphism $D_r^o(TS^n) \to TS^n$ by $(x,v) \mapsto (x, \rho(\|v\|)v)$. On the other hand there is a diffeomorphism between $\nu_{(r)}$ and γ_2 defined by

$$\nu_{(r)} \ni [x,v,z] \mapsto \left(\mathbb{R}\langle x,v\rangle, -\left(\mathrm{Im}(z)x + \mathrm{Re}(z)\frac{v}{\|v\|}\right)\right) \in \gamma_2.$$

When we use the embedding of $\nu_{(r)}$ in Lemma 4.1 and descriptions $X_r = D_r^o(TS^n) \cup D_{r_\varepsilon}^o(\nu_{(r)})$ and $X = TS^n \cup D_{r_\varepsilon}^o(\gamma_2)$ for $r_\varepsilon = \sqrt{\frac{\varepsilon}{r-\varepsilon}}$, two maps defined above give a diffeomorphism $X_r \cong X$. □

Remark 4.5. By the above description the normal bundle of $\mathrm{Gr}_2^{+}(\mathbb{R}^{n+1})$ (resp. S^n) in $X = X_r$ is isomorphic to γ_2 (resp. TS^n).

4.3. **Quadratic hypersurface.** Let Q_n be the quadratic hypersurface in $\mathbb{C}\mathrm{P}^{n+1}$ defined by

$$Q_n := \left\{ [z_0 : z_1 : \cdots : z_{n+1}] \;\Big|\; \sum_{i=0}^{n+1} z_i^2 = 0 \right\}.$$

Lemma 4.6. *X_r is diffeomorphic to Q_n for any $r > 0$.*

Proof. From Lemma 4.4 we have the diffeomorphism $X_r \cong \mathrm{Gr}_2^+(\mathbb{R}^{n+2})$. Let P be an oriented 2-plane in $\mathbb{R}^{n+2}$ with an oriented orthonormal 2-frame $\{u_0, u_1\}$. The point $[u_0 + \sqrt{-1}u_1]$ in $\mathbb{C}\mathrm{P}^{n+1}$ is well-defined and contained in Q_n. The map $X_r \to Q_n$ gives the required diffeomorphism. $\square$

Remark 4.7. Under the above identification the divisor $Q_{n-1} \subset Q_n$ defined by $z_{n+1} = 0$ corresponds to the submanifold $\mathrm{Gr}_2^+(\mathbb{R}^{n+1}) \subset \mathrm{Gr}_2^+(\mathbb{R}^{n+2})$. The normal bundle of Q_{n-1} in Q_n is the tautological bundle $\mathcal{O}(1)|_{Q_{n-1}}$, which is isomorphic to the tautological bundle $\gamma_2 \to \mathrm{Gr}_2^+(\mathbb{R}^{n+1})$ under the identification $Q_{n-1} = \mathrm{Gr}_2^+(\mathbb{R}^{n+1})$.

4.4. **Comparison of almost complex structures.**

4.4.1. *J_0, J_1 and J_2.* We first define three almost complex structures J_0, J_1 and J_2.

(0) J_0. Let J_0 be the standard almost complex structure on $V_0 = T^*S^n = TS^n$ determined by the Riemannian metric and the Levi-Civita connection of S^n. Let $p : TS^n \to S^n$ be the projection and $\iota_0 : p^*TS^n \to T(TS^n)$ be the splitting of the bundle map $T(TS^n) \to p^*TS^n$ determined by the Levi-Civita connection. Let $\iota_1 : p^*TS^n \to T(TS^n)$ be the natural embedding of p^*TS^n into the tangent bundle along fibers of $p : TS^n \to S^n$. We have a decomposition $T(TS^n) \cong p^*TS^n \oplus p^*TS^n$ via ι_0 and ι_1. The standard almost complex structure J_0 is characterized by the condition $J_0\iota_0 = \iota_1$. Note that J_0 is invariant under the S^1-action of the normalized geodesic flow on V_{01}.

Now we give more explicit description of J_0 using the natural embedding $TS^n \subset S^n \times \mathbb{R}^{n+1}$. We fix a non-zero tangent vector $(x, v) \in TS^n$. Under this embedding one has

$$T_{(x,v)}(TS^n) = \{(u_1, u_2) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \mid u_1 \cdot x = u_2 \cdot x + u_1 \cdot v = 0\}.$$

We give the description of the horizontal lift $T_xS^n \to T_{(x,v)}(TS^n)$ using the above description. For each $w \in T_xS^n$ the subspace $\mathbb{R}\langle x, v, w\rangle$ generated by x,v and w is a 2-plane if w is parallel to v or a 3-space otherwise. Since the unit sphere in $\mathbb{R}\langle x, v, w\rangle$ is totally geodesic in S^n, it suffices to consider the case $n = 2$. We fix an oriented orthonormal frame $\{e_1, e_2, e_3\}$ of $\mathbb{R}^3$ so that $x = e_1$, $v = \|v\|e_2$. Note that e_2 and e_3 form an oriented orthonormal basis of T_xS^2.

Lemma 4.8. *The horizontal lifts of $e_2, e_3 \in T_xS^2$ are given by $(e_2, -\|v\|e_1), (e_3, 0) \in T_xS^2 \times \mathbb{R}^3$ respectively.*

The natural embedding $T_xS^2 \to T_{(x,v)}(TS^2)$ of the fiber direction is given by $T_xS^2 \ni w \mapsto (0, w) \in T_xS^2 \times \mathbb{R}^3$. By definition of J_0 we have the following characterization.

Lemma 4.9. *J_0 is characterized by the condition*

$$J_0 : (e_2, -\|v\|e_1) \mapsto (0, e_2),\ \ (e_3, 0) \mapsto (0, e_3).$$

(1) J_1. We define an almost complex structure J_1 on $X = \mathrm{Gr}_2^+(\mathbb{R}^{n+2})$ as follows. For each oriented 2-plane $P \in X$, its orientation and Euclidean metric determine a canonical complex structure J_P on P. We define an almost complex structure $(J_1)_P$ on $T_PX = \mathrm{Hom}(P, P^\perp)$ by

$$(J_1)_P(f) := f \circ J_P^{-1}.$$

(2) J_2. Let J_2 be the standard complex structure on Q_n as a projective variety in $\mathbb{CP}^{n+1}$.

4.4.2. $J_0 \sim J_1 = J_2$ *on* TS^n. Let V_1 be the complement of S^n in $X = \mathrm{Gr}_2^+(\mathbb{R}^{n+2})$. We give two types of decompositions of $TX|_{V_0 \cap V_1}$.

For $P \in V_1 \subset \mathrm{Gr}_2^+(\mathbb{R}^{n+2})$ let P_2 be the image of P under the natural projection $\mathbb{R}^{n+2} \to \mathbb{R}^{n+1}$, which is a 2-plane in $\mathbb{R}^{n+1}$. Let $P^\perp$ be the orthogonal complement of P in $\mathbb{R}^{n+2}$ and $\check{P}_2$ the orthogonal complement of P_2 in $\mathbb{R}^{n+1}$. Note that $\check{P}_2 = \check{P}_2 \oplus \{0\}$ is a codimension 1 subspace of $P^\perp$. Let F_P be the orthogonal complement of $\check{P}_2$ in $P^\perp$. We put $E_P := \mathrm{Hom}(P, \check{P}_2)$ and $L_P := \mathrm{Hom}(P, F_P)$. In this way we have the first decomposition $TX|_{V_1} = E \oplus L$. Note that E and L are J_1-invariant subbundles.

For $(x, v) \in TS^n$ consider the subspace $\mathbb{R}\langle x, v\rangle$ in $\mathbb{R}^{n+1}$. Let $E'_{(x,v)}$ be the subspace of $T_{(x,v)}(TS^n)$ defined by $E'_{(x,v)} := \mathbb{R}\langle x, v\rangle^\perp \times \mathbb{R}\langle x, v\rangle^\perp$ in $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$. Let $L'_{(x,v)}$ be the subspace of $T_{(x,v)}(TS^n)$ defined by $L'_{(x,v)} = (\mathbb{R}\langle x, v\rangle \times \mathbb{R}\langle x, v\rangle) \cap T_{(x,v)}(TS^n)$. From Lemma 4.9 note that if $v = 0$, then $L'_{(x,0)} = \{0\}$ and if $v \neq 0$, then $L'_{(x,v)}$ has a natural basis $\left\{\left(\frac{v}{\|v\|}, -\|v\|x\right), \left(0, \frac{v}{\|v\|}\right)\right\}$. In this way we have the second decomposition $TX|_{V_0 \cap V_1} = E' \oplus L'$. Note that E' and L' are J_0-invariant subbundles.

Lemma 4.10. *Under the identification* $TS^n \setminus S^n = V_0 \cap V_1 = \mathrm{Gr}_2^+(\mathbb{R}^{n+2}) \setminus (S^n \cup \mathrm{Gr}_2^+(\mathbb{R}^{n+1}))$*, the subbundle* E *(resp.* L*) is identified with* E' *(resp.* L'*).*

Proof. It suffices to prove for the case $n = 2$. For $(x, v) \in TS^n \setminus S^n$ we use an oriented orthonormal frame $\{e_1, e_2, e_3\}$ of $\mathbb{R}^3$ as in Lemma 4.8 and Lemma 4.9. In this case we have $E'_{(x,v)} = \mathbb{R}\langle (e_3, 0), (0, e_3)\rangle$ and $L'_{(x,v)} = \mathbb{R}\langle (e_2, -\|v\|e_1), (0, e_2)\rangle$. We put $P = \mathbb{R}\langle x \oplus 0, v \oplus 1\rangle \in \mathrm{Gr}_2^+(\mathbb{R}^{n+2})$, and then we have a natural basis $e_2 \oplus (-\|v\|)$ of F_P. Under the identification $T_{(x,v)}(TS^n) = T_P\mathrm{Gr}_2^+(\mathbb{R}^{n+2})$ one can check the following correspondence by direct computations.

$$(e_3, 0) \mapsto f_{(e_3,0)},\ (0, e_3) \mapsto f_{(0,e_3)},\ (e_2, -\|v\|e_1) \mapsto f_{(e_2,-\|v\|e_1)},\ (0, e_2) \mapsto f_{(0,e_2)},$$

where linear maps $f_{(e_3,0)} \in T_P\mathrm{Gr}_2^+(\mathbb{R}^{n+2})$ etc. are defined by

$$f_{(e_3,0)} : \begin{cases} x \oplus 0 \mapsto e_3 \oplus 0 \\ v \oplus 1 \mapsto 0, \end{cases} \quad f_{(0,e_3)} : \begin{cases} x \oplus 0 \mapsto 0 \\ v \oplus 1 \mapsto e_3 \oplus 0, \end{cases}$$

$$f_{(e_2,-\|v\|e_1)} : \begin{cases} x \oplus 0 \mapsto \frac{1}{1+\|v\|^2}(e_2 \oplus (-\|v\|)) \\ v \oplus 1 \mapsto 0, \end{cases} \quad f_{(0,e_2)} : \begin{cases} x \oplus 0 \mapsto 0 \\ v \oplus 1 \mapsto \frac{1}{1+\|v\|^2}(e_2 \oplus (-\|v\|)). \end{cases}$$

It implies that $E'_{(x,v)}$ (resp. $L'_{(x,v)}$) is isomorphic to E_P (resp. L_P). $\square$

Remark 4.11. Though E (resp. L) and E' (resp. L') are isomorphic they are not isometric with respect to standard metrics of $\mathrm{Gr}_2^+(\mathbb{R}^{n+2})$ and $S^n \times \mathbb{R}^{n+1}$.

Using the above basis of $T_P\mathrm{Gr}_2^+(\mathbb{R}^{n+2})$ the almost complex structure J_1 is characterized as follows.

Lemma 4.12. $J_1(f_{(e_3,0)}) = \sqrt{1+\|v\|^2} f_{(0,e_3)}$, $J_1(f_{(e_2,-\|v\|e_1)}) = \sqrt{1+\|v\|^2} f_{(0,e_2)}$.

Consider the standard symplectic structure $d\alpha$ on $V_0 = TS^n$. It is easy to see that the frame $\{(e_3,0),\ (0,e_3),\ (e_2,-\|v\|e_1),\ (0,e_2)\}$ form a symplectic basis of $T(TS^n)$. Lemma 4.12 shows that J_1 is compatible with $d\alpha$ and the following fact.

Proposition 4.13. *Two almost complex structures J_0 and J_1 on V_0 are homotopic to each other in $d\alpha$-compatible almost complex structures.*

On the other hand one can check the following by definition of the diffeomorphism in the proof of Lemma 4.6.

Lemma 4.14. $J_1 = J_2$ *on* $\mathrm{Gr}_2^+(\mathbb{R}^{n+2}) = Q_n$.

4.5. **Localization and almost complex structures.** Hereafter we assume that r is a positive integer k. Recall the diffeomorphism $X_k \cong X$ as in Lemma 4.4. Since the diffeomorphism is equal to the identity map on the complement of the small S^1-invariant neighborhood $D^o_{r_\varepsilon}(\nu_{(k)})$ of $M_{(k)}$, Proposition 4.13 guarantees that there exists an almost complex structure J on X_k which satisfies the following conditions.

- J is homotopic to J_1 on X_k.
- J is S^1-invariant on $X_k \smallsetminus D_{r'_\varepsilon}(\nu_{(k)}) \subset D^o_r(TS^n)$.
- $J = J_0$ on $X_k \smallsetminus D_{r'_\varepsilon}(\nu_{(k)}) = D^o_{r-2\varepsilon}(TS^n)$.
- $J = J_1$ on $D^o_{r_\varepsilon}(\nu_{(k)})$,

where $r_\varepsilon = \sqrt{\frac{\varepsilon}{r-\varepsilon}}$ and $r'_\varepsilon = r_{2\varepsilon} = \sqrt{\frac{2\varepsilon}{r-2\varepsilon}}$.

Proposition 4.15. *We have the following localization formula.*

$$RR(X_k) = RR_{\mathrm{loc}}(S^n) + \sum_{l=1}^{k-1} RR_{\mathrm{loc}}(S_l(TS^n)) + RR_{\mathrm{loc}}(Q_n).$$

Proof. By the homotopy invariance of the Riemann-Roch number, we may use J to compute $RR(X_k)$. On the other hand when we take $\varepsilon > 0$ small enough so that there are no integers in the interval $[k-2\varepsilon, k)$, we may use J to define local Riemann-Roch numbers in the right hand side by Lemma 3.2 and the properties of J. The equality follows from the excision formula of local Riemann-Roch numbers. □

5. Local models and their local Riemann-Roch number

5.1. **Local models.** Let M_0 be a closed symplectic manifold. We assume that there exists a prequantizing line bundle (L_0, ∇_0) over M_0. Let $p_0 : P_0 \to M_0$ be a principal S^1-bundle over M_0 and α_0 a principal connection of P_0, which is a pure imaginary 1-form on P_0. We define two open prequantized symplectic manifolds M_D and M_{BS}.

5.1.1. M_D. Let $\mathbb{C}_1$ be the one-dimensional representation of S^1 of weight 1. Let $M_{\mathbb{C}} = P_0 \times_{S^1} \mathbb{C}_1$ be the quotient space by the diagonal S^1-action. For a complex coordinate $z = x + \sqrt{-1}y$ of $\mathbb{C}_1$ and $r := |z|$, the pure imaginary 1-form on $P_0 \times \mathbb{C}_1$

$$\widetilde{\alpha}_{\mathbb{C}} := \frac{1}{2} r^2 \alpha_0 + \frac{\sqrt{-1}}{2}(ydx - xdy)$$

is basic with respect to the S^1-action. Let $\alpha_{\mathbb{C}}$ be the 1-form on $M_{\mathbb{C}}$ whose pull-back to $P_0 \times \mathbb{C}_1$ is equal to $\widetilde{\alpha}_{\mathbb{C}}$. Define a Hermitian line bundle with connection $(L_{\mathbb{C}}, \nabla_{\mathbb{C}})$ by

$$(L_{\mathbb{C}}, \nabla_{\mathbb{C}}) := (p_{\mathbb{C}}^* L_0, p_{\mathbb{C}}^* \nabla_0 + \alpha_{\mathbb{C}}),$$

where $p_{\mathbb{C}} : M_{\mathbb{C}} \to M_0$ is the projection.

Lemma 5.1. *Fix $0 < \varepsilon < \sqrt{2}$ and we put $M_D := P_0 \times_{S^1} D^o_\varepsilon(\mathbb{C}_1)$.*

1. *If ε is small enough, then the restriction of $(L_{\mathbb{C}}, \nabla_{\mathbb{C}})$ to M_D is a prequantizing line bundle for a suitable symplectic structure on M_D.*
2. *The natural S^1-action on M_D using the P_0-component is Hamiltonian with the moment map $[u, z] \mapsto |z|^2/2$.*
3. *The holonomy of the S^1-orbit through $[u, z] \in M_D$ is trivial if and only if $z = 0$.*

Proof. (1) The restriction of $(L_{\mathbb{C}}, \nabla_{\mathbb{C}})$ to $M_0 = P_0 \times_{S^1} \{0\}$ is equal to (L_0, ∇_0), hence, the curvature 2-form of $\nabla_{\mathbb{C}}$ gives a symplectic structure for $\varepsilon > 0$ small enough. By definition $(L_{\mathbb{C}}, \nabla_{\mathbb{C}})|_{M_D}$ is a prequantizing line bundle for this symplectic structure. (2) The S^1-action on the P_0-component preserves $\nabla_{\mathbb{C}}$, and the evaluation of the infinitesimal action by $\alpha_{\mathbb{C}}$ is equal to $|z|^2/2$. (3) follows from (2). □

5.1.2. M_{BS}. Define a pure imaginary 1-form α_{BS} on $P_0 \times \mathbb{R}$ by

$$\alpha_{BS} = r\alpha_0,$$

where r is the coordinate of $\mathbb{R}$. Define a Hermitian line bundles with connection over $P_0 \times \mathbb{R}$ by

$$(L_{BS}, \nabla_{BS}) := (p_0^* L_0, p_0^* \nabla_0 + \alpha_{BS}).$$

By the similar argument as in the proof of Lemma 5.1 we have the following.

Lemma 5.2. *Fix $0 < \varepsilon < 1$ and we put $M_{BS} = P_0 \times (-\varepsilon, \varepsilon)$.*

1. *If ε is small enough, then the restriction of (L_{BS}, ∇_{BS}) to M_{BS} is a prequantizing line bundle for a suitable symplectic structure on M_{BS}.*
2. *The natural S^1-action on the P_0-component is Hamiltonian with the moment map $(u, r) \mapsto r$.*
3. *The holonomy of the S^1-orbit through $(u, r) \in M_{BS}$ is trivial if and only if $r = 0$.*

5.2. **Local Riemann-Roch numbers.** Using the free S^1-action on $M_D \smallsetminus M_0$ and $M_{BS} \smallsetminus P_0$ we can define their local Riemann-Roch number $RR_{\mathrm{loc}}(M_0)$ and $RR_{\mathrm{loc}}(P_0)$. On the other hand since M_0 is closed the usual Riemann-Roch number $RR(M_0)$ is defined.

Lemma 5.3. *We have the following equalities*

$$RR_{\mathrm{loc}}(M_0) = RR_{\mathrm{loc}}(P_0) = RR(M_0).$$

Proof. These equalities follows from the product formula and the facts $RR_{\mathrm{loc}}(D^o_\varepsilon) = RR_{\mathrm{loc}}(S^1 \times (-\varepsilon, \varepsilon)) = 1$ as S^1-equivariant local Riemann-Roch numbers. Note that we use the identification $P_0 \times (-\varepsilon, \varepsilon) = P_0 \times_{S^1} (S^1 \times (-\varepsilon, \varepsilon))$. □

5.3. **A formula for local Riemann-Roch numbers.** Let (M,L) be a prequantized symplectic manifold with a Hamiltonian S^1-action and its moment map $\mu : M \to \mathbb{R}$. We assume that 0 is a regular value of μ and $\mu^{-1}(0)$ is a compact submanifold of M. We also assume that the S^1-action on $\mu^{-1}(0)$ is free. Let $M_0 := \mu^{-1}(0)/S^1$ be the symplectic quotient at 0, which is a compact symplectic manifold with a prequantizing line bundle $L_0 := L|_{\mu^{-1}(0)}/S^1$. We put $P_0 := \mu^{-1}(0)$ and then the natural projection $P_0 \to M_0$ gives a structure of a principal S^1-bundle over M_0. Consider the symplectic cut of M at 0,

$$M_{\mathrm{cut}} := \mu^{-1}(-\infty, 0) \cup M_0.$$

For these data we have two local Riemann-Roch numbers $RR_{\mathrm{loc}}(P_0)$ and $RR_{\mathrm{loc}}(M_0)$.

Theorem 5.4. *We have the following equalities*

$$RR_{\mathrm{loc}}(P_0) = RR_{\mathrm{loc}}(M_0) = RR(M_0).$$

Proof. A version of Darboux's theorem ([3, Proposition 5.11]) implies that we can use local models M_{BS} and M_D for M_0 and P_0. The equalities follow from the product formula (Lemma 5.3). □

6. Proof of the main theorem

In this section we use following notations for a fixed positive integer k and an integer l with $0 < l \leq k$.

- $X_k = D_k^o(TS^n) \cup h_{01}^{-1}(k)/S^1 \cong \mathrm{Gr}_2^+(\mathbb{R}^{n+2}) \cong Q_n$
- $P_{(l)} = h_{01}^{-1}(l) = S_l(TS^n)$
- $M_{(l)} = P_{(l)}/S^1 \cong Q_{n-1}$
- $RR_n^l := \binom{n+l}{n} + \binom{n+l-1}{n}$, where $\binom{\cdot}{\cdot}$ is the binomial coefficient.

6.1. **Pre-quantizing line bundles.** Let $(L_0, d - 2\pi\sqrt{-1}\alpha)$ be the prequantizing line bundle over V_0, where L_0 is the trivial line bundle and α is the Liouville 1-form. Note that the weight of the S^1-action on the space of global parallel sections of $L_0|_{P_{(l)}}$ is equal to l, and hence, the prequantizing line bundle over $M_{(l)}$ is given by $P_{(l)} \times_{S^1} \mathbb{C}_l$, where $\mathbb{C}_l$ is the complex line with the standard S^1-action of weight l. On the other hand, as in Remark 4.5 and Remark 4.7, $P_{(l)}$ is isomorphic to the unit circle bundle of the tautological plane bundle $\gamma_2 \cong \mathcal{O}(1)|_{Q_{n-1}} \to \mathrm{Gr}_2^+(\mathbb{R}^{n+1}) \cong Q_{n-1}$. Summarizing we have the following.

Lemma 6.1. *Under the identification $M_{(l)} \cong Q_{n-1}$, $L_{(l)} := \mathcal{O}(l)|_{Q_{n-1}}$ gives a prequantizing line bundle over $M_{(l)}$.*

On the other hand, the symplectic cutting construction for (V_0, L_0) yields a prequantizing line bundle $L_k \to X_k$ whose restriction to the quotient $M_{(k)}$ is given by $(L_0|_{h_{01}^{-1}(k)} \otimes \mathbb{C}_k)/S^1 = L_{(k)}$.

Lemma 6.2. *Under the identification $X_k \cong Q_n$, the pre-quantizing line bundle L_k is isomorphic to $\mathcal{O}(k)$ as a prequantizing line bundle.*

Proof. Note that since Q_n is simply connected the isomorphism class of pre-quantizing line bundles is unique. The required isomorphism follows from the isomorphism $H^2(Q_n, \mathbb{Z}) \xrightarrow{\cong} H^2(Q_{n-1}, \mathbb{Z})$, $c_1(\mathcal{O}(k)) \mapsto c_1(L_{(k)})$. □

6.2. **Riemann-Roch number of the compactification.**

Lemma 6.3. *For each positive integer l, the dimension of the space of holomorphic sections of $\mathcal{O}(l)|_{Q_n}$ is equal to RR_n^l.*

Proof. The involution

$$[z_0 : z_1 : \dots : z_n : z_{n+1}] \mapsto [z_0 : z_1 : \dots : z_n : -z_{n+1}]$$

acts on Q_n and its orbit space is $\mathbb{C}\mathrm{P}^n$. The projection map $Q_n \to \mathbb{C}\mathrm{P}^n$ given by $[z_0 : \dots : z_n : z_{n+1}] \mapsto [z_0 : \dots : z_n]$ is a branched covering with branching locus $Q_{n-1} \subset \mathbb{C}\mathrm{P}^n$. The involution lifts to $\mathcal{O}(l)|_{Q_n}$ so that the quotient bundle is $\mathcal{O}(l) \to \mathbb{C}\mathrm{P}^n$. Consider the decomposition of the space of holomorphic sections with respect to the involution, $H^0(Q_n, \mathcal{O}(l)) = H^+_{n,l} \oplus H^-_{n,l}$, where the involution acts on $H^\pm_{n,l}$ by ± 1. The invariant part $H^+_{n,l}$ is isomorphic to $H^0(\mathbb{C}\mathrm{P}^n, \mathcal{O}(l))$, which is the vector space of homogeneous polynomial of degree l of $(n+1)$-variable $z_0, \dots, z_n$. Its dimension is given by the binomial coefficient $\binom{n+l}{n}$. On the other hand any section $s \in H^-_{n,l}$ can be divisible by z_{n+1}, and s/z_{n+1} defines a section in $H^+_{n,l-1}$. In particular we have $\dim H^-_{n,l} = \dim H^0(\mathbb{C}\mathrm{P}^n, \mathcal{O}(l-1)) = \binom{n+l-1}{n}$. ☐

Proposition 6.4. *The Riemann-Roch number of X_k with the prequantizing line bundle L_k is given by $RR(X_k) = RR_n^k$.*

Proof. We use the identifications $X_k = Q_n$, $L_k = \mathcal{O}(k)$ and the natural complex structure $J_2 = J_1$ of $Q_n \subset \mathbb{C}\mathrm{P}^{n+1}$. By the Kodaira vanishing theorem, $RR(X_k) = RR(Q_n)$ is equal to the dimension of the space of holomorphic sections $H^0(Q_n, \mathcal{O}(k)) = RR_n^k$. ☐

6.3. **Localization.** Recall that the complement of S^n and $h_{01}^{-1}(k)/S^1 \cong Q_{n-1}$ in X_k has a free Hamiltonian S^1-action induced by the geodesic flow. The holonomy representation of the restriction of L_k to a orbit is trivial if and only if the orbit is contained in $P_{(l)} = h_{01}^{-1}(l)$ for some integer l.

Proof of Theorem 3.3. The required formula follows from Theorem 5.4, Lemma 6.3 and the identification $M_{(l)} \cong Q_{n-1}$. ☐

We also have the local Riemann-Roch number $RR_{\mathrm{loc}}(M_{(k)})$ in X_k with the prequantizing line bundle $L_k|_{M_{(k)}}$.

Proposition 6.5. $RR_{\mathrm{loc}}(M_{(k)}) = RR_{n-1}^k$.

Proof. The required equality follows from the identifications $M_{(k)} \cong Q_{n-1}$, $L_k|_{M_{(k)}} \cong \mathcal{O}(k)|_{Q_{n-1}}$, Theorem 5.4 and Lemma 6.3. ☐

Lemma 6.6. *For $n \geq 1$, we have the following equality.*

$$RR_{n-1}^0 + RR_{n-1}^1 + \cdots + RR_{n-1}^k = RR_n^k.$$

Proof. Compare the coefficients of a^{n-1} in the equality

$$(a+1)^{n-1} + (a+1)^n + \cdots + (a+1)^{n+k-1} = (a+1)^{n-1}\left((a+1)^{k+1} - 1\right)/a.$$

☐

Proof of Theorem 3.1. By Proposition 4.15 we have

$$RR_{\mathrm{loc}}(S^n) + \sum_{l=1}^{k-1} RR_{\mathrm{loc}}(P_{(l)}) + RR_{\mathrm{loc}}(M_{(k)}) = RR(X_k),$$

and hence, $RR_{\mathrm{loc}}(S^n) = RR^0_{n-1}$ for $n \geq 1$ by Theorem 6.4, Theorem 3.3, Proposition 6.5 and Lemma 6.6. For $n = 0$ we have $RR_{\mathrm{loc}}(S^0) = 2$ by definition. □

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