

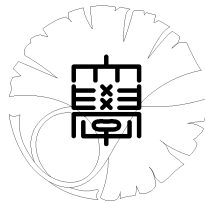
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**Second cohomology classes of the group of
 C^1 -flat diffeomorphisms of the line**

by

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SECOND COHOMOLOGY CLASSES OF THE GROUP OF C^1 -FLAT DIFFEOMORPHISMS OF THE LINE

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ABSTRACT. We study the cohomology of the group consisting of all C^∞ -diffeomorphisms of the line, which are C^1 -flat to the identity at the origin. We construct non-trivial two second real cohomology classes and uncountably many second integral homology classes of this group.

1. NOTATIONS AND MAIN RESULTS

We denote by \mathfrak{a}_1 the Lie algebra of all formal vector fields on \mathbb{R} with the Krull topology. For $k \geq 0$, we denote by \mathfrak{a}_1^k the Lie subalgebra of \mathfrak{a}_1 consisting of formal vector fields which are C^k -flat at the origin. Let $\text{Diff}_0^\infty(\mathbb{R})$ be the group of orientation-preserving C^∞ -diffeomorphisms of \mathbb{R} which fix the origin. Let $\mathcal{G}(1)$ be the group of germs of local C^∞ -diffeomorphisms at the origin of \mathbb{R} . Let $G^\infty(1)$ be the group of ∞ -jets of local C^∞ -diffeomorphisms at the origin of \mathbb{R} . For $k \geq 1$, we denote by $\text{Diff}_k^\infty(\mathbb{R})$, $\mathcal{G}_k(1)$ and $G_k^\infty(1)$ the subgroup of $\text{Diff}_0^\infty(\mathbb{R})$, $\mathcal{G}(1)$ and $G^\infty(1)$ respectively, consisting of elements which are C^k -flat to the identity at the origin. The groups $G^\infty(1)$ and $G_k^\infty(1)$ can be considered as infinite-dimensional Lie groups, whose Lie algebras are \mathfrak{a}_1^0 and \mathfrak{a}_1^k , respectively.

We define the Gel'fand-Fuks cohomology of \mathfrak{a}_1^1 in §2. It is known to be 2-dimensional for each degree ([2], [6]). Moreover, Millionschikov proved its generators in degree greater than 1 can be described by the Massey products ([4]). We carried out the calculation of the Massey products on $\text{Diff}_1^\infty(\mathbb{R})$, and we give two 2-cocycles of $\text{Diff}_1^\infty(\mathbb{R})$ in §3. For $l \geq k$ and $i \geq 2$, let α_l and α_l^i be the 1-cochains of $\text{Diff}_k^\infty(\mathbb{R})$ defined by

$$\alpha_l(f) = \frac{d^l}{dx^l} f(0) \quad \text{for } f \in \text{Diff}_k^\infty(\mathbb{R}),$$

and

$$\alpha_l^i(f) = \alpha_l(f)^i \quad \text{for } f \in \text{Diff}_k^\infty(\mathbb{R}),$$

respectively. Then the following proposition holds.

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Proposition 1.1. *The following γ_-^2 and γ_+^2 are 2-cocycles of the group $\text{Diff}_1^\infty(\mathbb{R})$.*

$$\begin{aligned}\gamma_-^2 &= -\alpha_2 \smile \beta_2 - \beta_1 \smile \tilde{\alpha}_3 \\ &= \left(\frac{1}{2}\alpha_4 - 3\alpha_2\alpha_3 + 3\alpha_2^3\right) \smile \left(\alpha_3 - \frac{3}{2}\alpha_2^2\right) - \frac{1}{2}\alpha_2 \smile \left(\alpha_3 - \frac{3}{2}\alpha_2^2\right)^2, \\ \gamma_+^2 &= -\alpha_2 \smile \beta_9 - \beta_1 \smile \beta_7 - \beta_5 \smile \beta_1 - \beta_8 \smile \tilde{\alpha}_3 \\ &= -\alpha_2 \smile \left(\frac{1}{10}\alpha_3\alpha_5 - \frac{1}{8}\alpha_4^2 - \frac{3}{20}\alpha_2^2\alpha_5\right. \\ &\quad \left. + \frac{1}{2}\alpha_2\alpha_3\alpha_4 - \frac{4}{9}\alpha_3^3 + \frac{1}{2}\alpha_2^2\alpha_3^2 - \frac{3}{4}\alpha_2^4\alpha_3 + \frac{3}{8}\alpha_2^6\right) \\ &\quad + \left(\frac{1}{2}\alpha_4 - 3\alpha_2\alpha_3 + 3\alpha_2^3\right) \smile \left(\frac{1}{10}\alpha_5 - \alpha_2\alpha_4 - \frac{1}{3}\alpha_3^2 + 4\alpha_2^2\alpha_3 - 3\alpha_2^4\right) \\ &\quad + \left(\frac{1}{5}\alpha_5 - \frac{3}{2}\alpha_2\alpha_4 - \alpha_3^2 + 6\alpha_2^2\alpha_3 - \frac{15}{4}\alpha_2^4\right) \smile \left(-\frac{1}{2}\alpha_4 + 3\alpha_2\alpha_3 - 3\alpha_2^3\right) \\ &\quad + \left(\frac{1}{30}\alpha_6 - \frac{3}{10}\alpha_2\alpha_5 - \frac{1}{2}\alpha_3\alpha_4 + \frac{3}{2}\alpha_2^2\alpha_4 + 2\alpha_2\alpha_3^2 - 5\alpha_2^3\alpha_3 + \frac{9}{4}\alpha_2^5\right) \\ &\quad \smile \left(\alpha_3 - \frac{3}{2}\alpha_2^2\right).\end{aligned}$$

Our main theorem is the following.

Theorem 1.2. *Let $\gamma^2 : H_2(\text{Diff}_1^\infty(\mathbb{R}); \mathbb{Z}) \longrightarrow \mathbb{R}^2$ be the homomorphism defined by*

$$\gamma^2(\xi) = (\gamma_-^2(\xi), \gamma_+^2(\xi)).$$

Then γ^2 is surjective.

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2. FIRST COHOMOLOGY OF $\text{Diff}_k^\infty(\mathbb{R})$

In this section, we review a result of Fukui([1]) and compute generators of $H^1(\text{Diff}_k^\infty(\mathbb{R}); \mathbb{R})$.

Definition 2.1. For a topological Lie algebra \mathfrak{g} , we denote by $A_C^*(\mathfrak{g})$ the differential graded algebra of all continuous alternating forms on \mathfrak{g} . The Gel'fand-Fuks cohomology of \mathfrak{g} is defined to be the cohomology of the complex $(A_C^*(\mathfrak{g}), d)$. Here, d is the ordinary differential mapping of cochain complexes of Lie algebras. We denote the Gel'fand-Fuks cohomology of \mathfrak{g} by $H_{GF}^*(\mathfrak{g})$.

In the case $\mathfrak{g} = \mathfrak{a}_1^k$, the complex $A_C^*(\mathfrak{a}_1^k)$ is an exterior algebra generated by $\delta^{(k+1)}, \delta^{(k+2)}, \dots$, where $\delta^{(l)}$'s are the 1-forms on \mathfrak{a}_1^k defined by

$$\delta^{(l)}\left(f(x)\frac{d}{dx}\right) = (-1)^l f^{(l)}(0) \quad \text{for } f(x) \in \mathbb{R}[[x]].$$

Since it is easily seen that $d\delta^{(l)} = 0$ if and only if $k+1 \leq l \leq 2k+1$, we obtain the following proposition.

Proposition 2.2. *For $k \geq 1$,*

$$H_{GF}^1(\mathfrak{a}_1^k) \cong \mathbb{R}^{k+1}.$$

Moreover, $\delta^{(k+1)}, \delta^{(k+2)}, \dots, \delta^{(2k+1)}$ generate $H_{GF}^1(\mathfrak{a}_1^k)$.

In particular, $H_{GF}^1(\mathfrak{a}_1^1)$ is generated by δ'' and δ''' .

On the other hand, Fukui proved a proposition about the homology of groups corresponding to \mathfrak{a}_1^k .

Theorem 2.3 (Fukui[1]). *For $k \geq 1$,*

$$H_1(\text{Diff}_k^\infty(\mathbb{R}); \mathbb{Z}) \cong \mathbb{R}^{k+1}.$$

Theorem 2.3 is obtained from the fact that the group homomorphism

$$\Psi_k : \text{Diff}_k^\infty(\mathbb{R}) \longrightarrow \mathbb{R}^{k+1}$$

defined by

$$\Psi_k(f) := \left(\frac{1}{(k+1)!} f^{(k+1)}(0), \frac{1}{(k+2)!} f^{(k+2)}(0), \dots, \frac{1}{(2k+1)!} f^{(2k+1)}(0) \right)$$

induces an isomorphism in the first homology. Here, \mathbb{R}^{k+1} means the group which is \mathbb{R}^{k+1} as a set, where the addition is defined by

$$\begin{aligned} (a_1, a_2, \dots, a_{k+1}) + (b_1, b_2, \dots, b_{k+1}) \\ = (a_1 + b_1, a_2 + b_2, \dots, a_k + b_k, a_{k+1} + b_{k+1} + (k+1)a_1b_1). \end{aligned}$$

Since Ψ_k is a group homomorphism, $\alpha_{k+1}, \alpha_{k+2}, \dots, \alpha_{2k}$ are 1-cocycles. Moreover, if we denote the cochain $\tilde{\alpha}_{2k+1}$ by

$$\tilde{\alpha}_{2k+1} = \alpha_{2k+1} - \frac{1}{2} \binom{2k+1}{k} \alpha_{k+1}^2,$$

then it is also a 1-cocycle. Hence the following proposition is obtained.

Proposition 2.4. $\alpha_{k+1}, \alpha_{k+2}, \dots, \alpha_{2k}$ and $\tilde{\alpha}_{2k+1}$ generate $H^1(\text{Diff}_k^\infty(\mathbb{R}); \mathbb{R}) \cong \mathbb{R}^{k+1}$.

In particular, $H^1(\text{Diff}_1^\infty(1); \mathbb{R}) \cong \mathbb{R}^2$ is generated by α_2 and $\tilde{\alpha}_3 = \alpha_3 - \frac{3}{2}\alpha_2^2$.

Remark 2.5. The same argument can be applied to the groups $\mathcal{G}_1(1)$ and $G_1^\infty(1)$ instead of $\text{Diff}_1^\infty(\mathbb{R})$. Hence Theorem 2.3 and Proposition 2.4 hold for $\mathcal{G}_1(1)$ and $G_1^\infty(1)$ by regarding α_l 's as the 1-cochains of $\mathcal{G}_1(1)$ and $G_1^\infty(1)$, respectively.

3. CONSTRUCTION OF THE 2-COCYCLES OF $\text{Diff}_1^\infty(\mathbb{R})$

In this section, we recall the definition of the Massey products following [3], and construct the 2-cocycles γ_\pm^2 of the group $\text{Diff}_1^\infty(\mathbb{R})$.

Definition 3.1 ([3]). Let $\mathcal{A} = (\mathcal{A}^n, d)$ be a differential graded algebra. For $u_i \in H^{p_i}(\mathcal{A})$, we set a_i a cocycle representative of u_i . We define $p(i, j)$ to be $\sum_{r=i}^j (p_r - 1)$. A collection of cochains $A = (a(i, j))$ for $1 \leq i \leq j \leq k$ and $(i, j) \neq (1, k)$ is a defining system of $\{a_1, \dots, a_k\}$ if

(i) $a(i, i) = a_i \in \mathcal{A}^{p_i}$,

(ii) $a(i, j) \in \mathcal{A}^{p(i, j)+1}$,

and

(iii) $da(i, j) = \sum_{r=i}^{j-1} (-1)^{\deg a(i, r)} a(i, r) a(r+1, j)$.

Definition 3.2 ([3]). When a defining system A of $\{a_1, \dots, a_k\}$ exists, we define $c(A) \in \mathcal{A}^{p(1, k)+2}$ by setting

$$c(A) = \sum_{r=1}^{k-1} (-1)^{\deg a(i, r)} a(1, r) a(r+1, k).$$

Then $c(A)$ is a cocycle and the set

$$\{\text{a cohomology class of } c(A); A \text{ is a defining system of } \{a_1, \dots, a_k\}\}$$

depends only on the cohomology classes u_1, \dots, u_k . We call the elements of $c(A)$ the Massey products of $\{u_1, \dots, u_k\}$.

By Goncharova's theorem([2], [6]), which gives $\dim H_{GF}^p(\mathfrak{a}_1^k)$ for any $p, k \geq 1$, we know

$$H_{GF}^p(\mathfrak{a}_1^1) \cong \mathbb{R}^2 \quad \text{for any } p.$$

Furthermore, the following theorem is known.

Theorem 3.3 (Millionschikov[4]). *For any $p \geq 2$, there exist generators $g_-^p, g_+^p \in H_{GF}^p(\mathfrak{a}_1^1)$ of $H_{GF}^p(\mathfrak{a}_1^1) \cong \mathbb{R}^2$, which are described by the Massey products. In particular, both the triple Massey product of $\{\delta'', \delta''', \delta'''\}$ and the 5-fold Massey product $\{\delta'', \delta''', \delta'', \delta'', \delta'''\}$ determine non-trivial cohomology classes in $H_{GF}^2(\mathfrak{a}_1^1)$, which are linearly independent.*

In fact, the defining systems of $\{\delta'', \delta''', \delta'''\}$ and $\{\delta'', \delta''', \delta'', \delta'', \delta'''\}$ can be written as

$$\begin{pmatrix} \delta'' & -\frac{1}{2}\delta^{(4)} & * \\ & \delta''' & 0 \\ & & \delta''' \end{pmatrix} \text{ and } \begin{pmatrix} \delta'' & -\frac{1}{2}\delta^{(4)} & -\frac{1}{5}\delta^{(5)} & -\frac{1}{30}\delta^{(6)} & * \\ & \delta''' & \frac{1}{2}\delta^{(4)} & \frac{1}{10}\delta^{(5)} & 0 \\ & & \delta'' & \frac{1}{3}\delta''' & \frac{1}{10}\delta^{(5)} \\ & & & \delta'' & -\frac{1}{2}\delta^{(4)} \\ & & & & \delta''' \end{pmatrix},$$

respectively.

Proof of Theorem 1.1. For $\text{Diff}_1^\infty(\mathbb{R})$ we checked that the defining systems of both of $\{\alpha_2, \tilde{\alpha}_3, \tilde{\alpha}_3\}$ and $\{\alpha_2, \tilde{\alpha}_3, \alpha_2, \alpha_2, \tilde{\alpha}_3\}$ also exist. In fact, they can be written as

$$\begin{pmatrix} \alpha_2 & \beta_1 & * \\ & \tilde{\alpha}_3 & \beta_2 \\ & & \tilde{\alpha}_3 \end{pmatrix} \text{ and } \begin{pmatrix} \alpha_2 & \beta_1 & \beta_5 & \beta_8 & * \\ & \tilde{\alpha}_3 & \beta_3 & \beta_6 & \beta_9 \\ & & \alpha_2 & \beta_4 & \beta_7 \\ & & & \alpha_2 & \beta_1 \\ & & & & \tilde{\alpha}_3 \end{pmatrix},$$

respectively. Here,

$$\begin{aligned} \beta_1 &= -\frac{1}{2}\alpha_4 + 3\alpha_2\alpha_3 - 3\alpha_2^3, & \beta_2 &= \frac{1}{2}\tilde{\alpha}_3^2, \\ \beta_3 &= \frac{1}{2}\alpha_4 - 2\alpha_2\alpha_3 + \frac{3}{2}\alpha_2^3, & \beta_4 &= \frac{1}{3}\alpha_3, \\ \beta_5 &= -\frac{1}{5}\alpha_5 + \frac{3}{2}\alpha_2\alpha_4 + \alpha_3^2 - 6\alpha_2^2\alpha_3 + \frac{15}{4}\alpha_2^4, \\ \beta_6 &= \frac{1}{10}\alpha_5 - \frac{1}{2}\alpha_2\alpha_4 - \frac{1}{3}\alpha_3^2 + \frac{3}{2}\alpha_2^2\alpha_3 - \frac{3}{4}\alpha_2^4, \\ \beta_7 &= \frac{1}{10}\alpha_5 - \alpha_2\alpha_4 - \frac{1}{3}\alpha_3^2 + 4\alpha_2^2\alpha_3 - 3\alpha_2^4, \\ \beta_8 &= -\frac{1}{30}\alpha_6 + \frac{3}{10}\alpha_2\alpha_5 + \frac{1}{2}\alpha_3\alpha_4 - \frac{3}{2}\alpha_2^2\alpha_4 - 2\alpha_2\alpha_3^2 + 5\alpha_2^3\alpha_3 - \frac{9}{4}\alpha_2^5, \\ \beta_9 &= \frac{1}{10}\alpha_3\alpha_5 - \frac{1}{8}\alpha_4^2 - \frac{3}{20}\alpha_2^2\alpha_5 + \frac{1}{2}\alpha_2\alpha_3\alpha_4 - \frac{4}{9}\alpha_3^3 + \frac{1}{2}\alpha_2^2\alpha_3^2 - \frac{3}{4}\alpha_2^4\alpha_3 + \frac{3}{8}\alpha_2^6. \end{aligned}$$

Following to the definition of the Massey products, we obtain Proposition 1.1 \square

4. PROOF OF THE MAIN THEOREM

In this section, we prove the non-triviality of γ_\pm^2 by constructing uncountably many 2-cycles $\xi_2^\pm \in \mathbb{Z}[\text{Diff}_1^\infty(\mathbb{R})^2]$ such that $\gamma_-^2(\xi_2^-) \neq 0$ and $\gamma_+^2(\xi_2^+) \neq 0$ and this proves Theorem 1.2. To construct ξ_2^\pm , we use the following lemma.

Lemma 4.1. *For any $k, l \geq 1 (k \neq l)$ and any $f \in \text{Diff}_{k+l}^\infty(\mathbb{R})$, there exist $g \in \text{Diff}_k^\infty(\mathbb{R})$ and $h \in \text{Diff}_l^\infty(\mathbb{R})$ such that $f = [g, h]$. Here, $[g, h]$ means $ghg^{-1}h^{-1}$.*

Moreover, for any $k, l \geq 1 (k \neq l)$ it is true that $[\text{Diff}_k^\infty(\mathbb{R}), \text{Diff}_l^\infty(\mathbb{R})] = \text{Diff}_{k+l}^\infty(\mathbb{R})$. In the case $k = l$, Fukui proved that $[\text{Diff}_k^\infty(\mathbb{R}), \text{Diff}_k^\infty(\mathbb{R})] = \text{Diff}_{2k+1}^\infty(\mathbb{R})$ for $k \geq 1$ in [1].

Proof. We may assume that $k > l$ and the ∞ -jet of f is written as

$$f(x) = x + \sum_{n=k+l+1}^{\infty} a_n x^n.$$

If we take $h \in \text{Diff}_l^\infty(\mathbb{R})$ so that h can be described as

$$h(x) = x + x^{l+1}$$

in a some neighborhood of 0, then the ∞ -jet of fh at 0 is written as

$$fh(x) = x + x^{l+1} + a_{k+l+1}x^{k+l+1} + \dots$$

Here, we apply the following proposition of the normal forms of diffeomorphisms of $(\mathbb{R}, 0)$.

Theorem 4.2 (Takens[5]). *For any $k \geq 1$ and $\psi \in \text{Diff}_k^\infty(\mathbb{R})$, there exists $\varphi \in \text{Diff}_0^\infty(\mathbb{R})$ such that*

$$\varphi\psi\varphi^{-1}(x) = x + \delta x^{k+1} + \alpha x^{2k+1},$$

in a some neighborhood of 0 for some $\delta = \pm 1$ and $\alpha \in \mathbb{R}$. Here δ and α are uniquely determined by the $(2k+1)$ -jet of ψ .

Because of the uniqueness of δ and α , there exists φ such that $\varphi^{-1}fh\varphi = h$ in some neighborhood U of 0. By Takens' construction of φ in Proposition 4.2, it is seen that one can choose φ to be C^k -flat to the identity at 0. We denote the composition $\varphi^{-1}fh\varphi$ by Φ . If we take h so that both of h and Φ have no fixed points except for 0, then Φ is conjugate to h . In the case k is even and $x < 0$, there exists an integer $n_x \geq 0$ such that $(\Phi^{-1})^n(x) \in U$ for any $n \geq n_x$ and we define $\tilde{\varphi}(x) = (h^{-1})^{n_x}\Phi^{n_x}(x)$. Otherwise, for any x there exists an integer $n_x \geq 0$ such that $\Phi^{n_x}(x) \in U$ for any $n \geq n_x$ and we define $\tilde{\varphi}(x) = \Phi^{n_x}(h^{-1})^{n_x}(x)$. Then $\tilde{\varphi}^{-1}\Phi\tilde{\varphi}$ coincides with h . If we set $g = \varphi\tilde{\varphi}$, then g is contained in $\text{Diff}_k^\infty(\mathbb{R})$ and Lemma 4.1 is proved. \square

Proof of Theorem 1.2. If $f, g \in \text{Diff}_1^\infty(\mathbb{R})$ and the ∞ -jet of them are written as

$$f(x) = x + \sum_{n=2}^{\infty} a_n x^n, \quad g(x) = x + \sum_{n=2}^{\infty} b_n x^n,$$

then

$$\gamma_-^2(f, g) = 36(b_3 - b_2^2)(2a_4 - 6a_2a_3 + 4a_2^3 - a_2b_3 + a_2b_2^2).$$

Thus if $f_i \in \text{Diff}_i^\infty(\mathbb{R})$ for $i = 1, 2, 3, 4$, then $\gamma_-^2(f_2, f_3) = \gamma_-^2(f_1, f_4) = \gamma_-^2(f_4, f_1) = 0$. On the other hand, if both of the coefficient of x^4 in the jet of f_3 and the coefficient of x^3 in the jet of f_2 are non-zero, then $\gamma_-^2(f_3, f_2) \neq 0$. Therefore, we assume $f_3 \in \text{Diff}_3^\infty(\mathbb{R}) \setminus \text{Diff}_4^\infty(\mathbb{R})$ and $f_2 \in \text{Diff}_2^\infty(\mathbb{R}) \setminus \text{Diff}_3^\infty(\mathbb{R})$. By Lemma 4.1,

we can choose $f_1 \in \text{Diff}_1^\infty(\mathbb{R})$ and $f_4 \in \text{Diff}_4^\infty(\mathbb{R})$ such that $[f_2, f_3] = [f_1, f_4]$. If we set

$$\begin{aligned} \xi_2^- := & (f_3, f_2) - (f_2, f_3) + ([f_2, f_3], f_3 f_2) \\ & - (f_4, f_1) + (f_1, f_4) - ([f_1, f_4], f_4 f_1) \in \mathbb{Z}[\text{Diff}_1^\infty(\mathbb{R})^2], \end{aligned}$$

then ξ_2^- is a cycle and

$$\gamma_2^-(\xi_2^-) = \gamma_2^-((f_3, f_2)) = \frac{1}{2} \alpha_4(f_3) \alpha_3(f_2) \neq 0.$$

Therefore, the non-triviality of γ_2^2 is proved.

The non-triviality of γ_+^2 can be proved similarly. Let $g_3 \in \text{Diff}_3^\infty(\mathbb{R}) \setminus \text{Diff}_4^\infty(\mathbb{R})$ and $g_4 \in \text{Diff}_4^\infty(\mathbb{R}) \setminus \text{Diff}_5^\infty(\mathbb{R})$. If we choose $g_1 \in \text{Diff}_1^\infty(\mathbb{R})$ and $g_6 \in \text{Diff}_6^\infty(\mathbb{R})$ such that $[g_3, g_4] = [g_1, g_6]$ and set

$$\begin{aligned} \xi_2^+ := & (g_4, g_3) - (g_3, g_4) + ([g_3, g_4], g_4 g_3) \\ & - (g_6, g_1) + (g_1, g_6) - ([g_1, g_6], g_6 g_1) \in \mathbb{Z}[\text{Diff}_1^\infty(\mathbb{R})^2], \end{aligned}$$

then ξ_2^+ is a cycle and

$$\gamma_+^2(\xi_2^+) = \gamma_+^2((g_4, g_3) - (g_3, g_4)) = -\frac{1}{20} \alpha_5(g_4) \alpha_4(g_3) \neq 0.$$

Consequently, γ_\pm^2 are non-trivial cohomology classes in $H^2(\text{Diff}_1^\infty(\mathbb{R}); \mathbb{R})$. Furthermore, clearly $\gamma_-^2(\xi_2^+) = 0$ and $\gamma_-^2(\xi_2^-)$ can take any real value by changing f_2 or f_3 . Similarly, $\gamma_+^2(\xi_2^+)$ also can take any value. This concludes the proof of Theorem 1.2. \square

Moreover, the following corollary holds.

Corollary 4.3. *There exist uncountably many isomorphism classes of flat \mathbb{R} -bundles on genus 2 surface Σ_2 , such that the image of their holonomy homomorphisms*

$$\pi_1(\Sigma_2) \longrightarrow \text{Diff}^\infty(\mathbb{R})$$

are contained in $\text{Diff}_1^\infty(\mathbb{R})$.

Remark 4.4. The same discussion in §3, and §4 can be applied to the group $\mathcal{G}_1(1)$ and $G_1^\infty(1)$ instead of $\text{Diff}_1^\infty(\mathbb{R})$. Therefore, we can regard γ_\pm^2 as the 2-cochains of $\mathcal{G}_1(1)$ and $G_1^\infty(1)$, and Theorem 1.2 also holds for $\mathcal{G}_1(1)$ and $G_1^\infty(1)$, respectively.

On the other hand, for any group G and commuting $f, g \in G$, the chain $(f, g) - (g, f)$ is the simplest 2-cycle of G . However, if we regard γ_\pm^2 as the 2-cocycles of $G_1^\infty(1)$, then it is seen that $\gamma_\pm^2((f, g) - (g, f)) = 0$ for any commuting $f, g \in G_1^\infty(1)$. Hence the following is true.

Proposition 4.5. *For any group homomorphism $\rho : \pi_1(T^2) \longrightarrow G_1^\infty(1)$,*

$$\rho^* \gamma_\pm^2 = 0.$$

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