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Abstract

Scattering theory between the fractional power $H_0 = \kappa^{-1}(-\Delta)^{\kappa/2}$ $(\kappa \geq 1)$ of negative Laplacian and the Hamiltonian $H = H_0 + V$ perturbed by short- and long-range potentials is developed. The existence and asymptotic completeness of wave operators are proved.

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1 Introduction

We consider a fractional power of negative Laplacian of the form

$$H_0 = \kappa^{-1}(-\Delta)^{\kappa/2} \tag{1.1}$$

defined in $\mathcal{H} = L^2(\mathbb{R}^n)$ for $\kappa \geq 1$. Here

$$\Delta = \sum_{k=1}^{n} \frac{\partial^2}{\partial x_k^2}$$

is Laplacian with domain $\mathcal{D}(\Delta) = H^2(\mathbb{R}^n)$, the Sobolev space of order two. The operator H_0 is considered a selfadjoint operator with domain

$$\mathcal{D}(H_0) = \{f |\ f \in \mathcal{H}, |\xi|^\kappa \mathcal{F} f = |\xi|^\kappa \hat{f} \in L^2(\mathbb{R}^n)\}$$

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with $\mathcal{F}$ denoting the Fourier transformation. It is obvious that H_0 is absolutely continuous on $\mathcal{H} = L^2(\mathbb{R}^n)$. The perturbed Hamiltonian we consider is

$$H = H_0 + V. \tag{1.2}$$

The potential $V = V_S(x) + V_L(x)$ is a sum of the real-valued measurable short-range potential $V_S(x)$ and the real-valued long-range potential $V_L(x)$ which satisfy the following assumptions. We use the notation: $\partial_x = (\partial/\partial_{x_1}, \cdots, \partial/\partial_{x_n})$, $\partial_x^\alpha = (\partial/\partial_{x_1})^{\alpha_1} \cdots (\partial/\partial_{x_n})^{\alpha_n}$ for a multi-index $\alpha = (\alpha_1, \cdots, \alpha_n)$ with $\alpha_j \geq 0$ being an integer, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, and $\langle y \rangle = (1 + |y|^2)^{1/2}$ for $y \in \mathbb{R}^d$ $(d \geq 1)$.

Assumption S There exist constants $C > 0$ and $0 < \delta < 1$ such that for all $x \in \mathbb{R}^n$

$$|V_S(x)| \leq C\langle x \rangle^{-1-\delta}. \tag{1.3}$$

Assumption L Let $\delta \in (0, 1)$ be the same constant as in Assumption S. For all multi-indices α there exists a constant $C_\alpha > 0$ such that for all $x \in \mathbb{R}^n$

$$|\partial_x^\alpha V_L(x)| \leq C_\alpha \langle x \rangle^{-|\alpha|-\delta}. \tag{1.4}$$

Under these assumptions, V defines a bounded operator, so that H is considered a selfadjoint operator with $\mathcal{D}(H) = \mathcal{D}(H_0)$. A concrete form of H_0 which will be useful is the expression by Fourier transform or by oscillatory integral. Namely for $f \in \mathcal{D}(H_0)$

$$\begin{aligned} H_0 f(x) &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\xi} \kappa^{-1} |\xi|^\kappa \hat{f}(\xi) d\xi \\ &= (2\pi)^{-n} \iint_{\mathbb{R}^{2n}} e^{i(x-y)\xi} \kappa^{-1} |\xi|^\kappa f(y) dy d\xi. \end{aligned}$$

We will use a convention $\widehat{d\xi} = (2\pi)^{-n} d\xi$. Then (1.5) is written as

$$H_0 f(x) = \iint_{\mathbb{R}^{2n}} e^{i(x-y)\xi} \kappa^{-1} |\xi|^\kappa f(y) dy \widehat{d\xi}.$$

We denote the symbol of H_0 by $H_0(\xi) = \kappa^{-1}|\xi|^\kappa$.

Example 1.1

i) When $\kappa = 2$ we have the usual Schrödinger scattering pair (H_0, H) with

$$H_0 = -\frac{1}{2}\Delta, \quad H = -\frac{1}{2}\Delta + V.$$

ii) When $\kappa = 1$ we have a pair (H_0, H) of relativistic Hamiltonians such that

$$H_0 = \sqrt{-\Delta}, \quad H = \sqrt{-\Delta} + V.$$

The scattering theory for the case $\kappa = 2$ is fairly well investigated, while concerning the relativistic Hamiltonians, it seems that only the work of Wei [9] has dealt with the short-range perturbations insofar as concerned with the existence and asymptotic completeness of wave operators. The immediate motivation of the present work was to find a proof of the asymptotic completeness in the case of long-range perturbations with respect to $H_0 = \sqrt{-\Delta}$. In doing so, it is noticed that the more general Hamiltonians $H_0 = \kappa^{-1}(-\Delta)^{\kappa/2}$ are possible to be handled.

Let $U_0(t) = e^{-itH_0}$ $(t \in \mathbb{R})$ be a unitary group generated by H_0.

We mention some basic concepts of micro-local analysis following [4] and [7].

We fix constants a, b with $0 < a < b < \infty$ arbitrarily and define a subspace $\mathcal{H}(a, b)$ of $\mathcal{H}$ by

$$\mathcal{H}(a, b) = E_0([a, b])\mathcal{H}, \tag{1.5}$$

where $E_0(B)$ is the spectral measure of the Hamiltonian H_0 for Borel sets $B \subset \mathbb{R}$.

Let $-1 < \theta_- < \theta_+ < 1$ and let $\rho_\pm^{\theta_-,\theta_+}(\tau) \in C^\infty(\mathbb{R})$ satisfy $0 \le \rho_\pm^{\theta_-,\theta_+}(\tau) \le 1$, $\rho_+^{\theta_-,\theta_+}(\tau) + \rho_-^{\theta_-,\theta_+}(\tau) \equiv 1$ and

$$\rho_+^{\theta_-,\theta_+}(\tau) = \begin{cases} 1 & (\tau \ge \theta_+), \\ 0 & (\tau \le \theta_-). \end{cases}$$

Further let $\chi_0(x) \in C^\infty(\mathbb{R}^n)$ with $0 \le \chi_0(x) \le 1$ satisfy

$$\chi_0(x) = \begin{cases} 1 & |x| \ge 2, \\ 0 & |x| \le 1. \end{cases} \tag{1.6}$$

For an interval $\Delta = [a, b] \subset \mathbb{R}$ $(0 < a < b < \infty)$ we let $\gamma_\Delta \in C_0^\infty((0, \infty))$ satisfy $0 \le \gamma_\Delta(\xi) \le 1$, $\gamma_\Delta(\lambda) = 1$ for $\lambda \in \Delta = [a, b]$, and $\operatorname{supp} \gamma_\Delta \subset [a/2, 2b]$.

For $x, \xi \in \mathbb{R}^n \backslash \{0\}$ we set $\omega_x = x/|x|$ and $\omega_\xi = \xi/|\xi|$. We then define a real-valued C^∞ function $p_\pm^{\theta_-,\theta_+}(x, \xi)$ by

$$p_\pm^{\theta_-,\theta_+}(x, \xi) = \rho_\pm^{\theta_-,\theta_+}(\omega_x \cdot \omega_\xi)\chi_0(x)\gamma_\Delta(H_0(\xi)). \tag{1.7}$$

We note that for $|x| \ge 2$ and $a \le H_0(\xi) = \kappa^{-1}|\xi|^\kappa \le b$

$$p_+^{\theta_-,\theta_+}(x, \xi) + p_-^{\theta_-,\theta_+}(x, \xi) = 1.$$

We denote by $\mathcal{S} = \mathcal{S}(\mathbb{R}^n)$ the totality of rapidly decreasing functions on $\mathbb{R}^n$. Then the pseudodifferential operators $P_\pm^{\theta_-,\theta_+}$ with symbol functions $p_\pm^{\theta_-,\theta_+}(x, \xi)$ are defined by

$$P_\pm^{\theta_-,\theta_+} f(x) = p_\pm^{\theta_-,\theta_+}(X, D_x) f(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\xi} p_\pm^{\theta_-,\theta_+}(x, \xi)\hat{f}(\xi) d\xi \tag{1.8}$$

for $f \in \mathcal{S}$, where $\hat{f}(\xi) = \mathcal{F}f(\xi)$ denotes the Fourier transform of $f \in \mathcal{S}$ as above. Using oscillatory integral, this is equivalently expressed as follows.

$$P_{\pm}^{\theta_-,\theta_+} f(x) = (2\pi)^{-n} \iint_{\mathbb{R}^{2n}} e^{i(x-y)\xi} p_{\pm}^{\theta_-,\theta_+}(x,\xi) f(y) dy d\xi. \tag{1.9}$$

The pseudodifferential operators $P_{\pm} = P_{\pm}^{\theta_-,\theta_+}$ with those symbols are extended to bounded linear operators from $\mathcal{H} = L^2(\mathbb{R}^n)$ into itself. We note that the adjoint operators of $P_{\pm}^{\theta_-,\theta_+}$ are given by

$$(P_{\pm}^{\theta_-,\theta_+})^* f(x) = (2\pi)^{-n} \iint_{\mathbb{R}^{2n}} e^{i(x-y)\xi} p_{\pm}^{\theta_-,\theta_+}(y,\xi) f(y) dy d\xi \tag{1.10}$$

for $f \in \mathcal{S}$. From the definition of the symbol functions we have a micro-local decomposition of the identity:

$$(P_{+}^{\theta_-,\theta_+} + P_{-}^{\theta_-,\theta_+}) f(x) = f(x) \tag{1.11}$$

for $|x| \geq 2$ and $f \in \mathcal{H}(a,b) = E_0([a,b])\mathcal{H}$.

2 Propagation estimates

We prove some estimate corresponding to Lemma 3.3 of [4], Theorem 4.2 in [5] or Theorem 5.7 of [6].

Theorem 2.1 *Let* $0 < \rho < 1$, $-1 < \theta_- - \rho < \theta_- < \theta_+ < \theta_+ + \rho < 1$. *Let* $P_+ = P_+^{\theta_+,\theta_++\rho}$ *and* $P_- = P_-^{\theta_--\rho,\theta_-}$ *be as above. Then we have for any* $s \geq 0$ *and* $\sigma \geq 0$

$$\|\langle x\rangle^{\sigma} P_- e^{-itH_0} P_+^* \langle x\rangle^{\sigma}\| \leq C_{s\sigma} \langle t\rangle^{-s} \quad (t \geq 0), \tag{2.1}$$

$$\|\langle x\rangle^{\sigma} P_+ e^{-itH_0} P_-^* \langle x\rangle^{\sigma}\| \leq C_{s\sigma} \langle t\rangle^{-s} \quad (t \leq 0), \tag{2.2}$$

where the constant $C_{s\sigma} > 0$ *is independent of* t.

Proof We prove (2.1). The inequality (2.2) is proved similarly. We note that

$$P_- e^{-itH_0} P_+^* f(x) = (2\pi)^{-n} \iint_{\mathbb{R}^{2n}} e^{i(x\xi - \kappa^{-1}t|\xi|^{\kappa} - y\xi)} p_-(x,\xi) p_+(y,\xi) f(y) dy d\xi$$

for $f \in \mathcal{S}$, where $p_-(x,\xi) = p_-^{\theta_--\rho,\theta_-}(x,\xi)$ and $p_+(y,\xi) = p_+^{\theta_+,\theta_++\rho}(y,\xi)$ with $-1 < \theta_- - \rho < \theta_- < \theta_+ < \theta_+ + \rho < 1$. For the sake of convenience, we write $p(x,\xi,y) = p_-(x,\xi)p_+(y,\xi)$. Then for $(x,\xi,y) \in \operatorname{supp} p$, we have $\omega_x \cdot \omega_\xi \leq \theta_-$, $\omega_y \cdot \omega_\xi \geq \theta_+$ and $0 < (2^{-1}\kappa a)^{1/\kappa} \leq |\xi| \leq (2\kappa b)^{1/\kappa} < \infty$. From these follows $\omega_\xi \cdot \omega_{x-y} \leq \theta_+ (< 1)$. If we define the differential operator L by

$$L = (1 + |\nabla_\xi(x\cdot\xi - \kappa^{-1}t|\xi|^{\kappa} - y\cdot\xi)|^2)^{-1}(1 - i\nabla_\xi(x\cdot\xi - \kappa^{-1}t|\xi|^{\kappa} - y\cdot\xi)\cdot\nabla_\xi),$$

we have

$$Le^{i(x\cdot\xi-\kappa^{-1}t|\xi|^\kappa-y\cdot\xi)} = e^{i(x\cdot\xi-\kappa^{-1}t|\xi|^\kappa-y\cdot\xi)}.$$

Setting $d = (2^{-1}\kappa a)^{(\kappa-1)/\kappa} \le (2\kappa b)^{(\kappa-1)/\kappa}$, we have for $(x, \xi, y) \in \text{supp}\, p$ and $t \ge 0$

$$\begin{aligned}
|\nabla_\xi(x\cdot\xi - \kappa^{-1}t|\xi|^\kappa - y\cdot\xi)|^2 &= |(x-y) - t|\xi|^{\kappa-1}\omega_\xi|^2 \\
&= |x-y|^2 - 2t|\xi|^{\kappa-1}\omega_\xi\cdot(x-y) + t^2|\xi|^{2(\kappa-1)} \\
&\ge |x-y|^2 - 2\theta_+ t|\xi|^{\kappa-1}|x-y| + t^2|\xi|^{2(\kappa-1)} \\
&\ge (1-\theta_+)(|x-y|^2 + t^2|\xi|^{2(\kappa-1)}) \\
&\ge (1-\theta_+)(|x-y|^2 + d^2t^2) \\
&\ge C(1-\theta_+)(|x| + dt + |y|)^2 \qquad (2.3)
\end{aligned}$$

for some constant $C > 0$. By integration by parts we have for an arbitrary integer $\ell \ge 0$

$$\begin{aligned}
P_- e^{-itH_0}P_+^* f &= (2\pi)^{-n}\iint_{\mathbb{R}^{2n}} e^{i(x\cdot\xi-\kappa^{-1}t|\xi|^\kappa-y\cdot\xi)}\ ({}^tL)^\ell\{p(x,\xi,y)f(y)\}dyd\xi \\
&= (2\pi)^{-n}\iint_{\mathbb{R}^{2n}} e^{i(x-y)\cdot\xi}[e^{-\kappa^{-1}t|\xi|^\kappa}({}^tL)^\ell\{p(x,\xi,y)f(y)\}]dyd\xi.
\end{aligned}$$

Here ${}^tL$ is the transposed operator of L. From (2.3) we have for any multi-indices α, β, γ

$$|\partial_x^\alpha\partial_\xi^\beta\partial_y^\gamma[e^{-\kappa^{-1}t|\xi|^\kappa}({}^tL)^\ell\{p(x,\xi,y)f(y)\}]| \le C_{\alpha\beta\gamma}\langle t\rangle^{|\beta|}\langle x\rangle^{-\ell/3}\langle t\rangle^{-\ell/3}\langle y\rangle^{-\ell/3}.$$

Taking ℓ large enough and Calderón-Vaillancourt theorem conclude the proof of (2.1). □

We next prove an estimate corresponding to Theorem 5.6 of [6].

Theorem 2.2 *Let $P_\pm = P_\pm^{\theta_-,\theta_+}$ $(-1 < \theta_- < \theta_+ < 1)$ be as above. Then we have for any $s \ge 0$ and $s \ge \sigma \ge 0$*

$$\|\langle x\rangle^{-s}e^{-itH_0}P_+^*\langle x\rangle^\sigma\| \le C_{s\sigma}\langle t\rangle^{-s+\sigma} \quad (t \ge 0), \qquad (2.4)$$

$$\|\langle x\rangle^{\sigma}P_- e^{-itH_0}\langle x\rangle^{-s}\| \le C_{s\sigma}\langle t\rangle^{-s+\sigma} \quad (t \ge 0), \qquad (2.5)$$

where the constant $C_{s\sigma} > 0$ is independent of t.

Proof We prove (2.5). The inequality (2.4) is proved similarly. It suffices to prove the case when $s \in \mathbb{N} = \{0, 1, 2, \dots\}$. For $f \in \mathcal{S}$, $P_- e^{-itH_0}f$ is written as follows.

$$P_- e^{-itH_0}f = (2\pi)^{-n}\iint_{\mathbb{R}^{2n}} e^{i(x-y)\cdot\xi}p_-(x,\xi)(\mathcal{F}^{-1}e^{-i\kappa^{-1}t|\xi|^\kappa}\mathcal{F}f)(y)dyd\xi.$$

Here $p_-(x,\xi) = p_-^{\theta_-,\theta_+}(x,\xi) = \rho_-^{\theta_-,\theta_+}(\omega_x\cdot\omega_\xi)\chi_0(x)\gamma_\Delta(H_0(\xi))$ and $\mathcal{F}^{-1}e^{-i\kappa^{-1}t|\xi|^\kappa}\mathcal{F}$ is extended to a unitary operator of $L^2(\mathbb{R}^n)$. Therefore the case $s = 0$ is obvious by the definition of $p_-(x,\xi)$ and the Calderón-Vaillancourt theorem.

We define

$$L = (1 + |\nabla_\xi(x\cdot\xi - \kappa^{-1}t|\xi|^\kappa)|^2)^{-1}(1 - i\nabla_\xi(x\cdot\xi - \kappa^{-1}t|\xi|^\kappa)\cdot\nabla_\xi).$$

Then we have

$$Le^{i(x\cdot\xi-\kappa^{-1}t|\xi|^\kappa)} = e^{i(x\cdot\xi-\kappa^{-1}t|\xi|^\kappa)}.$$

For $(x,\xi) \in \text{supp } p_-$ we have $\omega_x \cdot \omega_\xi \le \theta_+(<1)$. Therefore for $t \ge 0$

$$\begin{aligned}|\nabla_\xi(x\cdot\xi-\kappa^{-1}t|\xi|^\kappa)|^2 &= |x-t|\xi|^{\kappa-1}\omega_\xi|^2\\ &= |x|^2-2t|x||\xi|^{\kappa-1}\omega_x\cdot\omega_\xi+t^2|\xi|^{2(\kappa-1)}\\ &\ge 2^{-1}(1-\theta_+)(|x|+t|\xi|^{\kappa-1})^2.\end{aligned} \tag{2.6}$$

Integration by parts yields

$$\begin{aligned}P_-e^{-itH_0}f &= (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{i(x\cdot\xi-\kappa^{-1}t|\xi|^\kappa)}\,{}^tL\{p_-(x,\xi)\hat{f}(\xi)\}d\xi\\ &= (2\pi)^{-n}\sum_{k=1}^K \iint_{\mathbb{R}^{2n}} e^{i(x-y)\cdot\xi}q_{1tk}(x,\xi)(\mathcal{F}^{-1}e^{-i\kappa^{-1}t|\xi|^\kappa}\mathcal{F}P_k(y)f)(y)dyd\xi,\end{aligned}$$

where ${}^tL$ is the transposed operator of L, K is an integer, and $P_k(y)$ is a polynomial of y of degree 1. By virtue of (2.6) the symbol $q_{1tk}(x,\xi)$ on the right-hand side (RHS) satisfies the following estimate for arbitrary multi-indices α, β.

$$|(1+t+|x|)\{\partial_x^\alpha\partial_\xi^\beta q_{1tk}(x,\xi)\}| \le C_{1\alpha\beta},$$

where $C_{1\alpha\beta} > 0$ is a constant independent of t, x, ξ and $1 \le k \le K$.

Another application of integration by parts gives a symbol $q_{2tk}(x,\xi)$ which satisfies the following similar estimate.

$$|(1+t+|x|)^2\{\partial_x^\alpha\partial_\xi^\beta q_{2tk}(x,\xi)\}| \le C_{2\alpha\beta}.$$

Similarly we obtain for every integer $s \in \mathbb{N}$

$$|(1+t+|x|)^s\{\partial_x^\alpha\partial_\xi^\beta q_{stk}(x,\xi)\}| \le C_{s\alpha\beta}.$$

These and Calderón-Vaillancourt theorem yield

$$\|(1+t+|x|)^s P_-e^{-itH_0}(1+|x|)^{-s}\|_{L^2\to L^2} \le C_s$$

for some constant $C_s > 0$ for each $s \in \mathbb{N}$. $(1+t+|x|)^s = (1+t+|x|)^{s-\sigma}(1+t+|x|)^\sigma$ $(s \ge \sigma \ge 0)$ concludes the proof. □

We now prove an estimate corresponding to Theorem 5.5 of [6].

Theorem 2.3 *Let $q(\xi) \in C^\infty(\mathbb{R}^n)$ satisfy*

$$\begin{aligned}&\sup_{\xi\in\mathbb{R}^n}|\partial_\xi^\alpha q(\xi)| < \infty \quad (\forall\alpha),\\ &q(\xi) = 0 \quad (|\xi| < d) \quad (\exists d > 0).\end{aligned}$$

Then for any $s \ge 0$ we have

$$\|\langle x\rangle^{-s}q(D_x)e^{-itH_0}\langle x\rangle^{-s}\| \le C_s\langle t\rangle^{-s} \quad (t \in \mathbb{R}), \tag{2.7}$$

where the constant $C_s > 0$ is independent of $t \in \mathbb{R}$.

Proof We write for $f \in \mathcal{S}$

$$q(D_x)e^{-itH_0}f(x) = (2\pi)^{-n} \iint_{\mathbb{R}^{2n}} e^{i(x-y)\xi} e^{-i\kappa^{-1}t|\xi|^\kappa} q(\xi)f(y)dyd\xi. \tag{2.8}$$

Letting

$$L = i|\nabla_\xi(-\kappa^{-1}t|\xi|^\kappa)|^{-2}\nabla_\xi(-\kappa^{-1}t|\xi|^\kappa) \cdot \nabla_\xi,$$

we have

$$Le^{-i\kappa^{-1}t|\xi|^\kappa} = e^{-i\kappa^{-1}t|\xi|^\kappa}.$$

For $\xi \in \operatorname{supp} q$, we have

$$|\xi| \geq d(>0).$$

Thus integration by parts by using the operator L is possible inside the integral (2.8), and we get for any integer $\ell \geq 0$

$$\begin{aligned} q(D_x)e^{-itH_0}f(x) &= (2\pi)^{-n} \iint_{\mathbb{R}^{2n}} L^\ell(e^{-i\kappa^{-1}t|\xi|^\kappa})e^{i(x-y)\xi}q(\xi)f(y)dyd\xi \\ &= (2\pi)^{-n} \iint_{\mathbb{R}^{2n}} e^{-i\kappa^{-1}t|\xi|^\kappa}({}^tL)^\ell(e^{i(x-y)\xi}q(\xi))f(y)dyd\xi. \end{aligned} \tag{2.9}$$

From this and Calderón-Vaillancourt theorem, we obtain

$$\|\langle x\rangle^{-\ell}q(D_x)e^{-itH_0}\langle x\rangle^{-\ell}f\| \leq C_\ell\langle t\rangle^{-\ell}\|f\|,$$

which concludes the proof. □

3 Phase function

We will treat the problem of the existence and the asymptotic completeness of wave operators in the framework of two Hilbert spaces by the method we have developed in [4].

In the case of our Hamiltonian $H = H_0 + V$ in (1.2), the corresponding propagator is a unitary group $U(t) = e^{-itH}$, and the identification operator J between the two Hilbert spaces is a bounded operator from $\mathcal{H} = L^2(\mathbb{R}^n)$ into itself. The wave operators are defined as follows. Let $W_1(t)$ and $W_2(t)$ be defined by

$$W_1(t) = U(-t)JU_0(t), \quad W_2(t) = U_0(-t)J^{-1}U(t),$$

where $U_0(t) = \exp(-itH_0)$ and $U(t) = \exp(-itH)$. Then the wave operators are defined by

$$W_1^\pm f = \lim_{t\to\pm\infty} W_1(t)f$$

for $f \in \mathcal{H} = L^2(\mathbb{R}^n)$. The asymptotic completeness means that the range $\mathcal{R}(W_1^\pm)$ of $W_1^\pm$ is equal to the continuous spectral subspace $\mathcal{H}_c$ of H. This is equivalent to the existence of the limits

$$W_2^\pm f = \lim_{t\to\pm\infty} W_2(t)f$$

for $f \in \mathcal{H}_c$.

The identification operator J is defined as a Fourier integral operator as follows.

$$\begin{aligned} Jf(x) &= (2\pi)^{-n} \iint e^{i(\varphi(x,\xi)-y\xi)} f(y) dy d\xi \\ &= (2\pi)^{-n/2} \int e^{i\varphi(x,\xi)} \hat{f}(\xi) d\xi. \end{aligned} \tag{3.1}$$

Here the phase function $\varphi(x,\xi)$ is constructed as a solution of an eikonal equation

$$\kappa^{-1}|\nabla_x \varphi(x,\xi)|^\kappa + V_L(x) = \kappa^{-1}|\xi|^\kappa$$

in the forward and backward regions, in which $x \in \mathbb{R}^n$ and $\xi \in \mathbb{R}^n$ are almost parallel and anti-parallel to each other, respectively.

To construct the phase function $\varphi(x,\xi)$, we need to consider the classical orbits associated with the classical Hamiltonian with $\kappa \geq 1$:

$$H_\rho(t,x,\xi) = \kappa^{-1}|\xi|^\kappa + V_\rho(t,x). \tag{3.2}$$

Here $0 < \rho < 1$ and

$$V_\rho(t,x) = V_L(x)\chi_0(\rho x)\chi_0\left(\frac{\langle \log \langle t\rangle\rangle}{\langle t\rangle} x\right), \tag{3.3}$$

where $\chi_0(x)$ is a $C^\infty(\mathbb{R}^n)$ function defined by (1.6). Then V_ρ satisfies

$$|\partial_x^\alpha V_\rho(t,x)| \leq C_\alpha \rho^{\delta_0} \langle t\rangle^{-\ell} \langle x\rangle^{-m} \tag{3.4}$$

for $\ell, m \geq 0$ and $0 < \delta_0 < \delta$ such that $\delta_0 + \ell + m < |\alpha| + \delta$.

The corresponding classical orbit $(q,p)(t,s,y,\xi) = (q(t,s,y,\xi), p(t,s,y,\xi))$ is determined by the equation

$$\begin{cases} q(t,s) = y + \int_s^t \nabla_\xi H_\rho(\tau, q(\tau,s), p(\tau,s)) d\tau = y + \int_s^t |p(\tau,s)|^{\kappa-1} \omega_{p(\tau,s)} d\tau, \\ p(t,s) = \xi - \int_s^t \nabla_x H_\rho(\tau, q(\tau,s), p(\tau,s)) d\tau = \xi - \int_s^t \nabla_x V_\rho(\tau, q(\tau,s)) d\tau, \end{cases} \tag{3.5}$$

where $\omega_x = x/|x|$ for $x \in \mathbb{R}^n$. Letting $\delta_0, \delta_1 > 0$ be fixed as $0 < \delta_0 + \delta_1 < \delta$, we have the following estimates for $(q,p)(t,s,y,\xi)$, which are proved by solving the equation (3.5) by iteration. For $d > 0$, we use the notation

$$\mathbb{R}^n_d = \{\xi | \xi \in \mathbb{R}^n, |\xi| \geq d\}.$$

Proposition 3.1 *Let $d > 0$. Then there are constants $\rho > 0$ and $C_\ell > 0$ ($\ell = 0, 1, 2, \cdots$) such that for all $(y,\xi) \in \mathbb{R}^n \times \mathbb{R}^n_d$, $\pm t \geq \pm s \geq 0$ the solutions q, p of (3.5) exist and satisfy* *for all multi-index α:*

$$|p(s,t,y,\xi)-\xi|+|p(t,s,y,\xi)-\xi|\le C_0\rho^{\delta_0}\langle s\rangle^{-\delta_1}, \tag{3.6}$$

$$|\partial_y^\alpha[\nabla_y q(s,t,y,\xi)-I]|\le C_{|\alpha|}\rho^{\delta_0}\langle s\rangle^{-\delta_1}, \tag{3.7}$$

$$|\partial_y^\alpha[\nabla_y p(s,t,y,\xi)]|\le C_{|\alpha|}\rho^{\delta_0}\langle s\rangle^{-1-\delta_1}, \tag{3.8}$$

$$|\nabla_\xi q(t,s,y,\xi)-(t-s)I|\le C_0\rho^{\delta_0}\langle s\rangle^{-\delta_1}|t-s|, \tag{3.9}$$

$$|\nabla_\xi p(t,s,y,\xi)-I|\le C_0\rho^{\delta_0}\langle s\rangle^{-\delta_1}, \tag{3.10}$$

$$|\nabla_y q(t,s,y,\xi)-I|\le C_0\rho^{\delta_0}\langle s\rangle^{-1-\delta_1}|t-s|, \tag{3.11}$$

$$|\nabla_y p(t,s,y,\xi)|\le C_0\rho^{\delta_0}\langle s\rangle^{-1-\delta_1}, \tag{3.12}$$

$$\begin{aligned}&|\partial_\xi^\alpha[q(t,s,y,\xi)-y-(t-s)|p(t,s,y,\xi)|^{\kappa-1}\omega_{p(t,s,y,\xi)}]|\\&\qquad\le C_{|\alpha|}\rho^{\delta_0}\min(\langle t\rangle^{1-\delta_1},|t-s|\langle s\rangle^{-\delta_1}).\end{aligned} \tag{3.13}$$

Further for any α,β satisfying $|\alpha+\beta|\ge 2$, there is a constant $C_{\alpha\beta}>0$ such that

$$|\partial_y^\alpha\partial_\xi^\beta q(t,s,y,\xi)|\le C_{\alpha\beta}\rho^{\delta_0}|t-s|\langle s\rangle^{-\delta_1}, \tag{3.14}$$

$$|\partial_y^\alpha\partial_\xi^\beta p(t,s,y,\xi)|\le C_{\alpha\beta}\rho^{\delta_0}\langle s\rangle^{-\delta_1}. \tag{3.15}$$

We remark that the estimate (3.6) automatically holds by (3.4) and (3.5). Then we can take $\rho>0$ so small that $|p(s,t,y,\xi)|\ge c|\xi|>0$ is satisfied with some constant $c>0$ since we assume that $\xi\in\mathbb{R}^n_d$. So the second term on the RHS of the first equation of (3.5) is well-defined for $(y,\xi)\in\mathbb{R}^n\times\mathbb{R}^n_d$ and the equation (3.5) has the solution.

For the constant $C_0>0$ in this proposition, we take $\rho>0$ so small that $C_0\rho^{\delta_0}<1/2$ holds. Then the mapping $T_x(y)=x+y-q(s,t,y,\xi):\mathbb{R}^n\longrightarrow\mathbb{R}^n$ becomes a contraction. Therefore there is a unique fixed point $y\in\mathbb{R}^n$ for every $x\in\mathbb{R}^n$ such that $T_x(y)=y$, whence $x=q(s,t,y,\xi)$. Thus we obtain the following.

Proposition 3.2 *Take $\rho>0$ so that $C_0\rho^{\delta_0}<1/2$. Then for $\pm t\ge\pm s\ge 0$ one can construct a diffeomorphism of $\mathbb{R}^n$ for $\xi\in\mathbb{R}^n_d$*

$$x\mapsto y(s,t,x,\xi)$$

such that

$$q(s,t,y(s,t,x,\xi),\xi)=x. \tag{3.16}$$

The mapping $y(s,t,x,\xi)$ is C^∞ in $(x,\xi)\in\mathbb{R}^n\times\mathbb{R}^n_d$ and its derivatives $\partial_x^\alpha\partial_\xi^\beta y$ are C^1 in (t,s,x,ξ). Using this diffeomorphism we define for $\xi\in\mathbb{R}^n_d$

$$\eta(t,s,x,\xi)=p(s,t,y(s,t,x,\xi),\xi). \tag{3.17}$$

Then $\eta(t,s,x,\xi)$ is a C^∞ mapping from $\mathbb{R}^n\times\mathbb{R}^n_d$ into $\mathbb{R}^n$, and satisfies

$$p(t,s,x,\eta(t,s,x,\xi))=\xi. \tag{3.18}$$

They satisfy the relation

$$y(s,t,x,\xi) = q(t,s,x,\eta(t,s,x,\xi)) \tag{3.19}$$

and the estimates for any α, β

$$|\partial_x^\alpha \partial_\xi^\beta [\nabla_x y(s,t,x,\xi) - I]| \le C_{\alpha\beta}\rho^{\delta_0}\langle s\rangle^{-\delta_1}, \tag{3.20}$$
$$|\partial_x^\alpha \partial_\xi^\beta [\nabla_x \eta(t,s,x,\xi)]| \le C_{\alpha\beta}\rho^{\delta_0}\langle s\rangle^{-1-\delta_1}, \tag{3.21}$$
$$|\partial_\xi^\alpha [\eta(t,s,x,\xi) - \xi]| \le C_\alpha \rho^{\delta_0}\langle s\rangle^{-\delta_1}, \tag{3.22}$$
$$|\partial_\xi^\alpha [y(s,t,x,\xi) - x - (t-s)|\xi|^{\kappa-1}\omega_\xi]| \le C_\alpha \rho^{\delta_0}\min(\langle t\rangle^{1-\delta_1}, |t-s|\langle s\rangle^{-\delta_1}). \tag{3.23}$$

Further for any $|\alpha+\beta| \ge 2$

$$|\partial_x^\alpha \partial_\xi^\beta \eta(t,s,x,\xi)| \le C_{\alpha\beta}\rho^{\delta_0}\langle s\rangle^{-\delta_1}, \tag{3.24}$$
$$|\partial_x^\alpha \partial_\xi^\beta y(s,t,x,\xi)| \le C_{\alpha\beta}\rho^{\delta_0}\langle t-s\rangle\langle s\rangle^{-\delta_1}. \tag{3.25}$$

Here the constants $C_\alpha, C_{\alpha\beta} > 0$ *are independent of* t, s, x, ξ

The following illustration would be helpful to understand the meaning of the diffeomorphisms $y(s,t,x,\xi)$ and $\eta(t,s,x,\xi)$. Let $U(t,s)$ be the map that assigns the point $(q,p)(t,s,x,\eta)$ to the initial data (x,η). Then

$$\begin{array}{ccc} \text{time } s & & \text{time } t \\ \begin{pmatrix} x \\ \\ \eta(t,s,x,\xi) \end{pmatrix} & \overset{U(t,s)}{\longmapsto} & \begin{pmatrix} y(s,t,x,\xi) \\ \\ \xi \end{pmatrix} \end{array}$$

We now define $\phi(t,x,\xi)$ for $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n_d$ by

$$\phi(t,x,\xi) = u(t,x,\eta(t,0,x,\xi)),$$

where

$$u(t,x,\eta) = x\cdot\eta + \int_0^t (H_\rho - x\cdot\nabla_x H_\rho)(\tau, q(\tau,0,x,\eta), p(\tau,0,x,\eta))d\tau.$$

Then it is shown by a direct calculation that $\phi(t,x,\xi)$ satisfies the Hamilton-Jacobi equation

$$\begin{cases} \partial_t\phi(t,x,\xi) = \kappa^{-1}|\xi|^\kappa + V_\rho(t,\nabla_\xi\phi(t,x,\xi)), \\ \phi(0,x,\xi) = x\cdot\xi, \end{cases} \tag{3.26}$$

and the relation

$$\begin{cases} \nabla_x\phi(t,x,\xi) = \eta(t,0,x,\xi), \\ \nabla_\xi\phi(t,x,\xi) = y(0,t,x,\xi). \end{cases} \tag{3.27}$$

We define for $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n_d$

$$\phi_\pm(x,\xi) = \lim_{t\to\pm\infty}(\phi(t,x,\xi) - \phi(t,0,\xi)). \tag{3.28}$$

We will show the existence of the limits below. We set $d_1 = 2d$ and for $R > 0$, $d_2 > d_1 > 0$ and $\sigma_0 \in (-1,1)$

$$\begin{aligned}\Gamma_\pm &= \Gamma_\pm(R, d_1, d_2, \sigma_0)\\ &= \{(x,\xi) \in \mathbb{R}^{2n} | \ |x| \geq R, d_1 \leq |\xi| \leq d_2, \pm\cos(x,\xi) \geq \pm\sigma_0\},\end{aligned}$$

where we used a convention $\cos(x,\xi) = \omega_x \cdot \omega_\xi = (x\cdot\xi)/(|x||\xi|)$.

Proposition 3.3 *The limits (3.28) exist for all $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n_d$ and define C^∞ functions of $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n_d$. The limit $\phi_\pm(x,\xi)$ satisfies the eikonal equation: For any $d_2 > d_1 = 2d > 0$ and $\sigma_0 \in (-1,1)$, there is a constant $R = R_{d_1,d_2,\sigma_0} > 1$ such that for any $(x,\xi) \in \Gamma_\pm = \Gamma_\pm(R,d_1,d_2,\sigma_0)$, the following relation holds:*

$$\kappa^{-1}|\nabla_x\phi_\pm(x,\xi)|^\kappa + V_L(x) = \kappa^{-1}|\xi|^\kappa. \tag{3.29}$$

Further for any α, β we have the estimate:

$$|\partial_x^\alpha\partial_\xi^\beta(\phi_\pm(x,\xi) - x\cdot\xi)| \leq C_{\alpha\beta}|\xi|^{1-\kappa}\langle x\rangle^{1-|\alpha|-\delta}, \tag{3.30}$$

where $C_{\alpha\beta} > 0$ is independent of $(x,\xi) \in \Gamma_\pm$.

Proof: We consider ϕ_+ only. ϕ_- can be treated similarly. We first prove the existence of the limit (3.28) for $t \to +\infty$ and $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n_d$. To do so, setting

$$R(t,x,\xi) = \phi(t,x,\xi) - \phi(t,0,\xi),$$

we show the existence of the limits

$$\lim_{t\to\infty}\partial_x^\alpha\partial_\xi^\beta R(t,x,\xi) = \lim_{t\to\infty}\int_0^t \partial_x^\alpha\partial_\xi^\beta\partial_t R(\tau,x,\xi)d\tau + \partial_x^\alpha\partial_\xi^\beta(x\cdot\xi).$$

By Hamilton-Jacobi equation (3.26),

$$\begin{aligned}\partial_t R(t,x,\xi) &= \partial_t\phi(t,x,\xi) - \partial_t\phi(t,0,\xi)\\ &= V_\rho(t,\nabla_\xi\phi(t,x,\xi)) - V_\rho(t,\nabla_\xi\phi(t,0,\xi))\\ &= (\nabla_\xi\phi(t,x,\xi) - \nabla_\xi\phi(t,0,\xi))\cdot a(t,x,\xi)\\ &= (y(0,t,x,\xi) - y(0,t,0,\xi))\cdot a((t,x,\xi)\\ &= \nabla_\xi R(t,x,\xi)\cdot a(t,x,\xi),\end{aligned} \tag{3.31}$$

where

$$a(t,x,\xi) = \int_0^1 (\nabla_x V_\rho)(t, \nabla_\xi\phi(t,0,\xi) + \theta\nabla_\xi R(t,x,\xi))d\theta, \tag{3.32}$$

$$\nabla_\xi R(t,x,\xi) = x\cdot\int_0^1(\nabla_x y)(0,t,\theta x,\xi)d\theta. \tag{3.33}$$

By (3.20), we have for any α, β

$$|\partial_x^\alpha \partial_\xi^\beta \nabla_\xi R(t,x,\xi)| \le C_{\alpha\beta} \langle x \rangle. \tag{3.34}$$

By (3.23) and (3.27), for $|\beta| \ge 1$

$$|\partial_\xi^\beta \nabla_\xi \phi(t,0,\xi)| \le C_\beta |t| (1 + |\xi|^{\kappa-1-|\beta|}). \tag{3.35}$$

Noting

$$\partial_\xi^\beta (f(g(\xi)) = \sum_{k \le |\beta|} \sum_{\beta_1 + \cdots + \beta_k = \beta, |\beta_j| \ge 1} c_{k,\{\beta_j\}} \nabla_x^k f(g(\xi)) \prod_{j=1}^k \partial_\xi^{\beta_j} g(\xi), \tag{3.36}$$

we have from (3.32), (3.34) and (3.35)

$$|\partial_x^\alpha \partial_\xi^\beta a(t,x,\xi)| \le C_{\alpha\beta} \langle t \rangle^{-1-\delta/2} \langle x \rangle^{|\alpha|+|\beta|} \sum_{\substack{k \le |\beta| \\ \beta_1 + \cdots + \beta_k = \beta, |\beta_j| \ge 1}} \prod_{j=1}^k (1 + |\xi|^{\kappa-1-|\beta_j|}). \tag{3.37}$$

Thus by (3.31), (3.34) and (3.37), there exists the limit for any α, β and $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n_d$

$$\lim_{t\to\infty} \partial_x^\alpha \partial_\xi^\beta R(t,x,\xi) = \int_0^\infty \partial_x^\alpha \partial_\xi^\beta \left(\nabla_\xi R(t,x,\xi) \cdot a(t,x,\xi) \right) dt + \partial_x^\alpha \partial_\xi^\beta (x \cdot \xi).$$

In particular, $\phi_+(x,\xi) = \lim_{t\to\infty} R(t,x,\xi)$ and $\eta(\infty,0,x,\xi) = \lim_{t\to\infty} \nabla_x \phi(t,x,\xi)$ exist and are C^∞ in $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n_d$.

Next we show (3.29). By the arguments above, the following limit exist:

$$\begin{aligned} \nabla_x \phi_+(x,\xi) &= \lim_{t\to\infty} \nabla_x \phi(t,x,\xi) = \lim_{t\to\infty} \eta(t,0,x,\xi) \\ &= \lim_{t\to\infty} p(0,t,y(0,t,x,\xi),\xi). \end{aligned}$$

Thus for a sufficiently large $|x|$ (i.e. for $|\rho x| \ge 2$) we have

$$\kappa^{-1} |\nabla_x \phi_+(x,\xi)|^\kappa + V_L(x) = \kappa^{-1} \lim_{t\to\infty} |p(0,t,y(0,t,x,\xi),\xi)|^\kappa + V_\rho(0,x). \tag{3.38}$$

Set for $0 \le s \le t < \infty$

$$f_t(s,y,\xi) = \kappa^{-1} |p(s,t,y,\xi)|^\kappa + V_\rho(s, q(s,t,y,\xi)).$$

Then by (3.5) we have

$$\begin{aligned} \frac{\partial f_t}{\partial s}(s,y,\xi) &= |p(s,t,y,\xi)|^{\kappa-1} \omega_{p(s,t,y,\xi)} \cdot \partial_s p(s,t,y,\xi) \\ &\quad + (\nabla_x V_\rho)(s, q(s,t,y,\xi)) \cdot \partial_s q(s,t,y,\xi) + \frac{\partial V_\rho}{\partial t}(s, q(s,t,y,\xi)) \\ &= \frac{\partial V_\rho}{\partial t}(s, q(s,t,y,\xi)). \end{aligned}$$

On the other hand we have from (3.16), (3.17), (3.18) and (3.19)

$$\begin{aligned} q(s,t,y(0,t,x,\xi),\xi) &= q(s,t,q(t,0,x,\eta(t,0,x,\xi)),\xi) \\ &= q(s,0,x,\eta(t,0,x,\xi)), \\ p(s,t,y(0,t,x,\xi),\xi) &= p(s,t,q(t,0,x,\eta(t,0,x,\xi)),\xi) \\ &= p(s,0,x,\eta(t,0,x,\xi)). \end{aligned}$$

Now using Proposition 3.1, we have for $\cos(x,\xi) \geq \sigma_0$

$$\begin{aligned} &|q(s,t,y(0,t,x,\xi),\xi)| = |q(s,0,x,\eta(t,0,x,\xi))| \\ &\quad \geq |x + s|p(s,0,x,\eta(t,0,x,\xi))|^{\kappa-1}\omega_{p(s,0,x,\eta(t,0,x,\xi))}| - C_0\rho^{\delta_0}\langle s\rangle^{1-\delta_1} \\ &\quad = |x + s|p(s,t,y(0,t,x,\xi),\xi)|^{\kappa-1}\omega_{p(s,t,y(0,t,x,\xi),\xi)}| - C_0\rho^{\delta_0}\langle s\rangle^{1-\delta_1} \\ &\quad \geq c(|x| + s|\xi|^{\kappa-1}) - C_0\rho^{\delta_0}\langle s\rangle^{1-\delta_1} - C_0\rho^{\delta_0}\langle s\rangle^{1-\delta_1}, \end{aligned}$$

where $c > 0$ is a constant independent of s, t, x, ξ. By $(x,\xi) \in \Gamma_+(R, d_1, d_2, \sigma_0)$, we have $d_2 \geq |\xi| \geq d_1$, and from the definition (3.3) of $V_\rho(t,x)$

$$\operatorname{supp} \frac{\partial V_\rho}{\partial t}(s,x) \subset \{x | 1 \leq \langle \log\langle s\rangle\rangle |x| / \langle s\rangle \leq 2\}.$$

Thus there is a constant $S = S_{d_1,d_2,\sigma_0} > 1$ independent of t such that for any $s \in [S,t]$

$$\frac{\partial f_t}{\partial s}(s, y(0,t,x,\xi),\xi) = 0.$$

For $s \in [0,S]$, taking $R = R_S > 1$ large enough, we have for $|x| \geq R$ and $\cos(x,\xi) \geq \sigma_0$

$$\frac{\partial f_t}{\partial s}(s, y(0,t,x,\xi),\xi) = 0.$$

Therefore we have shown that for $(x,\xi) \in \Gamma_+(R, d_1, d_2, \sigma_0)$

$$f_t(s, y(0,t,x,\xi),\xi) = \text{constant for } 0 \leq s \leq t < \infty.$$

In particular we have

$$f_t(0, y(0,t,x,\xi),\xi) = f_t(t, y(0,t,x,\xi),\xi),$$

which means

$$\kappa^{-1}|p(0,t,y(0,t,x,\xi),\xi)|^\kappa + V_\rho(0,x) = \kappa^{-1}|\xi|^\kappa + V_\rho(t, y(0,t,x,\xi)).$$

Since $V_\rho(t,y) \to 0$ uniformly in $y \in \mathbb{R}^n$ when $t \to \infty$ by (3.4), we have from this and (3.38)

$$\kappa^{-1}|\nabla_x\phi_+(x,\xi)|^\kappa + V_L(x) = \kappa^{-1}|\xi|^\kappa \quad \text{for} \quad (x,\xi) \in \Gamma_+(R, d_1, d_2, \sigma_0),$$

if $R > 1$ is sufficiently large.

We finally prove the estimates (3.30). We first consider the derivatives with respect to ξ:

$$\partial_\xi^\beta(\phi_+(x,\xi)-x\cdot\xi)=\int_0^\infty \partial_\xi^\beta\partial_t R(t,x,\xi)dt, \tag{3.39}$$

where $R(t,x,\xi)=\phi(t,x,\xi)-\phi(t,0,\xi)$ as above. Set

$$\gamma(t,x,\xi)=y(0,t,x,\xi)-(x+t|\xi|^{\kappa-1}\omega_\xi)$$

for $(x,\xi)\in\Gamma_+(R,d_1,d_2,\sigma_0)$. Then by (3.23) we have for $\theta\in[0,1]$

$$\begin{aligned}|\nabla_\xi\phi(t,0,\xi)+\theta\nabla_\xi R(t,x,\xi)| &= |y(0,t,0,\xi)+\theta(y(0,t,x,\xi)-y(0,t,0,\xi))|\\ &= |t|\xi|^{\kappa-1}\omega_\xi+\gamma(t,0,\xi)+\theta(x+\gamma(t,x,\xi)-\gamma(t,0,\xi))|\\ &= |\theta x+t|\xi|^{\kappa-1}\omega_\xi+(1-\theta)\gamma(t,0,\xi)+\theta\gamma(t,x,\xi)|\\ &\geq c_0(\theta|x|+t|\xi|^{\kappa-1})-c_1\rho^{\delta_0}\min(\langle t\rangle^{1-\delta_1},|t|) \qquad (3.40)\end{aligned}$$

for some constants $c_0,c_1>0$ independent of x,ξ,θ and $t\geq 0$. Thus there are constants $\rho\in(0,d)$ and $T=T_{d_1,d_2,\sigma_0}>0$ such that for all $t\geq T$ and $(x,\xi)\in\Gamma_+(R,d_1,d_2,\sigma_0)$

$$\langle\nabla_\xi\phi(t,0,\xi)+\theta\nabla_\xi R(t,x,\xi)\rangle^{-1}\leq C\langle\theta|x|+t|\xi|^{\kappa-1}\rangle^{-1}.$$

Therefore $a(t,x,\xi)$ defined by (3.32) satisfies by (3.34)-(3.36)

$$\begin{aligned}|\partial_\xi^\beta a(t,x,\xi)|\leq C_\beta\int_0^1 \sum_{\substack{k\leq|\beta|\\ \beta_1+\cdots+\beta_k=\beta,|\beta_j|\geq 1}} &\langle\theta|x|+t|\xi|^{\kappa-1}\rangle^{-1-\delta-k}\times\\ &\times\prod_{j=1}^k\langle\theta|x|+t(1+|\xi|^{\kappa-1-|\beta_j|})\rangle d\theta. \qquad (3.41)\end{aligned}$$

Using (3.40), we see that (3.41) holds also for $t\in[0,T]$ if we take $\rho>0$ small enough. Therefore for all $(x,\xi)\in\Gamma_+(R,d_1,d_2,\sigma_0)$ we have from (3.31), (3.34), (3.39) and (3.41)

$$\begin{aligned}&|\partial_\xi^\beta(\phi_+(x,\xi)-x\cdot\xi)|\\ &\leq C_{T,\beta,d_1,d_2}\langle x\rangle\sum_{\substack{k\leq|\beta|\\ \ell_1+\cdots+\ell_k=|\beta|,\ell_j\geq 1}}\int_0^\infty\int_0^1\langle\theta|x|+t|\xi|^{\kappa-1}\rangle^{-1-\delta-k}\prod_{j=1}^k\langle\theta|x|+t(1+|\xi|^{\kappa-1-\ell_j})\rangle d\theta dt\\ &\leq C_{T,\beta,d_1,d_2}\langle x\rangle|\xi|^{1-\kappa}\int_0^1\int_0^\infty\langle\theta|x|+\tau\rangle^{-1-\delta}d\tau d\theta\\ &\leq C_{T,\beta,d_1,d_2}\langle x\rangle^{1-\delta}|\xi|^{1-\kappa}.\end{aligned}$$

We next consider

$$\begin{aligned}
\nabla_x\phi_+(x,\xi)-\xi &= \lim_{t\to\infty}(\nabla_x\phi(t,x,\xi)-\xi)\\
&= \lim_{t\to\infty}(p(0,t,y(0,t,x,\xi),\xi)-\xi)\\
&= \lim_{t\to\infty}\int_0^t(\nabla_x V_\rho)\,(\tau,q(\tau,t,y(0,t,x,\xi),\xi))\,d\tau\\
&= \lim_{t\to\infty}\int_0^t(\nabla_x V_\rho)\,(\tau,q(\tau,0,x,\eta(t,0,x,\xi)))\,d\tau\\
&= \int_0^\infty(\nabla_x V_\rho)\,(\tau,q(\tau,0,x,\eta(\infty,0,x,\xi)))\,d\tau.
\end{aligned}$$

By (3.6) and (3.13) of Proposition 3.1

$$\begin{aligned}
|q(\tau,0,x,\eta(\infty,0,x,\xi))| &\ge |x+\tau|p(\tau,0,x,\eta(\infty,0,x,\xi))|^{\kappa-1}\omega_{p(\tau,0,x,\eta(\infty,0,x,\xi))}|-C_0\rho^{\delta_0}\langle\tau\rangle^{1-\delta_1}\\
&\ge |x+\tau|p(\tau,\infty,y(0,\infty,x,\xi),\xi)|^{\kappa-1}\omega_{p(\tau,\infty,y(0,\infty,x,\xi),\xi)}|-C_0\rho^{\delta_0}\langle\tau\rangle^{1-\delta_1}\\
&\ge c_0(|x|+\tau|\xi|^{\kappa-1})-C_0\rho^{\delta_0}\langle\tau\rangle^{1-\delta_1}-C_0\rho^{\delta_0}\langle\tau\rangle^{1-\delta_1}
\end{aligned}$$

for some constant $c_1>0$ and for all $(x,\xi)\in\Gamma_+(R,d_1,d_2,\sigma_0)$. Thus taking $\rho>0$ sufficiently small and $R=R_{d_1,d_2,\sigma_0,\rho}>1$ sufficiently large, we have for $(x,\xi)\in\Gamma_+(R,d_1,d_2,\sigma_0)$

$$|q(\tau,0,x,\eta(\infty,0,x,\xi))|\ge c_0(|x|+\tau|\xi|^{\kappa-1})$$

for some constant $c_0>0$. Therefore we obtain

$$|\nabla_x\phi_+(x,\xi)-\xi|\le C\int_0^\infty\langle|x|+\tau|\xi|^{\kappa-1}\rangle^{-1-\delta}d\tau\le C|\xi|^{1-\kappa}\langle x\rangle^{-\delta}.$$

For higher derivatives, the proof is similar. For example let us consider

$$\begin{aligned}
\partial_\xi\partial_x\phi_+(x,\xi)-I &= \int_0^\infty\partial_\xi\{(\nabla_x V_\rho)\,(\tau,q(\tau,0,x,\eta(\infty,0,x,\xi)))\}d\tau\\
&= \int_0^\infty(\nabla_x\nabla_x V_\rho)\,(\tau,q(\tau,0,x,\eta(\infty,0,x,\xi)))\,\nabla_\xi q\cdot\nabla_\xi\eta d\tau,
\end{aligned}$$

where we abbreviated $q=q(\tau,0,x,\eta(\infty,0,x,\xi))$ and $\eta=\eta(\infty,0,x,\xi)$. The RHS is bounded by a constant times

$$\int_0^\infty\langle|x|+\tau|\xi|^{\kappa-1}\rangle^{-2-\delta}\langle\tau|\xi|^{\kappa-2}\rangle d\tau\le c_{d_1,d_2}|\xi|^{1-\kappa}\langle x\rangle^{-\delta}$$

for $(x,\xi)\in\Gamma_+(R,d_1,d_2,\sigma_0)$ by (3.9) and (3.22) of Propositions 3.1 and 3.2. Other estimates are proved similarly by using (3.9), (3.14), (3.22) and (3.24). □

Now let $-1<\sigma_-<\sigma_+<1$ and take two functions $\psi_\pm(\sigma)\in C^\infty([-1,1])$ such that

$$\begin{aligned}
&0\le\psi_\pm(\sigma)\le 1,\\
&\psi_+(\sigma)=\begin{cases}1, & \sigma_+\le\sigma\le 1\\ 0, & -1\le\sigma\le\sigma_-\end{cases},\\
&\psi_-(\sigma)=1-\psi_+(\sigma)=\begin{cases}0, & \sigma_+\le\sigma\le 1\\ 1, & -1\le\sigma\le\sigma_-\end{cases},
\end{aligned}$$

and set

$$\chi_\pm(x,\xi) = \psi_\pm(\cos(x,\xi)).$$

We then define the phase function $\varphi(x,\xi)$ by

$$\varphi(x,\xi) = \{(\phi_+(x,\xi) - x\cdot\xi)\chi_+(x,\xi) + (\phi_-(x,\xi) - x\cdot\xi)\chi_-(x,\xi)\}\,\chi_0(2\xi/d_1)\chi_0(2x/R) + x\cdot\xi, \tag{3.42}$$

where $\chi_0(x)$ is the function defined by (1.6). The function $\varphi(x,\xi)$ is a C^∞ function of $(x,\xi)\in\mathbb{R}^{2n}$ by $d_1 = 2d$.

Noting that $\chi_+(x,\xi) + \chi_-(x,\xi) \equiv 1$ for $x \neq 0, \xi \neq 0$, we have proved the following theorem.

Theorem 3.4 *Let the notations be as above. Then for any $d_2 > d_1 > 0$ and $-1 < \sigma_- < \sigma_+ < 1$, there is $R = R_{d_1,d_2,\sigma_\pm} > 1$ such that the following holds:*

i) For $d_2 \ge |\xi| \ge d_1$, $|x| \ge R$ and $\cos(x,\xi) \ge \sigma_+$ or $\cos(x,\xi) \le \sigma_-$

$$\kappa^{-1}|\nabla_x\varphi(x,\xi)|^\kappa + V_L(x) = \kappa^{-1}|\xi|^\kappa. \tag{3.43}$$

ii) For any multi-indices α,β there is a constant $C_{\alpha\beta} > 0$ such that for $d_2 \ge |\xi| \ge d_1$ and $x\in\mathbb{R}^n$

$$|\partial_x^\alpha\partial_\xi^\beta(\varphi(x,\xi) - x\cdot\xi)| \le C_{\alpha\beta}\langle x\rangle^{1-\delta-|\alpha|}\langle\xi\rangle^{1-\kappa}. \tag{3.44}$$

In particular for $|\alpha| \neq 0$, by virtue of (3.42) we have for $\delta_0,\delta_1 \ge 0$ with $\delta_0+\delta_1 = \delta$

$$|\partial_x^\alpha\partial_\xi^\beta(\varphi(x,\xi) - x\cdot\xi)| \le C_{\alpha\beta}R^{-\delta_0}\langle x\rangle^{1-\delta_1-|\alpha|}\langle\xi\rangle^{1-\kappa}. \tag{3.45}$$

iii) Set for $f\in\mathcal{S}$

$$Tf(x) = (HJ - JH_0)f(x). \tag{3.46}$$

Then we have

$$Tf(x) = \iint e^{i(\varphi(x,\xi)-y\xi)}\{a(x,\xi) + V_S(x)\}f(y)dyd\widehat{\xi}. \tag{3.47}$$

Here

$$a(x,\xi) = \kappa^{-1}|\nabla_x\varphi(x,\xi)|^\kappa + V_L(x) - \kappa^{-1}|\xi|^\kappa + r(x,\xi), \tag{3.48}$$

where

$$r(x,\xi) = -i\iint e^{i(x-y)\eta}\nabla_y\cdot\left(\int_0^1 |\widetilde{\nabla}_x\varphi(x,\xi,y)+\theta\eta|^{\kappa-2}(\widetilde{\nabla}_x\varphi(x,\xi,y)+\theta\eta)d\theta\right)dyd\widehat{\eta}, \tag{3.49}$$

and

$$\widetilde{\nabla}_x\varphi(x,\xi,y) = \int_0^1 \nabla_x\varphi(y+\theta(x-y),\xi)d\theta.$$

The symbol $a(x,\xi)$ satisfies for $d_2 \ge |\xi| \ge d_1$, $|x| \ge R$ and any α, β

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \le \begin{cases} C_{\alpha\beta}\langle x\rangle^{-1-\delta-|\alpha|}\langle\xi\rangle^{1-\kappa}, & \cos(x,\xi) \in [-1,\sigma_-] \cup [\sigma_+,1], \\ C_{\alpha\beta}\langle x\rangle^{-\delta-|\alpha|}, & \cos(x,\xi) \in [\sigma_-,\sigma_+]. \end{cases} \tag{3.50}$$

We remark that the factor $\langle\xi\rangle^{1-\kappa}$ in the bounds above is effective just in each region $d_1 \le |\xi| \le d_2$ and the constant $C_{\alpha\beta}$ depends on d_1 and d_2.

We recall the definition of J. It is defined for $f \in \mathcal{S}$

$$Jf(x) = \iint e^{i(\varphi(x,\xi)-y\xi)} f(y)dyd\widehat{\xi}.$$

Since the regions $\Gamma_\pm(R,d_1,d_2,\sigma_0)$ of definition for the phase function $\varphi(x,\xi)$ are enlarged if we wait enough until late or early time t near $+\infty$ or $-\infty$, they in total cover the whole region $\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$. Thus J is regarded to have been defined on the whole Hilbert space $\mathcal{H} = L^2(\mathbb{R}^n)$. When it is thought to be constructed in such a way, this J is known (Theorem 3.3 in [5]) to have a bounded inverse J^{-1}. Thus we can define $W_1(t)$ and $W_2(t)$ as follows:

$$W_1(t) = U(-t)JU_0(t), \quad W_2(t) = U_0(-t)J^{-1}U(t),$$

where $U_0(t) = e^{-itH_0}$ and $U(t) = e^{-itH}$.

We assume that $f \in \mathcal{H}_c(a,b) = E_H([a,b])\mathcal{H}_c$ with $0 < a < b < \infty$, where $E_H(B)$ denotes the spectral measure for the Hamiltonian H and $\mathcal{H}_c$ is the continuous spectral subspace for H. As our propagators $U(t)$ and $U_0(t)$ are unitary operators, we can consider the two limits

$$W_2^\pm = \text{s-}\lim_{t\to\pm\infty} W_2(t).$$

We consider the asymptotic behavior of $U(t)f$ for $f \in \mathcal{H}_c(a,b)$. Let the pseudodifferential operators $P_\pm = P_\pm^{\theta_-,\theta_+}$ $(-1 < \theta_- < \theta_+ < 1)$ be defined as in (1.8) or (1.9) with the same constants $0 < a < b < \infty$ as above. We take $d_1 = (2^{-1}\kappa a)^{1/\kappa}$ and $d_2 = (2\kappa b)^{1/\kappa}$ in Theorem 3.4 and define the phase function φ and the identification operator J accordingly. Using those, we calculate as follows for $t \in \mathbb{R}$.

$$\begin{aligned} U(t)P_\pm^* &= (U(t) - JU_0(t)J^{-1})P_\pm^* + JU_0(t)J^{-1}P_\pm^* \\ &= U(t)(I - U(-t)JU_0(t)J^{-1})P_\pm^* + JU_0(t)J^{-1}P_\pm^* \\ &= -U(t)\int_0^t \frac{d}{d\sigma}\left(U(-\sigma)JU_0(\sigma)J^{-1}\right)P_\pm^* d\sigma + JU_0(t)J^{-1}P_\pm^* \\ &= -iU(t)K_\pm(t) + JU_0(t)J^{-1}P_\pm^*, \end{aligned} \tag{3.51}$$

where

$$K_\pm(t) = \int_0^t U(-\sigma)(HJ - JH_0)U_0(\sigma)J^{-1}P_\pm^* d\sigma. \tag{3.52}$$

We note that we can write for $f \in \mathcal{S}$ with using the function $a(x,\xi)$ in Theorem 3.4-iii) (3.48)

$$(HJ - JH_0)f(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i\varphi(x,\xi)}\{a(x,\xi) + V_S(x)\}\hat{f}(\xi)d\xi. \tag{3.53}$$

Therefore, if we take $-1 < \theta_- = \sigma_+ + \rho < \theta_+ < 1$ for some $\rho > 0$ and the constant $\sigma_+ \in (-1,1)$ of Theorem 3.4, $K_+(t)$ defines a compact operator on $\mathcal{H}$ and converges to a compact operator K_+ of $\mathcal{H} = L^2(\mathbb{R}^n)$ in operator norm when $t \to +\infty$ by Theorems 2.1 – 2.3, Theorem 3.4 and the factor $\gamma_\Delta(H_0(\xi))$ in the symbol $p_\pm(x,\xi)$ in (1.7) with some calculation of Fourier integral and pseudodifferential operators (section 6.3 [6]). Similarly if we take $-1 < \theta_- < \theta_+ = \sigma_- - \rho < 1$ for some $\rho > 0$ and the constant $\sigma_- \in (-1,1)$ of Theorem 3.4, $K_-(t)$ converges to a compact operator K_- of $\mathcal{H} = L^2(\mathbb{R}^n)$ in operator norm when $t \to -\infty$ by the same reason. Therefore we have proved that for $t \in \mathbb{R}$

$$J^{-1}U(t)P_\pm^* = -iJ^{-1}U(t)K_\pm(t) + U_0(t)J^{-1}P_\pm^*, \tag{3.54}$$

where the first term is a compact operator on $\mathcal{H}$. This means that the operator $J^{-1}U(t)P_\pm^*$ behaves like $U_0(t)J^{-1}P_\pm^*$ except for a compact operator $-iJ^{-1}U(t)K_\pm(t)$.

Summarizing the arguments up to here we have proved the following theorem.

Theorem 3.5 *When $t \geq 0$, let $-1 < \theta_- = \sigma_+ + \rho < \theta_+ < 1$ for some $\rho > 0$ and the constant $\sigma_+ \in (-1,1)$ of Theorem 3.4, and when $t \leq 0$ let $-1 < \theta_- < \theta_+ = \sigma_- - \rho < 1$ for some $\rho > 0$ and the constant $\sigma_- \in (-1,1)$ of Theorem 3.4, and define $P_\pm = P_\pm^{\theta_-,\theta_+}$ by (1.8) or (1.9) with $0 < a = 2\kappa^{-1}(d_1)^\kappa < b = 2^{-1}\kappa^{-1}(d_2)^\kappa < \infty$. Then for $t \in \mathbb{R}$*

$$J^{-1}U(t)P_\pm^* = -iJ^{-1}U(t)K_\pm(t) + U_0(t)J^{-1}P_\pm^*, \tag{3.55}$$

where $K_\pm(t)$ in the first term on the right hand side is a compact operator on $\mathcal{H}$ and converges to a compact operator $K_\pm$ on $\mathcal{H}$ in operator norm as $t \to \pm\infty$.

4 Asymptotic behavior of scattering state

Finally we will see an asymptotic behavior of $U(t)f$ when $f \in \mathcal{H} = L^2(\mathbb{R}^n)$ is a scattering state, i.e. when it belongs to $\mathcal{H}_c(a,b)$ for $0 < a < b < \infty$. As the proof of Lemma 3.4 in [6] holds for the case of our Hamiltonian H in (1.2), we have that Theorem 3.2 of [6] holds for our case. Namely we have the following.

Lemma 4.1 *Let $\mathcal{H}$ be a separable Hilbert space, and let H be a selfadjoint operator in $\mathcal{H}$. We assume that $B(t)$ ($t \in \mathbb{R}$) is a continuous family of uniformly bounded operators with respect to t in uniform operator topology, and that K is a compact operator in $\mathcal{H}$. Let P_c be the orthogonal projection onto the continuous spectral subspace $\mathcal{H}_c(\subset \mathcal{H})$ of H. Then we have*

$$\lim_{T\to\pm\infty} \left\| \frac{1}{T}\int_0^T B(t)Ke^{-itH}P_c dt \right\| = 0. \tag{4.1}$$

Proof This is essentially Proposition 5.1 of Enss [3]. For the sake of completeness we repeat the proof. As the Hilbert space $\mathcal{H}$ is separable, the compact operator KP_c is approximated by operators of finite rank in operator norm. Thus we can assume that KP_c is a one dimensional operator $KP_cf = (f, \phi)\psi$ for $\phi \in \mathcal{H}_c$ and $\psi \in \mathcal{H}$. (Note that $Ke^{-itH}P_c = KP_ce^{-itH}$.) We compute with writing KP_c just as K

$$\begin{aligned}
\left\| \frac{1}{T} \int_0^T B(t)Ke^{-itH}P_c dt \right\|^2 &= \left\| \frac{1}{T} \int_0^T B(t)Ke^{-itH} dt \right\|^2 \\
&= \left\| \frac{1}{T} \int_0^T e^{itH}K^*B(t)^* dt \right\|^2 \\
&= \sup_{\|f\|=1} \left\| \frac{1}{T} \int_0^T e^{itH}K^*B(t)^* f dt \right\|^2 \\
&= \sup_{\|f\|=1} \frac{1}{T^2} \int_0^T \int_0^T (B(t)^*f, \psi)(\psi, B(s)^*f)(e^{-i(s-t)H}\phi, \phi) ds dt \\
&\leq C \frac{1}{T^2} \int_0^T \int_0^T |(e^{-i(s-t)H}\phi, \phi)| ds dt \\
&\leq C \frac{1}{T} \int_{-T}^T |(e^{-itH}\phi, \phi)| dt,
\end{aligned}$$

where $C = \|\psi\|^2 \sup_{t\in\mathbb{R}} \|B(t)\|^2$. By Schwarz inequality, the RHS is bounded by

$$\sqrt{2}C \left(\frac{1}{T} \int_{-T}^T |(e^{-itH}\phi, \phi)|^2 dt \right)^{\frac{1}{2}}.$$

Noting that the function $\mu(\lambda) = (E_H(\lambda)\phi, \phi)$ is monotonically increasing and bounded, we calculate the formula inside the parentheses as follows.

$$\frac{1}{T} \int_{-T}^T \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-i(\lambda-\lambda')t} d\mu(\lambda) d\mu(\lambda') dt = 2 \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{\sin\{(\lambda-\lambda')T\}}{(\lambda-\lambda')T} d\mu(\lambda) d\mu(\lambda').$$

Dividing the integration region $\mathbb{R}_\lambda \times \mathbb{R}_{\lambda'}$ into $|\lambda - \lambda'| \leq \epsilon$ and the other, we obtain a bound:

$$2 \int_{|\lambda-\lambda'|\leq\epsilon} d\mu(\lambda) d\mu(\lambda') + \frac{2}{\epsilon T}.$$

Utilizing the fact that the measure generated by $\mu(\lambda)$ is bounded continuous by $\phi \in \mathcal{H}_c$, we can show that the first term is arbitrarily small if we take $\epsilon > 0$ small enough. The second term goes to 0 when letting $T \to \infty$. □

From this follows the following.

Lemma 4.2 *For any $f \in \mathcal{H}_c(a,b)$ $(0 < a < b < \infty)$ with $\langle x\rangle^2 f \in \mathcal{H} = L^2(R^n)$, there exists a sequence $t_k \to \pm\infty$ as $k \to \pm\infty$ such that for any $\phi \in C_0^\infty(\mathbb{R})$ and $R > 0$*

$$\|\chi_{\{x\in\mathbb{R}^n||x|<R\}}U(t_k)f\| \to 0, \tag{4.2}$$

$$\|(\phi(H) - \phi(H_0))U(t_k)f\| \to 0, \tag{4.3}$$

$$\left\|\left(\frac{x}{t_k} - |D_x|^{\kappa-2}D_x\right)U(t_k)f\right\| \to 0 \tag{4.4}$$

as $k \to \pm\infty$, where $D = D_x = -i\partial_x$ and χ_B denotes the characteristic function of a set B.

Proof In fact the relation (4.2) is a consequence of Lemma 4.1 and the fact that $\chi_{\{x\in\mathbb{R}^n||x|<R\}}$ $E_H([a,b])$ is a compact operator. The relation (4.3) follows from (4.2). To prove the relation (4.4) we compute

$$\begin{aligned}
&\left\|\left(\frac{x}{t} - |D|^{\kappa-2}D\right)e^{-itH}f\right\|^2 \\
&= \left(f, e^{itH}\left(\frac{x}{t} - |D|^{\kappa-2}D\right)^2 e^{-itH}f\right) \\
&= \frac{1}{t^2}(f,(e^{itH}x^2e^{-itH} - x^2)f) - \frac{2}{t}(f, e^{itH}A_\kappa e^{-itH}f) + (f, e^{itH}|D|^{2(\kappa-1)}e^{-itH}f) + \frac{1}{t^2}(f, x^2 f),
\end{aligned} \tag{4.5}$$

where we set

$$A_\kappa = \frac{1}{2}(x\cdot D|D|^{\kappa-2} + |D|^{\kappa-2}D\cdot x).$$

By direct calculation we have

$$i[H_0, x^2] = i(H_0x^2 - x^2H_0) = 2A_\kappa.$$

This gives that the first term on the RHS of (4.5) is equal to

$$\frac{1}{t^2}\int_0^t (f, e^{isH}i[H_0, x^2]e^{-isH}f)ds = \frac{2}{t^2}\int_0^t (f, e^{isH}A_\kappa e^{-isH}f)ds.$$

Therefore the sum of the first term and the second term on the RHS of (4.5) is equal to

$$\begin{aligned}
D(t) &= \frac{2}{t^2}\left(\int_0^t (f, e^{isH}A_\kappa e^{-isH}f)ds - t(f, e^{itH}A_\kappa e^{-itH}f)\right) \\
&= \frac{1}{t^2}\int_0^t \frac{d(\tau^2 D)}{d\tau}(\tau)d\tau \\
&= -\frac{2}{t^2}\int_0^t s(f, e^{isH}i[H, A_\kappa]e^{-isH}f)ds. \\
&= -\frac{2}{t^2}\int_0^t s(f, e^{isH}i[H_0, A_\kappa]e^{-isH}f)ds - \frac{2}{t^2}\int_0^t s(f, e^{isH}i[V, A_\kappa]e^{-isH}f)ds.
\end{aligned} \tag{4.6}$$

We need the following lemma.

Lemma 4.3 *We have*

$$i[H_0, A_\kappa] = |D|^{2(\kappa-1)}.$$

Proof Using oscillatory integrals, we compute for $f \in \mathcal{S}$.

$$\begin{aligned} i[H_0, A_\kappa]f(x) = &\frac{i}{2}\kappa^{-1} \iint e^{i(x-y)\xi}|\xi|^\kappa \iint e^{i(y-z)\eta}(y+z)\cdot\eta|\eta|^{\kappa-2}f(z)dzd\widehat{\eta}dyd\widehat{\xi} \\ &-\frac{i}{2}\kappa^{-1} \iint e^{(x+y)\xi}(x+y)\cdot\xi|\xi|^{\kappa-2} \iint e^{i(y-z)\eta}|\eta|^\kappa f(z)dzd\widehat{\eta}dyd\widehat{\xi}. \end{aligned}$$

Changing the order of integrations, we write this as a sum of the three terms as follows.

$$i[H_0, A_\kappa]f(x) = \mathrm{I} + \mathrm{II} + \mathrm{III}, \tag{4.7}$$

where

$$\begin{aligned} \mathrm{I} &= \frac{i}{2}\kappa^{-1} \iint e^{i(x-z)\xi} \iint e^{i(y-z)(\eta-\xi)}(|\xi|^\kappa y\cdot\eta|\eta|^{\kappa-2} - |\xi|^{\kappa-2}y\cdot\xi|\eta|^\kappa)dyd\widehat{\eta}f(z)dzd\widehat{\xi}, \\ \mathrm{II} &= \frac{i}{2}\kappa^{-1} \iint e^{i(x-z)\xi} \iint e^{i(y-z)(\eta-\xi)}|\xi|^\kappa z\cdot\eta|\eta|^{\kappa-2}dyd\widehat{\eta}f(z)dzd\widehat{\xi}, \\ \mathrm{III} &= -\frac{i}{2}\kappa^{-1} \iint e^{i(x-z)\xi} \iint e^{i(y-z)(\eta-\xi)}|\xi|^{\kappa-2}x\cdot\xi|\eta|^\kappa dyd\widehat{\eta}f(z)dzd\widehat{\xi}. \end{aligned}$$

By Fourier's inversion formula, the second and the third terms are equal to

$$\begin{aligned} \mathrm{II} &= \frac{i}{2}\kappa^{-1} \iint e^{i(x-z)\xi}z\cdot\xi|\xi|^{2(\kappa-1)}f(z)dzd\widehat{\xi}, \\ \mathrm{III} &= -\frac{i}{2}\kappa^{-1} \iint e^{i(x-z)\xi}x\cdot\xi|\xi|^{2(\kappa-1)}f(z)dzd\widehat{\xi}. \end{aligned}$$

We have by direct calculation with using integration by parts

$$\begin{aligned} \mathrm{II} + \mathrm{III} &= -\frac{i}{2}\kappa^{-1} \iint D_\xi(e^{i(x-z)\xi})\cdot\xi|\xi|^{2(\kappa-1)}f(z)dzd\widehat{\xi} \\ &= \frac{1}{2}\kappa^{-1}(n+2(\kappa-1))|D|^{2(\kappa-1)}f(x). \end{aligned} \tag{4.8}$$

We decompose the first term as follows.

$$\mathrm{I} = \mathrm{M} + \mathrm{R},$$

where

$$\begin{aligned} \mathrm{M} &= \frac{i}{2}\kappa^{-1} \iint e^{i(x-z)\xi} \iint D_\eta(e^{i(y-z)(\eta-\xi)})\cdot\{\eta|\xi|^\kappa|\eta|^{\kappa-2} - \xi|\xi|^{\kappa-2}|\eta|^\kappa\}dyd\widehat{\eta}f(z)dzd\widehat{\xi}, \\ \mathrm{R} &= \frac{i}{2}\kappa^{-1} \iint e^{i(x-z)\xi} \iint e^{i(y-z)(\eta-\xi)}(|\xi|^\kappa|\eta|^{\kappa-2}\eta - |\xi|^{\kappa-2}|\eta|^\kappa\xi)\cdot zdyd\widehat{\eta}f(z)dzd\widehat{\xi}. \end{aligned}$$

The term R obviously vanishes so that we have the following by integration by parts and some computation.

$$\mathrm{I} = -\frac{1}{2}\kappa^{-1}(n-2)|D|^{2(\kappa-1)}f(x). \tag{4.9}$$

Taking the sum of (4.8) and (4.9), we obtain the lemma. □

Applying the lemma to (4.6) we obtain from (4.5)

$$\begin{aligned}
&\left\| \left(\frac{x}{t} - |D|^{\kappa-2}D \right) e^{-itH} f \right\|^2 \\
&= -\frac{2}{t^2}\int_0^t s(f, e^{isH}|D|^{2(\kappa-1)}e^{-isH}f)ds + (f, e^{itH}|D|^{2(\kappa-1)}e^{-itH}f) \\
&\quad -\frac{2}{t^2}\int_0^t s(f, e^{isH}i[V, A_\kappa]e^{-isH}f)ds + \frac{1}{t^2}(f, x^2 f)
\end{aligned} \tag{4.10}$$

By the assumption on the initial state f, the last term goes to 0 as $t \to \infty$. Further from the assumptions on potentials and the initial state f we have from Lemma 4.1 that the third term goes to 0 as $t \to \infty$. To treat the first and second terms, we set

$$H(t) = \frac{2}{t^2}\int_0^t s(f, e^{isH}|D|^{2(\kappa-1)}e^{-isH}f)ds.$$

Then the sum of the first and second terms on the RHS of (4.10) is equal to

$$\frac{t}{2}\frac{dH}{dt}(t).$$

From the definition of the function $H(t)$ and the assumptions $\kappa \geq 1$ and $f \in \mathcal{H}_c(a,b)$, we see that $H(t)$ $(t > 1)$ is a real-valued, continuously differentiable, uniformly bounded function, and that its derivative with respect to t tends to 0 as $t \to \infty$. Then we can apply Lemma 8.15 of [2] to find a sequence $T_k \to \infty$ as $k \to \infty$ for each positive constant $A > 1$ such that

$$\lim_{k\to\infty} \frac{1}{A}\int_{T_k}^{T_k+A} t\frac{dH}{dt}(t)dt = 0.$$

This together with the above-mentioned properties of the third and fourth terms yields that there is a sequence $t_k \to \infty$ as $k \to \infty$ such that

$$\left\| \left(\frac{x}{t_k} - |D|^{\kappa-2}D \right) e^{-it_k H} f \right\| \to 0$$

as $k \to \infty$. The case $t \to -\infty$ is treated similarly. The proof of Lemma 4.2 is complete. □

The relation (4.4) in particular implies that the configuration x is proportional to momentum $\pm\xi$ in phase space asymptotically as $t \to \pm\infty$. As a consequence, the relation

$$\lim_{t\to\pm\infty} P_\mp^* U(t) f = 0. \tag{4.11}$$

holds for $f \in \mathcal{H}_c(a,b)$ when t tends to $\pm\infty$ along the sequence $t = t_k \to \pm\infty$ $(k \to \pm\infty)$ given above. The relation (4.2) implies that $\text{w-}\lim_{k\to\pm\infty} U(t_k)f = 0$.

Summing up we have the following theorem.

Theorem 4.4 *For any $f \in \mathcal{H}_c(a,b)$ there is a sequence $t_k \to \pm\infty$ as $k \to \pm\infty$ such that*

$$\lim_{k\to\pm\infty} P_{\mp}^* U(t_k)f = 0, \tag{4.12}$$

$$\text{w-}\lim_{k\to\pm\infty} U(t_k)f = 0, \tag{4.13}$$

and for any $\phi \in C_0^\infty(\mathbb{R})$

$$\|(\phi(H) - \phi(H_0))U(t_k)f\| \to 0 \quad (k \to \pm\infty). \tag{4.14}$$

5 Asymptotic Completeness

We now prove the existence and asymptotic completeness of $W_1^\pm$. For this purpose, as we have stated at the beginning of section 3, it suffices to prove the existence of the two limits:

$$\begin{aligned} W_1^\pm f &= \lim_{t\to\pm\infty} W_1(t)f \quad (\forall f \in \mathcal{H} = L^2(\mathbb{R}^n)), \\ W_2^\pm g &= \lim_{t\to\pm\infty} W_2(t)g \quad (\forall g \in \mathcal{H}_c). \end{aligned}$$

The existence of $W_1^\pm$ is proved similarly to and more easily than that of the existence of $W_2^\pm$, and the case $t \to -\infty$ is treated similarly to the case $t \to \infty$. Therefore we will consider the existence of the limit

$$W_2^+ f = \lim_{t\to\infty} W_2(t)f \tag{5.1}$$

for $f \in \mathcal{H}_c(a,b)$ and $0 < a < b < \infty$. For this purpose we will prove that for $f \in \mathcal{H}_c(a,b)$

$$\begin{aligned} W_2(t+s)f - W_2(t)f &= U_0(-t-s)J^{-1}U(t+s)f - U_0(-t)J^{-1}U(t)f \\ &= \left\{U_0(-t-s)J^{-1}U(s) - U_0(-t)J^{-1}\right\} U(t)f \end{aligned} \tag{5.2}$$

converges to 0 uniformly in $s \geq 0$ as t goes to ∞ along the sequence $t = t_k \to \infty$ $(k \to \infty)$ specified in Theorem 4.4. If we have shown this, we have proved the existence of W_2^+. To prove this we let $P_\pm = P_\pm^{\theta_-,\theta_+}$ for $-1 < \theta_- = \sigma_+ + \rho < \theta_+ < 1$ for some $\rho > 0$ and the constant $\sigma_+ \in (-1,1)$ of Theorem 3.4. Then the state $U(t)f$ is decomposed

$$U(t)f = d(t) + e(t) + r(t),$$

where

$$d(t) = P_+^* U(t)f, \quad e(t) = P_-^* U(t)f, \quad r(t) = U(t)f - (P_+^* + P_-^*)U(t)f.$$

By (3.55) of Theorem 3.5, we have

$$J^{-1}U(s)P_+^* = -iJ^{-1}U(s)K_+(s) + U_0(s)J^{-1}P_+^*. \tag{5.3}$$

Thus

$$\begin{aligned}U_0(-t-s)J^{-1}U(s)d(t) &= U_0(-t-s)J^{-1}U(s)P_+^*U(t)f \\ &= -iU_0(-t-s)J^{-1}U(s)K_+(s)U(t)f + U_0(-t)J^{-1}P_+^*U(t)f.\end{aligned}$$

On the other hand we have

$$U_0(-t)J^{-1}d(t) = U_0(-t)J^{-1}P_+^*U(t)f.$$

The difference (5.2) is then equal to

$$\begin{aligned}&W_2(t+s)f - W_2(t)f \\ &\quad = \widetilde{K}_+(t,s)U(t)f + \left\{U_0(-t-s)J^{-1}U(s) - U_0(-t)J^{-1}\right\}(e(t)+r(t)),\end{aligned} \tag{5.4}$$

where

$$\widetilde{K}_+(t,s) = -iU_0(-t-s)J^{-1}U(s)K_+(s). \tag{5.5}$$

By (4.12) of Theorem 4.4

$$e(t_k) \to 0 \ \text{ as } \ k \to \infty. \tag{5.6}$$

From the definition of pseudodifferential operators $P_\pm$, it is easy to see that $P_\pm - P_\pm^*$ are compact operators on $\mathcal{H}$. From this fact and (4.13) in Theorem 4.4 we have

$$\|r(t_k) - \{I - (P_+ + P_-)\}U(t_k)f\| \to 0 \ \text{ as } \ k \to \infty. \tag{5.7}$$

From $f \in \mathcal{H}_c(a,b)$ and (4.14) in Theorem 4.4, we have

$$\|U(t_k)f - E_0([a,b])U(t_k)f\| \to 0 \quad \text{as} \quad k \to \infty. \tag{5.8}$$

By (5.7), (5.8), (1.11) and (4.13), we have

$$\|r(t_k)\| \to 0 \ \text{ as } \ k \to \infty. \tag{5.9}$$

From (5.4), (5.6) and (5.9), we have

$$\sup_{s\ge 0}\left\|W_2(t_k+s)f - W_2(t_k)f - \widetilde{K}_+(t_k,s)U(t_k)f\right\| \to 0 \ \text{ as } \ k \to \infty. \tag{5.10}$$

Here by Theorem 3.5, (5.5) and (4.13), we have

$$\sup_{s\ge 0}\left\|\widetilde{K}_+(t_k,s)U(t_k)f\right\| = \sup_{s\ge 0}\left\|J^{-1}U(s)K_+(s)U(t_k)f\right\| \to 0 \ \text{ as } \ k \to \infty. \tag{5.11}$$

The relations (5.10) and (5.11) imply that

$$\sup_{s\ge 0}\|W_2(t_k+s)f - W_2(t_k)f\| \to 0 \ \text{ as } \ k \to \infty. \tag{5.12}$$

This proves that the inverse wave operator W_2^+ exists on $\mathcal{H}_c$, and concludes the proof of the asymptotic completeness for the scattering problem with Hamiltonians (1.1) and (1.2). Namely we have proved the following theorem.

Theorem 5.1 *Under Assumptions S and L, we have the existence of the wave operators*

$$W_1^{\pm} = s\text{-}\lim_{t\to\pm\infty} e^{itH} J e^{-itH_0} \tag{5.13}$$

on $\mathcal{H} = L^2(\mathbb{R}^n)$ *and the asymptotic completeness*

$$\mathcal{R}(W_1^{\pm}) = \mathcal{H}_c, \tag{5.14}$$

where $\mathcal{H}_c$ *is the continuous spectral subspace for* H*. Moreover* $W_1^{\pm}$ *is an isometry and intertwines* H_0 *and* H*. Namely for any Borel set* $B \subset \mathbb{R}$

$$E_H(B) W_1^{\pm} = W_1^{\pm} E_0(B) \tag{5.15}$$

holds.

Proof We have already proved the existence of (5.13) and the asymptotic completeness (5.14). The intertwining property (5.15) is proved in the usual way with using the obvious relation for $s \in \mathbb{R}$

$$e^{-isH} W_1^{\pm} = W_1^{\pm} e^{-isH_0}.$$

We have only to show that $W_1^{\pm}$ is an isometry. For this purpose it suffices to see that for $f \in \mathcal{H}(a,b) = E_0([a,b])\mathcal{H}$ with the constants $0 < a < b < \infty$ which are assumed in the definition of each preliminary J

$$(W_1^{\pm})^* W_1^{\pm} f = f. \tag{5.16}$$

The norm of the difference of both sides is equal to

$$\lim_{t\to\pm\infty} \|(J^*J - I)e^{-itH_0} f\| = \lim_{t\to\pm\infty} \|(J^*J - I)E_0([a,b])e^{-itH_0} f\|.$$

Therefore the desired isometry follows from the fact that $(J^*J - I)E_0([a,b])$ is a compact operator, which is proved similarly to Lemma 3.7 of [4] with using the estimate (3.44). □

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