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Abstract

In this paper, we give several applications of the discrete game approach to partial differential equations. We first present a rigorous game-theoretic proof of fattening phenomenon for motion by curvature with figure-eight shaped initial curves without using parabolic PDE theory. The proof is based on a comparison between the game value and its inverse one. Accompanied with the example of figure eight, our second result shows, for the stationary equation of mean curvature type in an arbitrary region Ω, that fattening of positive curvature flow with initial surface $\partial\Omega$ causes loss of the weak comparison principle, which partially answers an open question posed by R. V. Kohn and S. Serfaty in 2006. In addition, we prove the existence of solutions of the stationary problem and its game approximation in the absence of comparison principles but under regularity conditions of the flow. The main difference between our games and those in other papers is that we take the domain perturbation into consideration.

Key words: viscosity solutions, deterministic games, curvature flow equations, fattening, weak comparison principle

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1 Introduction

In recent years, a game-theoretic approach, deterministic or stochastic, to various elliptic and parabolic equations [10, 27, 28, 30, 31, 35] attracts enormous attention. These results provide representation theorems for the solution through value functions of games.

In this paper, we aim to make use of the game theory to investigate the properties of the mean curvature flow equation, which is of great importance in applications. Mathematically, we consider a family $\{\Gamma_t\}_{t\geq 0}$ of compact

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hypersurfaces embedded in the Euclidean space $\mathbb{R}^n$. The *mean curvature flow* equation is

$$V = \kappa, \tag{1.1}$$

where V and κ denote respectively the normal velocity and (mean) curvature of Γ_t. With proper initial data Γ_0, the existence and uniqueness of such a smooth solution is well understood. The smooth solution, however, usually exists only in finite time and ends up becoming singular [17, 24]. Classical methods cannot be applied after the singularity and other approaches based on a definition of generalized solutions, are necessary. It is now widely known that there are at least three effective approaches of generalized solutions comprising one from geometric measure theory [8], another through phase transitions (see, for example, [15]) and the third of a level set method [9, 16], which is presented in great detail in the book [21].

In spite of the perfect existence of all these generalized solutions, the answer to the uniqueness problem remains incomplete. On some occasions, the enhanced varifold solution might be non-unique and the convergence for phase transition approach breaks down. By contrast, the level set method is supposed to ensure uniqueness of solutions because it is established on the ground of comparison principles in parabolic PDE theory, but, unfortunately, a level set at times turns out to develop an interior, which is thereby named *fattening*. To be more precise, we rewrite the curvature flow equation (1.1) as an initial value PDE problem:

$$\begin{cases} \partial_t u - |\nabla u| \mathrm{div}(\frac{\nabla u}{|\nabla u|}) = 0 & \text{in } \mathbb{R}^n \times (0, \infty), \\ u(x, 0) = u_0(x) & \text{in } \mathbb{R}^n. \end{cases} \tag{1.2}$$

Here u_0 is a bounded uniformly continuous function such that $\{x \in \mathbb{R}^n : u_0(x) = 0\} = \Gamma_0$. Using the viscosity solution theory, we get a unique solution u. We are then interested in the situation that the level set $\Gamma_t = \{x \in \mathbb{R}^n : u(x,t) = 0\}$ develops interior even when Γ_0 has empty interior. Examples are given in [16, 33, 25, 21] for Cauchy problems and in [18, 19, 5] for Neumann boundary problems. Although this difficulty can be overcome by imposing various nonfattening conditions on the initial data [6, 1, 34, 21] or by adding stochastic perturbation on the equation [14, 37], it is not well understood in general.

With regard to the game-theoretic approach, the pioneering work of Kohn and Serfaty [27] presents a type of deterministic and discrete games, whose value functions u^ε approximate the unique solution u of motion by curvature. (See [22, 23, 26, 28] for generalizations in different directions.) The mechanisms can be briefly explained as follows. Suppose that finitely many pairs of "inf sup" are arranged over proper sets for a prescribed function. The *dynamic programming principle* will subsequently admit a nonlinear semigroup. If its generator is close to that of mean curvature flow, then the approximation is obtained. We will review a few related results in Section 2.

In contrast to other approximation methods, an important feature of games is that the convergence keeps valid even when level sets develop interior. Our primary motivation is therefore to understand the game interpretation for fattening. For different purposes, we use two game-theoretic methods:

1. comparing with the *inverse games*; and
2. perturbing the *objective function*.

The first way, from the viewpoint of PDE, is to substitute u_0 with $-u_0$ and if the geometric flow is *orientation-free*, the solution essentially remains unchanged, but the corresponding game becomes different. For (1.2), not only can we discuss the normal games, but the inverse ones [27] can also be constructed by simply switching all "inf sup" to "sup inf." Notice that we do not change the order of the sets over which extrema are taken. Our changes formally make little difference to the original arguments and thus still yield the same equation. Nevertheless, we will later see that the optimal trajectories in the original and inverse games can be entirely different. Several examples in two dimensions with explicit game strategies are given in Section 3 to show this clearly. These examples of "figure eight" are all known to cause fattening ([16, 21, 33], etc). We intend to reveal on some level that fattening can be studied by observing and comparing the distinction between the optimal decisions of players for both types of games.

It is worth remarking that we employ only the game interpretation without using any parabolic PDE theory, which is usually resorted to when one tries to prove the existence of fattening rigorously [7, 21, 25, 33]. Our idea is close to the proof of fattening for first order equations in [6] and our computation for the examples is elementary.

It is an exclusive property of the second-order games that their inverse versions lead to the same equations. A large class of first-order Hamilton-Jacobi equations possess interpretation of continuous-time and deterministic differential games [4], but if one alters the players' objectives, other equations will be derived. Our method is consequently not applicable to the fattening phenomenon for Hamilton-Jacobi equations. Also, when dealing with the geometric flows which are not orientation-free such as the *signed mean curvature flow* equation

$$\begin{cases} \partial_t u - |\nabla u|\left(\operatorname{div}(\frac{\nabla u}{|\nabla u|}) \vee 0\right) = 0 & \text{in } \mathbb{R}^n \times (0,\infty), \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^n, \end{cases} \tag{1.3}$$

or those with driving forces, one will find it inconvenient to construct the inverse games. It is easier to use our second method based on perturbed objective function, which is more natural from the PDE point of view. We will not develop the method for parabolic equations. Rather, we would like to apply our point of view to elliptic equations such as

$$\begin{cases} -|\nabla U|\operatorname{div}(\frac{\nabla U}{|\nabla U|}) - 1 = 0 & \text{in } \Omega, \\ U = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

We at first remark that the equation in question should be nonhomogeneous. An example [31] shows that the solutions of $|\nabla U|\operatorname{div}(\frac{\nabla U}{|\nabla U|}) = 0$ are not unique even in a two-dimensional disk with smooth boundary data. We thus have to turn our attention to (1.4), which looks more reasonable.

The study of (1.4) is usually conducted in a bounded and (strictly) *mean convex* domain Ω, initiated by [16], in which the existence and uniqueness of continuous solutions are clarified. The convexity assumption of Ω is a non-degeneracy condition guaranteeing that the boundary condition holds in the strict sense. See results of this type for more general equations in [13] etc. About the regularity, since the equation is degenerate elliptic, Krylov-Safonov theory cannot be applied directly. Ilmanen [25] shows that the solution does not necessarily belong to the class C^2 in general. For a strictly convex domain, it is recently known that the solution is of class $C^3(\overline{\Omega})$ in two dimensions [27] but is not necessarily the case in higher dimensions [32].

On the other hand, if Ω is not mean convex, little is known about the wellposedness, since there may be loss of boundary condition. We thus have to relax the Dirichlet boundary condition to a weak sense. The paper [27] establishes a family of *exit time* games with values U^ε and shows the *relaxed limits* in $\overline{\Omega}$

$$\limsup_{\varepsilon\to 0}{}^{*} U^\varepsilon \text{ and } \liminf_{\varepsilon\to 0}{}_{*} U^\varepsilon$$

are respectively an upper semicontinuous subsolution and a lower semicontinuous supersolution with the Dirichlet boundary condition interpreted in the viscosity sense. By assuming the domain to be *star-shaped* and then establishing a *weak comparison principle*, they prove the existence and uniqueness of solutions which are not necessarily continuous. By weak comparison principle, we mean the property that any subsolution $W_1 \in USC(\overline{\Omega})$ and supersolution $W_2 \in LSC(\overline{\Omega})$ of (1.4) satisfy

$$(W_1)_* \leq W_2 \quad \text{and } W_1 \leq (W_2)^* \quad \text{in } \overline{\Omega},$$

where $(W_1)_*$ and $(W_2)^*$ stand respectively for the lowersemicontinuous envelope of W_1 and uppersemicontinuous envelope of W_2. However, it is an open question whether comparison principle and uniqueness of solutions hold in a more general domain.

We attempt to answer the question by investigating the effect of fattening on its elliptic versions. Heuristically speaking, since the formation of fat level sets is explained as nonuniqueness of solutions, it is tempting to see whether the nonuniqueness for parabolic equations can bring us loss of uniqueness for elliptic cases. The figure-eight shaped region, well analyzed in Section 2, turns out to be an immediate counterexample to disprove the existence of comparison principle even in the weak sense.

For a more general Ω, we utilize the corresponding version of our second method, perturbing Ω from inside and outside so that we have value functions $U_+^{\varepsilon;\delta}$ under the same game rules for the equation (1.3), where $\delta \in \mathbb{R}$ is a parameter standing for the perturbation exerted on Ω. Notice that a finite horizon game for (1.3) and an exit-time game for (1.4) share exactly the same rules except for the form of cost functions. If fattening happens at a certain $x_0 \in \Omega$ to the signed mean curvature flow (1.3) starting from $\partial\Omega$, we are able to obtain optimal strategies for perturbed games of (1.3) and implement them in the exit time games of (1.4) to get

$$\limsup_{\varepsilon\to 0,\ \delta\to 0+}{}^{*}\ U_{+}^{\varepsilon,\delta} > \liminf_{\varepsilon\to 0,\ \delta\to 0-}{}_{*}\ U_{+}^{\varepsilon,\delta}$$

in a neighborhood of x_0. Since the left hand side is a subsolution and the right hand side is a supersolution, we prove the loss of comparison principle.

Theorem 1.1. *Assume that Ω is an open bounded subset of $\mathbb{R}^n$. Let u_0 be the signed distance function of Ω (with values being negative in Ω). If the zero level set of the solution of signed mean curvature flow (1.3) develops interior in Ω, then the weak comparison principle for (1.4) fails to hold.*

Any example of fattening becomes a counterexample for the existence of a weak comparison principle. Our counterexample of figure eight is now a special case of Theorem 1.1. As little is known about the positive mean curvature flow, we do not know whether or not the level set of the motion starting from smooth surface $\partial\Omega$ may develop nonempty interior in Ω, although there is evidence to show that the fattening of mean curvature flow (1.2) does have chance to take place from a smooth initial surface in dimensions $n \geq 3$ ([2, 3, 38]).

On the other hand, we are also curious about the situation when there is no fattening during the evolution. We cannot prove the comparison principle holds in this case but we still obtain the existence of a solution when the motion (1.3) is regular enough.

Theorem 1.2. *Assume that Ω is an open bounded subset of $\mathbb{R}^n$. Let u_0 be the signed distance function (with values being negative in Ω) and u be the viscosity solution of (1.3). If the evolution (1.3) started from $\partial\Omega$ is regular in the sense that u satisfies*

$$\overline{\{(x,t)\in\mathbb{R}^n\times[0,\infty) : u(x,t)>0\}} = \{(x,t)\in\mathbb{R}^n\times[0,\infty) : u(x,t)\geq 0\} \tag{1.5}$$

and

$$\overline{\{(x,t)\in\mathbb{R}^n\times[0,\infty) : u(x,t)<0\}} = \{(x,t)\in\mathbb{R}^n\times[0,\infty) : u(x,t)\leq 0\}, \tag{1.6}$$

then

$$\left(\limsup_{\varepsilon\to 0,\ \delta\to 0+}{}^{*}\ U_{+}^{\varepsilon,\delta}\right)_{*} \leq \liminf_{\varepsilon\to 0,\ \delta\to 0-}{}_{*}\ U_{+}^{\varepsilon,\delta}\ \text{ in } \overline{\Omega}$$

and

$$\limsup_{\varepsilon\to 0,\ \delta\to 0+}{}^{*}\ U_{+}^{\varepsilon,\delta} \leq \left(\liminf_{\varepsilon\to 0,\ \delta\to 0-}{}_{*}\ U_{+}^{\varepsilon,\delta}\right)^{*}\ \text{ in } \overline{\Omega}.$$

This theorem enables us to get the existence of a solution of (1.4) which is continuous except at a nowhere dense subset in $\overline{\Omega}$ and verify the game approximation (in a weaker sense) without using any comparison principle. The regularity assumptions (1.5) and (1.6) say that neither the interior motion nor the exterior motion develops nonempty level sets. The uniqueness of solutions is not clear in this case because comparison results are still unknown.

Since the starshapedness of Ω implies regularity of (1.3) ([21, Theorem 4.5.9]), we actually generalizes the game interpretation in [27]. Our idea for Theorem 1.2 is inspired by [36], in which a game approach for the first order Hamilton-Jacobi equations is exploited.

One may wonder why the motion by positive mean curvature (1.3) instead of the original motion by curvature (1.2) is involved. We prefer the former because of a certain monotonicity along trajectories in its game representation. It is not clear whether we can substitute all assumptions on (1.3) in the theorems above with similar ones on (1.2) and obtain the same conclusions.

To conclude our introduction, we pose the following two questions:

1. The loss of weak comparison principle of (1.4) is closely related to the fattening for (1.3). Are they actually equivalent?

2. Does the solution of positive mean curvature flow equation (1.3) have fattening behavior when and only when (1.2) does? Note that it is true for the initial curve of a figure-eight.

If we can give affirmative answers to the above, we believe that the weak comparison principle for a smooth domain should hold in two dimensions but does not need to be true in higher dimensions.

We will not tackle other geometric flows but our results in this paper can be extended for more general geometric motions.

This paper is organized in the following way. In Section 2, we review the game setting we are relying on and investigate some special game strategies. In Section 3, we prove the fattening phenomenon for two dimensional curvature flow equation with figure-eight initial curves by using the game interpretation. In Section 4, we use the examples to show that the weak comparison theorem of (1.4) is not necessarily true in general and prove Theorems 1.1 and 1.2.

Notations:

For any $p = \begin{pmatrix} p_1 \\ \vdots \\ p_n \end{pmatrix}$ and $q = \begin{pmatrix} q_1 \\ \vdots \\ q_n \end{pmatrix} \in \mathbb{R}^n$, $p^\top$ denotes the transpose of p, $p \cdot q$ denotes the inner product in $\mathbb{R}^n$, i.e., $p \cdot q = p^\top q = q^\top p = \sum_{i=1}^n p_i q_i$, and $p \otimes q$ represents the tensor product in $\mathbb{R}^n$.

For every $z \in \mathbb{R}^n$ and $r > 0$, we denote by $B_r(z)$ the open ball with center at z and radius r.

For any $a, b \in \mathbb{R}$, $a \vee b := \max\{a, b\}$ and $a \wedge b := \min\{a, b\}$. The bracket $[a]$ stands for the largest integer less than or equal to a.

2 Two-person Games

We begin with a review of the game setting given in [27]. We are most interested in the so-called Paul and Carol game. A marker, representing the *game state*, is initialized at a position $x \in \overline{\Omega}$ from time 0. The maturity time given is denoted by t. Let the step size for space be $\varepsilon > 0$. Time ε^2 is consumed for every step. Then the total number of game steps N can be regarded as $[\frac{t}{\varepsilon^2}]$. Two players, Paul and Carol participate the game. Paul intends to minimize at the final state an *objective function*, which in our case is u_0, while the other, Carol, is to maximize it. At each round,

(1) Paul chooses in $\mathbb{R}^n$ $n-1$ unit vectors $v_1, v_2, \ldots, v_{n-1}$ pairwise perpendicular, i.e., $v_i \cdot v_j = 0$ for all $1 \le i < j \le n-1$;

(2) Carol has the right to reverse Paul's choice, which determines $b^{(i)} = \pm 1$ for $i = 1, 2, \ldots, n-1$;

(3) The marker is moved from the present state x to $x + \sqrt{2}\varepsilon \sum_{i=1}^{n} b^{(i)} v_i$.

To express the rules in a more mathematical way, set

$$\mathcal{Q} = \{Q = (v_1, \ldots, v_{n-1}) \in \mathbb{R}^{n\times(n-1)} : |v_i| = 1 \text{ and } v_i \cdot v_j = 0, \\ \text{for } 1 \le i < j \le n\}$$

and $\mathcal{B} = \{b = (b^{(1)}, \ldots, b^{(n-1)})^\top \in \mathbb{R}^{n-1} : b^{(i)} = \pm 1 \text{ for all } i = 1, 2, \ldots, n-1\}$. Then the inductive *state equation* writes as

$$\begin{cases} y_{k+1} = y_k + \sqrt{2}\varepsilon Q_k b_k, & k = 0, 1, \ldots, N-1; \\ y_0 = x, \end{cases}$$

where $Q_k \in \mathcal{Q}$ and $b_k \in \mathcal{B}$. We denote by α and β respectively the *nonanticipating strategies* of Paul and Carol. Hereafter, for any $x \in \mathbb{R}^n$ and $s \in [0, \infty)$, $y(x, s; \alpha, \beta)$ stands for the game state at the step $[s/\varepsilon^2]$ starting from x under the competing strategies α and β so that our games look like continuous ones. We also use the notation $y(x, s)$ for short if there is no ambiguity in strategies.

Note that we here insist, for simplicity, considering the game in time period $[0, t]$ so that the associated equation is exactly (1.2) instead of a backward-in-time one as in [27]. No essential changes are made. More precisely, the *value function* is defined as

$$u_1^\varepsilon(x, t) := \min_{Q_1 \in \mathcal{Q}} \max_{b_1 \in \mathcal{B}} \ldots \min_{Q_N \in \mathcal{Q}} \max_{b_N \in \mathcal{B}} u_0(y(x, t)), \tag{2.1}$$

or, by using the notation of strategies, briefly and equivalently expressed as

$$u_1^\varepsilon(x, t) = \min_{\alpha} \max_{\beta} u_0(y(x, t; \alpha, \beta)). \tag{2.2}$$

By the *dynamic programming*:

$$u_1^\varepsilon(x, t) = \min_{Q \in \mathcal{Q}} \max_{b \in \mathcal{B}} u_1^\varepsilon(x + \sqrt{2}\varepsilon Q b, t - \varepsilon^2) \tag{2.3}$$

with $u_1^\varepsilon(x, 0) = u_0(x)$, we may prove $u_1^\varepsilon(x, t)$ converges locally uniformly to the solution u of (1.2), as $\varepsilon \to 0$.

On the other hand, an *inverse game value* u_2^ε is defined as

$$u_2^\varepsilon(x, t) = \max_{Q_1 \in \mathcal{Q}} \min_{b_1 \in \mathcal{B}} \ldots \max_{Q_N \in \mathcal{Q}} \min_{b_N \in \mathcal{B}} u_0(y(x, t)) = \max_{\alpha} \min_{\beta} u_0(y(x, t; \alpha, \beta)), \tag{2.4}$$

which, via similar arguments, can also be shown to converge to u. A rigorous mathematical statement is as follows.

Theorem 2.1 (Game approximation for mean curvature flow [27]). *Assume that u_0 is a bounded and uniformly continuous function in $\mathbb{R}^n$. Let u_1^ε and u_2^ε be the value functions defined by* (2.1) *and* (2.4) *respectively. Then both u_1^ε and u_2^ε converge, as $\varepsilon \to 0$, to the unique viscosity solution of* (1.2) *uniformly on compact subsets of $\mathbb{R}^n \times (0, \infty)$.*

A variant for *positive (negative) mean curvature flow* equation is to use

$$\mathcal{Q}' = \{Q = (v_1, \ldots, v_{n-1}) \in \mathbb{R}^{n\times(n-1)} : |v_i| \le 1 \text{ and } v_i \cdot v_j = 0 \\ \text{for } 1 \le i < j \le n\}$$

and let

$$\begin{aligned} u_+^\varepsilon(x,t) &= \max_{Q_1\in\mathcal{Q}'} \min_{b_1\in\mathcal{B}} \ldots \max_{Q_N\in\mathcal{Q}'} \min_{b_N\in\mathcal{B}} u_0(y(x,t)) \\ &= \max_{\alpha} \min_{\beta} u_0(y(x,t;\alpha,\beta)), \end{aligned} \tag{2.5}$$

then we get another convergence theorem.

Theorem 2.2 (Game approximation for signed mean curvature flow [27]). *Assume that u_0 is a bounded and uniformly continuous function in $\mathbb{R}^n$. Let u_2^ε be the value functions defined by* (2.5). *Then u_+^ε converge, as $\varepsilon \to 0$, to the unique viscosity solution of (1.3) uniformly on compact subsets of $\mathbb{R}^n \times (0, \infty)$.*

We remark that the original theorem is only established for u_0 which is constant outside a certain compact set. The extension to our general initial data here seems direct. With the aid of a comparison theorem, the proof for the convergence u_1^ε, for instance, rests on showing the half relaxed limits

$$\overline{u}_1(x,t) := \limsup_{\varepsilon\to 0}{}^* u_1^\varepsilon(x,t) = \lim_{\delta\to 0} \sup\{u^\varepsilon(y,s) : 0 < \varepsilon < \delta,\ |x-y| + |t-s| < \delta\}$$

and

$$\underline{u}_1(x,t) := \liminf_{\varepsilon\to 0}{}_* u_1^\varepsilon(x,t) = \lim_{\delta\to 0} \inf\{u^\varepsilon(y,s) : 0 < \varepsilon < \delta,\ |x-y| + |t-s| < \delta\}.$$

are respectively a subsolution and a supersolution of (1.2).

Heuristically speaking, the core of proofs lies on the following observation for smooth functions. Let $\phi \in C^\infty(\mathbb{R}^n \times (0,\infty))$ in place of u_1^ε satisfy the dynamic programming principle (2.3) for all $x \in \mathbb{R}^n$ and $t \ge \varepsilon^2$. Then by Taylor expansion with ε taken small, we have

$$\varepsilon^2 \partial_t \phi(x) + \min_{Q\in\mathcal{Q}} \max_{b\in\mathcal{B}} \left(\sqrt{2}\varepsilon \nabla\phi(x) \cdot Qb + \varepsilon^2 \left(\nabla^2\phi(x)Qb\right) \cdot Qb\right) = O(\varepsilon^3).$$

It follows immediately that

$$\partial_t \phi - |\nabla\phi| \operatorname{div}\left(\frac{\nabla\phi}{|\nabla\phi|}\right) = 0 \quad \text{in } \mathbb{R}^n \times (0,\infty),$$

on the basis of the fundamental lemma below. (Notice that the replacement of $\min\max$ by $\max\min$ does not change the equation.)

Lemma 2.3. *For any $\xi \in \mathbb{R}^n \setminus \{0\}$ and $X \in \mathbb{R}^{n\times n}$ symmetric, the following inequalities hold:*

$$\min_{Q\in\mathcal{Q}} \max_{b\in\mathcal{B}} \left(\frac{1}{\varepsilon}\xi^\top Qb + (Qb)^\top XQb\right) \leq tr\left((I - \frac{\xi\otimes\xi}{|\xi|^2})X\right). \tag{2.6}$$

$$\min_{Q\in\mathcal{Q}} \max_{b\in\mathcal{B}} \left(\frac{1}{\varepsilon}\xi^\top Qb + (Qb)^\top XQb\right) \geq tr\left((I - \frac{\xi\otimes\xi}{|\xi|^2})X\right) - C\varepsilon^2, \tag{2.7}$$

where the constant $C = C(n, X) > 0$ is bounded whenever $X \in \mathbb{R}^{n\times n}$ is bounded. In particular,

$$\min_{Q\in\mathcal{Q}} \max_{b\in\mathcal{B}} \left(\frac{1}{\varepsilon}\xi^\top Qb + (Qb)^\top XQb\right) \to tr\left((I - \frac{\xi\otimes\xi}{|\xi|^2})X\right) \quad \text{as } \varepsilon \to 0 \tag{2.8}$$

locally uniformly with respect to $\xi \in \mathbb{R}^n \setminus \{0\}$ and $X \in \mathbb{R}^{n\times n}$.

Proof. Since $\mathcal{Q}$ is invariant under rotation and $\mathcal{B}$ is symmetric with respect to 0, we assume throughout the proof $\xi = (0, \dots, 0, |\xi|)^\top$. Let us first consider the case $|\xi| = 1$. Then we have a convenient equivalence $I - \xi\otimes\xi = \begin{pmatrix} I_{n-1} & 0 \\ 0 & 0\end{pmatrix}$, where I_{n-1} denotes the identity in $\mathbb{R}^{(n-1)\times(n-1)}$.

Part 1. To show (2.6), we ake a specific Q such that $Q = \begin{pmatrix} \tilde{Q} \\ 0 \end{pmatrix}$, where $\tilde{Q} \in \mathbb{R}^{(n-1)\times(n-1)}$ fulfills $\tilde{Q}^\top = \tilde{Q}^{-1}$, and then we get

$$(Qb)^\top XQb = b^\top \tilde{Q}^\top (I_{n-1}, 0)X \begin{pmatrix} I_{n-1} \\ 0 \end{pmatrix} \tilde{Q}b.$$

We choose $\tilde{Q}$ to diagonalize the matrix $(I_{n-1}, 0)X \begin{pmatrix} I_{n-1} \\ 0 \end{pmatrix}$ so that

$$(Qb)^\top XQb = \mathrm{tr}\left((I - \frac{\xi\otimes\xi}{|\xi|^2})X\right)$$

and obtain (2.6).

Part 2. To prove (2.7), we claim that there exists $C_n > 0$ depending only on n such that for any $Q = (v_1, \dots, v_{n-1}) \in \mathcal{Q}$ and $\varepsilon > 0$

$$\max_{b\in\mathcal{B}} \left(\frac{1}{\varepsilon}\xi^\top Qb + (Qb)^\top XQb\right) \geq \frac{1}{\varepsilon}C_n|Q^\top\xi| + \sum_{i=1}^{n-1} v_i^\top X v_i. \tag{2.9}$$

We postpone the proof of this claim. From Part 1, we get $|Q^\top\xi| \leq C\varepsilon$ for some constant $C > 0$ depending on n and X; in other words, we have $|v_i \cdot \xi| \leq C\varepsilon$ for all $i = 1, ..., n-1$. Take a vector $\eta \in \mathbb{R}^n$ with $|\eta| = 1$ and $\eta \cdot v_i = 0$ for any $i = 1, ..., n-1$. Then we get $|\eta - \xi| \leq Cn\varepsilon$, which implies

$$\sum_{i=1}^{n-1} (v_i)^\top X v_i = \mathrm{tr}X - \eta^\top X\eta \geq \mathrm{tr}\big((I - \xi\otimes\xi)X\big) - C\varepsilon^2 \tag{2.10}$$

with $C = C(n, X) > 0$ updated. Combining (2.9) and (2.10), we are led to (2.7) in the case $|\xi| = 1$.

We next prove the claim (2.9). It is clear that one can take $b_{(1)} = \pm 1$ such that $(\xi \cdot v_1)b_{(1)} = |\xi \cdot v_1|$. Then in this case we have

$$\begin{aligned}&\frac{1}{\varepsilon}\xi^\top Qb + (Qb)^\top XQb\\ =&\frac{1}{\varepsilon}|\xi \cdot v_1| + \sum_{i=1}^{n-1} v_i^\top X v_i + \frac{1}{\varepsilon}\sum_{j=2}^{n-1}(\xi \cdot v_j)b_{(j)} + 2\sum_{j=2}^{n-1}\sum_{i=1}^{j-1} v_i^\top X v_j b_{(i)} b_{(j)}.\end{aligned}$$

Choosing $b_{(j)} (j = 2, ..., n-1)$ one after another, we can make the last two terms in the above inequality nonnegative. We thus reach the conclusion that

$$\max_{b\in\mathcal{B}}\left(\frac{1}{\varepsilon}\xi^\top Qb + (Qb)^\top XQb\right) \geq \frac{1}{\varepsilon}|\xi \cdot v_1| + \sum_{i=1}^{n-1} v_i^\top X v_i.$$

We similarly obtain other inequalities with the first term on the right hand side above replace by $\frac{1}{\varepsilon}|\xi \cdot v_i|$ for all $i = 2, ..., n-1$. It then suffices to take the average of these inequalities to get (2.9).

For the general case that $|\xi| \neq 1$, we take $\lambda = |\xi| > 0$ and then (2.6) and (2.7) hold with ξ/λ and X/λ. It follows that

$$\begin{aligned}&\operatorname{tr}\left((I - \frac{\xi\otimes\xi}{|\xi|^2})X\right) - \lambda C(n, X/\lambda)\varepsilon^2\\ \leq&\min_{Q\in\mathcal{Q}}\max_{b\in\mathcal{B}}\left(\frac{1}{\varepsilon}\xi^\top Qb + (Qb)^\top XQb\right) \leq \operatorname{tr}\left((I - \frac{\xi\otimes\xi}{|\xi|^2})X\right)\end{aligned}$$

and the rest of our statements hold. □

There is an elliptic version of the games above, which we review briefly in what follows. We need a domain in which solutions of an elliptic equation can be defined. For our particular purpose, we relax the notion of a domain to a more general open set Ω. In fact, there is no obvious reason to restrict our study in a domain, especially from the game-theoretic point of view and we are curious about the solutions on a finite union of open, bounded and connected subsets, say, a set shaped like a figure eight.

The equation we are concerned with is (1.4). We follow the same rules as in (1)–(3) above, but this time we are interested in the exit time. Namely, for each $x \in \mathbb{R}^n$, we denote by $T^\varepsilon(x; \alpha, \beta)$ the first time of exit from Ω and by $\hat{T}^\varepsilon(x; \alpha, \beta)$ the first time of exit from $\overline{\Omega}$ under alternate controls $Q \in \mathcal{Q}$ and $b \in \mathcal{B}$ determined by both players. Define

$$U_1^\varepsilon(x) = \max_\alpha \min_\beta \hat{T}^\varepsilon(x; \alpha, \beta) \quad \text{and} \quad U_2^\varepsilon(x) = \min_\alpha \max_\beta T^\varepsilon(x; \alpha, \beta) \tag{2.11}$$

and let

$$\overline{U_i} = \limsup_{\varepsilon\to 0}{}^* U_i^\varepsilon \quad \text{and} \quad \underline{U_i} = \liminf_{\varepsilon\to 0}{}_* U_i^\varepsilon \quad \text{in } \overline{\Omega} \tag{2.12}$$

for $i = 1, 2$. We then have

Theorem 2.4 (Game approximation for the elliptic problem)**.** *Suppose Ω is a bounded open set. Then $\overline{U_i}$ and $\underline{U_i}$ defined in* (2.12) *are respectively viscosity subsolutions and supersolutions of* (1.4) *for $i = 1, 2$ with the boundary conditions interpreted in the viscosity sense.*

Interestingly, we still get the subsolutions and supersolutions of (1.4) when taking $\mathcal{Q}'$ in place of $\mathcal{Q}$, which is different from the former parabolic case. The elliptic problem therefore seems to have more solutions and be more complicated than the parabolic problem. Again, we skip the proof, which follows [27]. See also Propositions 4.2 and 4.3.

At the end of this section, we give two preliminary lemmas on a very special type of strategies, which we call *concentric sphere strategy*, They essentially play the role of comparison with evolution of spheres and will be often applied later.

Lemma 2.5 (Concentric sphere strategy of Paul)**.** *For the games with control set $\mathcal{Q}$, Paul has a strategy α_c such that for all $t \geq 0$ and Carol's strategy β the following results hold:*

(i) There exists a function $\omega_0^1 : [0, \infty) \to \mathbb{R}$ satisfying $y(x, t + \omega_0^1(\varepsilon); \alpha_c, \beta) \in B_{r(t)}(x_0)$ if $x \in B_{r_0}(x_0)$;

(ii) There exists a function $\omega_0^2 : [0, \infty) \to \mathbb{R}$ satisfying $y(x, t + \omega_0^2(\varepsilon); \alpha_c, \beta) \in B_{r(t)}(x_0)^c$ if $x \in B_{r_0}(x_0)^c$,

where $r(t) = (r_0^2 + 2t)^{\frac{1}{2}}$ and $|\omega_0^i(\varepsilon)| \leq \epsilon^2$ for each $\varepsilon > 0$ and $i = 1, 2$.

Proof. We ask Paul to choose the directions tangential to the concentric spheres where the marker is located. More specifically, for every step, suppose the marker is at $y \in \mathbb{R}^n$. Then Paul should take $Q \in \mathcal{Q}$ so that $(y - x_0) \cdot Q = 0$. This choice is of feedback and enables us to get

$$|y(x, t, \alpha_c, \beta) - x_0|^2 = r_0^2 + 2\varepsilon^2[\frac{t}{\varepsilon^2}]$$

by inductive applications of the Pythagoras Theorem. Let $\omega_0^1(\varepsilon) = t - [\frac{t}{\varepsilon^2}]\varepsilon^2$ and $\omega_0^2(\varepsilon) = t + \varepsilon^2 - [\frac{t}{\varepsilon^2}]\varepsilon^2$. Then (i) and (ii) follow easily and it is also clear that $|\omega_t^i(\varepsilon)| \leq \epsilon^2$ for $i = 1, 2$. $\square$

Lemma 2.6 (Concentric sphere strategy of Carol)**.** *For the games with control set $\mathcal{Q}$, Carol has a strategy β_c such that for all $t \geq 0$ and Paul's strategy α the following results hold:*

(i) There exists a function $\omega_0^1 : [0, \infty) \to \mathbb{R}$ satisfying $y(x, t + \omega_0^1(\varepsilon); \alpha, \beta_c) \in B_{r(t)}(x_0)$ if $x \in B_{r_0}(x_0)$;

(ii) There exists a function $\omega_0^2 : [0, \infty) \to \mathbb{R}$ satisfying $y(x, t + \omega_0^2(\varepsilon); \alpha, \beta_c) \in B_{r(t)}(x_0)^c$ if $x \in B_{r_0}(x_0)^c$,

where $r(t) = (r_0^2 + 2t)^{\frac{1}{2}}$ and $|\omega_0^i(\varepsilon)| \leq \epsilon^2$ for each $\varepsilon > 0$ and $i = 1, 2$.

Proof. The proof is as follows: For every game position y, no matter what $Q \in \mathcal{Q}$ is, take b satisfying $(y - x_0) \cdot Qb \leq 0$, which is certainly possible. Then for every step

$$|y + \sqrt{2}\varepsilon Qb - x_0|^2 - |y - x_0|^2 \leq 2\varepsilon^2,$$

which implies (i) and (ii). The choices of ω_0^1 and ω_0^2 are the same as those in Lemma 2.5. □

Remark 2.1. We need $\omega_0^1(\varepsilon)$ and $\omega_0^2(\varepsilon)$, which actually depend on t too, since our computation is on the discrete level and errors are caused by discretization. One can also merely consider the times which can be divided by ε^2 exactly and define the game values by interpolation. In this paper we choose to add this tiny adjustment in our computation. Hereafter we do not distinguish ω_0^1 and ω_0^2 and only use the notation ω_0 to denote either of them.

Remark 2.2. It is easy to see that the conclusions in Lemma 2.5 and Lemma 2.6(i) are also true for the modified games in Theorem 2.2 with Paul's control set $\mathcal{Q}$ replaced by $\mathcal{Q}'$, but that of Lemma 2.6(ii) is not.

3 Examples of Fattening for Curve Shortening

In this section, we present several examples of our game approach to the fattening phenomenon. The initial data are all like figure eight, which are known very well to give rise to fat level sets. Our explanation however is in a very different style.

Throughout this section, we take $n = 2$ and let Γ_t denote the zero level set of a solution of (1.2). It is obvious that Γ_t is a closed set.

3.1 Crossing Straight Lines

Let us make the first step with a simple example, which is discussed in [16] and [33]. We consider the curvature flow initialized from two straight lines crossing perpendicularly in a plane. Without loss, we may think of the lines as x_1-axis and x_2-axis and then denote the origin by O. It is certainly possible for one to endow this initial curve with an initial function u_0, which fulfills the requirement in Theorem 2.1. Indeed, arbitrarily decide a cone from the two open areas divided by axes, say the union of the first and third quadrants, denote it by Ω_- and name the other $\Omega_+ = \mathbb{R}^2 \setminus \overline{\Omega_-}$. Then we may use the signed distance of Ω_-

$$d(x) = \mathrm{dist}(x, \overline{\Omega_-}) - \mathrm{dist}(x, \overline{\Omega_+}) \tag{3.1}$$

to meet our needs, razing those too high and too low places by taking a minimum and a maximum with certain constants, i.e.,

$$u_0(x) = (d(x) \wedge M) \vee (-M), \quad \text{for all } x \in \mathbb{R}^2, \tag{3.2}$$

where $M > 0$ is sufficiently large. We take such a constant M just to assure that u_0 is bounded. It is only the neighborhood of these two axes that really counts.

Such an initial curve is known to develop interior instantly, so our consequence is certainly as follows.

Theorem 3.1 (Fattening from crossing lines)**.** *Let Ω_- be defined as above and the initial data u_0 of* (1.2) *be given as in* (3.2)*. Then the zero level set Γ_t of the solution u has nonempty interior for every $t > 0$.*

Proof. (1) We consider u_1^ε first and take the starting point $x \in \Omega_-$. Without loss of generality, we assume $x = (x_1, x_2)$ with $x_1 > 0$ and $x_2 > 0$. Since Paul tries his best to minimize the value of u_0 where the marker finally is, he probably could use the following strategy, denoted by α_o.

For every step, he picks a feedback control $v = x/|x|$. Let us call this constant strategy *origin-oriented strategy.* It guarantees that the marker never leaves $\Omega_- \cup \{0\}$, no matter which decision is made by Carol. Paul might have better options but this one is enough for us to deduce

$$u_1^\varepsilon(x,t) \leq 0 \quad \text{for all } t > 0. \tag{3.3}$$

(2) On the other hand, suppose that Paul wants to get out of Ω_- as soon as possible, and then he should rely on the strategy α_c described in Lemma 2.5 instead. To be more precise, set $r_0 = x_1 + x_2 + \sqrt{2x_1x_2}$ and $x_0 = (r_0, r_0)$, and then it is clear that the circle $B_{r_0}(x_0)$ passes x. We then use Lemma 2.5(ii) to make sure that despite Carol's best hinderance, the marker can leave Ω_- with consumption of time t_1 at most $r_0^2/2 + \omega_0(\varepsilon)$, noticing that the marker cannot keep staying in Ω_- if it is expelled from $B_{\sqrt{2}r_0}(x_0)$.

One may ask whether the marker can enter Ω_- again after it leaves. The answer is negative. Actually, Paul is supposed to alter his strategy no sooner than the marker exits. The new strategy he adopts is exactly the same as what was described in (1). Assume he is now at $y \in (\Omega_-)^c$, then take $v = y/|y|$ if $|y| \neq 0$ and $v = (0,1)$ if $y = 0$. It follows easily that the $u_0(y(x,t)) \geq 0$ for $t \geq t_1$. There is nothing his opponent can do about this, again due to the particularity of such strategies.

Paul's strategy is summarized to be a combination of the concentric α_c and the origin-oriented α_o. (See Figure 1.) Since $t_1 \leq r_0^2/2 + \omega_0(\varepsilon)$, we are led to

$$u_2^\varepsilon(x,t) \geq -\sqrt{2}\varepsilon \quad \text{for all } t > t_1 = \frac{1}{2}\left(x_1 + x_2 + \sqrt{2x_1x_2}\right)^2 + \omega_0(\varepsilon). \tag{3.4}$$

It follows from Theorem 2.1 that

$$u(x,t) = 0 \quad \text{for all } x \in A_t^1, \tag{3.5}$$

where $A_t^1 := \{x = (x_1,x_2) \in \mathbb{R}^2 : x_1 > 0,\ x_2 > 0 \text{ and } x_1 + x_2 + \sqrt{2x_1x_2} < \sqrt{2t}\}$. That is to say $A_t^1 \subset \Gamma_t$. Noticing that A_t^1 has interior for every $t > 0$, we have proved in a very concise manner that u has fat level sets. □

Notice further that since Γ_t is closed, we have $\overline{A_t^1} \subset \Gamma_t$, which can also be obtained by repeating the whole argument above for all $x \in \partial A_t^1$. By symmetry, we obtain, without difficulty, that

$$A_t := \{x = (x_1,x_2) \in \mathbb{R}^2 : |x_1| + |x_2| + \sqrt{2|x_1x_2|} < \sqrt{2t}\} \subset \Gamma_t$$

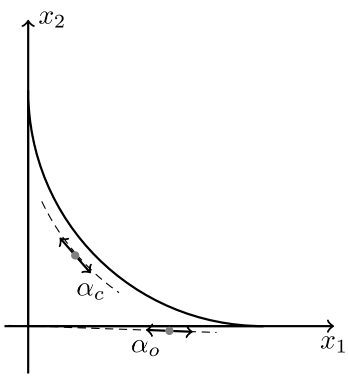

Figure 1: Paul's strategies α_c and α_o

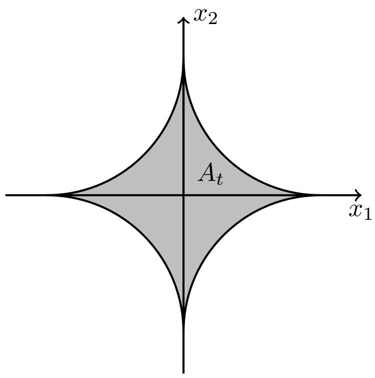

Figure 2: Subset A_t of the fat level set

and thus $\overline{A_t} \subset \Gamma_t$ (Figure 2).

Our computation above is straightforward, which is close to the study of characteristics for Hamilton-Jacobi equations. We neither used the parabolic PDE theory as was done in [33] and [25] nor directly calculated the solution of (1.2).

We however are only able to get a lower bound for Γ_t so far because we keep standing on Paul's side; in other words, we only consider suboptimal strategies. One might think that an upper bound will become available if we turn to seek Carol's optimal strategies. It is true but working on that is harder for the reason that Carol's controls ($b = \pm 1$) are zero-dimensional and hence less powerful than Paul's $(n-1)$-dimensional ones ($|v| = 1$).

3.2 Figure Eight

We continue to investigate a little more complicated situation, again proposed by Evans and Spruck [16]. The initial surface is a real figure-eight.

Fix a constant $R > 0$. Let $P_1 = (-R, 0)$, $P_2 = (R, 0)$ and $\Omega_- = B_R(P_1) \cup B_R(P_2)$. We take the initial value of (1.2) in the same way as in (3.2) and set subsequently $\Omega_+ := \overline{\Omega}_-^c$. In these circumstances, we obtain the following.

Theorem 3.2 (Fattening from a figure-eight curve)**.** *Assume that u_0 is defined in (3.2) and (3.1) with the above choice of Ω_-. Then the zero level set Γ_t of the solution u of (1.2) has nonempty interior for every $0 < t < R^2/2$.*

To prove this theorem, it is necessary to estimate the values of u_1^ε and u_2^ε at a fixed time $t > 0$, as is done in Section 3.1. The proof is useful for our discussion in Section 4 as well.

We find it better to first look into the points lying on the segment between P_1 and the origin $(0, 0)$.

Proposition 3.3. *There exist a constant $C > 0$ and $h \in C[0, +\infty)$ with $|h(\rho)| \leq C\rho$ such that*

$$u_1^\varepsilon((x_1, 0), t - h(\varepsilon)) \leq 0, \quad \text{for all } t < R^2/2 \text{ and } -R < x_1 < 0. \tag{3.6}$$

Proof. Suppose that Paul uses the origin-oriented strategy α_o as in the proof of Theorem 2.1. Carol has three options to run the game.

Case A. She moves the marker towards P_1. Then after finite many steps, the marker will reach the nearest point of P_1, which is assumed to be $(-l, 0)$, where $l \in \mathbb{R}$ satisfies $|l - R| < \sqrt{2}\varepsilon$. If Paul turns his strategy to α_c, concentric to P_1, then the estimate desired holds. Indeed, denote the game result $I_a(x, t)$ under the strategy for a starting point x and maturity time t. Let $t_1 = \frac{(x+l)\varepsilon}{\sqrt{2}}$ and $t_2 = t - t_1$. Then by Lemma 2.5(i), we may roughly get

$$I_a((x_1, 0), t) \leq 0 \quad \text{if } t_2 < \frac{(R - \sqrt{2}\varepsilon)^2}{2} + \omega_0(\varepsilon). \tag{3.7}$$

We can take a continuous function h_1 with $|h_1(\varepsilon)| \leq C\varepsilon$ for some $C > 0$ so that

$$I_a((x_1, 0), t - h_1(\varepsilon)) \leq 0 \quad \text{if } t < \frac{1}{2}R^2. \tag{3.8}$$

Case B. Carol lets the marker get close to P_2. This virtually leads to the same situation with Case A. Paul can take a concentric circle strategy with respect to P_2 this time, which yields again an estimate like (3.8):

$$I_b((x_1, 0), t - h_2(\varepsilon)) \leq 0 \quad \text{if } t < \frac{1}{2}R^2. \tag{3.9}$$

Here V_b denotes the game value in this case and h_2 plays the same role as in (3.8).

Case C. Carol has the marker wander between P_1 and P_2. The it is clear that the game value V_c on this occasion satisfies the following.

$$I_c((x_1, 0), t) \leq 0 \quad \text{for all } t \geq 0. \tag{3.10}$$

Combining the three cases above and letting $h = \max\{h_1, h_2\}$, we conclude that

$$u_1^\varepsilon((x_1, 0), t - h(\varepsilon)) \leq \max\{I_a, I_b, I_c\} \leq 0, \quad \text{for all } t < \frac{1}{2}R^2.$$

□

The additional function h contains not only the error caused by discretization but also the time cost for the origin-oriented strategy.

Proposition 3.4. *For any $-R < x_1 < 0$ and $t > \frac{1}{2}(R^2 - (R + x_1)^2)$, the inverse game value satisfies*

$$u_2^\varepsilon((x_1, 0), t) \geq -\sqrt{2}\varepsilon.$$

Proof. We define a region

$$L := \{(x_1, x_2) \in \mathbb{R}^2 : |x_1| \leq \sqrt{2}\varepsilon\}.$$

Let Paul take α_c as described in Lemma 2.5 with center P_1 and then the marker starting from $x = (x_1, 0)$ will leave the left open disc before the time

$t_3 := R^2 - (R - x_1)^2/2 + \omega_0(\varepsilon)$. However there is possibility of getting into the right disc immediately, which we are afraid to see. To prevent it from happening, we devise a new strategy which is described in the following. Set $t_4 := \min\{t \in \mathbb{R} : y(x,t) \in L\}$. For the game after t_4, we ask Paul to implement a strategy α_p "parallel to the x_2-axis;" namely, $v = (0,1)$ for every $t \geq t_4$. Then it follows that

$$y(x,t) \notin \Omega_- \text{ or } y(x,t) \in L, \quad \text{for all } t \geq \min\{t_3, t_4\},$$

indifferent to Carol's decisions. Thus the definitions of u_0 and u_2^ε imply

$$u_2^\varepsilon((x_1,0),t) \geq -\sqrt{2}\varepsilon, \quad \text{for } t \geq \frac{1}{2}\left(R^2 - (R+x_1)^2\right) + \omega_0(\varepsilon).$$

□

Remark 3.1. We emphasize that if one thinks about the minimal exit time under the same rules, then $U_2^\varepsilon((x_1,0)) \leq \frac{1}{2}(R^2 - (R+x_1)^2) + \omega_0(\varepsilon)$ holds. There is no need to consider the maintenance strategy α_p in this case because our game gets over whenever the marker touches the boundary. This is a spectacular difference between games for parabolic and elliptic problems.

It is certain that Proposition 3.4 can be extended for more points on the plane. For instance, we easily observe that the points on the segment between the origin and P_2 own a similar estimate as well. A generalized version is written below without proofs.

Proposition 3.5. *There exist a constant $C > 0$ and a continuous function $h \in C[0,+\infty)$ with $|h(\rho)| \leq C\rho$ such that for all $x = (x_1,x_2) \in \mathbb{R}^2 \subset \{0\}$ and $t < \frac{x_1^2 R^2}{2(x_1^2+x_2^2)}$*

$$u_1^\varepsilon(x, s - h(\varepsilon)) \leq 0 \quad \textit{whenever } s \leq t. \tag{3.11}$$

Proposition 3.6. *The inverse game value satisfies*

$$u_2^\varepsilon(x,t) \geq -\sqrt{2}\varepsilon,$$

for all $x = (x_1,x_2) \in \mathbb{R}^2$ and $t \geq 0$ satisfying $x_1^2 + x_2^2 - R|x_1| < 0$ and $t > R|x_1| - \frac{1}{2}(x_1^2 + x_2^2) + \omega_0(\varepsilon)$.

Putting the above two propositions together and sending $\varepsilon \to 0$ with an application of Theorem 2.1, we get the following lemma.

Lemma 3.7. *Assume that u is the solution of* (1.2) *with initial value u_0. Then*

$$u(x,t) = 0, \tag{3.12}$$

for all $0 < t < \frac{1}{2}R^2$ and $x = (x_1,x_2) \in E_t^1$, where

$$E_t^1 := \Big\{(x_1,x_2) \in \mathbb{R}^2 \setminus \{0\} : x_1^2 + x_2^2 - R|x_1| < 0 \textit{ and } R|x_1| - \frac{1}{2}\left(x_1^2 + x_2^2\right) < t < \frac{x_1^2 R^2}{2(x_1^2 + x_2^2)}\Big\}.$$

This lemma amounts to saying that $E_t^1 \subset \Gamma_t$. Notice that for every t, E_t^1 has interior, which already enables us to complete the proof of Theorem 3.2. See Figure 3. However it is still unsatisfactory since the origin $(0,0)$ is supposed to be an interior point of the level set Γ_t. We next show it by means of our game interpretation.

Lemma 3.8. *Let u be the solution of* (1.2) *with initial value u_0. Then $E_t^2 \subset \Gamma_t$ for all $0 < t < \frac{1}{2}R^2$, where*

$$E_t^2 := \left\{ (x_1, x_2) \in \mathbb{R}^2 \setminus \{0\} : \ x_1^2 + \left(x_2 - \frac{R\sqrt{2t}}{\sqrt{R^2 - 2t}} \right)^2 > \frac{4t^2}{R^2 - 2t}, \right.$$
$$\left. x_1^2 + x_2^2 < 2t \ \text{and} \ \frac{x_1^2 R^2}{x_1^2 + x_2^2} < 2t \right\}.$$

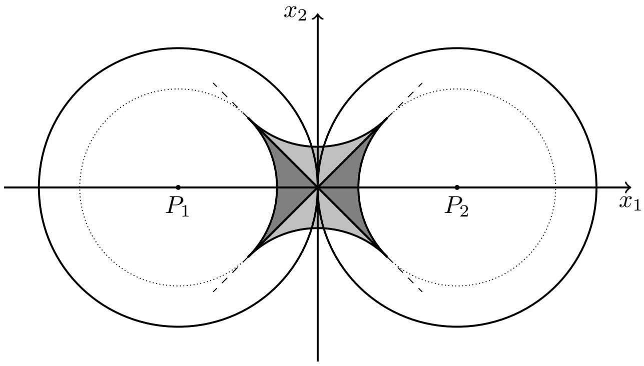

Figure 3: Dark gray region E_t^1 and light gray region E_t^2

To prove the lemma, we show the next two propositions. The first one is obvious.

Proposition 3.9. *For any $0 < t < \frac{1}{2}R^2$ and $x \in E_t^2$,* $\operatorname{limsup}^*_{\varepsilon \to 0} u_2^\varepsilon(x,t) \geq 0$.

We skip its proof, which is the same as that of Proposition 3.4. The other proposition is given as follows.

Proposition 3.10. *For any $0 < t < \frac{1}{2}R^2$ and $x \in E_t^2$,* $\operatorname{liminf}_{*\varepsilon \to 0} u_1^\varepsilon(x,t) \leq 0$.

Proof. Take the function h as in Proposition 3.5. Paul may adopt the concentric circle strategy α_c with the center P, a particular point coordinated as $(0, \frac{R\sqrt{2t}}{\sqrt{R^2-2t}})$. If Carol makes choices leading to the existence of $s \leq t - h(\varepsilon)$ such that $y(x,s;\alpha_c,\beta) \notin E_t^2$ and thus $y(x,t) \in \overline{E_t^1}$, then Paul can utilize the strategy described in Proposition 3.3 and its generalization Proposition 3.5 after this moment. We consequently obtain that $u_1^\varepsilon(x, t-h(\varepsilon)) \leq 0$.

If $y(x,s) \in E_t^2$ for all $s \leq t - h(\varepsilon)$, then by Lemma 2.5(ii) Paul's strategy yields, with no regard to Carol's response, that

$$|y(x, t-h(\varepsilon)) - P|^2 > \frac{2R^2 t}{R^2 - 2t} - 4C\varepsilon, \quad \text{for } \varepsilon > 0 \text{ sufficiently small,}$$

which implies that $|y(x, t - h(\varepsilon))|$ has to tend to 0 as $\varepsilon \to 0$. Either of the cases gives $\liminf_{\varepsilon \to 0} u_1^\varepsilon(x, t - h(\varepsilon)) \le 0$ and our conclusion follows. □

We can take advantage of this point of view to understand fattening for more geometric evolutions, whose game interpretations are given in [22], besides the two examples above. The fattening example of motion by curvature proposed in [25] for noncompact surfaces can actually be interpreted in the similar way. Another example with Neumann boundary condition, formerly studied by Barles [5], is revisited in [29]. In general, however, it is not always easy to find strategies, explicit and nearly optimal.

4 Comparison Principle for Stationary Problem

In this section, we intend to provide another view of the comparison principle for (1.4), the mean curvature type of elliptic equation with Dirichlet boundary condition in the viscosity sense.

As we have already seen, the inconsistency of optimal strategies for evolutive games results in the fattening of level sets. What happens if the same situation appears in the stationary case? How can we apply the preceding idea to an elliptic equation? We will find in a moment that it is related to the weak comparison principle of (1.4) for a general smooth domain, which was left as an open problem in [27].

Before proceeding to the discussion about weak comparison pinciple, we recall the definitions of subsolutions and supersolutions of (1.4) with the boundary condition interpreted in the viscosity sense.

Definition 4.1. An upper semicontinuous function U defined on $\overline{\Omega}$ ($U \in USC(\overline{\Omega})$) is called a viscosity subsolution of (1.4) provided that any test function $\phi \in C^2(\overline{\Omega})$ such that $U - \phi$ attains a unique maximum at $x_0 \in \overline{\Omega}$ satisfies the following:

1. If $x_0 \in \Omega$, then the viscosity inequalities hold:

$$-|\nabla \phi| \mathrm{div}(\frac{\nabla \phi}{|\nabla \phi|}) - 1 \le 0 \text{ at } x_0 \tag{4.1}$$

   when $\nabla \phi(x_0) \ne 0$ and

$$-1 - \mathrm{tr}\left((I - \xi \otimes \xi)\nabla^2 \phi(x_0)\right) \le 0 \tag{4.2}$$

   for some $\xi \in \mathbb{R}^n$ with $\|\xi\| = 1$ when $\nabla \phi(x_0) = 0$.

2. If $x_0 \in \partial\Omega$, then either $U(x_0) \le 0$ or the viscosity inequalities hold, i.e., (4.1) holds for $\nabla \phi(x_0) \ne 0$ and (4.2) holds with some $\|\xi\| = 1$ for $\nabla \phi(x_0) = 0$.

Definition 4.2. A lower semicontinuous function U defined on $\overline{\Omega}$ ($U \in LSC(\overline{\Omega})$)is called a viscosity supersolution of (1.4) provided that any test function $\phi \in C^2(\overline{\Omega})$ such that $U - \phi$ attains a unique minimum at $x_0 \in \overline{\Omega}$ satisfies the following:

1. If $x_0 \in \Omega$, then the viscosity inequalities hold:

$$-|\nabla\phi|\mathrm{div}(\frac{\nabla\phi}{|\nabla\phi|}) - 1 \geq 0 \text{ at } x_0 \tag{4.3}$$

   when $\nabla\phi(x_0) \neq 0$ and

$$-1 - \mathrm{tr}\,\big((I - \xi \otimes \xi)\nabla^2\phi(x_0)\big) \geq 0 \tag{4.4}$$

   for some $\xi \in \mathbb{R}^n$ with $\|\xi\| = 1$ when $\nabla\phi(x_0) = 0$.

2. If $x_0 \in \partial\Omega$, then either $U(x_0) \geq 0$ or the viscosity inequalities hold, i.e., (4.1) holds for $\nabla\phi(x_0) \neq 0$ and (4.2) holds with some $\|\xi\| = 1$ for $\nabla\phi(x_0) = 0$.

Definition 4.3. A bounded function U defined on $\overline{\Omega}$ is said to be the viscosity solution of (1.4) if U^* is a subsolution and U_* is a supersolution.

Remark 4.1. It is worth mentioning that by $\phi \in C^2(\overline{\Omega})$ we mean ϕ have C^2 extension in the whole space $\mathbb{R}^n$, following the choice of test functions in the User's Guide [12].

The comparison principle we are concerned with is as follows.

Weak Comparison Principle: If $W_1 \in USC(\overline{\Omega})$ and $W_2 \in LSC(\overline{\Omega})$ are respectively a subsolution and a supersolution of (1.4), then

$$(W_1)_* \leq W_2 \quad \text{and } W_1 \leq (W_2)^* \quad \text{in } \overline{\Omega}.$$

Comparison principles of this type are also named *proper comparison principle* and have important applications in other contexts [20].

Let us again consider the case of figure eight first. Of course, the open region enclosed by the figure eight shaped curve is not really a domain, but if we try to solve the equation (1.4) any way, we will lose the comparison principle.

Theorem 4.1 (Loss of comparison in a figure-eight type region)**.** *Let $\Omega = \Omega_-$ as in Section 3.2. Then the weak comparison principle fails to hold.*

Proof. Fix a small $\theta > 0$ and denote $P_\theta = (-\theta, 0) \in \mathbb{R}^2$. Then for any $t < R^2/2$, we may take a positive quantity $\rho < \theta$ small enough to get $B_\rho(P_\theta) \subset E_t^1$, where E_t^1 is given in Lemma 3.7. For every $x \in B_\rho(P_\theta) \cap \Omega$, the game strategies in the proofs of Propositions 3.3, 3.5 and 3.10 yield $U_1^\varepsilon(x) \geq t$ while those in Propositions 3.4, 3.6 and 3.9 give $U_2^\varepsilon(x) \leq \theta + \rho$. We are therefore led to $(\overline{U_1})_*(P_\theta) \geq R^2/2$ and $\underline{U_2}(P_\theta) \leq \theta(R - 2\theta)$. Our assertion hence follows immediately from Theorem 2.4. $\square$

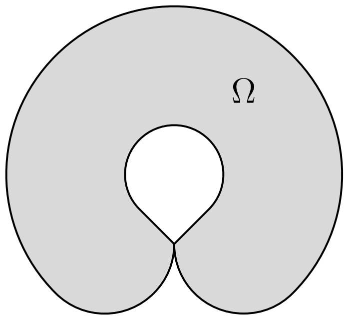

Figure 4: A domain Ω in which the weak comparison fails

Remark 4.2. The open set enclosed by the figure-eight type curve is certainly not connected. However, one can let it become a domain by slight modification without changing the essence of the proof above. The weak comparison principle does not hold for the domain Ω in Figure 4.

To generalize Theorem 4.1, we follow the conventional way of characterizing fattening, perturbing the set Ω a little bit before playing the games again. Set, for each $\delta \in \mathbb{R}$,

$$\Omega_\delta := \{x \in \mathbb{R}^n : d(x) < \delta\}.$$

Let $T^{\varepsilon,\delta}$ denote the corresponding exit times from Ω_δ. In contrast to the former sections, since we take the region perturbation into account, it is sufficient to consider the min max games of arrival time towards the boundary only. In order to handle the positive mean curvature flow, we convexify the control set of Paul; in other words, we use $\mathcal{Q}' = \{Q = (v_1, \ldots, v_{n-1}) \in \mathbb{R}^{n\times(n-1)} : |v_i| \leq 1 \text{ and } v_i \cdot v_j = 0\}$ instead of $\mathcal{Q}$. We define the value functions for every $x \in \mathbb{R}^n$

$$U_+^{\varepsilon,\delta}(x) := \min_\alpha \max_\beta T^{\varepsilon,\delta}(x; \alpha, \beta).$$

and take relaxed limits

$$V_1 := \operatorname*{limsup^*}_{\varepsilon\to 0,\ \delta\to 0+} U_+^{\varepsilon,\delta} \quad \text{and} \quad V_2 := \operatorname*{liminf_*}_{\varepsilon\to 0,\ \delta\to 0-} U_+^{\varepsilon,\delta}. \tag{4.5}$$

Proposition 4.2. *V_1 is a viscosity subsolution of (1.4).*

Proof. We first notice that $V_1(x) = 0$ for all $x \in \overline{\Omega}^c$.[1] Assume first that there are $x_0 \in \partial\Omega$ and a function $\phi \in C^2(\overline{\Omega} \cap B_r(x_0))$ with $r > 0$ such that $V_1 - \phi$ attains at x_0 a unique maximum in $\overline{\Omega} \cap B_r(x_0)$ with $V_1(x_0) = \phi(x_0) > 0$. Since ϕ can be extended to a function in $C^2(\mathbb{R}^n)$, we denote this new test function still by ϕ. It is consequently easy to see that $V_1 - \phi$ attains a maximum at x_0 in $B_r(x_0)$. Then by definition of V_1, there exist sequences $\varepsilon_k, \delta_k > 0$ and $x_k \in B_r(x_0)$ such that $\varepsilon_k \to 0$, $\delta_k \to 0$ and $x_k \to x_0$ as $k \to \infty$ and

$$U_1^{\varepsilon_k,\delta_k}(x_k) - \phi(x_k) \geq \sup_{B_r(x_0)} (U_1^{\varepsilon_k,\delta_k} - \phi) - \varepsilon_k^3. \tag{4.6}$$

Denote for brevity $U_1^k = U_1^{\varepsilon_k,\delta_k}$. By the dynamic programming principle, we have

$$U_1^k(x_k) = \min_{Q\in\mathcal{Q}'} \max_{b\in\mathcal{B}} U_1^k(x_k + \sqrt{2}\varepsilon_k Qb) + \varepsilon_k^2,$$

which, combined with (4.6), implies that

$$\min_{Q\in\mathcal{Q}'} \max_{b\in\mathcal{B}} \phi(x_k + \sqrt{2}\varepsilon_k Qb) - \phi(x_k) + \varepsilon_k^2 + \varepsilon_k^3 \geq 0. \tag{4.7}$$

If $\nabla\phi(x_0) \neq 0$, then $\nabla\phi(x_k)$ is bounded away from 0 for all k. Using Taylor expansion and an analogue of Lemma 2.3 and then sending $k \to \infty$, we obtain

$$-|\nabla\phi| \left(\mathrm{div}\Big(\frac{\nabla\phi}{|\nabla\phi|}\Big) \wedge 0\right) - 1 \leq 0 \quad \text{at } x_0,$$

which yields

$$-|\nabla\phi|\mathrm{div}\Big(\frac{\nabla\phi}{|\nabla\phi|}\Big) - 1 \leq 0 \quad \text{at } x_0.$$

(An alternative way to get this is noticing that (4.7) implies

$$\min_{Q\in\mathcal{Q}} \max_{b\in\mathcal{B}} \phi(x_k + \sqrt{2}\varepsilon_k Qb) - \phi(x_k) + \varepsilon_k^2 + \varepsilon_k^3 \geq 0.$$

and applying Lemma 2.3 directly.)

If, on the other hand, $\nabla\phi(x_0) = 0$, then $\nabla\phi(x_k) \to 0$ as $k \to \infty$. We discuss two cases. In the case that $\nabla\phi(x_k) \neq 0$ for some subsequence, Taylor expansion of (4.7) with application of (2.6) in Lemma 2.3 yields

$$-\mathrm{tr}\left(\Big(I - \frac{\nabla\phi(x_k)\otimes\nabla\phi(x_k)}{|\nabla\phi(x_k)|^2}\Big)\nabla^2\phi(x_k)\right) - 1 \leq \varepsilon_k.$$

Passing to a subsequence x_{k_j} such that $\nabla\phi(x_{k_j})/|\nabla\phi(x_{k_j})|$ converges to some $\xi \in \mathbb{R}^n$ with $|\xi| = 1$, we get (4.2). The remaining case is that $\nabla\phi(x_k) = 0$ for all k. We deduce from (4.7) that

$$\mathrm{tr}\,\big((Q_k b_k)\otimes(Q_k b_k)\nabla^2\phi(x_k)\big) + 1 \geq -\varepsilon_k,$$

where Q_k and b_k are respectively the minimizer and maximizer among the controls in $\mathcal{Q}'$ and $\mathcal{B}$. Since it is not difficult to find $\xi_k \in \mathbb{R}^n$ such that $\|\xi_k\| = 1$ and $\xi_k^\top Q_k = 0$, we let $k \to 0$, taking a subsequence if necessary, to get $\xi = \lim_{k\to\infty} \xi_k$, which implies (4.2) again.

We can use the same argument to handle the easier case $x_0 \in \Omega$. □

Proposition 4.3. *V_2 is a viscosity supersolution of (1.4).*

We skip the proof since the boundary condition is satisfied in the classical sense and thus the proof is almost the same as the subsolution part presented above.

A result more general than Theorem 4.1 is given below.

Theorem 4.4 (Loss of comparison due to fattening)**.** *Suppose that Ω is a bounded open subset of $\mathbb{R}^n$ and take*

$$u_0 = (d(x) \wedge M) \vee (-M) \in BUC(\mathbb{R}^n)$$

as a defining function of $\partial\Omega$, where $M > 0$ is a large constant and d is the signed distance of Ω, i.e.,

$$d(x) = dist(x, \overline{\Omega}) - dist(x, \Omega^c).$$

Let u be the unique solution of (1.3). If the zero level set of u fattens at some $x_0 \in \Omega$; that is, there exist $\rho > 0$ and $t_0 > 0$ such that

$$u(x, t_0) = 0 \quad \text{for all } x \in B_\rho(x_0) \subset \Omega,$$

then

$$V_1(x) \geq V_2(x) + \frac{3\rho^2}{8} \quad \text{for all } x \in B_{\rho/2}(x_0). \tag{4.8}$$

The theorem says that the occurrence of fattening in the region enclosed by an initial surface gives rise to the existence of discrepancy between the relaxed limits of minimal exit time with boundary perturbation involved. We therefore can adopt examples of fattening to disprove the general existence of comparison principle for (1.4).

Corollary 4.5. *Let u_0 be defined as in Theorem 4.4. If the zero level set of the viscosity solution of (1.3) fattens, then the weak comparison principle for (1.4) fails to hold.*

On the other hand, we get a solution of (1.4) which is continuous except at a nowhere dense subset of $\overline{\Omega}$ provided that the flow is regular.

Theorem 4.6 (Convergence of game values due to nonfattening)**.** *Under the same assumptions of Theorem 4.4 on Ω and the choice of u_0, let u be the unique solution of (1.3). If the zero level set of u satisfies (1.5) and (1.6), then*

$$(V_1)_* \leq V_2 \text{ and } V_1 \leq (V_2)^* \quad \text{in } \overline{\Omega}. \tag{4.9}$$

We are essentially able to prove, as an immediate consequence, the convergence in games with no use of comparison principles.

Corollary 4.7. *Assume that the solution u of (1.3) satisfies (1.5) and (1.6). Then there exists a possibly discontinuous solution V of (1.4) which satisfies $(V^*)_* = V_*$ and $(V_*)^* = V^*$ and the game values $U_+^{\varepsilon,\delta}$ and $U_+^{\varepsilon,-\delta}$ converge as $\varepsilon, \delta \downarrow 0$ to V, in the sense that*

$$\operatorname*{liminf_*}_{\varepsilon\to 0,\ \delta\to 0-} U_+^{\varepsilon,\delta} = V_* \text{ and } \operatorname*{limsup^*}_{\varepsilon\to 0,\ \delta\to 0+} U_+^{\varepsilon,\delta} = V^*.$$

We next prove Theorems 4.4 and 4.6.

Proof of Theorem 4.4. Since $u(x,t_0)=0$ and $u_+^\varepsilon \to u$ uniformly in $B_\rho(x_0)\times\{t_0\}$, for any $\delta>0$, there exists $\varepsilon_0>0$ such that

$$u_+^\varepsilon(x,t_0)<\delta \tag{4.10}$$

and

$$u_+^\varepsilon(x,t_0)\geq -\delta \tag{4.11}$$

for all $\varepsilon\leq\varepsilon_0$ and $x\in B_\rho(x_0)$. The first inequality (4.10) means that for every $z\in B_\rho(x_0)$ there exists a strategy β_z satisfying

$$u_0(y(z,t_0;\alpha,\beta_z))<\delta \tag{4.12}$$

no matter what strategy α Paul adopts. Note further that with this strategy put to use, Carol can also guarantee that the marker never departs from Ω_δ in the whole process, i.e.,

$$u_0(y(z,t;\alpha,\beta_z))<\delta \tag{4.13}$$

for all $t\leq t_0$ and α, for otherwise Paul can make the marker stop moving after the departure moment so that (4.12) is violated.

Now if we start games from $x\in B_{\rho/2}(x_0)$, Carol can use the strategy of concentric spheres in Lemma 2.6 to guarantee that the time needed for exit from $B_\rho(x_0)$ is greater than $\tau_0=3\rho^2/8+\omega_0(\varepsilon)$. We therefore must have $y(\tau_0)\in B_\rho(x_0)$ and then we can go on applying the strategy α_z for $z=y(\tau_0)$ to obtain, for all $0\leq t\leq t_0+\tau_0$ and $\varepsilon\leq\varepsilon_0$,

$$u_0(y(x,t))<\delta,$$

which further implies $U_+^{\varepsilon,\delta}(x)\geq t_0+\tau_0$ and consequently $V_1(x)\geq t_0+3\rho^2/8$.

On the other hand, it follows immediately from (4.11) that $U_+^{\varepsilon,-\delta}\leq t_0$ and hence $V_2\leq t_0$ in $B_\rho(x_0)$. □

Proof of Theorem 4.6. (i) Fix an arbitrary $x_0\in\overline{\Omega}$. Let $t_0=V_2(x_0)$. We first claim that $u(x_0,t_0)=0$. Indeed, there are sequences $x_k\to x_0$, $\varepsilon_k\to 0$ and $\delta_k\to 0$ fulfilling $t_k=U_+^{\varepsilon_k,-\delta_k}(x_k)\to t_0$ as $k\to\infty$, and therefore by definition we have

$$|u_+^{\varepsilon_k}(x_k,t_k)+\delta_k|\leq\sqrt{2}\varepsilon_k.$$

Sending $k\to\infty$, we get $u(x_0,t_0)=0$ by Theorem 2.2.

Since the level set of u satisfies (1.5), i.e.,

$$\{(x,t)\in\mathbb{R}^n\times[0,\infty):u(x,t)=0\}\subset\overline{\bigcup_{\delta>0}\{(x,t)\in\mathbb{R}^n\times[0,\infty):u(x,t)>\delta\}}.$$

We can take $\delta'_k\to 0$, $y_k\to x_0$ and $s_k\to t_0$ as $k\to\infty$ such that $u(y_k,s_k)>3\delta'_k$ for all $k\geq 1$. In what follows, we discuss in detail for every k and thus suppress the index k for ease of notation. We again use Theorem 2.2 to find $0<\varepsilon'\leq\delta'$ so that $u^\varepsilon(y,s)\geq 2\delta'$ for all $\varepsilon\leq\varepsilon'$.

This means that, for every $\varepsilon \le \varepsilon'$, Paul has a strategy to leave $\Omega_{2\delta'}$ from y by the time s in spite of Carol's obstruction. We follow this strategy to ensure that one is able to exit $\Omega_{\delta'}$ from all $x \in B_{\delta'}(y)$ by the time s, which is expressed as

$$U_+^{\varepsilon,\delta'}(x) \le s \quad \text{for all } x \in B_{\delta'}(y) \text{ and } \varepsilon \le \varepsilon'.$$

We then easily get

$$U_+^{\varepsilon,\delta}(x) \le s \quad \text{for all } x \in B_{\delta'}(y),\ \varepsilon \le \varepsilon' \text{ and } \delta \le \delta',$$

which implies $V_1(y) \le s$. Recalling that $y = y_k$ and $s = s_k$ actually depend on k and passing to the limit $k \to \infty$, we finally obtain $(V_1)_*(x_0) \le t_0 = V_2(x_0)$.

(ii) We prove $V_1 \le (V_2)^*$ in $\overline{\Omega}$. Fix $x_0 \in \overline{\Omega}$ and set $t_0 = V_1(x_0)$ this time. We may use the similar argument for (i) to show $u(x_0, t_0) = 0$. The condition (1.6) indicates that

$$\{(x,t) \in \mathbb{R}^n \times [0,\infty) : u(x,t) = 0\} \subset \overline{\bigcup_{\delta>0} \{x \in \mathbb{R}^n \times [0,\infty) : u(x,t) < -\delta\}}$$

and then we can again pick sequences $\delta'_k > 0$ with $\delta'_k \to 0$, $y_k \to x_0$ and $s_k \to t_0$ as $k \to \infty$ satisfying

$$u(y_k, s_k) < -3\delta'_k. \tag{4.14}$$

Again we use brief notation without the index k.

We take $0 < \varepsilon' \le \delta'$ such that $u_+^\varepsilon(y,s) < -2\delta'$ for all $\varepsilon \le \varepsilon'$, which indicates that Carol has an effective strategy to make the marker appear in $\Omega_{-2\delta'}$ at the time s. In fact, this strategy also prevents the exit from $\Omega_{-2\delta'}$ before s for the reason that once the exit occurs at any moment before s, Paul will choose $Q = 0$ for every step later on to keep himself outside $\Omega_{-2\delta'}$, which leads to a contradiction to the situation (4.14). Considering the game strategies for starting points $x \in B_{\delta'}(y)$, we are led to

$$U_+^{\varepsilon,-\delta}(x) \ge s \quad \text{for all } x \in B_{\delta'}(y),\ \varepsilon \le \varepsilon' \text{ and } \delta \le \delta'.$$

We thus deduce $V_2(y_k) \ge s_k$ and conclude by letting $k \to \infty$. □

Remark 4.3. Our result shows that the game relaxed limits satisfy the weak comparison principle as long as the interior of the zero level set in parabolic problem keeps empty. The fattening on other levels has no influence on it.

The above proof of (i) actually works for the mean curvature flow equation (1.2) too. We state it for a smooth domain in two dimensions.

Proposition 4.8. *Suppose that Ω is a smoothly bounded domain in $\mathbb{R}^2$. Then*

$$(\overline{U_i})_* \le \underline{U_i} \ \text{in } \overline{\Omega}, \tag{4.15}$$

where $\overline{U_i}$ and $\underline{U_i}$ for $i = 1, 2$ are defined in (2.11) and (2.12).

Proof. As $\partial\Omega$ is compact and smooth, by the results of Gage-Hamilton [17] and Grayson [24] on curve shortening, we obtain the regularity conditions (1.5) and (1.6) for motion by curvature (1.2) and the proof for Theorem 4.6(i) works. □

We obtain merely half of what is supposed to hold by the weak comparison principle. It is not obvious whether the other half is true. Our proof (ii) of Theorem 4.6 is not valid in this case because the regularity condition for (1.2) is not sufficient. The issue we cannot get through is that even though we have in hand strategies to guarantee that the marker is in $\Omega_{-\delta}$ ($\delta > 0$) at some time t, we are not sure for each strategy whether or not the marker has already left Ω and just come back into Ω again by the time t. We are looking for a certain strategy to keep it staying in Ω until t_0, which is indispensable in our argument to assert that the minimal exit time cannot be less than t_0. The case of signed curvature flow is simpler at this aspect.

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