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by

Shigeo Kusuoka

UNIVERSITY OF TOKYO
GRADUATE SCHOOL OF MATHEMATICAL SCIENCES
KOMABA, TOKYO, JAPAN

A remark on Malliavin Calculus : Uniform Estimates and Localization

Shigeo KUSUOKA
*Graduate School of Mathematical Sciences
The University of Tokyo
Komaba 3-8-1, Meguro-ku, Tokyo 153-8914, Japan

1 Introduction

Let $W_0 = \{w \in C([0,\infty);\mathbf{R}^d);\ w(0) = 0\}$, $\mathcal{F}$ be the Borel algebra over W_0 and μ be the standard Wiener measure on $(W_0, \mathcal{F})$. Let $B^i : [0,\infty) \times W_0 \to \mathbf{R}$, $i = 1,\dots,d$, be given by $B^i(t,w) = w^i(t)$, $(t,w) \in [0,\infty) \times W_0$. Then $\{(B^1(t),\dots,B^d(t)); t \in [0,\infty)\}$ is a d-dimensional Brownian motion under μ. Let $B^0(t) = t$, $t \in [0,\infty)$. Let $\mathcal{F}_s^t$, $t \geqq s \geqq 0$, be a sub-σ-algebra generated by $\{B^i(r) - B^i(s); r \in [s,t],\ i = 1,\dots,d\}$. Then $\{\mathcal{F}_0^t\}_{t\geqq 0}$ is the Brownian filtration.

Let Λ be a set. We denote by $U_\Lambda C_b^\infty(\mathbf{R}^N;\mathbf{R}^M)$, $N, M \geqq 1$, the set of families of smooth functions $\{f_\lambda\}_{\lambda\in\Lambda}$ from $\mathbf{R}^N$ to $\mathbf{R}^M$ such that

$$\sup_{\lambda\in\Lambda, x\in\mathbf{R}^N} |\frac{\partial^\alpha}{\partial x^\alpha} f_\lambda(x)| < \infty$$

for any multi-index $\alpha \in \mathbf{Z}^N_{\geqq 0}$.

Let $\{V_i^\lambda\}_{\lambda\in\Lambda} \in U_\Lambda C_b^\infty(\mathbf{R}^N;\mathbf{R}^N)$, $i = 0,1,\dots,d$. We regard V_i^λ's as vector fields on $\mathbf{R}^N$. Let $X^\lambda(t,x)$, $t \in [0,\infty)$, $x \in \mathbf{R}^N$, $\lambda \in \Lambda$, be the solution to the Stratonovich stochastic integral equation

$$X^\lambda(t,x) = x + \sum_{i=0}^d \int_0^t V_i^\lambda(X^\lambda(s,x)) \circ dB^i(s). \tag{1}$$

Then there is a unique strong solution to this equation. Moreover we may assume that $X^\lambda(t,x)$ is continuous in t and smooth in x, and that $X^\lambda(t,\cdot) : \mathbf{R}^N \to \mathbf{R}^N$, $t \in [0,\infty)$, is a diffeomorphism with probability one.

Let $A = A_d = \{v_0, v_1, \dots, v_d\}$, be an alphabet, a set of letters, and A^* be the set of words consisting of A including the empty word which is denoted by 1. For $u = u^1 \cdots u^k \in A^*$, $u^j \in A$, $j = 1,\dots,k$, $k \geqq 0$, we denote by $n_i(u)$, $i = 0,\dots,d$, the cardinal of $\{j \in \{1,\dots,k\}; u^j = v_i\}$. Let $|u| = n_0(u) + \ldots + n_d(u)$, a length of u, and $\parallel u \parallel = |u| + n_0(u)$ for $u \in A^*$. Let $\mathbf{R}\langle A\rangle$ be the $\mathbf{R}$-algebra of noncommutative polynomials on A, $\mathbf{R}\langle\langle A\rangle\rangle$ be

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the $\mathbf{R}$-algebra of noncommutative formal series on A, $\mathcal{L}(A)$ be the free Lie algebra over $\mathbf{R}$ on the set A, and $\mathcal{L}((A))$ be the $\mathbf{R}$-Lie algebra of free Lie series on the set A.

Let $r : A^* \setminus \{1\} \to \mathcal{L}(A)$ denote the right normed bracketing operator inductively given by

$$r(v_i) = v_i, \qquad i = 0, 1, \ldots, d,$$

and

$$r(v_i u) = [v_i, r(u)], \qquad i = 0, 1, \ldots, d, \ u \in A^* \setminus \{1\}.$$

For any $w_1 = \sum_{u \in A^*} a_{1u} u, \in \mathbf{R}\langle\langle A \rangle\rangle$ and $w_2 = \sum_{u \in A^*} a_{2u} u, \in \mathbf{R}\langle A \rangle$, we define a kind of an inner product $\langle w_1, w_2 \rangle$ by

$$\langle w_1, w_2 \rangle = \sum_{u \in A^*} a_{1u} a_{2u} \in \mathbf{R}.$$

We can regard vector fields V_i^λ, $i = 0, 1, \ldots, d$, $\lambda \in \Lambda$, as first differential operators over $\mathbf{R}^N$. Let $\mathcal{DO}(\mathbf{R}^N)$ denote the set of smooth differential operators over $\mathbf{R}^N$. Then $\mathcal{DO}(\mathbf{R}^N)$ is a noncommutative algebra over $\mathbf{R}$. Let $\Phi^\lambda : \mathbf{R}\langle A \rangle \to \mathcal{DO}(\mathbf{R}^N)$, $\lambda \in \Lambda$, be a homomorphism given by

$$\Phi^\lambda(1) = Identity, \qquad \Phi^\lambda(v_{i_1} \cdots v_{i_n}) = V_{i_1}^\lambda \cdots V_{i_n}^\lambda,$$

for any $n \geqq 1, \ i_1, \ldots, i_n = 0, 1, \ldots, d, \ \lambda \in \Lambda$. Then we see that

$$\Phi^\lambda(r(v_i u)) = [V_i^\lambda, \Phi^\lambda(r(u))], \qquad i = 0, 1, \ldots, d, \ u \in A^* \setminus \{1\}.$$

Let $A_m^* = \{u \in A^*; \ \| u \| = m\}$, $m \geqq 0$, and let $\mathbf{R}\langle A \rangle_m = \sum_{u \in A_m^*} \mathbf{R}u$, and $\mathbf{R}\langle A \rangle_{\leqq m} = \sum_{k=0}^m \mathbf{R}\langle A \rangle_k$, $m \geqq 0$. Let $\mathcal{L}(A)_m = \mathcal{L}(A) \cap \mathbf{R}\langle A \rangle_m$, and $\mathcal{L}(A)_{\leqq m} = \mathcal{L}(A) \cap \mathbf{R}\langle A \rangle_{\leqq m}$, $m \geqq 1$. Let $A^{**} = \{u \in A^*; \ u \neq 1, v_0\}$, and $A_{\leqq m}^{**} = \{u \in A^{**}; \ \| u \| \leqq m\}$, $m \geqq 1$.

Now we introduce a condition (U$_\Lambda$FG) on the family of vector field $\{V_i^\lambda, i = 0, 1, \ldots, d, \lambda \in \Lambda\}$, as follows.
(U$_\Lambda$FG) There are an integer ℓ_0 and $\{\varphi_{u,u'}^\lambda\} \in U_\Lambda C_b^\infty(\mathbf{R}^N; \mathbf{R})$, $u \in A_{\leqq \ell_0+2}^{**}$, $u' \in A_{\leqq \ell_0}^{**}$, satisfying the following ondition.

$$\Phi^\lambda(r(u)) = \sum_{u' \in A_{\leqq \ell_0}^{**}} \varphi_{u,u'} \Phi^\lambda(r(u')), \qquad u \in A_{\leqq \ell_0+2}^{**}.$$

Now let us define a semigroup of linear operators $\{P_t^\lambda\}_{t \geqq 0}$ on $C_b^\infty(\mathbf{R}^N)$ by

$$(P_t^\lambda f)(x) = E^\mu[f(X^\lambda(t, x))], \qquad f \in C_b^\infty(\mathbf{R}^N).$$

We prove the following in this paper.

Theorem 1 *Assume* (U$_\Lambda$FG) *holds. Then for any $n, m \geqq 0$ with $n + m \geqq 1$ and $u_1, \ldots, u_{n+m} \in A^{**}$, there exists a $C > 0$ such that*

$$\sup_{\lambda \in \Lambda, x \in \mathbf{R}^N} |\Phi^\lambda(r(u_1) \cdots r(u_n))(P_t^\lambda(\Phi^\lambda(r(u_{n+1}) \cdots r(u_{n+m}))f))(x)|$$

$$\leqq C t^{-(||u_1|| + \cdots + ||u_{n+m}||)/2} \sup_{x \in \mathbf{R}^N} |f(x)|$$

for any $f \in C_b^\infty(\mathbf{R}^N)$.

Now let $\tilde{V}_i^\lambda : \mathbf{R}^N \to \mathbf{R}^N$, $\lambda \in \Lambda$, $i = 0, \dots, d$, be C^2 functions for which their derivatives are bounded. Let $\tilde{X}^\lambda(t,x)$, $t \in [0,\infty)$, $x \in \mathbf{R}^N$, be a solution to the following SDE

$$\tilde{X}^\lambda(t,x) = x + \sum_{i=0}^d \int_0^t \tilde{V}_i^\lambda(\tilde{X}^\lambda(s,x)) \circ dB^i(s). \tag{2}$$

Let us define a semigroup of linear operators $\{\tilde{P}_t^\lambda\}_{t \geqq 0}$ on $C_b(\mathbf{R}^N)$ by

$$(\tilde{P}_t^\lambda f)(x) = E^\mu[f(\tilde{X}^\lambda(t,x))], \qquad f \in C_b(\mathbf{R}^N).$$

Then we have the following localization result.

Theorem 2 *Let $x_0 \in \mathbf{R}^N$ and $\varepsilon_0 > 0$. Assume that $\{V_i^\lambda\}_{\lambda \in \Lambda}$, $i = 0, 1, \dots, d$, belongs to $U_\Lambda C_b^\infty(\mathbf{R}^N; \mathbf{R}^N)$ and satisfies* (U_ΛFG)*. Assume moreover that*

$$\tilde{V}_i^\lambda(x) = V_i^\lambda(x), \qquad x \in B(x_0; 2\varepsilon_0),\ \lambda \in \Lambda,\ i = 0, 1, \dots, d.$$

Then for any $\varphi \in C_0^\infty(B(x_0; \varepsilon_0))$ and $u_1, \dots, u_n \in A^{**}$, $n \geqq 1$, there exists a $C > 0$ such that

$$\sup_{\lambda \in \Lambda, x \in \mathbf{R}^N} |\Phi^\lambda(r(u_1) \cdots r(u_n))(\varphi \tilde{P}_t^\lambda f))(x)| \leqq C t^{-(||u_1|| + \cdots + ||u_n||)/2} \sup_{x \in \mathbf{R}^N} |f(x)|$$

and

$$\sup_{\lambda \in \Lambda, x \in \mathbf{R}^N} |(\tilde{P}_t^\lambda(\Phi^\lambda(r(u_1) \cdots r(u_n))(\varphi f)))(x)| \leqq C t^{-(||u_1|| + \cdots + ||u_n||)/2} \sup_{x \in \mathbf{R}^N} |f(x)|$$

for any $f \in C_b^\infty(\mathbf{R}^N; \mathbf{R})$. Here $B(x_0, \varepsilon_0)$ denotes ε_0-neighborhood of x_0.

We use Malliavin calculus to prove above theorems, and use the notation in Shigekawa [5] for Malliavin calculus. We regard $(W_0, \mathcal{F}, \mu, \{\mathcal{F}_0^t\}_{t \geqq 0})$ as a filtered probability space, and use the following notation. $\mathcal{S}$ denotes the set of continuous $\{\mathcal{F}_0^t\}_{t \geqq 0}$-semimartingales. $S : \mathcal{S} \times A^* \to \mathcal{S}$ and $\hat{S} : \mathcal{S} \times A^* \to \mathcal{S}$ are defined inductively by

$$S(Z; 1)(t) = Z(t), \quad t \geqq 0,$$

and

$$\hat{S}(Z; 1)(t) = Z(t), \quad t \geqq 0, \qquad Z \in \mathcal{S},$$

and

$$S(Z; uv_i)(t) = -\int_0^t S(Z, u)(s) \circ dB^i(s), \quad \hat{S}(Z; v_i u)(t) = -\int_0^t \tilde{S}(Z, u)(s) \circ dB^i(s), \quad t \geqq 0,$$

for any $Z \in \mathcal{S},\ i = 0, 1, \dots, d,\ u \in A^*$.

Also, we denote $S(1, u)(t)$ and $\hat{S}(1, u)$, $u \in A^*$, by $B(t; u)$ and $\hat{B}(t; u)$ respectvely.

2 Semimartingale on $\mathbf{R}\langle\langle A\rangle\rangle$

We say that $X : [0,\infty) \times W_0 \to \mathbf{R}\langle\langle A\rangle\rangle$ is an $\mathbf{R}\langle\langle A\rangle\rangle$-valued continuous semimartingale, if there are continuous semimartingales X_u, $u \in A^*$, such that $X(t) = \sum_{u\in A^*} X_u(t)u$. For $\mathbf{R}\langle\langle A\rangle\rangle$-valued continuous semimartingale $X(t), Y(t)$, we can define $\mathbf{R}\langle\langle A\rangle\rangle$-valued continuous semimartingales $\int_0^t X(s) \circ dY(s)$ and $\int_0^t \circ dX(s)Y(s)$ by

$$\int_0^t X(s) \circ dY(s) = \sum_{u,w\in A^*} (\int_0^t X_u(s) \circ dY_w(s))uw,$$

$$\int_0^t \circ dX(s)Y(s) = \sum_{u,w\in A^*} (\int_0^t Y_w(s) \circ dX_u(s))uw,$$

where

$$X(t) = \sum_{u\in A^*} X_u(t)u, \qquad Y(t) = \sum_{w\in A^*} Y_w(t)w.$$

Then we have

$$X(t)Y(t) = X(0)Y(0) + \int_0^t X(s) \circ dY(s) + \int_0^t \circ dX(s)Y(s).$$

Since $\mathbf{R}$ is regarded a vector subspace in $\mathbf{R}\langle\langle A\rangle\rangle$, we can define $\int_0^t X(s) \circ dB^i(s)$, $i = 0, 1, \ldots, d$, naturally.

Now let us consider the following SDE on $\mathbf{R}\langle\langle A\rangle\rangle$

$$\hat{X}(t) = 1 + \sum_{i=0}^d \int_0^t \hat{X}(s)v_i \circ dB^i(s), \qquad t \geqq 0. \tag{3}$$

One can easily solve this SDE and obtains

$$\hat{X}(t) = \sum_{u\in A^*} B(t;u)u.$$

We also have the following (c.f. [1]).

Proposition 3 $\log \hat{X}(t) \in \mathcal{L}((A))$, $t \geqq 0$, *with probability one.*

Note that

$$d(\hat{X}(t)^{-1}) = -\hat{X}(t)^{-1} d\hat{X}(t)\hat{X}(t)^{-1} = -\sum_{i=0}^d v_i\hat{X}(t)^{-1} \circ dB^i(t)$$

and so

$$\hat{X}(t)^{-1} = 1 - \sum_{i=0}^d v_i\hat{X}(t)^{-1} \circ dB^i(t)$$

3 Uniform Estimates

We assume the condition (U_ΛFG) throughout this section. The argument in this section is essentially the same as in Sections 2 and 3 in [1], or [2], and so we state results sometimes without proofs.

Proposition 4 *There are* $\{\varphi^\lambda_{u,u'}\}_{\lambda\in\Lambda} \in U_\Lambda C^\infty_b(\mathbf{R}^N)$, $u \in A^{**}$, $u' \in A^{**}_{\leqq \ell_0}$ *such that*

$$\Phi^\lambda(r(u)) = \sum_{u'\in A^{**}_{\leqq \ell_0}} \varphi^\lambda_{u,u'}\Phi^\lambda(r(u')), \qquad u \in A^{**}.$$

Proof. It is obivious that our assetion is valid for $u \in A^{**}_{\leqq \ell_0+2}$. Suppose that our assertion is valid for any $u \in A^{**}_{\leqq m}$, $m \geqq \ell_0$. Then we have for any $i = 0, 1, \ldots, d$ and $u \in A^{**}_{\leqq m}$,

$$\Phi^\lambda(r(v_i u)) = [V^\lambda_i, \Phi^\lambda(r(u))] = \sum_{u'\in A^{**}_{\leqq \ell_0}} [V^\lambda_i, \varphi_{u,u'}\Phi^\lambda(r(u'))]$$

$$= \sum_{u'\in A^{**}_{\leqq \ell_0}} (V^\lambda_i \varphi^\lambda_{u,u'})\Phi^\lambda(r(u')) + \sum_{u',u''\in A^{**}_{\leqq \ell_0}} \varphi^\lambda_{u,u'}\varphi^\lambda_{u',u''}\Phi^\lambda(r(u''))$$

So we see that our assertion is valid for any $u \in A^{**}_{\leqq m+1}$. Thus by induction we have our Proposition. ∎

For any C^∞ vector field W on $\mathbf{R}^N$, we see that

$$d(X^\lambda(t)^{-1}_* W)(x) = \sum_{i=0}^d (X^\lambda(t)^{-1}_*[V^\lambda_i, W])(x) \circ dB^i(t),$$

where $X^\lambda(t)_*$ is a push-forward operator with respect to the diffeomorphism $X^\lambda(t,\cdot) : \mathbf{R}^N \to \mathbf{R}^N$. So we have

$$d(X^\lambda(t)^{-1}_*\Phi^\lambda(r(u)))(x)$$

$$= \sum_{i=0}^d ((X(t)^\lambda_*)^{-1}\Phi^\lambda(r(v_i u)))(x) \circ dB^i(t)$$

for any $u \in A^* \setminus \{1\}$.

Let $m \geqq 3\ell_0$. Let $\{c^{\lambda,m}_i(\cdot, u, u')\}_{\lambda\in\Lambda} \in U_\Lambda C^\infty_b(\mathbf{R}^N, \mathbf{R})$, $i = 0, 1, \ldots, d$, $u, u' \in A^{**}_{\leqq m}$, be given by

$$c^{\lambda,m}_i(x; u, u') = \begin{cases} 1, & \text{if } ||v_i u|| \leqq m \text{ and } u' = v_i u, \\ \varphi^\lambda_{v_i u, u'}(x), & \text{if } ||v_i u|| > m \text{ and } ||u'|| \leqq \ell_0, \\ 0, & \text{otherwise.} \end{cases}$$

Here $\varphi^\lambda_{u,u'}$'s are as in Proposition 4. Then we have

$$d(X^\lambda(t)^{-1}_*\Phi^\lambda(r(u)))(x) = \sum_{i=0}^d \sum_{u'\in A^{**}_{\leqq m}} (c^{\lambda,m}_i(X^\lambda(t,x); u, u')(X^\lambda(t)^{-1}_*\Phi^\lambda(r(u')))(x) \circ dB^i(t)$$

for any $u \in A^{**}_{\leqq m}$.

Let $a^{\lambda,m}(t,x;u,u')$, $u,u'\in A^{**}_{\leqq m}$, be the solution to the following SDE

$$da^{\lambda,m}(t,x;u,u')$$

$$=\sum_{i=0}^{d}\sum_{u''\in A^{**}_{\leqq m}} c_i^{\lambda,m}(X^\lambda(t,x);u,u'')a^{\lambda,m}(t,x;u'',u')dB^i(t)$$

$$+\frac{1}{2}\sum_{i=1}^{d}\sum_{u''\in A^{**}}(V_i^\lambda(X^\lambda(t,x);u'',u')^{\lambda,m}(t,x;u'',u')dt$$

$$+\frac{1}{2}\sum_{i=1}^{d}\sum_{u_1,u_2'\in A^{**}_{\leqq m}}(c_i^{\lambda,m}(X^\lambda(t,x);u,u_1)c_i^{\lambda,m}(X^\lambda(t,x);u_1,u_2)a^{\lambda,m}(t,x;u_2,u')dt,$$

$$a^{\lambda,m}(0,x;u,u')=\langle u,u'\rangle.$$

Such a solution exists uniquely, and moreover, we may assume that $a^{\lambda,m}(t,x;u,u')$ is smooth in x with probability one. Then we have

$$\sup_{\lambda\in\Lambda,x\in\mathbf{R}^N}E^\mu[\sup_{t\in[0,T]}|\frac{\partial^{|\alpha|}}{\partial x^\alpha}a^{\lambda,m}(t,x;u,u')|^p]<\infty,\qquad p\in[1,\infty),\ T>0$$

for any multi-index α. One can easily see that

$$da^{\lambda,m}(t,x;u,u')=\sum_{i=0}^{d}\sum_{u''\in A^{**}_{\leqq m}}(c_i^{\lambda,m}(X^\lambda(t,x);u,u'')a^{\lambda,m}(t,x;u'',u'))\circ dB^i(t).\qquad(4)$$

Then the uniqueness of SDE implies

$$(X^\lambda(t)_*^{-1}\Phi^\lambda(r(u)))(x)=\sum_{u'\in A^{**}_{\leqq m}}a^{\lambda,m}(t,x;u,u')\Phi^\lambda(r(u'))(x),\ u\in A^{**}_{\leqq m}.$$

Similarly we see that there exists a unique solution $b^{\lambda,m}(t,x;u,u')$, $u,u'\in A^{**}_{\leqq m}$, to the SDE

$$b^{\lambda,m}(t,x;u,u')$$

$$=\langle u,u'\rangle-\sum_{i=0}^{d}\sum_{u''\in A^{**}_{\leqq m}}\int_0^t(b^{\lambda,m}(s,x;u,u''))(c_i^{(m)}(X^\lambda(s,x);u'',u'))\circ dB^i(t).\qquad(5)$$

Then we see that

$$\sum_{u''\in A^{**}_{\leqq m}}a^{\lambda,m}(t,x,u,u'')b^{\lambda,m}(t,x,u'',u)=\langle u,u'\rangle,\qquad u,u'\in A^{**}_{\leqq m},$$

$$\Phi^\lambda(r(u))(x)=\sum_{u'\in A^{**}_{\leqq m}}b^{\lambda,m}(t;x;u,u')(X(t)_*^{-1}\Phi^\lambda(r(u')))(x),\ u\in A^{**}_{\leqq m},$$

and

$$\sup_{\lambda\in\Lambda,x\in\mathbf{R}^N} E^{\mu}[\sup_{t\in[0,T]}|\frac{\partial^{|\alpha|}}{\partial x^{\alpha}}b^{\lambda,m}(t,x;u,u')|^p]<\infty, \qquad p\in[1,\infty),\ T>0$$

for any multi-index α. Let

$$R_m^* = \{v_0u; u\in A^*, ||u||=m-1\}\cup\bigcup_{i=0}^{d}\{v_iu; u\in A^*, ||u||=m\}.$$

Then we have the following.

Proposition 5 *For any $m\geqq 3\ell_0$,*

$$\begin{aligned} &a^{\lambda,m}(t,x,u,u') \\ &= \sum_{u_1\in A^*_{\leqq m}}\langle u_1u,u'\rangle B(t,u_1) \\ &+\sum_{u_1\in A^*:u_1u\in R_m^*}\sum_{u_2\in A^*_{\leqq\ell_0}} S(\varphi_{u_1u,u_2}(X^{\lambda}(\cdot,x))a^{\lambda,m}(\cdot,x,u_2,u'),u_1)(t)\end{aligned}$$

for any $t\in[0,\infty)$, $x\in\mathbf{R}^N$, and $u,u'\in A^{**}_{\leqq m}$.

Proof. The assertion is obvious from the definition, if $||u||=m$. Note that

$$\begin{aligned} &a^{\lambda,m}(t,x;u,u') \\ &=\langle u,u'\rangle+\sum_{i=0}^{d}\sum_{u_1\in A^{**}_{\leqq m}} S(c_i^{\lambda,m}(X^{\lambda}(\cdot,x);u,u_1)a^{\lambda,m}(\cdot,x;u_1,u'),v_i)(t).\end{aligned}$$

Therefore, if $||u||=m-1$, we have

$$\begin{aligned} &a^{\lambda,m}(t,x;u,u') \\ &=\langle u,u'\rangle+\sum_{i=1}^{d}S(\langle v_iu,u'\rangle a^{\lambda,m}(\cdot,x;v_iu,u'),v_i)(t) \\ &+\sum_{u_1\in A^{**}_{\leqq\ell_0}} S(\varphi_{v_0u,u_1}(X(\cdot,x))a^{\lambda,m}(\cdot,x,u_1,u'),v_0)(t) \\ &=\langle u,u'\rangle+\sum_{i=1}^{d}\langle v_iu,u'\rangle B(t,v_i) \\ &+\sum_{i=1}^{d}\sum_{j=0}^{d}\sum_{u_1\in A^{**}_{\leqq\ell_0}} S(S(\varphi_{v_jv_iu,u_1}(X(\cdot,x))a^{\lambda,m}(\cdot,x,u_1,u'),v_j),v_i)(t) \\ &+\sum_{u_1\in A^{**}_{\leqq\ell_0}} S(\varphi_{v_0u,u_1}(X(\cdot,x))a^{\lambda,m}(\cdot,x,u_1,u'),v_0)(t).\end{aligned}$$

So we have our assertion. Similarly by induction in $m-||u||$ we have our assertion. ∎

Corollary 6 *For any $m \geqq 3\ell_0$,*

$$a^{\lambda,m}(t,x;u,u')$$
$$= \langle \hat{X}(t)u,u'\rangle$$
$$+ \sum_{u_1\in A^*:u_1u\in R^*_m}\ \sum_{u_2\in A^*_{\leqq \ell_0}} S(\varphi_{u_1u,u_2}(X(\cdot,x))a^{\lambda,m}(\cdot,x;u_2,u'),u_1)(t)$$

for any $t\in[0,\infty)$, $x\in\mathbf{R}^N$, and $u,u'\in A^{**}_{\leqq m}$. In particular,

$$a^{\lambda,m}(t,x;v_i,u)$$
$$= \langle \hat{X}(t)v_i,u\rangle + \sum_{u_1\in A^*:u_1v_i\in R^*_m}\ \sum_{u_2\in A^*_{\leqq \ell_0}} S(\varphi^{\lambda}_{u_1v_i,u_2}(X^{\lambda}(\cdot,x))\langle \hat{X}(\cdot)u_2,v_i\rangle,u_1)(t)$$
$$+ \sum_{u_1\in A^*:u_1u\in R^*_m}\ \sum_{u_2\in A^*_{\leqq \ell_0}}\ \sum_{u_3\in A^*:u_3u_2\in R^*_m}\ \sum_{u_4\in A^*_{\leqq \ell_0}}$$
$$S(\varphi^{\lambda}_{u_1v_i,u_2}(X^{\lambda}(\cdot,x))S(\varphi^{\lambda}_{u_3u_2,u_4}(X^{\lambda}(\cdot,x))a^{\lambda,m}(\cdot,x,u_4,u),u_3),u_1)(t).$$

Here $\hat{X}(t)$ is a solution to SDE (3)*.*

Proposition 7 *Let $m \geqq 3\ell_0$.*
(1) *For any $u\in A^{**}_{\leqq m}$, $u'\in A^*$, $i=0,1,\dots,d$ with $v_iu'\in A^{**}_{\leqq m}$, if $||v_iu'||>\ell_0$, then*

$$b^{\lambda,m}(t,x,u,v_iu') = \tilde{S}(b^{\lambda,m}(\cdot,x,u,u');v_i) + \langle u,v_iu'\rangle,$$

and if $||v_iu'||\leqq \ell_0$, then

$$b^{\lambda,m}(t,x,u,v_iu') = \tilde{S}(b^{\lambda,m}(\cdot,x,u,u');v_i)(t) + \langle u,v_iu'\rangle$$
$$+\sum_{j=0}^{d}\ \sum_{u_1\in A^{**}_{\leqq m},v_ju_1\in R^*_m} \tilde{S}(b^{\lambda,m}(\cdot,x,u,v_ju_1)\varphi^{\lambda}_{v_ju_1,v_iu'}(X^{\lambda}(\cdot,x));v_j)(t)$$

for any $t\in[0,\infty)$, $x\in\mathbf{R}^N$, and $\lambda\in\Lambda$.
(2) *For any $u,u_2\in A^{**}_{\leqq m}$, $u_1\in A^*$ with $||u_2||\geqq \ell_0$, $||u||\leqq||u_2||$ and $||u_1u_2||\leqq m$,*

$$b^{\lambda,m}(t,x,u,u_1u_2) = \tilde{S}(b^{\lambda,m}(\cdot,x,u,u_2);u_1).$$

Proof. Since we have

$$b^{\lambda,m}(t,x,u,v_iu')$$
$$= \langle u,v_iu'\rangle + \sum_{j=0}^{d}\ \sum_{u_1\in A^{**}_{\leqq m}} \tilde{S}(b^{\lambda,m}(\cdot,x,u,u_1)c^{\lambda,m}_j(X^{\lambda}(\cdot,x));u_1,v_iu');v_j)(t),$$

we have the assertion (1) from the definition of $c^{\lambda,m}_j$.

The assertion (2) is an easy consequence of the first part of the assertion (1). ∎

Let E be a separable real Hilbert space and $r \in \mathbf{R}$. Let us denote by $W^{\infty,\infty-}(E)$ $\bigcap_{s\geqq 0,p\in(1,\infty)} W^{s,p}(E))$. Let $\mathcal{K}_\Lambda(E)$ denote the set of families $\{f_\lambda\}_{\lambda\in\Lambda}$ of functionals f_λ : $(0,1]\times\mathbf{R}^N \to W^{\infty,\infty-}(E)$ satisfying the following two conditions.
(1) $f_\lambda(t,x)$ is smooth in x and $\frac{\partial^\alpha}{\partial x^\alpha} f_\lambda(t,x)$ is continuous in $(t,x)\in(0,1]\times\mathbf{R}^N$ for any multi-index α.
(2) $\sup_{\lambda\in\Lambda,t\in(0,1],x\in\mathbf{R}^N} \| \dfrac{\partial^\alpha}{\partial^\alpha x} f_\lambda(t,x) \|_{W^{s,p}(E)} < \infty$, for any multi-index α,$s\in\mathbf{R}$ and $p\in(1,\infty)$.
We denote $\mathcal{K}_\Lambda(\mathbf{R})$ by $\mathcal{K}_\Lambda$.

By checking carefully the estimates discussed in Chapter 6 in Shigekawa [5], we see that $\{a^{\lambda,m}(t,x;u,u')\}_{\lambda\in\Lambda}$ and $\{b^{\lambda,m}(t,x;u,u')\}_{\lambda\in\Lambda}$ belong to $\mathcal{K}_\Lambda$ for any $u,u'\in A^*_{\leqq m}$.

Then by Corollary 6, we have the following.

Proposition 8 *For any $u,u'\in A^*_{\leqq m}$, $\{t^{-m/2}(a^{\lambda,m}(t,x;u,u') - \langle \hat{X}(t)u,u'\rangle)\}_{\lambda\in\Lambda}$ belong to $\mathcal{K}_\Lambda$. In particular, $\{t^{-((\|u'\|-\|u\|)\vee 0)/2}a^{\lambda,m}(t,x;u,u')\}_{\lambda\in\Lambda}$ belong to $\mathcal{K}_\Lambda$.*

Similarly by Proposition 7 we have the following.

Proposition 9 *For any $u,u'\in A^*_{\leqq m}$, $\{t^{-((\|u'\|-\|u\|)\vee 0)/2}b^{\lambda,m}(t,x;u,u')\}_{\lambda\in\Lambda}$ belong to $\mathcal{K}_\Lambda$.*

Now let $k^{\lambda,m}(t,x;u)\in H$, $\lambda\in\Lambda$, $(t,x)\in[0,\infty)\times\mathbf{R}^N$, $u\in A^{**}_{\leqq m}$, be given by

$$k^{\lambda,m}(t,x;u) = (\int_0^{t\wedge\cdot} a^{\lambda,m}(s,x;v_i,u)ds)_{i=1,\ldots,d}.$$

Then we have the following.

Proposition 10 *For any $u\in A^*_{\leqq m}$, $\{t^{-\|u\|/2}k^{\lambda,m}(t,x;u)\}_{\lambda\in\Lambda}$ belong to $\mathcal{K}_\Lambda(H)$.*

Let $M^{\lambda,m}(t,x;u,u')$, $(t,x)\in[0,\infty)\times\mathbf{R}^N$, $u,u'\in A^{**}_{\leqq m}$, be given by

$$\begin{aligned} M^{\lambda,m}(t,x;u,u') &= t^{-(\|u\|+\|u'\|)/2}(k^{\lambda,m}(t,x;u),k^{\lambda,m}(t,x;u'))_H \\ &= t^{-(\|u\|+\|u'\|)/2}\sum_{i=1}^d \int_0^t a^{\lambda,m}(s,x;v_i,u)a^{\lambda,m}(s,x;v_i,u')ds. \end{aligned} \tag{6}$$

Also, let $\hat{M}^{(m)}(t;u,u')$, $(t,x)\in[0,\infty)\times\mathbf{R}^N$, $u,u'\in A^{**}_{\leqq m}$, be given by

$$\hat{M}^{(m)}(t;u,u') = t^{-(\|u\|+\|u'\|)/2}\sum_{i=1}^d \int_0^t \langle\hat{X}(t)v_i,u\rangle\langle\hat{X}(t)v_i,u'\rangle. \tag{7}$$

We can prove the following from Propositions 8 and 9 by the exactly same method as in [1] Section 4 .

Proposition 11 (1) *For any $p\in(1,\infty)$,*

$$\sup_{\lambda\in\Lambda,t\in(0,1],x\in\mathbf{R}^N} E^\mu[det(M^{\lambda,m}(t,x;u,u'))^{-p}_{u,u'\in A^{**}_{\leqq m}}] < \infty.$$

(2) *For any $p\in(1,\infty)$,*

$$\sup_{t\in(0,1]} E^\mu[det(\hat{M}^{(m)}(t;u,u'))^{-p}_{u,u'\in A^{**}_{\leqq m}}] < \infty.$$

(3) $\{t^{-1/2}(M^{\lambda,m}(t,x;u,u') - \hat{M}^{(m)}(t;u,u'))\}_{\lambda\in\Lambda}$ *belong to $\mathcal{K}_\Lambda$ for any $u,u'\in A^{**}_{\leqq m}$*

Let $(\check{M}^{\lambda,m}(t,x;u,u'))_{u,u'\in A^{**}_{\leqq m}}$ be the inverse matrix of $(M^{\lambda,m}(t,x;u,u'))_{u,u'\in A^{**}_{\leqq m}}$ and $(\tilde{M}^{(m)}(t;u,u'))_{u,u'\in A^{**}_{\leqq m}}$ be the inverse matrix of $(\hat{M}^{(m)}(t,x;u,u'))_{u,u'\in A^{**}_{\leqq m}}$.

Then we have the following.

Corollary 12 *$\{\check{M}^{\lambda,m}(t,x;u,u')\}_{\lambda\in\Lambda}$ and $\{\tilde{M}^{(m)}(t;u,u')\}_{\lambda\in\Lambda}$ belong to $\mathcal{K}_\Lambda$ for any $u,u'\in A^{**}_{\leqq m}$. Moreover, $\{t^{-1/2}(\check{M}^{\lambda,m}(t,x;u,u')-\tilde{M}^{(m)}(t;u,u'))\}_{\lambda\in\Lambda}$ belong to $\mathcal{K}_\Lambda$ for any $u,u'\in A^{**}_{\leqq m}$.*

Note that

$$\begin{aligned}X^\lambda(t)^{-1}_* DX^\lambda(t,x) &= (\int_0^{t\wedge\cdot}(X^\lambda(s)^{-1}_* V^\lambda_i)(x)ds)_{i=1,\dots,d}\\ &= \sum_{u\in A^{**}_{\leqq m}} k^{\lambda,m}(t,x;u)\Phi^\lambda(r(u))(x)\end{aligned}$$

for $(t,x)\in[0,\infty)\times\mathbf{R}^N$ (c.f.[3]). Let $f\in C^\infty_b(\mathbf{R}^N)$. Since we have

$$D(f(X^\lambda(t,x))) = {}_{T^*_x}\langle (X^\lambda(t)^* df)(x), X^\lambda(t)^{-1}_* DX^\lambda(t,x)\rangle_{T_x},$$

we see that

$$\begin{aligned}&(D(f(X^\lambda(t,x))), k^{\lambda,m}(t,x;u))_H\\ =& \sum_{u'\in A^{**}_{\leqq m}} \langle (X^\lambda(t)^* df)(x), \Phi^\lambda(r(u'))\rangle_x t^{(||u||+||u'||)/2} M^{\lambda,m}(t,x;u,u').\end{aligned}$$

So we have

$$\begin{aligned}&t^{||u||/2}\Phi^\lambda(r(u))(f(X^\lambda(t,\cdot)))(x) = {}_{T^*_x}\langle (X^\lambda(t)^* df)(x), \Phi^\lambda(r(u))\rangle_{T_x}\\ =& \sum_{u'\in A^{**}_{\leqq m}} \check{M}^{\lambda,m}(t,x;u,u')t^{-||u'||/2}(D(f(X^\lambda(t,x)), k^{\lambda,m}(t,x;u'))_H \qquad (8)\end{aligned}$$

and

$$\begin{aligned}&t^{||u||/2}(\Phi^\lambda(r(u))f)(X^\lambda(t,x))\\ =& \sum_{u_1,u_2\in A^{**}_{\leqq m}} \check{M}^{\lambda,m}(t,x;u_1,u_2)t^{-(||u_1||-||u||)/2}b^{\lambda,m}(t,x;u,u_1)\\ &\times t^{-||u_2||/2}(D(f(X^\lambda(t,x)), k^{\lambda,m}(t,x;u_2))_H \qquad (9)\end{aligned}$$

Therefore we have the following.

Theorem 13 *Let $f\in C^\infty_b(\mathbf{R}^N)$. Then we have the following.*
(1) *For any $u\in A^{**}_{\leqq m}$, $p\in(1,\infty)$ and $r>0$,*

$$\sup_{t\in(0,1],\lambda\in\Lambda,x\in\mathbf{R}^N} ||t^{||u||/2}(\Phi^\lambda(r(u))f)(X^\lambda(t,\cdot))(x)||_{W^{r,p}} < \infty.$$

(2) *For any $F\in W^{\infty,\infty-}$ and $u\in A^{**}_{\leqq m}$, we have*

$$t^{||u||/2}\Phi^\lambda(r(u))(E^\mu[Ff(X^\lambda(t,\cdot))](x) = E^\mu[(\mathcal{R}^\lambda_0(t,x;u)F)f(X^\lambda(t,x))]$$

and

$$E^\mu[Ft^{||u||/2}\Phi^\lambda(r(u))f)(X^\lambda(t,x))] = E^\mu[(\mathcal{R}_1^\lambda(t,x;u)F)f(X^\lambda(t,x))].$$

Here

$$\mathcal{R}_0^\lambda(t,x;u)F$$
$$= \sum_{u' \in A^{**}_{\leqq m}} D^*(\check{M}^{\lambda,m}(t,x;u,u')t^{-||u'||/2}k^{\lambda,m}(t,x;u')F)$$

and

$$\mathcal{R}_1^\lambda(t,x;u)F$$
$$= \sum_{u_1,u_2 \in A^{**}_{\leqq m}} D^*(\check{M}^{\lambda,m}(t,x;u_1,u_2)t^{-(||u_1||-||u||)/2}b^{\lambda,m}(t,x;u,u_1)t^{-||u_2||/2}k^{\lambda,m}(t,x;u_2)F).$$

One can easily prove the following.

Proposition 14 *If $\{F_\lambda(t,x)\}_{\lambda\in\Lambda}$ belongs to $\mathcal{K}_\Lambda$, then $\{\mathcal{R}_0^\lambda(t,x;u)(F_\lambda(t,x))\}_{\lambda\in\Lambda}$ and $\{\mathcal{R}_1^\lambda(t,x;u)(F_\lambda(t,x))\}_{\lambda\in\Lambda}$ belong to $\mathcal{K}_\Lambda$.*

Now Theorem 1 is an easy consequence of Theorem 13 and the above Proposition.

4 Localization

First, we remind the following result (c.f. Stroock-Varadhan [6] Theorem 2.1.3)

Proposition 15 *Let E be a mormed space. Let $T, B > 0$ $\beta \in (0,1)$, and $p \in (2/\beta, \infty)$. Suppose that a continuous function $\phi : [0,T] \to E$ satisfies*

$$\int_0^T \int_0^T (\frac{||\phi(t)-\phi(s)||_E}{|t-s|^\beta})^p dsdt \leqq B.$$

Then we have

$$||\phi(t)-\phi(s)||_E \leqq \frac{8\beta(4B)^{1/p}}{\beta - 2/p}|t-s|^{\beta-2/p}, \qquad t,s \in [0,T].$$

Now let $x_0 \in \mathbf{R}^N$, $\varepsilon_0 > 0$. $\tilde{V}_i^\lambda : \mathbf{R}^N \to \mathbf{R}^N$, and $V_i^\lambda : \mathbf{R}^N \to \mathbf{R}^N$, $\lambda \in \Lambda$, $i = 0, \ldots, d$, be as in Theorem 2. Also, let $X^\lambda(t,x)$ and $\tilde{X}^\lambda(t,x)$ be solutions to Equation (1) and (2) respectively. We may assume that $x_0 = 0$, and $\varepsilon_0 < 1$.

By checking the computation in Shigekawa [5] Section 6, we see that for any $n \geqq 1$, $k \geqq 0$ and multi-index $\alpha \in \mathbf{Z}^N_{\geqq 0}$, there is a $C > 0$ such that

$$\sup_{\lambda\in\Lambda, x\in\mathbf{R}^N} E^\mu[||D^k\frac{\partial^\alpha}{\partial x^\alpha}X^\lambda(t,x) - D^k\frac{\partial^\alpha}{\partial x^\alpha}X^\lambda(s,x)||^{2n}_{H^{\otimes k}\otimes(\mathbf{R}^N)^{\otimes k+1}}] \leqq C|t-s|^n$$

for all $t,s \in [0,1]$.

Let $\tilde{Y}^\lambda(T) : W_0 \to [0,\infty)$, $T \in (0,1]$ given by

$$\tilde{Y}^\lambda(T)$$

$$= \int_0^T \int_0^T dt\, ds \int_{|x|<2} dx \frac{|X^\lambda(t,x) - X^\lambda(s,x)|^{2(N+2)} + |\nabla_x X^\lambda(t,x) - \nabla_x X^\lambda(s,x)|^{2(N+2)}}{|t-s|^{N+2}}$$

$\tilde{Y}^\lambda(T)$ is $\mathcal{F}_0^T$ measurable. Also, we see that for any $k \geqq 0$ and $p \in (1,\infty)$ there is a $C > 0$ such that

$$\sup_{\lambda\in\Lambda} ||\tilde{Y}^\lambda(T)||_{W^{k,p}} \leqq CT^2, \qquad T \in (0,1].$$

Thus we see that

$$\sup_{\lambda\in\Lambda, T\in(0,1]} T^{-2} ||\tilde{Y}^\lambda(T)||_{W^{r,p}} < \infty \tag{10}$$

for any $r > 0$ and $p \in (1,\infty)$.

Let us take a $\rho \in C_0^\infty(\mathbf{R};\mathbf{R})$ such that $0 \leqq \rho \leqq 1$, $\rho(z) = 1$, $|z| \leqq 1$, and $\rho(z) = 0$, $|z| > 2$.

Then we have the following.

Proposition 16 (1) *There is a* $C_0 > 0$ *such that*

$$E^\mu[\rho(T^{-1}\tilde{Y}^\lambda(T)), \sup_{x\in B(0,2), t\in[0,T]} |X^\lambda(t,x) - x| \geqq C_0 T^{1/3}] = 0$$

for any $\lambda \in \Lambda$, $T \in (0,1]$.
(2) *For any* $r > 1$

$$\sup_{\lambda\in\Lambda, T\in(0,1]} T^{-r} (\sum_{k=1}^n E^\mu[1 - \rho(T^{-1}\tilde{Y}^\lambda(T))]) < \infty.$$

(3) *For any* $n \geqq 1$ $p \in (1,\infty)$ *and* $r > 1$,

$$\sup_{\lambda\in\Lambda, T\in(0,1]} T^{-r} (\sum_{k=1}^n E^\mu[||D^k(\rho(T^{-1}\tilde{Y}^\lambda(T)))||_{H^{\otimes k}}^p]^{1/p}) < \infty.$$

Proof. Let E_N be a normed space such that $E_N = C^\infty(B(0,2);\mathbf{R}^N)$ as a set and the norm $||\ ||_{E_N}$ of E_N is given by

$$||f||_{E_N} = (\int_{B(0,2)} (|f(x)|^{2(N+2)} + |\nabla f(x)|^{2(N+2)}) dx)^{1/(2(N+2))}, \qquad f \in E_N.$$

Then by Sobolev's iequality, there is a constant $C_N > 0$ such that

$$\sup_{x\in B(0,2)} |f(x)| \leqq C_N ||f||_{E_N}, \qquad f \in E_N.$$

Note that

$$\tilde{Y}^\lambda(T) = \int_0^T \int_0^T dt\, ds (\frac{||X^\lambda(t,\cdot) - X^\lambda(s,\cdot)||_{E_N}}{|t-s|^{1/2}})^{2(N+2)}.$$

So, applying Proposition 15 for $p = 2(N+2)$, $B = T$, and $\beta = 1/2$, we see that if $\tilde{Y}^\lambda(T) \leqq 2T$, then

$$\sup_{x\in B(0,2)} |X^\lambda(t,x) - X^\lambda(s,x)| \leqq C_N ||X^\lambda(t,\cdot) - X^\lambda(s,\cdot)||_{E_N}$$

$$\leqq \frac{4C_N(8T)^{1/p}}{\beta - 2/p}|t-s|^{\beta-2/p}, \qquad t, s, \in [0,T],$$

which implies

$$\sup_{x\in B(0,2), t\in[0,T]} |X^\lambda(t,x) - x| \leqq \frac{4C_N 8(2N+4)}{N} T^{(N+1)/(2N+4)}.$$

Since $(N+1)/(2N+4) \geqq 1/3$, we have the assetion (1).

Note that

$$E^\mu[1 - \rho(T^{-1}\tilde{Y}^\lambda(T))] \leqq \mu(T^{-1}\tilde{Y}^\lambda(T)) \geqq 1) \leqq T^{-r}E^\mu[\tilde{Y}^\lambda(T)^r].$$

This and Equation (10) imply the assertion (2).

Since we have

$$D(\rho(T^{-1}\tilde{Y}^\lambda(T))) = T\rho'(T^{-1}\tilde{Y}^\lambda(T)))D(T^{-2}\tilde{Y}^\lambda(T))),$$

we see that

$$E^\mu[||D(\rho(T^{-1}\tilde{Y}^\lambda(T)))||_H^p]^{1/p} \leqq (\sup_{z\in\mathbf{R}}|\rho'(z)|)\mu(T^{-1}\tilde{Y}^\lambda(T) > 1)^{1/p}||\tilde{Y}^\lambda(T)||_{W^{1,p}}$$

So we have the assetion (3) for $n = 1$. Similarly, we have the assertion (3) for $n \geqq 2$ also. ∎

Proposition 17 *Suppose that $U_j \in W^{\infty,\infty-}$, $j = 1,\dots,m$, and assume that $|U_j| \leqq 1$ $\mu - a.s.$ $j = 1,\dots,m$. Then for any $n \geqq 1$*

$$||D^n(\prod_{j=1}^m U_j)||_{H^{\otimes n}} \leqq n^n \sum_{k=1}^n (\sum_{j=1}^m ||D^k U_j||_{H^{\otimes k}})^{n/k}.$$

Proof. Note that

$$||D^n(\prod_{j=1}^m U_j)||_{H^{\otimes n}}$$

$$\leqq \sum_{k=1}^n \sum_{1\leqq i_1<\cdots<i_k\leqq m} \sum_{\ell_1,\dots,\ell_k\geqq 1, \ell_1+\cdots \ell_k=n} \frac{n!}{\ell_1!\dots\ell_k!}(\prod_{j\neq i_1,\dots i_n}|U_j|)||D^{\ell_1}U_{i_1}||_{H^{\otimes \ell_1}}\cdots||D^{\ell_k}U_{i_k}||_{H^{\otimes \ell_k}}$$

$$\leqq \sum_{k=1}^n \sum_{\ell_1,\dots,\ell_k\geqq 1, \ell_1+\cdots \ell_k=n} \frac{n!}{\ell_1!\dots\ell_k!}(\sum_{i=1}^m ||D^{\ell_1}U_i||_{H^{\otimes \ell_1}})\cdots(\sum_{i=1}^m ||D^{\ell_k}U_i||_{H^{\otimes \ell_k}})$$

$$\leqq \sum_{k=1}^n \sum_{\ell_1,\dots,\ell_k\geqq 1, \ell_1+\cdots \ell_k=n} \frac{n!}{\ell_1!\dots\ell_k!}((\sum_{i=1}^m ||D^{\ell_1}U_i||_{H^{\otimes \ell_1}})^{n/\ell_1} + \cdots + (\sum_{i=1}^m ||D^{\ell_k}U_i||_{H^{\otimes \ell_k}})^{n/\ell_k}).$$

This implies our assertion. ∎

Let $\theta_T : W_0 \to W_0$, $T \geqq 0$, be given by

$$\theta_T(w)(t) = w(T+t) - w(T), \qquad w \in W_0.$$

Then $\mu \circ \theta_T^{-1} = \mu$.

Let $T_n = \sum_{k=n}^{\infty} 8^{-k} = 8^{-n+1}/7$, $n \geqq 0$, and let $Z_{n,m}^{\lambda} \in W^{\infty}, \infty-$, $n > m \geqq 1$, by

$$Z_{n,m}^{\lambda} = \prod_{k=m}^{n} \rho(8^k \tilde{Y}(8^{-k}; \theta_{T_{k+1}} w)).$$

Note that $\rho(8^k \tilde{Y}(8^{-k}; \theta_{T_{k+1}} w))$ is $\mathcal{F}_{T_{k+1}}^{T_k}$ and so $Z_{n,m}^{\lambda}$ is $\mathcal{F}_{T_{n+1}}^{T_m}$.

Proposition 18 (1) *Let $C_0 > 0$ be as in Proposition 16 and m_0 be an integer such that $C_0 2^{-m_0+1} < \varepsilon_0/2$. Then for any $n > m \geqq m_0$,*

$$E^{\mu}[Z_{n,m}^{\lambda}, \sup_{x \in B(0,1), t \in [0, T_m - T_n]} |X^{\lambda}(t, x; \theta_{T_n} w) - x| \geqq \varepsilon_0/2] = 0.$$

(2) *For any $r > 0$ and $p \in (1, \infty)$ we see that*

$$\sup_{\lambda \in \Lambda, n > m \geqq 1} ||Z_{n,m}^{\lambda}||_{W^{r,p}} < \infty$$

Proof. Note that

$$X^{\lambda}(t + s, x; \theta_{T_{n+1}} w) = X^{\lambda}(t, X^{\lambda}(s, x; \theta_{T_{n+1}} w)); \theta_{T_n + s} w).$$

Thereofore we have

$$\begin{aligned} &\sup_{x \in B(0,1), t \in [0, T_m - T_n]} |X^{\lambda}(t, x; \theta_{T_n} w) - x| \\ &\leqq \sup_{x \in B(0,1), t \in [0, T_{m+1} - T_n]} |X^{\lambda}(t, x; \theta_{T_n} w) - x| \\ &+ \sup_{x \in B(0,1), t \in [0, 8^{-m}]} |X^{\lambda}(t, X^{\lambda}(T_{m+1} - T_n, x; \theta_{T_n} w)); \theta_{T_{m+1}} w) - X^{\lambda}(T_{m+1} - T_n, x; \theta_{T_n} w))|. \end{aligned}$$

and so if $n > m \geqq m_0$

$$\begin{aligned} &\{\sup_{x \in B(0,1), t \in [0, T_m - T_n]} |X^{\lambda}(t, x; \theta_{T_n} w) - x| > C_0 2^{-m+1}\} \\ &\subset \{\sup_{x \in B(0,1), t \in [0, T_{m+1} - T_n]} |X^{\lambda}(t, x; \theta_{T_n} w) - x| > C_0 2^{-m}\} \\ &\qquad \cup \{\sup_{x \in B(0,2), t \in [0, 8^m]} |X^{\lambda}(t, x; \theta_{T_{m+1}} w) - x| > C_0 2^{-m}\}. \end{aligned}$$

. Therefore we see that

$$\begin{aligned} &E^{\mu}[Z_{n,m}^{\lambda}, \sup_{x \in B(0,1), t \in [0, T_m - T_n]} |X^{\lambda}(t, x; \theta_{T_n} w) - x| > C_0 2^{-m+1}] \\ &\leqq \sum_{k=m}^{n} E^{\mu}[Z_{n,m}^{\lambda}, \sup_{x \in B(0,2), t \in [0, 8^k]} |X^{\lambda}(t, x; \theta_{T_{k+1}} w) - x| > C_0 2^{-k}] = 0. \end{aligned}$$

This implies the assertion (1).

By Propositions 16 (3) we see that

$$\sum_{k=1}^{\infty} \sup_{\lambda \in \Lambda} E^{\mu}[||D^{\ell}\rho(8^k \tilde{Y}(8^{-k};\theta_{T_{k+1}}w))||^p_{H^{\otimes k}}] < \infty$$

for any $\ell \geqq 1$ and $p \in (1,\infty)$. Since $0 \leqq \rho \leqq 1$, we see by Propositions 17 that

$$\sum_{k=1}^{\ell} \sup_{\lambda \in \Lambda, n > m \geqq 1} E^{\mu}[||D^k Z^{\lambda}_{n,m}||^p_{H^{\otimes k}}] < \infty$$

for any $\ell \geq 1$ and $p \in (1,\infty)$. Since $|Z^{\lambda}_{n,m}| \leqq 1$, we have the assertion (2). ∎

Let $Z^{\lambda}_m = \lim_{n\to\infty} Z^{\lambda}_{n,m}$ for $\lambda \in \Lambda$ and $m \geqq 1$.

Then we have the following.

Proposition 19 (1) *Let $C_0 > 0$ be as in Proposition 16 and m_0 be an integer such that $C_0 2^{-m_0+1} < \varepsilon_0/2$. Then for any $m \geqq m_0$,*

$$E^{\mu}[Z^{\lambda}_m, \sup_{x \in B(0,1), t \in [0,T_m]} |X^{\lambda}(t,x) - x| \geqq \varepsilon_0/2] = 0.$$

(2) *$Z^{\lambda}_m \in W^{\infty,\infty-}$ for any $\lambda \in \Lambda$ and $m \geqq 1$, and moreover we see that for any $r > 0$ and $p \in (1,\infty)$*

$$\sup_{\lambda \in \Lambda, m \geqq 1} ||Z^{\lambda}_m||_{W^{r,p}} < \infty.$$

Now let

$$g_k(x; f, \lambda) = E^{\mu}[(1 - \rho(8^{k-1}\tilde{Y}(8^{-k};w)) f(\tilde{X}^{\lambda}(t - T_k, x))], \qquad x \in \mathbf{R}^N,\ k \geqq m_0$$

for any $f \in C^{\infty}_b(\mathbf{R}^N)$.

Then we see that

$$|g_k(x; f, \lambda)| \leqq E^{\mu}[(1 - \rho(8^{k-1}\tilde{Y}(8^{-k};w))^2]^{1/2} \sup_{x \in \mathbf{R}^N} |f(x)|. \tag{11}$$

By Proposition 16 (2) we see that that

$$\sup_{k \geqq 0,\ \lambda \in \Lambda} 8^{\gamma k} E^{\mu}[(1 - \rho(8^{k-1}\tilde{Y}(8^{-k};w))^2]^{1/2} < \infty \tag{12}$$

for any $\gamma > 0$.

For each $t \in (0,1]$, let $m = m(t)$ be a minimum integer m such that $m \geqq m_0$ and $T_m < t$. Then we see that $T_m \geqq T_{m_0} \wedge (t/8)$. Note that for any $\varphi \in C^{\infty}_0(B(0,\varepsilon_0))$

$$(\varphi \tilde{P}^{\lambda}_t f)(x) = \varphi(x) E^{\mu}[f(\tilde{X}^{\lambda}(t,x))]$$

$$= \varphi(x) E^{\mu}[Z^{\lambda}_m f(\tilde{X}^{\lambda}(t,x))] + \sum_{k=m+1}^{\infty} \varphi(x) E^{\mu}[Z^{\lambda}_k (1 - \rho(8^{k-1}\tilde{Y}(8^{-k};\theta_{T_k}w))) f(\tilde{X}^{\lambda}(t,x))].$$

$$= \varphi(x) E^{\mu}[Z^{\lambda}_m f(X^{\lambda}(t,x))] + \sum_{k=m+1}^{\infty} \varphi(x) E^{\mu}[Z^{\lambda}_k g_k(X^{\lambda}(T_k,x); f, \lambda)].$$

Then by Theorem 13 and Proposition 19 we see that for any $u_1, u_2, \ldots, u_n \in A^{**}$ there is a constant $C > 0$ independent of $\lambda \in \Lambda$ or $t \in (0,1]$ such that

$$\sup_{x\in\mathbf{R}^N} |(\Phi^\lambda(r(u_1)\ldots r(u_n))\varphi\tilde{P}_t^\lambda f)(x)|$$

$$\leqq Ct^{-||u_1u_2\ldots u_n||/2} \sup_{x\in\mathbf{R}^N}|f(x)| + \sum_{k=m+1}^\infty CT_k^{-||u_1u_2\ldots u_n||/2} \sup_{x\in\mathbf{R}^N}|g_k(x;f,\lambda)|.$$

Then Equations (11) and (12) imply the first part of Theorem 2.

Let $\hat{Z}_{n,m}^\lambda \in W^{\infty,\infty-}$, $n > m \geqq 1$, by

$$\hat{Z}_{n,m}^\lambda = \prod_{k=m}^n \rho(8^k\tilde{Y}(8^{-k};\theta_{T_n-T_{k+1}}w)),$$

and let $\hat{Z}_m^\lambda = \lim_{n\to\infty}\hat{Z}_{n,m}^\lambda$, $m \geqq 1$. For each $t \in (0,1]$, let $m = m(t)$ be a minimum integer m such that $m \geqq m_0$ and $T_m < t$. Then we have

$$(\tilde{P}_t^\lambda f)(x) = E^\mu[f(\tilde{X}^\lambda(t,x))]$$

$$= E^\mu[\hat{Z}_m^\lambda(\theta_{t-T_m}w)f(\tilde{X}^\lambda(t,x))]$$

$$+ \sum_{k=m+1}^\infty \varphi(x)E^\mu[\hat{Z}_k^\lambda(\theta_{t-T_k}w)(1-\rho(8^{k-1}\tilde{Y}(8^{-k};\theta_{t-T_{k-1}}w))f(\tilde{X}^\lambda(t,x))].$$

So we have the last assertion similarly.

This completes the proof of Theorem 2.

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ADDRESS:
Graduate School of Mathematical Sciences, The University of Tokyo
3–8–1 Komaba Meguro-ku, Tokyo 153-8914, JAPAN
TEL +81-3-5465-7001 FAX +81-3-5465-7012