UTMS 2011–17 September 12, 2011

Homeomorphism groups of commutator width one

by

Takashi Tsuboi

UNIVERSITY OF TOKYO

GRADUATE SCHOOL OF MATHEMATICAL SCIENCES

KOMABA, TOKYO, JAPAN

HOMEOMORPHISM GROUPS OF COMMUTATOR WIDTH ONE

TAKASHI TSUBOI

Abstract. We show that every element of the identity component $\mathrm{Homeo}(S^n)_0$ of the group of homeomorphisms of the n-dimensional sphere S^n can be written as one commutator. We also show that every element of the group $\mathrm{Homeo}(\mu^n)$ of homeomorphisms of the n-dimensional Menger compact space μ^n can be written as one commutator.

1. Introduction

The algebraic property of the group of homeomorphisms or diffeomorphisms are studied by many people. The identity component of the group of homeomorphisms or diffeomorphisms of compact manifolds are known to be perfect and moreover simple ([23], [1], [12], [10], [13], [16], [19], [11], [2]). Many of them, for example the group of homeomorphisms of the n-dimensional sphere, are known to be uniformly perfect ([1],[7], [20], [22]). A group is uniformly perfect if every element is written as a product of a bounded number of commutators. The least number of such bound is called the commutator width of the group.

In this paper we show that the commutator width of the identity component $\mathrm{Homeo}(S^n)_0$ of the group of homeomorphisms of the n-dimensional sphere S^n is one.

Theorem 1.1. *Any element of* $\mathrm{Homeo}(S^n)_0$ *can be written as one commutator.*

We also show that the commutator width of the group $\mathrm{Homeo}(\mu^n)$ of homeomorphisms of the n-dimensional Menger compact space μ^n is one.

Theorem 1.2. *Any element of* $\mathrm{Homeo}(\mu^n)$ *can be written as one commutator.*

Anderson showed that in the group $\mathrm{Homeo}_c(\mathbf{R}^n)$ of homeomorphisms of the n-dimensional Euclidean space with compact support, any element can be written as one commutator ([1], [15]). Since any element f of $\mathrm{Homeo}_0(S^n)$ can be written as a product $f = gh$ such that g and h are the identity on some nonempty open sets, the fact that the commutator width of $\mathrm{Homeo}_c(\mathbf{R}^n)$ is one implies that f can be written as a product of two commutators.

It is worth recalling the construction by Anderson ([1]). For given $f \in \mathrm{Homeo}_c(\mathbf{R}^n)$, we find a bounded ball U such that the support $\mathrm{supp}(f) \subset U$. Then we can find an element $g \in \mathrm{Homeo}_c(\mathbf{R}^n)$ such that $g^n(U)$ $(n \in \mathbf{Z})$ are disjoint and $\lim\limits_{n\to\infty} \mathrm{diam}(g^n(U)) = 0$. Put $F = \prod\limits_{n=0}^{\infty} g^n f g^{-n}$, then we have $gFg^{-1} = f^{-1}F$. Thus $f = FgF^{-1}g^{-1}$.

2010 *Mathematics Subject Classification.* Primary 54H15; 54H20; 57S05, Secondary 20F65; 37B05; 57N50.

Key words and phrases. homeomorphism group, uniformly perfect, commutator subgroup.

The author is partially supported by Grant-in-Aid for Scientific Research (A) 20244003, Grant-in-Aid for Exploratory Research 21654009, Japan Society for Promotion of Science, and by the Global COE Program at Graduate School of Mathematical Sciences, the University of Tokyo.

We understand the meaning that the commutator width is one as follows. In the case of $\mathrm{Homeo}_c(\mathbf{R}^n)$, we see that for any element f, there exist g such that g and fg are conjugate. That is, g is dynamically so strong that fg and g have the same dynamics, and hence they are conjugate.

In the case of $\mathrm{Homeo}_0(S^n)$ or $\mathrm{Homeo}(\mu^n)$, we have the candidate which has the strong dynamics. The candidate is the topologically hyperbolic homeomorphism. A topologically hyperbolic homeomorphism is a homeomorphism h with one source s_+ and one sink s_- such that $\lim_{n\to+\infty} h^n(x) = s_-$ and $\lim_{n\to-\infty} h^n(x) = s_+$ for $x \notin \{s_-, s_+\}$. It seems true that the orientation preserving topologically hyperbolic homeomorphisms are all conjugate, but we are not able to show it at the present. The topologically hyperbolic homeomorphisms we construct later are conjugate because the are constructed with nice fundamental domains outside the fixed points.

Hence what we do in this paper is for a given homeomorphism f to construct a topologically hyperbolic homeomorphism g which is so strong that fg is topologically hyperbolic.

We will show Theorem 1.1 in Section 2 and Theorem 1.2 in Section 3.

The author would like to thank Professors Koji Fujiwara, Kazuhiro Kawamura, Sadayoshi Kojima, Shigenori Matsumoto and Hiromichi Nakayama for valuable comments during the preparation of this paper.

2. The group of homeomorphisms of the n-dimensional sphere

For the proof of Theorem 1.1, we need the following deep theorems.

Theorem 2.1 (Generalized Schoenflies Theorem, [4], [5]). *Let Σ be a locally flat $(n-1)$-dimensional sphere in the n-dimensional sphere S^n. Then the closures of the complementary domains of Σ are homeomorphic to the n-dimensional disk D^n.*

Here, an $(n-1)$-dimensional submanifold L^{n-1} in an n-dimensional manifold M^n is locally flat if each point of L^{n-1} has a neighborhood U in M^n such that the pair $(U, U \cap L^{n-1})$ is homeomorphic to $(\mathbf{R}^n, \mathbf{R}^{n-1})$.

Theorem 2.2 (Annulus conjecture, [14], [17]). *Let Σ_0 and Σ_1 be disjoint locally flat $(n-1)$-dimensional spheres in the n-dimensional sphere S^n. Then the closure of the region between them is homeomorphic to $S^{n-1} \times [0,1]$.*

Let f be an orientation preserving homeomorphism of the n-dimensional sphere S^n which is not the identity. Then we can find small n-dimensional closed disks D_0^n and D_1^n in S^n such that ∂D_0^n, ∂D_1^n are locally flat and the four disks $f^{-1}(D_0^n)$, D_0^n, D_1^n, $f(D_1^n)$ are disjoint.

By Theorem 2.1, D_0^n and $S^n \setminus \mathrm{int}(D_1^n)$ are homeomorphic and D_1^n and $S^n \setminus \mathrm{int}(D_0^n)$ are homeomorphic. Hence there exist an orientation preserving homeomorphism g of S^n such that $g(D_0^n) = S^n \setminus \mathrm{int}(D_1^n)$ and $g(S^n \setminus \mathrm{int}(D_0^n)) = D_1^n$.

Let $\Sigma = \partial D_0^n$, then we have four disjoint $(n-1)$-dimensional spheres $f^{-1}(\Sigma)$, Σ, $g(\Sigma)$, $(fg)(\Sigma)$ which are the boundaries of $f^{-1}(D_0^n)$, D_0^n, D_1^n, $f(D_1^n)$, respectively. Then we see that $(gf^{-1})(\Sigma)$, $(g^2)(\Sigma)$, $(gfg)(\Sigma)$ are contained in D_1^n, $(fgf^{-1})(\Sigma)$, $(fg^2)(\Sigma)$, $(fgfg)(\Sigma)$ are contained in $f(D_1^n)$, $(g^{-1}f^{-1})(\Sigma)$, $(g^{-1})(\Sigma)$, $(g^{-1}fg)(\Sigma)$ are contained in D_0^n and $(f^{-1}g^{-1}f^{-1})(\Sigma)$, $(f^{-1}g^{-1})(\Sigma)$, $(f^{-1}g^{-1}f)(\Sigma)$ are contained in $f^{-1}(D_0^n)$. See Figure 1, where $f^{-1}(D_0^n)$ is lower left, D_0^n is lower right, D_1^n is upper left and $f(D_1^n)$ is upper right and f translates $f^{-1}(D_0^n)$ and D_1^n to D_0^n and $f(D_1^n)$, respectively.

We require the homeomorphism g to satisfy the following conditions.

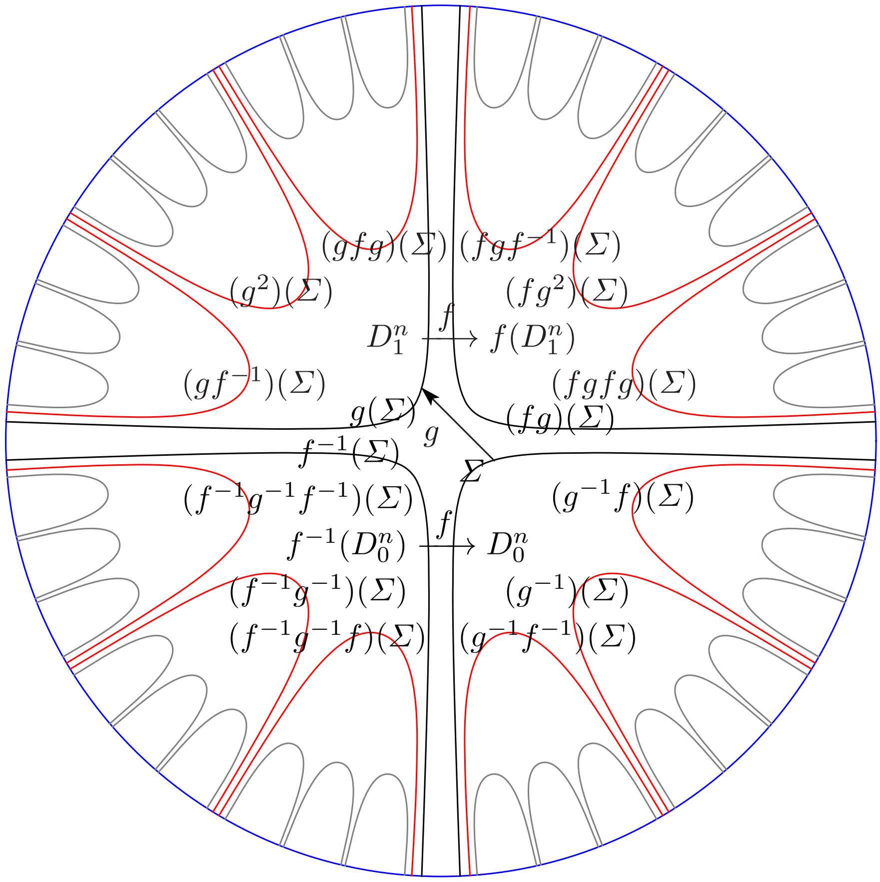

FIGURE 1. The actions of g and fg

(1) $\lim_{k\to\pm\infty} \operatorname{diam}(g^k(\Sigma)) = 0$.

(2) $\lim_{k\to\pm\infty} \operatorname{diam}((fg)^k(\Sigma)) = 0$,

Let D^n denote the n-dimensional standard disk. Let $\psi_{D_0^n} : D^n \longrightarrow D_0^n$ and $\psi_{D_1^n} : D^n \longrightarrow D_1^n$ be homeomorphisms. Then by Theorem 2.1, we have homeomorphisms $\psi_{S^n\setminus\mathrm{int}(D_0^n)} : D^n \longrightarrow S^n \setminus \mathrm{int}(D_0^n)$ extending $\psi_{D_0^n}|\partial D^n$ and $\psi_{S^n\setminus\mathrm{int}(D_1^n)} : D^n \longrightarrow S^n \setminus \mathrm{int}(D_1^n)$ extending $\psi_{D_1^n}|\partial D^n$. We first define a homeomorphism g by

$$g = \begin{cases} \psi_{S^n\setminus\mathrm{int}(D_1^n)}\psi_{D_0^n}{}^{-1} & \text{on } D_0^n \\ \psi_{D_1^n}\psi_{S^n\setminus\mathrm{int}(D_0^n)}{}^{-1} & \text{on } S^n \setminus \mathrm{int}(D_0^n) \end{cases},$$

and then we modify g so that (1) and (2) are satisfied. For this purpose we notice the following fact.

Lemma 2.3. *For any compact set K in the interior $\mathrm{int}(D^n)$ of the standard disk D^k and any positive real number ε, there is a homeomorphism $\varphi_{K,\varepsilon} : D^n \longrightarrow D^n$ which is the identity on ∂D^n such that $\operatorname{diam}(\varphi_{K,\varepsilon}(K)) \leq \varepsilon$.*

For a continuous map ψ between compact metric spaces $\psi : X \longrightarrow Y$, let μ_ψ denote the modulus of continuity of ψ. This means that $\mathrm{dist}_Y(\psi(x), \psi(y)) \leq \mu_\psi(\mathrm{dist}_X(x, y))$ for $x, y \in X$.

The modification of g is done step by step.

First we look at $g^2(\Sigma)$ and take $K_1 = {\psi_{D_1^n}}^{-1}((g^2)(\Sigma))$ and ε_1 such that $\mu_{\psi_{D_1^n}}(\varepsilon_1) \leq 2^{-2}$, where $\mu_{\psi_{D_1^n}}$ is the modulus of continuity of $\psi_{D_1^n}$. Then we look at $(fg)^2(\Sigma)$ and take $K_2 = {\psi_{D_1^n}}^{-1}((gfg)(\Sigma))$. We take δ such that $\mu_f(\delta) \leq 2^{-2}$ and take ε_2 such that $\mu_{\psi_{D_1^n}}(\varepsilon_2) \leq \delta$. Then for $K = K_1 \cup K_2$ and $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$, by Lemma 2.3, we have $\varphi_{K,\varepsilon} : D^n \longrightarrow D^n$. We replace $g|(S^n \setminus \mathrm{int}(D_0^n))$ by $\psi_{D_1^n}\varphi_{K,\varepsilon}{\psi_{D_1^n}}^{-1}(g|(S^n \setminus \mathrm{int}(D_0^n)))$. Then $\mathrm{diam}(g^2(\Sigma)) \leq 2^{-2}$ and $\mathrm{diam}((fg)^2(\Sigma)) \leq 2^{-2}$. The first step is done.

In the second step, we modify g on $D_1^n \cup f(D_1^n)$. We look at $g^3(\Sigma)$ and take $K_1 = (g \circ \psi_{D_1^n})^{-1}((g^3)(\Sigma))$ and ε_1 such that $\mu_{g\circ\psi_{D_1^n}}(\varepsilon_1) \leq 2^{-3}$, where $g \circ \psi_{D_1^n} : D^n \longrightarrow g(D_1^n)$. We also look at $(fg)^3(\Sigma)$ and take $K_2 = ((gf) \circ \psi_{D_1^n})^{-1}((g(fg)^2)(\Sigma))$, where $(gf) \circ \psi_{D_1^n} : D^n \longrightarrow (gf)(D_1^n)$. We take δ such that $\mu_f(\delta) \leq 2^{-3}$ and take ε_2 such that $\mu_{(gf)\circ\psi_{D_1^n}}(\varepsilon_2) \leq \delta$. We replace $g|D_1^n$ by $(g \circ \psi_{D_1^n})\varphi_{K_1,\varepsilon_1}(g \circ \psi_{D_1^n})^{-1}(g|D_1^n)$ and replace $g|f(D_1^n)$ by $((gf) \circ \psi_{D_1^n})\varphi_{K_2,\varepsilon_2}((gf) \circ \psi_{D_1^n})^{-1}(g|f(D_1^n))$, where $\varphi_{K_1,\varepsilon_1}$ and $\varphi_{K_2,\varepsilon_2}$ are given by Lemma 2.3. Then $\mathrm{diam}(g^3(\Sigma)) \leq 2^{-3}$ and $\mathrm{diam}((fg)^3(\Sigma)) \leq 2^{-3}$.

In the k-th step, we modify g on $g^{k-2}(D_1^n) \cup (fg)^{k-2}f(D_1^n)$. We look at $g^{k+1}(\Sigma)$ and take $K_1 = (g^{k-1} \circ \psi_{D_1^n})^{-1}((g^{k+1})(\Sigma))$ and ε_1 such that $\mu_{g^{k-1}\circ\psi_{D_1^n}}(\varepsilon_1) \leq 2^{-(k+1)}$, where $g^{k-1} \circ \psi_{D_1^n} : D^n \longrightarrow g^{k-1}(D_1^n)$. We also look at $(fg)^{k+1}(\Sigma)$ and take $K_2 = ((gf)^{k-1} \circ \psi_{D_1^n})^{-1}((g(fg)^k)(\Sigma))$, where $(gf)^{k-1} \circ \psi_{D_1^n} : D^n \longrightarrow (gf)^{k-1}(D_1^n)$. We take δ such that $\mu_f(\delta) \leq 2^{-(k+1)}$ and take ε_2 such that $\mu_{(gf)^{k-1}\circ\psi_{D_1^n}}(\varepsilon_2) < \delta$. We replace $g|g^{k-2}(D_1^n)$ by

$$(g^{k-1} \circ \psi_{D_1^n})\varphi_{K_1,\varepsilon_1}(g^{k-1} \circ \psi_{D_1^n})^{-1}(g|g^{k-2}(D_1^n))$$

and replace $g|((fg)^{k-2}f)(D_1^n)$ by

$$((gf)^{k-1} \circ \psi_{D_1^n})\varphi_{K_2,\varepsilon_2}((gf)^{k-1} \circ \psi_{D_1^n})^{-1}(g|((fg)^{k-2}f)(D_1^n)),$$

where $\varphi_{K_1,\varepsilon_1}$ and $\varphi_{K_2,\varepsilon_2}$ are given by Lemma 2.3. Then $\mathrm{diam}(g^k(\Sigma)) \leq 2^{-k}$ and $\mathrm{diam}((fg)^k(\Sigma)) \leq 2^{-k}$.

In this way, we modify g successively and we obtain a homeomorphism g such that $\lim\limits_{k\to\infty} \mathrm{diam}(g^k(\Sigma)) = 0$ and $\lim\limits_{k\to\infty} \mathrm{diam}((fg)^k(\Sigma)) = 0$, because the modification is done in finite stage for any point except those in $\bigcap\limits_{k=2}^{\infty} g^{k-2}(D_1^n) \cup \bigcap\limits_{k=2}^{\infty}((fg)^{k-2}f)(D_1^n)$ and it is ensured that $\bigcap\limits_{k=2}^{\infty} g^{k-2}(D_1^n)$ and $\bigcap\limits_{k=2}^{\infty}((fg)^{k-2}f)(D_1^n)$ are one-point sets.

Now we need to modify g so that the limit as k tends to $-\infty$ also satisfy the condition. We look at g^{-1} and $(fg)^{-1}$.

For the negative iteration of g and fg, we look at $g^{-1}(\Sigma)$ and take $K_1 = {\psi_{D_0^n}}^{-1}(g^{-1}(\Sigma))$ and ε_1 such that $\mu_{\psi_{D_0^n}}(\varepsilon_1) \leq 2^{-2}$. Then we look at $(fg)^{-1}(\Sigma)$ and take $K_2 = {\psi_{D_0^n}}^{-1}((fg)^{-1}(\Sigma))$. We take δ such that $\mu_{f^{-1}}(\delta) \leq 2^{-2}$ and take ε_2 such that $\mu_{\psi_{D_0^n}}(\varepsilon_2) \leq \delta$. Then for $K = K_1 \cup K_2$ and $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$, by Lemma 2.3, we have $\varphi_{K,\varepsilon} : D^n \longrightarrow D^n$. We replace $g^{-1}|(S^n \setminus \mathrm{int}(D_1^n))$ by $\psi_{D_0^n}\varphi_{K,\varepsilon}{\psi_{D_0^n}}^{-1}(g^{-1}|(S^n \setminus \mathrm{int}(D_1^n)))$. Note that we did not change $g|D_0^n : D_0^n \longrightarrow S^n \setminus \mathrm{int}(D_1^n)$ when we modified g for the positive iterations of g and fg. Then $\mathrm{diam}(g^{-1}(\Sigma)) \leq 2^{-2}$ and $\mathrm{diam}((fg)^{-1}(\Sigma)) \leq 2^{-2}$. The first step for g^{-1} and f^{-1} is done.

In the second step for g^{-1} and $(fg)^{-1}$, we modify g^{-1} on $D_0^n \cup f^{-1}(D_0^n)$. We look at $g^{-2}(\Sigma)$ and take $K_1 = (g^{-1} \circ \psi_{D_0^n})^{-1}(g^{-2}(\Sigma))$ and ε_1 such that $\mu_{g^{-1}\circ\psi_{D_0^n}}(\varepsilon_1) \leq 2^{-3}$, where $g^{-1} \circ \psi_{D_0^n} : D^n \longrightarrow g^{-1}(D_0^n)$. We also look at $(fg)^{-2}(\Sigma)$ and take $K_2 = ((fg)^{-1} \circ \psi_{D_0^n})^{-1}((fg)^{-2}(\Sigma))$, where $(fg)^{-1} \circ \psi_{D_0^n} : D^n \longrightarrow (fg)^{-1}(D_0^n)$. We take ε_2 such that

$\mu_{(fg)^{-1}\circ\psi_{D_0^n}}(\varepsilon_2) \leq 2^{-3}$. We replace $g^{-1}|D_0^n$ by $(g^{-1}\circ\psi_{D_0^n})\varphi_{K_1,\varepsilon_1}(g^{-1}\circ\psi_{D_0^n})^{-1}(g^{-1}|D_0^n)$ and replace $g|f^{-1}(D_0^n)$ by $((fg)^{-1}\circ\psi_{D_0^n})\varphi_{K_2,\varepsilon_2}((fg)^{-1}\circ\psi_{D_0^n})^{-1}(g^{-1}|f^{-1}(D_0^n))$, where $\varphi_{K_1,\varepsilon_1}$ and $\varphi_{K_2,\varepsilon_2}$ are given by Lemma 2.3. Then $\mathrm{diam}(g^{-2}(\Sigma)) \leq 2^{-3}$ and $\mathrm{diam}((fg)^{-2}(\Sigma)) \leq 2^{-3}$.

In the k-th step for g^{-1} and $(fg)^{-1}$, we modify g^{-1} on $g^{-k+2}(D_0^n)\cup(f^{-1}(fg)^{-k+2})(D_0^n)$. We look at $g^{-k}(\Sigma)$ and take $K_1 = (g^{-k+1}\circ\psi_{D_0^n})^{-1}((g^{-k})(\Sigma))$ and ε_1 such that $\mu_{g^{-k+1}\circ\psi_{D_0^n}}(\varepsilon_1) \leq 2^{-k-1}$, where $g^{-k+1}\circ\psi_{D_0^n} : D^n \longrightarrow g^{-k+1}(D_0^n)$. We also look at $(fg)^{-k}(\Sigma)$ and take $K_2 = ((fg)^{-k+1}\circ\psi_{D_0^n})^{-1}((fg)^{-k}(\Sigma))$, where $(fg)^{-k+1}\circ\psi_{D_0^n} : D^n \longrightarrow (fg)^{-k+1}(D_0^n)$. We take ε_2 such that $\mu_{(fg)^{-k+1}\circ\psi_{D_1^n}}(\varepsilon_2) < 2^{-k-1}$. We replace $g|g^{-k+2}(D_0^n)$ by

$$(g^{-k+1}\circ\psi_{D_0^n})\varphi_{K_1,\varepsilon_1}(g^{-k+1}\circ\psi_{D_0^n})^{-1}(g|g^{-k+2}(D_0^n))$$

and replace $g|(f^{-1}(fg)^{-k+2})(D_0^n)$ by

$$((fg)^{-k+1}\circ\psi_{D_0^n})\varphi_{K_2,\varepsilon_2}((fg)^{-k+1}\circ\psi_{D_0^n})^{-1}(g|(f^{-1}(fg)^{-k+2})(D_0^n)),$$

where $\varphi_{K_1,\varepsilon_1}$ and $\varphi_{K_2,\varepsilon_2}$ are given by Lemma 2.3. Then $\mathrm{diam}(g^{-k}(\Sigma)) \leq 2^{-k-1}$ and $\mathrm{diam}((fg)^{-k}(\Sigma)) \leq 2^{-k-1}$. In this way, we modify g^{-1} successively and we obtain a homeomorphism g^{-1} such that $\lim_{k\to-\infty}\mathrm{diam}(g^k(\Sigma)) = 0$ and $\lim_{k\to-\infty}\mathrm{diam}((fg)^k(\Sigma)) = 0$, because the modification is done in finite stage for any point except those in $\bigcap_{k=2}^{\infty} g^{-k+2}(D_0^n) \cup \bigcap_{k=2}^{\infty}(f^{-1}(fg)^{-k+2})(D_0^n)$ and it is ensured that $\bigcap_{k=2}^{\infty} g^{-k+2}(D_0^n)$ and $\bigcap_{k=2}^{\infty}(f^{-1}(fg)^{-k+2})(D_0^n)$ are one-point sets.

Thus we can construct the desired g.

Lemma 2.4. *The homeomorphisms g and fg are topologically conjugate. Namely, there is an orientation preserving homeomorphism $h : S^n \longrightarrow S^n$ such that $fg = hgh^{-1}$.*

Proof. This follows from Theorem 2.2. Since $\Sigma = \partial D_0^n$ and $g(\Sigma) = \partial D_1^n$ are locally flat $(n-1)$-dimensional spheres, there is a homeomorphism $\Phi_1 : S^{n-1}\times[0,1] \longrightarrow S^n$ to its image such that

$$\begin{aligned}&\Phi_1(S^{n-1}\times[0,1])\cap D_0^n = \Phi_1(S^{n-1}\times\{0\}) = \Sigma \quad \text{and}\\ &\Phi_1(S^{n-1}\times[0,1])\cap D_1^n = \Phi_1(S^{n-1}\times\{1\}) = g(\Sigma).\end{aligned}$$

Here by Theorem 2.2, there is Φ_1 such that $(\Phi_1|S^{n-1}\times\{1\})T(\Phi_1|S^{n-1}\times\{0\})^{-1} = g$, where $T(x,0) = (x,1)$ for $x\in S^{n-1}$. In the same way, since $\Sigma = \partial D_0^n$ and $(fg)(\Sigma) = \partial f(D_1^n)$ are locally flat $(n-1)$-dimensional spheres, there is a homeomorphism $\Phi_2 : S^{n-1}\times[0,1] \longrightarrow S^n$ to its image such that

$$\begin{aligned}&\Phi_2(S^{n-1}\times[0,1])\cap D_0^n = \Phi_2(S^{n-1}\times\{0\}) = \Sigma \quad \text{and}\\ &\Phi_2(S^{n-1}\times[0,1])\cap D_1^n = \Phi_2(S^{n-1}\times\{1\}) = (fg)(\Sigma).\end{aligned}$$

By Theorem 2.2, there is Φ_2 such that $(\Phi_2|S^{n-1}\times\{1\})T(\Phi_2|S^{n-1}\times\{0\})^{-1} = fg$.

The conjugating homeomorphism h is defined as follows.

$$\begin{cases} h(x) = \;(fg)^k((\Phi_2{\Phi_1}^{-1})(g^{-k}(x))) & \text{for } x\in g^k(\Phi_1(S^{n-1}\times[0,1])) \quad (k\in\mathbf{Z})\\ h(x) \in \displaystyle\bigcap_{k=2}^{\infty}(fg)^{k-2}f(D_1^n) & \text{for } x\in \displaystyle\bigcap_{k=2}^{\infty} g^{k-2}(D_1^n)\\ h(x) \in \displaystyle\bigcap_{k=2}^{\infty}(fg)^{-k+2}(D_0^n) & \text{for } x\in \displaystyle\bigcap_{k=2}^{\infty} g^{-k+2}(D_0^n)\end{cases}$$

Since $(\Phi_1|S^{n-1}\times\{1\})T(\Phi_1|S^{n-1}\times\{0\})^{-1} = g$, $\Phi_2(S^{n-1}\times[0,1])\cap D_1^n = \Phi_2(S^{n-1}\times\{1\}) = (fg)(\Sigma)$ and $\bigcap_{k=2}^{\infty}(fg)^{k-2}f(D_1^n))$, $\bigcap_{k=2}^{\infty} g^{k-2}(D_1^n))$, $\bigcap_{k=2}^{\infty}(fg)^{-k+2}(D_0^n))$ and $\bigcap_{k=2}^{\infty} g^{-k+2}(D_0^n))$ are one-point sets, h is well defined. Since h^{-1} can be defined in a similar way, h is a homeomorphism. By the definition of h, we have $fg = hgh^{-1}$.

By this lemma, we showed our main theorem 1.1.

3. The group of homeomorphisms of the n-dimensional Menger compact space

The proof of our main theorem uses Theorems 2.1 and 2.2 and Lemma 2.3. Similar theorems hold for the n-dimensional Menger compact space μ^n ([3], [18]). For the n-dimensional Menger space and Menger manifolds, we refer the reader to [3], [9] and [18].

The main tool to show the corresponding results for the n-dimensional Menger space is Bestvina's Z-set unknotting theorem.

Put $I^k = [0,1]^k$. A closed subset A of a k-dimensional Menger manifold M is a Z-set, if for any continuous map $f : I^k \longrightarrow M$ and any positive real number ε, there exists a continuous map $f' : I^k \longrightarrow M$ which is an ε-approximation of f and $f'(I^k)\cap A = \emptyset$ ([3], [9]). This is equivalent to that for any positive real number ε, there exists a continuous map $g : M \longrightarrow M \setminus A$ which is an ε-approximation of the identity. A Z-embedding is a homeomorphism onto a Z-set of a Menger manifold.

Theorem 3.1 (Z-set unknotting theorem, [3])**.** *Let A be a Z-set in a k-dimensional Menger manifold M. Any Z-embedding $A \longrightarrow M$ which is $(k-1)$-homotopic to the inclusion $A \subset M$ extends to a homeomorphism $M \longrightarrow M$ which is $(k-1)$-homotopic to the identity.*

Here $(k-1)$-homotopy is defined as follows ([8]). Two maps f_0 and $f_1 : X \longrightarrow Y$ are $(k-1)$-homotopic if $f_0\circ\alpha$ and $f_1\circ\alpha$ are homotopic for any continuous map $\alpha : Z \longrightarrow X$ from an arbitrary space Z of dimension less than k. When X and Y have the $(k-1)$-homotopy types of countable simplicial complexes, f_0 and f_1 are $(k-1)$-homotopic if and only if they induce the same homomorphisms in the homotopy groups of dimension less than k for each connected component. Note that compact k-dimensional Menger manifolds have the $(k-1)$-homotopy types of finite simplicial complexes ([8]).

In [18], we used Theorem 3.1 to construct topologically hyperbolic homeomorphism of the Menger compact space μ^n. We can reformulate what was used in the construction in [18] and what we are going to use as follows.

Proposition 3.2. *Let A be a closed set in the compact n-dimensional Menger space μ^n such that*

1. *A is homeomorphic to the compact $(n-1)$-dimensional Menger space μ^{n-1},*
2. *$\mu^n \setminus A = U_1 \cup U_2$, $U_1 \neq \emptyset$, $U_2 \neq \emptyset$ and $U_1 \cap U_2 = \emptyset$, and*
3. *$U_1 \cup A$ and $U_2 \cup A$ are n-dimensional Menger manifolds and A is a Z-set in $U_1 \cup A$ and in $U_2 \cup A$.*

Then $U_1 \cup A$ and $U_2 \cup A$ are homeomorphic to μ^n.

Proposition 3.3. *Let A_1 and A_2 be a closed set in the compact n-dimensional Menger space μ^n such that*

1. *A_1 and A_2 are homeomorphic to the disjoint union of two compact $(n-1)$-dimensional Menger space μ^{n-1},*

(2) $\mu^n \setminus A_i = U_{i1} \cup U_{i2} \cup U_{i3}$ *(disjoint union of nonempty open sets; $i = 1, 2$) and*
(3) $\overline{U_{i1}}$, $\overline{U_{i2}}$ *and* $\overline{U_{i3}}$ *are n-dimensional Menger manifolds and* $A_i \subset \overline{U_{i2}}$ *is a Z-set as well as* $A_i \cap \overline{U_{i1}} \subset \overline{U_{i1}}$ *and* $A_i \cap \overline{U_{i3}} \subset \overline{U_{i3}}$.

Then any homeomorphism $A_1 \longrightarrow A_2$ *extends to a homeomorphism* $h : \mu^n \longrightarrow \mu^n$ *such that* $h(U_{1j}) = U_{2j}$ *after changing the indices* $_{i1}$ *and* $_{i3}$ *if necessary.*

Proposition 3.4. *Under the assumption of Proposition 3.2, for any compact set K in U_1 and any positive real number ε, there is a homeomorphism $\varphi_{K,\varepsilon}$ of μ^n such that* $\varphi_{K,\varepsilon}|(U_2 \cup A) = \mathrm{id}_{U_2 \cup A}$, $\mathrm{diam}(\varphi_{K,\varepsilon}(K)) \leq \varepsilon$.

The proof of Theorem 1.2 is done by the same argument as that of Theorem 1.1. It is clear that Propositions 3.2, 3.3 and 3.4 can play the same role of Theorems 2.1 and 2.2 and Lemma 2.3. Thus we showed that the commutator width of Homeo(μ^n) is one.

References

[1] R. D. Anderson, *The algebraic simplicity of certain groups of homeomorphisms*, Amer. J. Math. **80** (1958) 955–963.

[2] A. Banyaga, *The structure of classical diffeomorphism groups, Mathematics and its Applications*, vol. 400, Kluwer Academic Publishers Group, Dordrecht (1997) xii+197 pp.

[3] M. Bestvina, *Characterizing k-dimensional universal Menger compacta*, Mem. Amer.Math. Soc. 71 (1988).

[4] M. Brown, *A proof of the generalized Schoenflies theorem*, Bull. Amer. Math. Soc. **66** (1960), 74–76.

[5] M. Brown, *Locally flat imbeddings of topological manifolds*, Ann. of Math., **75** (1962), 331–341.

[6] M. Brown and H. Gluck, *Stable Structures on Manifolds: I, Homeomorphisms of S^n*, Annals of Math., 2nd ser., **79** (1), (1964), 1–17.

[7] D. Burago, S. Ivanov and L. Polterovich, *Conjugation-invariant norms on groups of geometric origin*, Adv. Stud. Pure Math. **52**, Groups of Diffeomorphisms (2008) 221–250.

[8] A. Chigogidze, *Compact spaces lying in an n-dimensional universal Menger compact space having homeomorphic complement in it*, Math. USSR-Sb. 61 (1988), 471–484.

[9] A. Chigogidze, K. Kawamura and E. D. Tymchatyn, *Menger manifolds*, Continua with Houston Problem Book, Marcel Dekker, New York, 1995, pp. 37-88.

[10] D. B. A. Epstein, *The simplicity of certain groups of homeomorphisms*, Compositio Math. 22 (1970), 165–173.

[11] A. Fathi, *Structure of the group of homeomorphisms preserving a good measure on a compact manifold*, Ann. Sci. École Norm. Sup. (4) **13** (1980), no. 1, 45–93.

[12] G. M. Fisher, *On the group of all homeomorphisms of a manifold*, Trans. Amer. Math. Soc., **97** (1960) 193–212.

[13] M. Herman, *Sur la conjugaison différentiable des difféomorphismes du cercle à des rotations*, Publ. Math. Inst. Hautes Études Sci. **49** (1979), 5–234.

[14] R. Kirby *Stable homeomorphisms and the annulus conjecture*, Ann. of Math., 2nd Ser., **89**, (3) (1969), 575–582.

[15] J. Mather, *The vanishing of the homology of certain groups of homeomorphisms*, Topology **10** (1971), 297–298.

[16] J. Mather, *Commutators of diffeomorphisms* I, II and III, Comment. Math. Helv. **49** (1974), 512–528, **50** (1975), 33–40 and **60** (1985), 122–124.

[17] F. Quinn, *Ends of maps, III: Dimensions 4 and 5*, J. Differential Geometry, **17** (1982), 503–521.

[18] V. Sergiescu and T. Tsuboi, *Acyclicity of the groups of homeomorphisms of the Menger compact spaces*, American Journal of Mathematics **118** (1996) 1299–1312.

[19] W. Thurston, *Foliations and groups of diffeomorphisms*, Bull. Amer. Math. Soc. **80** (1974), 304–307.

[20] T. Tsuboi, *On the uniform perfectness of diffeomorphism groups*, Adv. Stud. Pure Math. **52**, Groups of Diffeomorphisms (2008) 505–524.

[21] T. Tsuboi, *On the uniform simplicity of diffeomorphism groups*, Differential geometry, 43–55, World Sci. Publ., Hackensack, NJ, 2009.

[22] T. Tsuboi, *On the uniform perfectness of the groups of diffeomorphisms of even-dimensional manifolds*, to appear in Comment. Math. Helve.

[23] S. M. Ulam and J. von Neumann, *On the group of homeomorphisms of the surface of the sphere*, (Abstract) Bull. Amer. Math. Soc. **53** (1947) 508.

Graduate School of Mathematical Sciences, University of Tokyo, Komaba Meguro, Tokyo 153-8914, Japan

E-mail address: tsuboi@ms.u-tokyo.ac.jp

UTMS

The Graduate School of Mathematical Sciences was established in the University of Tokyo in April, 1992. Formerly there were two departments of mathematics in the University of Tokyo: one in the Faculty of Science and the other in the College of Arts and Sciences. All faculty members of these two departments have moved to the new graduate school, as well as several members of the Department of Pure and Applied Sciences in the College of Arts and Sciences. In January, 1993, the preprint series of the former two departments of mathematics were unified as the Preprint Series of the Graduate School of Mathematical Sciences, The University of Tokyo. For the information about the preprint series, please write to the preprint series office.

ADDRESS:
Graduate School of Mathematical Sciences, The University of Tokyo
3–8–1 Komaba Meguro-ku, Tokyo 153-8914, JAPAN
TEL +81-3-5465-7001 FAX +81-3-5465-7012