UTMS 2010–11 August 27, 2010

Torus fibrations and localization of index III – equivariant version and its applications

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TORUS FIBRATIONS AND LOCALIZATION OF INDEX III
- EQUIVARIANT VERSION AND ITS APPLICATIONS

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Abstract. This paper is the third of the series concerning the localization of the index of Dirac-type operators. In our previous papers we gave a formulation of index of Dirac-type operators on open manifolds under some geometric setting, whose typical example was given by the structure of a torus fiber bundle on the ends of the open manifolds. We introduce two equivariant versions of the localization. As an application we give a proof of Guillemin-Sternberg's quantization conjecture in the case of torus action.

Contents

2000 *Mathematics Subject Classification.* Primary 19K56; Secondary 58J05, 53D50, 58J20.
Key words and phrases. equivariant index, localization, quantization conjecture.
[1]Partly supported by Grant-in-Aid for Young Scientists (Start-up) 21840045.
[2]Partly supported by Grant-in-Aid for Scientific Research (A) 19204003 and Grant-in-Aid for Scientific Research (B) 19340015.
[3]Partly supported by Grant-in-Aid for Young Scientists (B) 22740046, and Fujyukai Foundation.

1. Introduction

This is the third of the series concerning a localization of the index of elliptic operators. The localization of integral is a mechanism by which the integral of a differential form on a manifold becomes equal to the integral of another differential form on a submanifold, which has been formulated under various geometric settings. The *submanifold* is either an open submanifold or a closed submanifold. When it is an open submanifold, the localization is closely related to some *excision formula*. When it is a closed submanifold, the localization is usually obtained by applying some *product formula* to the normal bundle of the submanifold after the localization to the open tubular neighborhood.

A typical geometric setting for such localization is given by action of compact Lie group, and a localization is formulated in terms of the equivariant de Rham cohomology groups. An example is Duistermaat and Heckman's formula on a symplectic manifold. It is formally possible to replace the equivariant de Rham cohomology groups with the equivariant K cohomology groups, and the resulting localization in terms of the equivariant K-cohomology groups is known as Atiyah-Segal's Lefschetz formula of equivariant index.

In our previous papers [3, 4] the geometric setting ensuring our localization is typically given by the structure of a torus fiber bundle. Under this setting we consider the Riemann-Roch number or the index of the Dolbeault operator or the Dirac-type operator, associated to an almost complex structure or a spin^c structure, which is twisted by some vector bundle. We do not assume any global group action. Instead, on the vector bundle, we assume a family of flat connections of the fibers of the torus bundle. Our setting has generalized from the setting of a single torus bundle structure to the case that we have a finite open covering and a family of torus bundle structures on the open sets which satisfy some compatibility condition. The dimensions of the fibers of the family of torus bundle structures can vary. This generalization was necessary to formulate a product formula in a full form [4], and the product formula is used to compute the local contribution in some examples.

In this paper we introduce the equivariant version of our localization. When a compact Lie group G acts on everything, it is straightforward to generalize our previous argument. The index takes values in the character ring $R(G)$ and we have the Riemann-Roch character. We go further. Suppose two compact Lie group G and K acts on everything simultaneously and assume that their actions are commutative. In this paper we formulate another type of equivariant version as follows. The main assumption of our previous papers in our geometric setting was the vanishing of the de Rham cohomology groups with some local coefficients on each fiber of the torus bundles. Our new setting is given by weaken the assumption. Roughly speaking our new assumption is that only the G-invariant part of the de Rham cohomology groups vanish. Under this new weaker assumption, the full $G \times K$-equivariant index is not well defined. Instead only the G-invariant part of the $G \times K$-equivariant index is well defined as an element of the character ring of K.

As an application of the latter equivariant version we give a proof of Guillemin-Sternberg's quantization conjecture in the case of torus action.

Our localization is basically a purely topological statement. It would be required to formulate it as the equality between topological index and analytical index. The definition of topological index is, however, not given at the present.

In this paper we work in the smooth category. In Section 2 we first describe the orbifold version of our localization in the previous papers [3, 4]. We give several definitions under the same names as in the previous papers, though the notions are generalized as well as the propositions there. In the latter part of Section 2 we introduce group actions and give our main theorem (Theorem 2.43). In Section 3, as a typical example of our setting, we explain the construction using an action of a torus G with a simultaneous action of a compact Lie group K on an almost complex manifold. In Section 4, as a preparation of the proof of quantization conjecture, we show a vanishing property of the G-invariant part of the equivariant Riemann-Roch number when G is S^1 under some condition. In Section 5 we give a proof of quantization conjecture for torus action.

2. Equivariant local index

In this section we introduce several versions of indices and describe their localization theorems. Note that the definitions of compatible fibrations and compatible systems given in this section are generalized versions of those given in [4].

2.1. Compatible fibration.

Let M be a manifold.

Definition 2.1. A *compatible fibration on M* is a collection of the data $\{V_\alpha, \mathscr{F}_\alpha\}_{\alpha\in A}$ consisting of an open covering $\{V_\alpha\}_{\alpha\in A}$ of M and a foliation $\mathscr{F}_\alpha$ on V_α with compact leaves which satisfies the following properties.

(1) The holonomy group of each leaf of $\mathscr{F}_\alpha$ is finite.
(2) For each α and β, if a leaf $L \in \mathscr{F}_\alpha$ has non-empty intersection $L\cap V_\beta \neq \emptyset$, then, $L \subset V_\beta$.

Let $\{V_\alpha, \mathscr{F}_\alpha\}_{\alpha\in A}$ be a compatible fibration on M.

Definition 2.2. An open covering $\{V'_\alpha\}_{\alpha\in A}$ of M is said to be *admissible* if it satisfies the following conditions

(1) for each $\alpha\in A$, $V'_\alpha \subset V_\alpha$,
(2) for each α and β all leaves $L\in\mathscr{F}_\beta$ with $L\cap V'_\alpha\neq\emptyset$ are contained in V'_α.

Later we take and fix an admissible open covering satisfying some good property. See Assumption 2.7.

Suppose there exists an admissible open covering $\{V'_\alpha\}_{\alpha\in A}$ of M. We take and fix it. By the condition (2) in Definition 2.2 we can restrict the foliation $\mathscr{F}_\alpha$ to V'_α for each $\alpha\in A$. We set

$$\mathscr{F}_\alpha|_{V'_\alpha} = \{L\in\mathscr{F}_\alpha \mid L\cap V'_\alpha\neq\emptyset\}.$$

Definition 2.3. A subset C of M is said to be *admissible for* $\{V'_\alpha\}$ if, on each $V'_\alpha\cap C\neq\emptyset$, C contains all leaves $L\in\mathscr{F}_\alpha|_{V'_\alpha}$ which intersect with C.

For an admissible subset C we define the foliation $\mathscr{F}_\alpha|_C$ on $C\cap V'_\alpha$ by

$$\mathscr{F}_\alpha|_C = \{L\in\mathscr{F}_\alpha|_{V'_\alpha} \mid L\cap C\neq\emptyset\}.$$

Proposition 2.4. *Let C be an admissible submanifold for $\{V'_\alpha\}$ of M. Then, $\{C\cap V'_\alpha, \mathscr{F}_\alpha|_C\}_\alpha$ is a compatible fibration on C.*

Definition 2.5. A function $f : M \to \mathbb{R}$ is said to be *admissible for* $\{V'_\alpha\}$ if f is constant along leaves of $\mathscr{F}_\alpha|_{V'_\alpha}$ for all $\alpha \in A$.

Definition 2.6. An *averaging operation for* $\{V'_\alpha\}$ is a linear map $I\colon C^\infty(M) \to C^\infty(M)$ which satisfies the following properties.

(1) $I(f)$ is an admissible function for all $f \in C^\infty(M)$.
(2) If f is a constant function, then $I(f) = f$.
(3) If f is a non-negative function, so is $I(f)$.
(4) There exists an open covering $\{V''_\alpha\}_{\alpha\in A}$ of M which satisfies the following properties.
- For all $\alpha \in A$ we have $\overline{V''_\alpha} \subset V_\alpha$.
- For all $\alpha \in A$ if a leaf $L \in \mathscr{F}_\alpha$ has non-empty intersection $L \cap V''_\alpha \neq \emptyset$, then, $L \subset V''_\alpha$.
- For all $f \in C^\infty(M)$ and $x \in M$ there exists some $\alpha \in A$ such that $x \in V''_\alpha$ and
$$\min_{y\in L_\alpha} f(y) \leq I(f)(x) \leq \max_{y\in L_\alpha} f(y),$$
where $L_\alpha \in \mathscr{F}_\alpha$ is the leaf which contains x.

(5) Let $f : M \to \mathbb{R}$ be a smooth function. If $\operatorname{supp} f$ is contained in V'_α for some $\alpha \in A$, then $\operatorname{supp} I(f)$ is also contained in V'_α.

In the rest of this article we impose the following technical assumptions on $\{V_\alpha, \mathscr{F}_\alpha\}_{\alpha\in A}$ unless otherwise stated.

Assumption 2.7. (1) The index set A is finite.
(2) There exists an admissible open covering $\{V'_\alpha\}_{\alpha\in A}$ of M such that $\overline{V'_\alpha} \subset V_\alpha$, and we fix it. Hereafter we say *admissible* (resp. an *averaging operation*) instead of admissible (resp. an *averaging operation*) for $\{V'_\alpha\}$.
(3) There is an averaging operation $I : C^\infty(M) \to C^\infty(M)$.

Remark 2.8. (1) [4, Appendix B] we give a sufficient condition in order that an admissible open covering as in (2) of Assumption 2.7 exists for the smooth case. Although it would be true for the orbifold case, we do not pursue it here.
(2) For a torus action with finite isotropy types we can construct a compatible fibration that satisfies Assumption 2.7 by [4, Proposition 2.12 and Proposition 2.28].

Then, as is shown in [4, Lemma 2.13], we have the following *admissible partition of unity.*

Lemma 2.9 (Existence of admissible partition of unity [4, Lemma 2.13])**.** *Let M be a manifold with a compatible fibration $\{V_\alpha, \mathscr{F}_\alpha\}$. There is a smooth partition of unity $\{\rho_\alpha^2\}$ subordinate to the open covering $M = \cup_\alpha V'_\alpha$ such that each ρ_α is admissible.*

2.2. **Compatible system and strong acyclicity.** Let (M, g) be a Riemannian manifold, W a $\mathbb{Z}/2$-graded $Cl(TM)$-module bundle on M, and V an open subset of M equipped with a compatible fibration $\{V_\alpha, \mathscr{F}_\alpha\}_{\alpha\in A}$. In the rest of this paper we impose the following conditions on the Riemannian metric g.

Assumption 2.10. Let $\nu_\alpha = \{u \in TV_\alpha \mid g(u, v) = 0 \text{ for all } v \in T\mathscr{F}_\alpha\}$ be the normal bundle of $\mathscr{F}_\alpha$. Then, $g|_{\nu_\alpha}$ is invariant under holonomy, and gives a transverse invariant metric on ν_α.

Remark 2.11. The above Riemannian metric is an orbifold version of a compatible Riemannian metric which is actually used in [4].

Definition 2.12. A *compatible system* on $(\{V_\alpha, \mathscr{F}_\alpha\}, W|_V)$ is a data $\{D_\alpha\}_{\alpha\in A}$ satisfying the following properties.

(1) $D_\alpha\colon \Gamma(W|_{V_\alpha}) \to \Gamma(W|_{V_\alpha})$ is an order-one formally self-adjoint differential operator of degree-one.
(2) D_α contains only the derivatives along leaves of $\mathscr{F}_\alpha$.
(3) The principal symbol $\sigma(D_\alpha)$ of D_α is given by $\sigma(D_\alpha) = c \circ p_\alpha \circ \iota_\alpha^*\colon T^*V_\alpha \to \mathrm{End}(W|_{V_\alpha})$, where $\iota_\alpha\colon T\mathscr{F}_\alpha \to TV_\alpha$ is the natural inclusion from the tangent bundle along leaves of $\mathscr{F}_\alpha$ to TV_α, $p_\alpha\colon T^*\mathscr{F}_\alpha \to T\mathscr{F}_\alpha$ is the isomorphism induced by the Riemannian metric and $c\colon T\mathscr{F}_\alpha \to \mathrm{End}(W|_{V_\alpha})$ is the Clifford multiplication.
(4) For a leaf $L \in \mathscr{F}_\alpha$ let $\widetilde{u} \in \Gamma(\nu_\alpha|_L)$ be a section of $\nu_\alpha|_L$ parallel along L. $\widetilde{u}$ acts on $W|_L$ by the Clifford multiplication $c(\widetilde{u})$. Then D_α and $c(\widetilde{u})$ anti-commute each other, i.e.
$$0 = \{D_\alpha, c(\widetilde{u})\} := D_\alpha \circ c(\widetilde{u}) + c(\widetilde{u}) \circ D_\alpha$$
as an operator on $W|_L$.

The above definitions of the compatible fibration and the compatible system are introduced to avoid dealing with the orbifold singularities directly which come from finite isotropies of a torus action. The following lemma guarantees that we have an orbifold chart for each leaf, and analytic estimates for the compatible system holds as in [4, Section 4] by considering the pull-back of D_α's by q_L in next Lemma 2.13.

Lemma 2.13. *Suppose that $\mathscr{F}$ is a foliation on a manifold V and L is a leaf with finite holonomy group. Take a small open tubular neighborhood V_L of L which is a union of leaves. Then, there is a finite covering $q_L\colon \widetilde{V}_L \to V_L$ whose covering transformation is given by the holonomy representation. Moreover, such covering is unique up to isomorphism.*

Remark 2.14. By taking and fixing a point $x \in L$ we can obtain a holonomy representation of the fundamental group $\pi_1(L, x)$ of L with the base point x. The covering $q_L\colon \widetilde{V}_L \to V_L$ is constructed by using this holonomy representation. By construction the induced foliation on $\widetilde{V}_L$ is a bundle foliation. The covering $q_L\colon \widetilde{V}_L \to V_L$ in Lemma 2.13 is characterized by the following condition: The foliation on $\widetilde{V}_L$ is a bundle foliation and a generic leaf in V_L is diffeomorphic to some leaf in $\widetilde{V}_L$ by q_L.

Remark 2.15. Suppose that in Lemma 2.13 all holonomy groups of $\mathscr{F}$ are finite. By Lemma 2.13, on a neighborhood of each leaf L the covering q_L is determined up to isomorphism. But, these coverings depend on the choices of the base points of the holonomy representations and are not determined canonically. In particular these coverings are not necessarily patched together globally.

We will use the notations V_L, $\widetilde{V}_L$, and q_L in the following definition. We also denote the projection map of the fiber bundle structure by $\pi_L\colon \widetilde{V}_L \to \widetilde{U}_L$.

Definition 2.16. A compatible system $\{D_\alpha\}_{\alpha\in A}$ on $(\{V_\alpha, \mathscr{F}_\alpha\}, W)$ is said to be *strongly acyclic* if it satisfies the following conditions.

(1) The Dirac type operator $q_L^* D_\alpha|_{\pi_L^{-1}(\widetilde{b})}$ has zero kernel for each $\alpha \in A$, leaf $L \in \mathscr{F}_\alpha$ and $\widetilde{b} \in \widetilde{U}_L$.

(2) If $V_\alpha \cap V_\beta \neq \emptyset$, then the anti-commutator $\{D_\alpha, D_\beta\}$ is a non-negative operator on $V_\alpha \cap V_\beta$.

Remark 2.17. The second condition in Definition 2.16 gives a strong restriction on how the leaves $L_\alpha \in \mathscr{F}_\alpha$ and $L_\beta \in \mathscr{F}_\beta$ intersect on $V_\alpha \cap V_\beta$. See [4, Subsection 2.3] for a concrete example.

2.3. **Definition of** $\mathrm{ind}(M, V, W)$**.** One of the main result of [4] is the following theorem.

Theorem 2.18 ([4])**.** *Let (M, g) be a possibly non-compact Riemannian manifold and W a $\mathbb{Z}/2$-graded $Cl(TM)$-module bundle on M. Suppose that V is an open set of M that satisfies the following conditions.*

(1) *$M \smallsetminus V$ is compact.*
(2) *V is equipped with a compatible fibration $\{V_\alpha, \mathscr{F}_\alpha\}_{\alpha \in A}$.*
(3) *There exists a strongly acyclic compatible system $\{D_\alpha\}_{\alpha \in A}$ on $(\{V_\alpha, \mathscr{F}_\alpha\}, W|_V)$.*

Then there exists an integer $\mathrm{ind}(M, V, W) = \mathrm{ind}(M, V, W, \{V_\alpha, \mathscr{F}_\alpha\}, \{D_\alpha\})$ *depending on the all data such that* $\mathrm{ind}(M, V, W)$ *has the following properties.*

(1) $\mathrm{ind}(M, V, W)$ *is invariant under continuous deformations of the data.*
(2) *If M is closed, then* $\mathrm{ind}(M, V, W)$ *is equal to the index* $\mathrm{ind}\, D$ *of a Dirac-type operator D on W.*
(3) *If V' is an admissible open subset of V with complement $M \smallsetminus V'$ compact, then we have*
$$\mathrm{ind}(M, V, W) = \mathrm{ind}(M, V', W).$$
(4) *If M' is an open neighborhood of $M \smallsetminus V$ with $V \cap M'$ admissible, then* $\mathrm{ind}(M, V, W)$ *has the following excision property*
$$\mathrm{ind}(M, V, W) = \mathrm{ind}(M', V \cap M', W|_{M'}).$$
(5) *Suppose M is a disjoint union $M = M_1 \coprod M_2$. Then we have the following sum formula*
$$\mathrm{ind}(M, V, W) = \mathrm{ind}(M_1, V \cap M_1, , W|_{M_1}) + \mathrm{ind}(M_2, V \cap M_2, W|_{M_2}).$$
(6) *We have a product formula for* $\mathrm{ind}(M, V, W)$*. For the precise statement see* [4, Theorem 5.8]*. See also Section 4.2.*

In the rest of this paper, for simplicity, we sometimes use the notation $\mathrm{ind}(M, W)$ for $\mathrm{ind}(M, V, W)$. It would be no confusion since $\mathrm{ind}(M, V, W)$ have the excision property.

Remark 2.19. From the properties 2 and 5 in Theorem 2.18 we can show that $\mathrm{ind}(M, V, W)$ also has the following vanishing property.

(7) *If $M = V$, then we have*
$$\mathrm{ind}(M, V, W) = 0.$$

The outline of the construction of $\mathrm{ind}(M, V, W)$ is as follows. First we deform (M, V, W) in order to get the completeness of the Riemannian manifold. Then, we construct a Fredholm operator on $\Gamma(W)$ by perturbing a Dirac-type operator on $\Gamma(W)$ with D_α's. We define $\mathrm{ind}(M, V, W)$ to be the index of the Fredholm operator.

Here, we extend the argument in [4, Section 4] to our generalized setting. The precise construction is as follows. Since the argument here is parallel to that in [4, Section 4] we omit proofs. See [4, Section 4] for more details. Take a Dirac-type operator D on $\Gamma(W)$ and a non-negative real number $t \geq 0$. We define the operator D_t acting on $\Gamma(W)$ by

$$D_t := D + t \sum_{\alpha \in A} \rho_\alpha D_\alpha \rho_\alpha, \tag{2.1}$$

where $\{\rho_\alpha^2\}_{\alpha \in A}$ is an admissible partition of unity subordinate to $\{V_\alpha\}_{\alpha \in A}$ as in Lemma 2.9. By [4, Proposition 4.5] D_t is elliptic for all $t \geq 0$.

First we give a definition of $\mathrm{ind}(M, V, W)$ for the special case that M has a cylindrical end V and every data is translationally invariant on the end.

Definition 2.20. Suppose that M has a cylindrical end $V = N \times (0, \infty)$. The compatible fibration $\{V_\alpha, \mathscr{F}_\alpha\}$ is said to be *translationally invariant* if there exists a compatible fibration $\{N_\alpha, \hat{\mathscr{F}}_\alpha\}_{\alpha \in A}$ on N such that $\{V_\alpha, \mathscr{F}_\alpha\}$ is the product of $\{N_\alpha, \hat{\mathscr{F}}_\alpha\}_{\alpha \in A}$ and the compatible fibration $\{(0, \infty), \mathscr{F}_{(0,\infty)}\}$ on $(0, \infty)$, where $\mathscr{F}_{(0,\infty)}$ is the trivial foliation consisting of 0-dimensional leaves.

Definition 2.21. Suppose that M has a cylindrical end $V = N \times (0, \infty)$ and the compatible fibration $\{V_\alpha, \mathscr{F}_\alpha\}$ is translationally invariant. The compatible system $\{D_\alpha\}_{\alpha \in A}$ on $(\{V_\alpha, \mathscr{F}_\alpha\}, W)$ is said to be *translationally invariant* if $W|_{V_\alpha}$ and the operator D_α are invariant under the translation action of the $(0, \infty)$-dierction for each $\alpha \in A$.

Proposition 2.22. *Under the assumption in Theorem 2.18 suppose that M has a cylindrical end $V = N \times (0, \infty)$ and the compatible fibration $\{V_\alpha, \mathscr{F}_\alpha\}$ and the strongly acyclic compatible system $\{D_\alpha\}_{\alpha \in A}$ on $(\{V_\alpha, \mathscr{F}_\alpha\}, W)$ are translationally invariant. Then for a sufficiently large $t \gg 0$, the space of L^2-solutions of $D_t s = 0$ is finite dimensional and its super-dimension is independent of a sufficiently large $t \gg 0$ and any other continuous deformations of data.*

Definition 2.23. In the case of Proposition 2.22 we define the $\mathrm{ind}(M, V, W)$ to be the super-dimension of the space of L^2-solutions of $D_t s = 0$

$$\mathrm{ind}(M, V, W) := \dim \ker D_t^0 \cap L^2(M, W) - \dim \ker D_t^1 \cap L^2(M, W)$$

for a sufficiently large $t \gg 0$.

For the general end case, we have the following proposition.

Proposition 2.24. *For given (M, V, W) and the compatible fibration $\{V_\alpha, \mathscr{F}_\alpha\}$ and the strongly acyclic compatible system $\{D_\alpha\}_{\alpha \in A}$ on $(\{V_\alpha, \mathscr{F}_\alpha\}, W)$ we can deform them to (M', V', W') so that it has a cylindrical end with a translationally invariant strongly acyclic compatible system.*

Definition 2.25. For the general case we define the $\mathrm{ind}(M, V, W)$ to be $\mathrm{ind}(M', V', W')$ for the deformed data (M', V', W').

Note that $\mathrm{ind}(M, V, W)$ is well-defined, i.e, it does not depend on various choice of the construction.

To obtain a product formula we need to formulate and define $\mathrm{ind}(M, V, W)$ for a manifold whose end is the total space of a fiber bundle such that both of its base space and its fiber are manifolds with cylindrical end. A similar generalization is necessary for $\mathrm{ind}^G(M, V, W)$ which will be defined in Subsection 2.5. For more details, see [4].

As a corollary of Theorem 2.18 we have a localization theorem. See [4, Theorem 4.21].

Corollary 2.26. *Under the assumption of Theorem 2.18, suppose that there exists an open covering $\{O_i\}_{i=1}^m$ of $M \smallsetminus V$ which satisfies the following properties.*

1. *$\{O_i\}_{i=1}^m$ are mutually disjoint.*
2. *Each O_i is admissible.*

Then we have

$$\mathrm{ind}(M,V,W) = \sum_i \mathrm{ind}(O_i, O_i \cap V, W_{O_i}).$$

2.4. **Definition of** $\mathrm{ind}_K(M,V,W)$**.** Let K be a compact Lie group. In this section we consider the K-equivariant case. First we rigorously describe the assumption on a K-action since we deal with the orbifold setting. Let (M,g), W, and V be the data satisfying the assumption (1) in Theorem 2.18. Suppose that there exists an action of K on M which preserves all these data.

Definition 2.27. Let $\{V_\alpha, \mathscr{F}_\alpha\}_{\alpha\in A}$ be a compatible fibration on V. $\{V_\alpha, \mathscr{F}_\alpha\}_{\alpha\in A}$ is said to be K-*equivariant* if it satisfies the following conditions.

1. The K-action preserves V_α's.
2. The K-action preserves the foliation $\mathscr{F}_\alpha$ on V_α. We allow that the K-action sends a leaf to another leaf.
3. The K-action preserves the admissible open covering $\{V'_\alpha\}$ in (2) of Assumption 2.7.
4. We can take the averaging operation I in 3 of Assumption 2.7 to be K-equivariant.

Remark 2.28. The condition (4) in Definition 2.27 is realized if on each $V_\alpha \cap V_\beta \neq \emptyset$ $L_\alpha \in \mathscr{F}_\alpha$ and $L_\beta \in \mathscr{F}_\beta$ have non-empty intersection, then, $L_\alpha \subset L_\beta$, or $L_\beta \subset L_\alpha$.

Definition 2.29. Let $\{D_\alpha\}_{\alpha\in A}$ be a compatible system on $(\{V_\alpha, \mathscr{F}_\alpha\}, W)$. $\{D_\alpha\}_{\alpha\in A}$ is said to be K-*equivariant* if it satisfies the following conditions.

1. $\{V_\alpha, \mathscr{F}_\alpha\}_{\alpha\in A}$ is K-equivariant.
2. For each $\alpha \in A$ D_α commutes with the K-action on $\Gamma(W|_{V_\alpha})$ given by pull-back.

Let $\{V_\alpha, \mathscr{F}_\alpha\}_{\alpha\in A}$ be a K-equivariant compatible fibration on V and $\{D_\alpha\}_{\alpha\in A}$ a K-equivariant strongly acyclic compatible system on $(\{V_\alpha, \mathscr{F}_\alpha\}, W)$.

Proposition 2.30. *Suppose that M has a cylindrical end $V = N \times (0,\infty)$ and the compatible fibration $\{V_\alpha, \mathscr{F}_\alpha\}$ and the strongly acyclic compatible system $\{D_\alpha\}_{\alpha\in A}$ on $(\{V_\alpha, \mathscr{F}_\alpha\}, W)$ are translationally invariant. Suppose also that K acts trivially on all the data on the $(0,\infty)$-factor. Then,* $\mathrm{ind}(M,V,W)$ *is defined by Definition 2.23 and it becomes a virtual K-representation. We denote it by* $\mathrm{ind}_K(M,V,W)$.

For the general end case, we have the following proposition.

Proposition 2.31. *For given (M,V,W) and the compatible fibration $\{V_\alpha, \mathscr{F}_\alpha\}$ and the strongly acyclic compatible system $\{D_\alpha\}_{\alpha\in A}$ on $(\{V_\alpha, \mathscr{F}_\alpha\}, W)$, we can deform them to (M',V',W') so that M' has a cylindrical end $V' = N' \times (0,\infty)$ with a translationally invariant strongly acyclic compatible system and K acts trivially on all the data on the $(0,\infty)$-factor.*

Definition 2.32. By Proposition 2.30 and Proposition 2.31 we obtain the virtual K-representation $\mathrm{ind}_K(M',V',W')$. We define the virtual K-representation $\mathrm{ind}_K(M,V,W)$ to be $\mathrm{ind}_K(M',V',W')$.

For these data, we obtain an equivariant version of [4, Theorem 4.21] for $\mathrm{ind}_K(M,V,W)$.

Theorem 2.33. *Let (M,g) be a possibly non-compact Riemannian manifold, W a $\mathbb{Z}/2$-graded $Cl(TM)$-module bundle on M, and V an open set of M with complement $M\smallsetminus V$ compact. Let K be a compact Lie group which acts on M preserving all these data. Let $\{V_\alpha,\mathscr{F}_\alpha\}_{\alpha\in A}$ be a K-equivariant compatible fibration on V and $\{D_\alpha\}_{\alpha\in A}$ a K-equivariant strongly acyclic compatible system on $(\{V_\alpha,\mathscr{F}_\alpha\},W)$. Under the assumption suppose that there exists an open covering $\{O_i\}_{i=1}^m$ of $M\smallsetminus V$ which satisfies the following properties.*

1. *$\{O_i\}_{i=1}^m$ are mutually disjoint.*
2. *Each O_i is K-invariant.*
3. *Each O_i is admissible.*

Then we have the following localization formula for $\mathrm{ind}_K(M,V,W)$

$$\mathrm{ind}_K(M,V,W)=\sum_i \mathrm{ind}_K(O_i,O_i\cap V,W_{O_i})\in R(K), \tag{2.2}$$

where $R(K)$ is the representation ring of K.

When another compact Lie group G acts on M that satisfies the same assumption as that on the K-action in Theorem 2.33 and that commutes with the K-action, $\mathrm{ind}(M,V,W)$ becomes a virtual $G\times K$-representation. In particular, the G-invariant part of $\mathrm{ind}(M,V,W)$ is also a virtual K-representation. Then, by taking the G-invariant part of (2.2), we have a localization for the G-invariant part of $\mathrm{ind}(M,V,W)$ as the virtual K-representation. In the next subsection, we introduce another condition on acyclicity of $\{D_\alpha\}_{\alpha\in A}$ which is weaker than strongly acyclic condition. Under this weaker acyclic condition we give a localization formula for the G-invariant part of $\mathrm{ind}(M,V,W)$.

2.5. **G-acyclic compatible system and $\mathrm{ind}^G(M,V,W)$.** Let (M,g), W, and V be the data satisfying the assumption (1) in Theorem 2.18. Suppose that there is an action of a compact Lie group G on M which preserves all these data. We introduce the notion of a G-acyclic compatible system. Let $\{V_\alpha,\mathscr{F}_\alpha\}_{\alpha\in A}$ be a G-equivariant compatible fibration on V. For each $(V_\alpha,\mathscr{F}_\alpha)$ let G_α be the subgroup of G consisting of the elements preserving each leaf of $\mathscr{F}_\alpha$.

Lemma 2.34. *Let L_α be a leaf of $\mathscr{F}_\alpha$. Let V_{L_α}, $\widetilde{V}_{L_\alpha}$, and q_{L_α} be the data as in Lemma 2.13. Then, $q_{L_\alpha}\colon \widetilde{V}_{L_\alpha}\to V_{L_\alpha}$ has the unique structure of G_α-equivariant covering such that for any generic leaf $L'_\alpha\subset V_{L_\alpha}$, the G_α-action preserves $q_\alpha^{-1}(L'_\alpha)$, and the diffeomorphism $q_{L_\alpha}|_{q_{L_\alpha}^{-1}(L'_\alpha)}\colon q_{L_\alpha}^{-1}(L'_\alpha)\to L'_\alpha$ is G_α-equivariant.*

Proof. From Remark 2.14, $\widetilde{V}_{L_\alpha}$ is identified with the fiber product of the covering $L'_\alpha\subset V_{L_\alpha}\to L_\alpha$ and the projection $V_{L_\alpha}\to L_\alpha$. The G_α-action is constructed using the G_α-action on L'_α and V_{L_α}. ☐

Definition 2.35. Let $\{D_\alpha\}_{\alpha\in A}$ a G-equivariant compatible system on $(\{V_\alpha,\mathscr{F}_\alpha\},W)$. $\{D_\alpha\}_{\alpha\in A}$ is said to be *G-acyclic* if it satisfies the following conditions.

1. The G_α-invariant part $\ker(q_L^*D_\alpha|_{\pi_L^{-1}(\widetilde{b})})^{G_\alpha}$ is trivial for each $\alpha\in A$, leaf $L\in\mathscr{F}_\alpha$ and $\widetilde{b}\in\widetilde{U}_L$.
2. If $V_\alpha\cap V_\beta\neq\emptyset$, then the anti-commutator $\{D_\alpha,D_\beta\}$ restricted on $\Gamma(W|_{V_\alpha\cap V_\beta})^G$ is a non-negative operator over $V_\alpha\cap V_\beta$.

Remark 2.36. A G-equivariant strongly acyclic compatible system is G-acyclic.

Suppose that we have a G-acyclic compatible system $\{D_\alpha\}_{\alpha\in A}$ on $(\{V_\alpha, \mathscr{F}_\alpha\}, W)$. For any non-negative real number $t \geq 0$, any G-invariant Dirac-type operator D on $\Gamma(W)$, and $\{D_\alpha\}_{\alpha\in A}$ we consider the perturbation (2.1) in Subsection 2.3. By Assumption (4) in Definition 2.27 we can take the admissible partition of unity $\{\rho_\alpha^2\}_{\alpha\in A}$ to be G-invariant. Then, the same argument as that used to define $\mathrm{ind}(M, V, W)$ in Subsection 2.3 holds for the G-invariant part of $\ker D_t$.

Proposition 2.37. *Under the above assumption suppose that M has a cylindrical end $V = N \times (0, \infty)$ and the compatible fibration $\{V_\alpha, \mathscr{F}_\alpha\}$ and the G-acyclic compatible system $\{D_\alpha\}_{\alpha\in A}$ on $(\{V_\alpha, \mathscr{F}_\alpha\}, W)$ are translationally invariant. Suppose also that G acts trivially on all the data on the $(0, \infty)$-factor. Then for a sufficiently large $t \gg 0$, the space of G-invariant L^2-solutions of $D_t s = 0$ is finite dimensional and its super-dimension is independent of a sufficiently large $t \gg 0$ and any other continuous deformations of data.*

Definition 2.38. In the case of Proposition 2.37 we define $\mathrm{ind}^G(M, V, W)$ to be the super-dimension of the space of G-invariant L^2-solutions of $D_t s = 0$

$$\mathrm{ind}^G(M, V, W) := \dim(\ker D_t^0)^G \cap L^2(M, W) - \dim(\ker D_t^1)^G \cap L^2(M, W)$$

for a sufficiently large $t \gg 0$.

For the general end case, we have the following proposition.

Proposition 2.39. *For given (M, V, W) and the compatible fibration $\{V_\alpha, \mathscr{F}_\alpha\}$ and the G-acyclic compatible system $\{D_\alpha\}_{\alpha\in A}$ on $(\{V_\alpha, \mathscr{F}_\alpha\}, W)$ we can deform them to (M', V', W') so that M' has a cylindrical end $V' = N' \times (0, \infty)$ with a translationally invariant G-acyclic compatible system and G acts trivially on all the data on the $(0, \infty)$-factor.*

Definition 2.40. For the general case we define $\mathrm{ind}^G(M, V, W)$ to be $\mathrm{ind}^G(M', V', W')$ for the deformed data (M', V', W').

Note that $\mathrm{ind}^G(M, V, W)$ is well-defined, i.e, it does not depend on various choice of the construction.

For $\mathrm{ind}^G(M, V, W)$ we have the following localization.

Theorem 2.41. *Under the above assumption suppose that there exists an open covering $\{O_i\}_{i=1}^m$ of $M \smallsetminus V$ which satisfies the following properties.*

1. *$\{O_i\}_{i=1}^m$ are mutually disjoint.*
2. *Each O_i is G-invariant.*
3. *Each O_i is admissible.*

Then we have

$$\mathrm{ind}^G(M, V, W) = \sum_i \mathrm{ind}^G(O_i, O_i \cap V, W|_{O_i}).$$

Remark 2.42. For a G-equivariant strongly acyclic compatible system Theorem 2.41 is a consequence of Theorem 2.33. See Remark 2.36. However, for a general G-acyclic Theorem 2.41 is not obtained from Theorem 2.33.

2.6. **K-equivariant G-acyclic compatible system and $\mathrm{ind}_K^G(M,V,W)$.** Let (M,g), W, and V be the data satisfying the assumption (1) in Theorem 2.18. Let G and K be compact Lie groups. Suppose that there is a $G\times K$-action on M which preserves all these data. Let $\{V_\alpha,\mathscr{F}_\alpha\}_{\alpha\in A}$ be a $G\times K$-equivariant compatible fibration on V and $\{D_\alpha\}_{\alpha\in A}$ a $G\times K$-equivariant compatible system on $(\{V_\alpha,\mathscr{F}_\alpha\},W)$. Suppose that $\{D_\alpha\}_{\alpha\in A}$ is G-acyclic. Then, we have $\mathrm{ind}^G(M,V,W)$ and it becomes a virtual representation of K. We denote it by $\mathrm{ind}_K^G(M,V,W)$. In this case, we have a K-equivariant version of Theorem 2.41.

Theorem 2.43. *Under the above assumption suppose that there exists an open covering $\{O_i\}_{i=1}^m$ of $M\smallsetminus V$ which satisfies the following properties.*

1. *$\{O_i\}_{i=1}^m$ are mutually disjoint.*
2. *Each O_i is $G\times K$-invariant.*
3. *Each O_i is admissible.*

Then we have

$$\mathrm{ind}_K^G(M,V,W)=\sum_i \mathrm{ind}_K^G(O_i,O_i\cap V,W|_{O_i}).$$

In particular, we have the following corollary.

Corollary 2.44. *Under the above assumption suppose that V is equal to M itself. Then we have*

$$\mathrm{ind}_K^G(M,V,W)=0.$$

3. The case of torus action

Let G be an n-dimensional torus. We endow G with a *rational* flat Riemannian metric. Precisely speaking we take a Euclidean metric on the Lie algebra of G such that the intersection of the integral lattice and the lattice generated by some orthonormal basis is a sublattice of rank n. We extend the metric to the whole G. Let H be a closed subgroup of G. We denote by H^o the identity component of H. Let $H^\perp$ be the orthogonal complement of H^o defined as the image of the orthogonal complement of the Lie algebra of H by the exponential map. Since the metric is rational $H^\perp$ is well-defined as a compact connected subgroup of G and it has only finitely many intersection points $H\cap H^\perp$.

Let K be a compact Lie group. Let V be a smooth manifold equipped with an action of $G\times K$ and a $G\times K$-invariant almost complex structure J. We take and fix a Riemannian metric g on V which is invariant by the almost complex structure J and the $G\times K$-action. Let $L\to V$ be a $G\times K$-equivariant Hermitian line bundle over V equipped with a $G\times K$-invariant Hermitian connection ∇. Note that since ∇ is invariant under the torus action the restriction of (L,∇) to each G-orbit is a flat line bundle. We denote by G_x the stabilizer subgroup of G at a point $x\in V$. Let A be the set of subgroups of G which appears as the identity component of the stabilizer group at some point $x\in V$. We assume the following four conditions.

1. Each G-orbit is totally real, i.e., any subspace ξ of the tangent space of the orbit satisfies $\xi\cap J\xi=0$.
2. Each G-orbit has positive dimension.
3. A is a finite set.

(4) For each $x \in V$ there exists an open neighborhood V_x of x such that $gV_x = V_{gx}$ for all $g \in G$ and the restriction of L to $G_x^{\perp}$-orbit $G_x^{\perp}y$ has no $G_x^{\perp}$-invariant nontrivial parallel sections for all $y \in V_x$.

Note that since each orbit is a torus the last condition is equivalent to the vanishing of the $G_x^{\perp}$-invariant part of cohomologies with local coefficient, $H^*(G_x^{\perp}y, L|_{G_x^{\perp}y})^{G_x^{\perp}} = 0$. See [4, Lemma 2.29] for example. In this section we show that V is equipped with a structure of K-equivariant G-acyclic compatible system with open subsets parameterized by A.

3.1. **$G\times K$-equivariant compatible fibration.** Recall that there is a *good open covering* with respect to the G-action.

Lemma 3.1 (Existence of a good open covering, Lemma 2.31 in [4])**.** *There exists an open covering $\{V_H\}_{H\in A}$ of V parameterized by A satisfying the following properties.*

(1) *Each V_H is G-invariant.*
(2) *For each $x \in V_H$ we have $G_x^o \subset H$.*
(3) *If $V_H \cap V_{H'} \neq \emptyset$, then we have $H \subset H'$ or $H \supset H'$.*
(4) *$V_H \subset \bigcup_{G_x^o=H} V_x$.*

Let $\{V_H\}_{H\in A}$ be the good open covering as in Lemma 3.1. Since each V_H is constructed from an open neighborhood of the closed subset consisting of points whose stabilizer is equal to H, we may assume that V_H is $G \times K$-invariant. Note that since each V_H is an G-invariant open subset V_H has a structure of a foliation $\mathscr{F}_H$ via the decomposition into the union of $H^{\perp}$-orbits. Each leaf of $\mathscr{F}_H$ is a $H^{\perp}$-orbit through some point $x \in V_H$ and its holonomy group is equal to $G_x \cap H^{\perp}$. Note that $G_x \cap H^{\perp}$ is a finite group because G_x is a subgroup of H by the property (2) in Lemma 3.1. Then, $\{V_H, \mathscr{F}_H\}_{H\in A}$ is a $G \times K$-equivariant compatible fibration on V. Note that since the Riemannian metric g is G-invariant it satisfies the Assumption 2.10.

Remark 3.2. If G is the circle group S^1, then the finite set A consists of a single element. In this case we use the open covering $\{V_H\}_{H\in A}$ consisting of the single open set V.

3.2. **K-equivariant G-acyclic compatible system.** Now we construct a K-equivariant G-acyclic compatible system. Let $T\mathscr{F}_H \to V_H$ be the tangent bundle along leaves of $\mathscr{F}_H$. Since each G-orbit is totally real we have a canonical injection $T\mathscr{F}_H \otimes_{\mathbb{R}} \mathbb{C} \to TV_H$ via the almost complex structure on X. Let $T'V_H$ be the orthogonal complement of $T\mathscr{F}_H \otimes_{\mathbb{R}} \mathbb{C}$ in TV_H, which is canonically isomorphic to $TV_H/(T\mathscr{F}_H \otimes \mathbb{C})$. There is a canonical isomorphism $TV_H \cong (T\mathscr{F}_H \otimes \mathbb{C}) \oplus T'V_H$ as Hermitian vector bundles. Using the isomorphism we have an isomorphism

$$\wedge^{0,\bullet}T^*V_H \cong (\wedge^{\bullet}T^*\mathscr{F}_H \otimes \mathbb{C}) \otimes \wedge^{0,\bullet}(T'V_H)^*.$$

We define a $\mathbb{Z}/2$-graded Clifford module bundle W by

$$W := \wedge^{0,\bullet}T^*V \otimes L.$$

Note that there is the canonical isomorphism

$$W|_{V_H} \cong (\wedge^{\bullet}T^*\mathscr{F}_H \otimes \mathbb{C}) \otimes \wedge^{0,\bullet}(T'V_H)^* \otimes L|_{V_H}.$$

Let $D_H : \Gamma(W|_{V_H}) \to \Gamma(W|_{V_H})$ be the Dirac operator along leaves of $\mathscr{F}_H$ which is defined by the de Rham operator with coefficient in $L|_{V_H}$. Strictly speaking D_H is a differential operator acting on $\Gamma((\wedge^{\bullet}T^*\mathscr{F}_H \otimes \mathbb{C}) \otimes L|_{V_H})$ which is defined by the de Rham operator along leaves of $\mathscr{F}_H$ and the Hermitian connection ∇ of L. Since the restriction of $T'V_H$ to each leaf of $\mathscr{F}_H$ has a canonical flat structure induced by the $H^\perp$-action, we can regard D_H as a differential operator acting on $\Gamma((\wedge^\bullet T^*\mathscr{F}_H \otimes \mathbb{C}) \otimes \wedge^{0,\bullet}(T'V_H)^* \otimes L|_{V_H})$. Then $\{D_H\}_{H\in A}$ is a $G \times K$-equivariant compatible system on $\{V_H, \mathscr{F}_H\}_{H\in A}$.

Proposition 3.3. *The data $\{D_H\}$ is a K-equivariant G-acyclic compatible system on V.*

Proof. Suppose that $x \in V$ is contained in V_H for some $H \in A$. Let $H^\perp x$ be the $H^\perp$-orbit through the point x and $\widetilde{U}_x$ the slice at x with respect to the $H^\perp$-action. We put $\widetilde{V}_x := H^\perp \times \widetilde{U}_x$. Note that the natural map $q_x : \widetilde{V}_x \to V_x := H^\perp \widetilde{U}_x$ is a finite covering whose covering transformation group is equal to $G_x \cap H^\perp$. Let $\pi_x : \widetilde{V}_x \to \widetilde{U}_x$ be the projection to the second factor. The finite covering $q_x : \widetilde{V}_x \to V_x$ and the $H^\perp$-bundle $\pi_x : \widetilde{V}_x \to \widetilde{U}_x$ are the data in Lemma 2.13. In the above setting $H^\perp$ is the subgroup which preserves each leaf of $\mathscr{F}_H$. By our assumption and the property (4) of the good covering $\{V_H\}$ there exists $x' \in V$ such that $x \in V_{x'}$, $G_{x'} = H$ and $(\ker D_H)^{H^\perp} = H^*(G_{x'}^\perp x; L|_{G_{x'}})^{G_{x'}^\perp} = 0$. Moreover if $V_H \cap V_{H'} \neq \emptyset$ and $H' \subset H$, then the anti-commutator $\{D_H, D_{H'}\}$ is the Laplacian on the $H^\perp$-orbit (Lemma 2.35 in [4]). Then we complete the proof. $\square$

3.3. **Cotangent bundle case.** Let T^*G be the cotangent bundle of the torus G. Consider the canonical symplectic form ω on T^*G defined by the canonical 1-form α on T^*G. Note that T^*G has a natural G-action induced by the multiplication of G. We fix a G-equivariant trivialization $T^*G \cong G \times \mathfrak{g}^*$, where $\mathfrak{g}^*$ is the dual of the Lie algebra of G. Let L be the G-equivariant Hermitian line bundle $T^*G \times \mathbb{C}$ over T^*G with Hermitian connection ∇ defined by $\nabla = d - \sqrt{-1}\alpha$, where the G-action on the $\mathbb{C}$-factor is trivial. Let $\mathfrak{g}_\mathbb{Z}$ be the integral lattice and $\mathfrak{g}_\mathbb{Z}^*$ the weight lattice, i.e.,

$$\mathfrak{g}_\mathbb{Z} = \{\xi \in \mathfrak{g} \mid \exp \xi = e \in G\},$$
$$\mathfrak{g}_\mathbb{Z}^* = \{\eta^* \in \mathfrak{g}^* \mid \langle \xi, \eta^* \rangle \in \mathbb{Z}\ \forall \xi \in \mathfrak{g}_\mathbb{Z}\}.$$

Take an integral weight $\xi \in \mathfrak{g}_\mathbb{Z}^*$ and define $L(\xi)$ by $L(\xi) = L \otimes \underline{\xi}$, where $\underline{\xi}$ is the trivial line bundle equipped with the G-action whose weight is given by ξ. The definition of the canonical 1-form implies that $\eta \in \mathfrak{g}^*$ is in weight lattice $\mathfrak{g}_\mathbb{Z}^*$ if and only if the restriction of (L, ∇) to $G \times \{\eta\}$ is trivially flat. Moreover the proof of [4, Lemma 2.29] shows the following lemma.

Lemma 3.4. *We have the following isomorphism between representations of G.*

$$H^*(G \times \{\eta\}, (L(\xi), \nabla)|_{G\times\{\eta\}}) \cong \begin{cases} 0 & (\eta \notin \mathfrak{g}_\mathbb{Z}^*) \\ H^*(G \times \{\eta\}, \mathbb{C}) \otimes \xi & (\eta \in \mathfrak{g}_\mathbb{Z}^*). \end{cases}$$

Let M be a small G-invariant open neighborhood of the zero section in T^*G so that the image of M under the projection $T^*G \to \mathfrak{g}^*$ does not contain any non-zero integral points. Let V the complement of the zero section in M. Consider the natural complex structure on T^*G induced from the trivialization of T^*G and the metric of $\mathfrak{g}$, which is compatible with the symplectic structure. By Lemma 3.4 and the construction in the previous subsections we have a G-equivariant strongly acyclic compatible system on V for the Clifford module bundle $W(\xi) := \wedge^{0,\bullet} T^*M \otimes L(\xi)$, and we can define the G-equivariant index $\mathrm{ind}_G(M, V, W(\xi)) \in R(G)$. Note that since the G-action on T^*G is free, we use the open covering consisting of the single open set V. In [3, Remark 6.10] we give explicit solutions of the equation $D_t s = 0$ in the case of $\dim G = 1$. Using the explicit description we have the following, which will be used in the proof of Theorem 5.1.

Proposition 3.5. *If G is the circle group S^1, then we have* $\mathrm{ind}_G(M, V, W(\xi)) = \xi$.

Remark 3.6. It is expected that a similar argument is possible to calculate the equivariant index for higher dimensional cases. The numerical index is already calculated in our previous paper [3, Theorem 6.11] and is equal to 1. In the calculation there we used the embedding $T^*G \subset G \times G$ derived from the one-point compactification $\mathbb{R} \to S^1$. This compactification, however, is not G-equivariant and hence is not available to the calculation of the equivariant index. Another possible approach to the higher dimensional case would be to use the product structure $G = (S^1)^n$ and apply the product formula of equivariant index. However, it would be necessary to compare the compatible system given by the product structure with the one given by the G-action on T^*G. Since we have not shown such comparizon, this approach is not completed yet. Because the G-action on $H^*(G, \mathbb{C})$ is trivial, we have at least the next vanishing property of the G-invariant part from Lemma 3.4 and the vanishing of G-invariant index.

Proposition 3.7. *For $\xi \neq 0$ we have* $\mathrm{ind}^G(M, V, W(\xi)) = 0$.

4. Vanishing theorem for S^1-acyclic compatible systems

In this section we show the vanishing theorem of $\mathrm{ind}_K^{S^1}(M, V, W)$ for equivariant S^1-acyclic compatible system under the setting in Section 3.

Let K be a compact Lie group. Let M be a smooth manifold equipped with an action of $S^1 \times K$ and $S^1 \times K$-invariant almost complex structure. We take and fix an $S^1 \times K$-invariant Hermitian metric on M. Suppose that the fixed point set M^{S^1} of the S^1-action is a closed connected submanifold of M. Let $L \to M$ be an $S^1 \times K$-equivariant Hermitian line bundle over M equipped with an $S^1 \times K$-invariant Hermitian connection ∇ such that the fixed point set L^{S^1} of the S^1-action is equal to the image of the zero section of M^{S^1} to $L|_{M^{S^1}}$. Note that the restriction of (L, ∇) to each S^1-orbit is a flat line bundle.

In the next subsection we show that there is an $S^1 \times K$-invariant open neighborhood M' of M^{S^1} and a K-equivariant S^1-acyclic compatible system on $M' \smallsetminus M^{S^1}$. Here we use the Clifford module bundle $W := \wedge^{0,\bullet} T^*M' \otimes L$. The main theorem in this section is the following vanishing theorem.

Theorem 4.1.

$$\mathrm{ind}_K^{S^1}(M', M' \smallsetminus M^{S^1}, W) = 0 \in R(K).$$

4.1. S^1-acyclic compatible system.

In this subsection we show that there is an open subset of M on which we have a K-equivariant S^1-acyclic compatible system. To show it we first show the following.

Lemma 4.2. *For each $x \in M$ let $(L_{(x)}, \nabla_{(x)})$ be the restriction of the pull-back of (L, ∇) to $S^1 \times \{x\}$ by the multiplication map $S^1 \times M \to M$. There exists an $S^1 \times K$-invariant open neighborhood M' of M^{S^1} such that for a each $x \in M'$ the S^1-invariant part of the de Rham cohomology with local coefficient $H^*(S^1 \times \{x\}; (L_{(x)}, \nabla_{(x)}))^{S^1}$ is zero.*

Proof. For each $x \in M^{S^1}$ we have the canonical isomorphism

$$H^*(S^1 \times \{x\}; (L_{(x)}, \nabla_{(x)})) \cong H^*(S^1; \mathbb{C}) \otimes L_{(x)},$$

and the S^1-invariant part of the right hand side is zero because $L_{(x)}$ is a non-trivial representation of S^1 by our assumption. By the semi-continuity of the cohomology there exists an $S^1\times K$-invariant open neighborhood M' of M^{S^1} such that $H^*(S^1\times\{x\};(L_{(x)},\nabla_{(x)}))^{S^1}$ is zero for all $x\in M'$. $\square$

Using the open subset M' obtained in Lemma 4.2 we put $V:=M'\smallsetminus M^{S^1}$. There exist the structure of a compatible fibration on V and a K-equivariant S^1-acyclic compatible sytem on it as in Section 3.

4.2. **Product formula.** In this subsection we recall a product formula of local indices, which we will use in the proof of Theorem 4.1.

Let Y_0 be a closed manifold equipped with a K-action and a K-invariant Hermitian structure. Let $L_0\to Y_0$ be a K-equivariant Hermitian line bundle with a K-invariant Hermitian connection. Define W_0 to be $W_0:=\wedge^{0,\bullet}T^*Y_0\otimes L_0$, which has a natural structure of a K-equivariant $\mathbb{Z}/2$-graded Clifford module bundle over Y_0. Let K_1 be a compact Lie group and $Q\to Y_0$ a K-equivariant principal K_1-bundle over Y_0. Let Y_1 be a unitaty representation of $K_1\times S^1$ such that $Y_1^{S^1}=\{0\}$. Let R be a 1-dimansional unitary representation of S^1 and L_1 the $K_1\times S^1$-equivariant line bundle $Y_1\times R\to Y_1$. Here K_1 acts on R trivially and we consider the product connection on L_1. Define W_1 by $W_1:=\wedge^{0,\bullet}T^*Y_1\otimes L_1$, which has a natural structure of $K_1\times S^1$-equivariant $\mathbb{Z}/2$-graded Clifford module bundle over Y_1. We put $Y:=Q\times_{K_1}Y_1$ and $W:=W_0\otimes(Q\times_{K_1}W_1)$.

Note that Y_1 has a structure of a K-equivariant S^1-acyclic compatible system by taking $M'=Y_1$ and $M^{S^1}=\{0\}$ in Lemma 4.2. Using the product formula [4, Theorem 5.8], we have the following equality.

Proposition 4.3.

$$\mathrm{ind}_K^{S^1}(Y,W_Y)=\mathrm{ind}_K(Y_0,W_0\otimes(Q\times_{K_1}\mathrm{ind}_{K_1}^{S^1}(Y_1,W_1))).$$

Remark 4.4. The meaning of the right hand side in the above equality is as follows. As a character of K_1 we write $\mathrm{ind}_{K_1}^{S^1}(Y_1,W_1)$ as $\mathrm{ind}_{K_1}^{S^1}(Y_1,W_1)=[F_0]-[F_1]$, where F_0 and F_1 are finite dimensionl representations of K_1. Then we put

$$\mathrm{ind}_K(Y_0,W_0\otimes(Q\times_{K_1}\mathrm{ind}_{K_1}^{S^1}(Y_1,W_1))):=\mathrm{ind}_K(Y_0,W_0\otimes(Q\times_{K_1}F_0))-\mathrm{ind}_K(Y_0,W_0\otimes(Q\times_{K_1}F_1)).$$

4.3. **Model of the neighborhood of M^{S^1}.** Now we come back to the setting in Subsection 4.1. Let $\nu\to M^{S^1}$ be the normal bundle of M^{S^1} in M'. Then the fibers of ν are unitary representation of S^1. Since we assume that M^{S^1} is connected they are mutually isomorphic unitary representations. We take and fix a copy R_ν of the unitary representation of S^1 on ν. As in the same way we take and fix a copy R_L of the one-dimensional unitary representation of S^1 on $L|_{M^{S^1}}$.

Let K_1 be the group of unitary transformations of R_ν which commute with the S^1-action. Let $Q\to M^{S^1}$ be the K-equivariant principal K_1-bundle whose fiber Q_x at $x\in M^{S^1}$ is defined by the set of isomorphisms between R_ν and ν_x as S^1-representations. Note that ν is equal to the associated vector bundle $Q\times_{K_1}R_\nu$. Let L_0 be the K-equivariant Hermitian line bundle with connection over M^{S^1} defined by

$$L_0:=\mathrm{Hom}_{S^1}(R_L,L|_{M^{S^1}}).$$

Note that L_0 is abstractly isomorphic to $L|_{M^{S^1}}$ as Hermitian line bundles with connection but L_0 does not have S^1-action.

Let L_1 be the $K_1 \times S^1$-equivariant Hermitian line bundle with connection over R_ν defined by

$$L_1 := R_\nu \times R_L$$

with the product conneciton. Note that K_1 acts trivially on the second factor. Let (L_Y, ∇_Y) be the Hermitian line bundle with connection over ν defined by

$$L_Y := L_0 \otimes (Q \times_{K_1} L_1)$$

and ∇_Y is the tensor product connection. Applying Proposition 4.3 for $Y_0 = M^{S^1}$, $Y_1 = R_\nu$, $Y = \nu$ and the associated $\mathbb{Z}/2$-graded equivariant Clifford module bundles we have

$$\mathrm{ind}_K^{S^1}(\nu, W_\nu) = \mathrm{ind}_K(M^{S^1}, W_0 \otimes (Q \times_{K_1} \mathrm{ind}_{K_1}^{S^1}(R_\nu, W_1))). \tag{4.1}$$

4.4. **Comparison with the model.** In this subsection we show the following in the setting in Subsection 4.1.

Proposition 4.5.

$$\mathrm{ind}_K^{S^1}(M', W|_{M'}) = \mathrm{ind}_K^{S^1}(\nu, W_\nu).$$

As as a corollary we have the following by (4.1).

Proposition 4.6.

$$\mathrm{ind}_K^{S^1}(M', W|_{M'}) = \mathrm{ind}_K(M^{S^1}, W_0 \otimes (Q \times_{K_1} \mathrm{ind}_{K_1}^{S^1}(R_\nu, W_1))).$$

To show Proposition 4.5 we construct a one parameter family which connects two K-equivariant S^1-acyclic compatible systems on neighborhoods of $\nu^{S^1} = M^{S^1}$ in (ν, W_ν) and $(M', W|_{M'})$.

Let Y' be an $S^1 \times K$-invariant tubular neighborhood of ν^{S^1} in ν. We may assume the exponential map $\phi : Y' \to M'$ is a diffeomorphism. Let $g'_{M'}$ (resp. $J'_{M'}$) be the pull back of the Riemannanian metric $g_{M'}$ (resp. the almost complex structure $J_{M'}$) on M' by ϕ. On the other hand there is a Riemannian metric $g_{Y'}$ and a compatible almost complex strucutre $J_{Y'}$ on ν induced by the one in TM. Note that $g'_{M'}$ (resp. $J'_{M'}$) coincides with $g_{Y'}$ (resp. $J_{Y'}$) on the zero section $\nu^{S^1} = M^{S^1}$.

For $t \in [0,1]$ we define a family $(g_t, J_t, L_t, \nabla_t)$ on Y' which connects $(g'_{M'}, J'_{M'}, \phi^* L|_{M'}, \phi^*\nabla|_{M'})$ and $(g_{Y'}, J_{Y'}, L_Y|_{Y'}, \nabla_Y|_{Y'})$. The Riemannian metric g_t is defined by $g_t := t g'_{M'} + (1-t) g_{Y'}$. Note that for each $y \in Y'$ the set of almost complex structures of T_yY' which are compatible with g_t is a closed submanifold of $\mathrm{End}(TY')_y$. We can take Y' small enough so that the endmorphism $J'_t := tJ'_{M'} + (1-t)J_{Y'}$ is contained in a small normal disk bundle of the above closed submanifold with respect to the metric g_t, and there exists the unique compatible almost complex structure $(J_t)_y$ which minimizes the distance from $(J'_t)_y$ in $\mathrm{End}(TY')_y$. Then $J_t = \{(J_t)_y\}_{y \in Y'}$ is the reqiured family of almost complex structures on Y' compatible with g_t. Let $\phi_t : Y' \to M'$ be the map defined by $\phi_t(v) := \phi(tv)$, which connects the projection $\phi_0 : Y' \to M^{S^1}$ and the exponential map $\phi_1 = \phi : Y' \to M'$. Using this family of maps we put $(L_t, \nabla_t) := \phi_t^*(L, \nabla)|_{M'}$. We denote the Hermitian manifold Y' with the Hermitian structure (g_t, J_t) by Y_t. Then we have a family of $S^1 \times K$-equivariant $\mathbb{Z}/2$-graded Clifford module bundle W_t defined by $W_t := \wedge^{0,\bullet} T^* Y_t \otimes L_t$.

Lemma 4.7. *For each $(y,t) \in Y \times [0,1]$ let $(L_{(y,t)}, \nabla_{(y,t)})$ be the restriction of the pull-back of $(L \times [0,1], \nabla)$ to $S^1 \times \{(y,t)\}$ by the multiplication map $S^1 \times Y \times [0,1] \to Y \times [0,1]$. There exixts an $S^1 \times K$-invariant open neighborhood Y'' of Y^{S^1} in ν^{S^1} such that for each $(y,t) \in$*

$(Y''\smallsetminus Y^{S^1})\times[0,1]$ *the* S^1*-invariant part of the de Rham cohomology* $H^*(S^1;(L_{(y,t)},\nabla_{(y,t)}))^{S^1}$ *is zero.*

Proof. This lemma follows from Lemma 4.2 for $M := Y' \times [0,1]$. □

Proof of Proposition 4.5. By Lemma 4.7 and the same construction in Lemma 3.3 we have a one parameter family of K-equivariant S^1-acyclic system on Y''. Then the poropsition follows from the defomation invariance of the index and the exicition prpoerty. □

4.5. **Key proposition.** Recall the data K_1, R and (Y_1, L_1) considered in Subsection 4.2. In the next subsection we will show the following proposition which is a special case of Theorem 4.1.

Proposition 4.8.

$$\mathrm{ind}^{S^1}_{K_1}(Y_1, W_1) = 0.$$

Theorem 4.1 can be proved by using Proposition 4.8 as follows.

Proof of Thereom 4.1 assuming Proposition 4.8. By taking $Y_1 = R_\nu$ in Proposition 4.8 we have $\mathrm{ind}^{S^1}_K(M', W|_{M'}) = 0$ by Proposition 4.6. □

4.6. **Proof of the key proposition:one-dimensional case.** In this subsection we give the proof of Proposition 4.8 in the case of $\dim Y_1 = 1$. All the following construction can be carried out K_1-equivariantly. We fix an isomorpshim $z : Y_1 \to \mathbb{C}$ as Hermitian vector spaces. Define $w : Y_1 \smallsetminus \{0\} \to \mathbb{C}/2\pi\sqrt{-1}\mathbb{Z}$, $\sigma : Y_1 \smallsetminus \{0\} \to \mathbb{R}$ and $\theta : Y_1 \smallsetminus \{0\} \to \mathbb{R}/2\pi\mathbb{Z}$ by $w = \log z = \sigma + \sqrt{-1}\theta$. Let Y_1' be a manifold Y_1 whose complex structure is given by z and Kähler metric is given by $|dz| = \sqrt{2}$ on $\sigma < -1$ and $|dw| = \sqrt{2}$ on $\sigma > 1$. Let D' be the Dolbeault operator on $W_1' = \wedge^{0,\bullet}T^*Y_1' \otimes R_1$. We consider an S^1-acyclic comptible sytem on $V := \{\sigma > 0\}$ defined by S^1-orbits and the de Rham operator D'_V along fibers. Note that as in Remark 3.2 we use the open covering consisting of the single open set V to define the structure of a compatible fibration. Fix a positive number ρ_∞. Let $\rho : \mathbb{R} \to [0,\infty)$ be a non-negative smooth function such that $\rho(t) = 0$ for $t < 0$ and $\rho(t) = \rho_\infty$ for $t \gg 0$. For $t \geq 0$ we define a self adjoint elliptic operator D'_t by

$$D'_t := D' + t\rho D'_V : \Gamma(W') \to \Gamma(W').$$

Lemma 4.9. *For all* $t \geq 0$ *a section* $s \in \Gamma(W')$ *satisfies* $D'_t s = 0$ *and* $\lim_{\sigma\to\infty} s = 0$ *if and only of* $s = 0$*.*

Proof. We first give the proof for $t = 0$. If $s \in \Gamma(W')$ is a degree zero section such that $D's = 0$ and $\lim_{\sigma\to\infty} s = 0$, then s is a bounded holomorphic section, and hence $s = 0$. Note that if a degree one section s of W satisfies $D's = 0$, then s has a form $s = f(\bar{z})d\bar{z}$ for some anti-holomorphic function f. We show that if $s = f(\bar{z})d\bar{z}$ satisfies $\lim_{\sigma\to\infty} s = 0$, then $s = 0$. Since we have $s = f(\bar{z})d\bar{z} = f(\bar{z})\bar{z}d\bar{w}$ and s converges to zero at infinity, $f(\bar{z})\bar{z}$ converges to zero at infinity. We put $u := e^{-w} = 1/z$ and $h(u) := f(z)z$. Then $h(u)$ has zero of order at least one at $u = 0$, and hence,

$$s = f(\bar{z})d\bar{z} = \frac{h(\bar{u})}{\bar{u}}d\bar{u}$$

is an anti-holomorphic 1-form on the one point compactification $\mathbb{P}^1$ of Y' by Riemann's removable singularity theorem. Then we have $s = 0$.

We reduce the case $t > 0$ to the case $t = 0$. Let $\hat{\sigma}$ be a smooth increasing function on σ such that $\hat{\sigma}(\sigma) = \sigma$ for $\sigma < 0$ and

$$\frac{d\hat{\sigma}}{d\sigma} = 1 + t\rho(\sigma).$$

Let $\hat{z} : Y_1 \to \mathbb{C}$ be a coordinate function defined by

$$\hat{z} = e^{\hat{w}}, \ \hat{w} = \hat{\sigma} + \sqrt{-1}\theta.$$

Let Y'' be a manifold Y_1 whose complex structure is given by $\hat{z}$ and Kähler metric is given by $|d\hat{z}| = \sqrt{2}$ on $\sigma < -1$ and $|d\hat{w}| = \sqrt{2}$ on $\sigma > 1$. Let D'' be the Dolbeault operator on $W_1'' = \wedge^{0,\bullet}T^*Y'' \otimes R_1$. Since Kähler structures of Y' and Y'' can be identified on $\{\sigma < 0\}$, W' and W'' can be also identified on $\{\sigma < 0\}$. We extend this isomorphism as $\phi : W' \to W''$ by defining $1 \mapsto 1$ on the degree zero part and $\bar{w} \mapsto \overline{\hat{w}}$ on the degree one part. By the direct computation one can check that

$$(1 + t\rho)D'' = \phi \circ D_t' \circ \phi^{-1}.$$

Then $D_t's = 0$ and $\lim_{\sigma\to\infty} s = 0$ if and only if $D''(\phi(s)) = 0$ and $\lim_{\sigma\to\infty} \phi(s) = 0$. From the above argument for $t = 0$ we have $\phi(s) = 0$ and hence $s = 0$. $\square$

Using Lemma 4.9 Proposition 4.8 can be proved as follows.

Proof of Proposition 4.8. By the deformation invariance of local indices we have $\mathrm{ind}(Y_1, W_1) = \mathrm{ind}(Y_1', W_1')$. Note that $\mathrm{ind}(Y_1', W_1')$ is defined as the super-dimension of the space of L^2-solutions of the equation $D_t's = 0$. On the other hand any L^2-solution of $D_t's = 0$ satisfies the boundary condition $\lim_{\sigma\to\infty} s = 0$, and hence, we have $\mathrm{ind}(Y_1', W_1') = 0$ by Lemma 4.9. In particular we have $\mathrm{ind}_K^{S^1}(Y_1, W_1) = 0$. $\square$

4.7. **Proof of the key proposition:general case.** In this subsection we give the proof of Proposition 4.8 for general Y_1. Since we assume $Y_1^{S^1} = \{0\}$ we have the decomposition of Y_1 into the positive weight part E_+ and the negative weight part E_- as a representation of S^1,

$$Y_1 = E_+ \oplus E_-.$$

Define a unitarty representation E of S^1 by

$$E := E_+ \oplus \mathbb{C} \oplus E_-,$$

where we consider the trivial S^1-action on $\mathbb{C}$. Now we define a $\mathbb{C}^*$-action on E as follows: $\mathbb{C}^*$ acts by the complexification of the S^1-action on E_+ , the complexifiction of the S^1-action defined by $g \mapsto g^{-1}$ on E_- and the standard multiplication of weight 1 on $\mathbb{C}$. Note that all weights of the $\mathbb{C}^*$-action on E are positive and the $\mathbb{C}^*$-action commutes with the S^1-action. The quotient space

$$Z := (E \smallsetminus \{0\})/\mathbb{C}^*$$

is a weighted projective space with the induced $K_1 \times S^1$-action.

Lemma 4.10. (1) *We have*

$$H^{0,i}(Z, \mathcal{O}_Z) = 0 \qquad (i > 0)$$

for the structure sheaf $\mathcal{O}_Z$ of Z.

(2) *Let R be a non-trivial 1-dimensional representation of $K_1 \times S^1$. Then for $W_Z = \wedge^{0,\bullet}T^*Z \otimes R$ we have*

$$\mathrm{ind}_{K_1}^{S^1}(Z, W_Z) = 0 \in R(K_1).$$

Proof. (1) The first statement follows from the negativity of the canonical bundle $\det T^*Z$ of Z and the Bochner trick. We can show the negativity by the following (standard) argument. Let L_E be the trivial line bundle $E \times \mathbb{C}$ with a $\mathbb{C}^*$-action and the product holomorphic structure. Here $\mathbb{C}^*$ acts by weight one on the fiber $\mathbb{C}$. Let S^1 be the unit circle in $\mathbb{C}^*$. We define a S^1-invariant Hermitian structure on L_E by $|\underline{v}|^2 := e^{-\pi r^2}|v|^2$ for a constant section with value $v \in \mathbb{C}$, where $r : E \to \mathbb{R}$ is the distance from the origin of E. Combining with the holomorphic structure and this Hermitian structure of L_E we have the S^1-invariant Hermitian connection ∇_E on L_E whose $(0,1)$-part coincides with the Dolbeault operator of L_E and the curvature form coincides with the symplectic form on E. Since the S^1-action on E lifts to (L_E, ∇), the S^1-action is Hamiltonian. Let $\mu : E \to \sqrt{-1}\mathbb{R}$ be its moment map. Note that μ is proper because all weights of the S^1-action are positive. In this setting the complex quotient Z can be identified with the symplectic quotient $\mu^{-1}(0)/S^1$. In particular Z is a Kähler manifold with the induced prequantizing line bundle (L_Z, ∇_Z), and we have an isomorphic as holomorphic Hermitian line bundles

$$\det TZ \cong (E \smallsetminus \{0\}) \times_{\mathbb{C}^*} \det E \cong L_Z^d,$$

where $d > 0$ is the weight of S^1-action on $\det E$. Since L_Z is positive, the canonical bundle $\det T^*Z$ is negative.

(2) Let R be a non-trivial 1-dimensional representation of $K_1 \times S^1$. Since the Kähler structure of Z is $K_1 \times S^1$-equivariant we have a $K_1 \times S^1$-equivariant isomorphism

$$H^{0,i}(Z, \mathcal{O}_Z \otimes R) \cong H^{0,i}(Z, \mathcal{O}_Z) \otimes R$$

for $i \geq 0$. By the first statement and the Hodge theory we have

$$\mathrm{ind}_{K_1 \times S^1}(Z, W_Z) = \sum_i (-1)^i H^{0,i}(Z, \mathcal{O}_Z \otimes R) = R \in R(K_1 \times S^1).$$

Since R is a non-trivial representaion of S^1 we have $\mathrm{ind}_{K_1}^{S^1}(Z, W_Z) = 0$. □

Note that we have a decomposition of the fixed point set Z^{S^1} as

$$Z^{S^1} = Z_- \sqcup Z_0 \sqcup Z_+,$$

where we put $Z_- := (E_+ \smallsetminus \{0\})/\mathbb{C}^*$, $Z_0 := (\mathbb{C} \smallsetminus \{0\})/\mathbb{C}^*$ and $Z_+ := (E_- \smallsetminus \{0\})/\mathbb{C}^*$. Then by the localization formula Theorem 2.43, $\mathrm{ind}_{K_1}^{S^1}(Z, W_Z)$ is equal to the sum of the contribution from Z_-, Z_0 and Z_+:

$$\mathrm{ind}_{K_1}^{S^1}(Z, W_Z) = \mathrm{ind}_- + \mathrm{ind}_0 + \mathrm{ind}_+.$$

By definition the contribution ind_0 from Z_0 is equal to $\mathrm{ind}_{K_1}^{S^1}(Y_1, W_1)$ and hence we have

$$\mathrm{ind}_{K_1}^{S^1}(Y_1, W_1) = -(\mathrm{ind}_- + \mathrm{ind}_+)$$

from Lemma 4.10. Then we can show Proposition 4.8 as follows.

Proof of Proposition 4.8. It is enough to show $\mathrm{ind}_- = \mathrm{ind}_+ = 0$.

(0) If the complex codimension of M^{S^1} in M is 1, then we have $\mathrm{ind}_K^{S^1}(M', M' \setminus M^{S^1}) = 0$ by the product formula (Propostion 4.3) and Proposition 4.9.

(1) If $E_- = 0$, then we have $\mathrm{ind}_- = 0$ and the codimension of Z_+ is one. In this case we have $\mathrm{ind}_+ = 0$ from the above case (0).

(2) If $E_+ = 0$, we have $\mathrm{ind}_- = \mathrm{ind}_+ = 0$ as in the same way for (1).

(3) In the general case note that all the weights of the S^1-action on the normal direction on Z_- are positive. By (1) and the product formula we have $\mathrm{ind}_- = 0$. As in the same way we have $\mathrm{ind}_+ = 0$ by (2) and the product formula. $\square$

5. Localization of equivariant Riemann-Roch numbers

Let (M, ω) be a closed prequantized symplectic manifold with prequantizing line bundle (L, ∇). Let G be a torus and K a compact Lie group. Suppose that $G \times K$ acts effectively on (M, ω) and the $G \times K$-action on M lifts to L preserving ∇ and the Hermitian metric of L. Then, the action is Hamiltonian and each G-orbit is an isotropic torus in M. We denote the moment map for G-action associated with the lift by $\mu_G : M \to \mathfrak{g}^*$. We assume that $0 \in \mathfrak{g}^*$ is a regular value of μ.

For these data one can define the $G \times K$ equivariant Riemann-Roch number $RR_{G\times K}(M, L)$ as the index of the Dolbeault operator whose coefficient is in L. We denote its G-invariant part by $RR_K^G(M, L)$. Note that $RR_K^G(M, L)$ is an element of the character ring $R(K)$ of K.

On the other hand we have the quotient space $M_G = \mu_G^{-1}(0)/G$, the symplectic reduction of M at 0. Since we assume that 0 is a regular value M_G is a closed symplectic orbifold and has the natural induced K-action and the K-equivariant prequantizing line bundle $(L_G, \nabla_G) = (L|_{\mu_G^{-1}(0)}, \nabla|_{\mu_G^{-1}(0)})/G$. In this section we show the following theorem by induction on dimension of G.

Theorem 5.1.

$$RR_K^G(M, L) = RR_K(M_G, L_G) \in R(K).$$

As a special case we have a proof of Guillemin-Sternberg conjecture for the torus action.

Theorem 5.2 ([2, 6, 5, 9, 10, 11, 13, 14], etc.)**.**

$$RR^G(M, L) = RR(M_G, L_G) \in \mathbb{Z}.$$

Remark 5.3. The Guillemin-Sternberg conjecture itself is valid not only for a torus but also for a compact Lie group.

5.1. Acyclic compatible system and local Riemann-Roch number.

Definition 5.4 (L-acyclic point and (L, G)-acyclic point)**.** A point $\xi \in \mu_G(M)$ is called *L-acyclic* if the restriction of (L, ∇) to each orbit in $\mu^{-1}(\xi)$ does not have any non-trivial parallel sections. If the restriction does not have any non-trivial G-invariant parallel sections, then we call ξ a *(L, G)-acyclic point*.

Remark 5.5. Since each G-orbit is an isotropic torus, by [4, Lemma 2.29] and the Hodge theory, a point $\eta \in \mathfrak{g}^*$ is L-acyclic if and only if, for each orbit $\mathcal{O}$ which is contained in $\mu_G^{-1}(\eta)$, the de Rham operator on $\mathcal{O}$ with coefficients in $L|_{\mathcal{O}}$ has zero kernel. As in the similar way, $\eta \in \mathfrak{g}^*$ is (L, G)-acyclic if and only if the G-invariant part of the de Rham operator on the orbit is trivial.

Let $\mathfrak{g}_{\mathbb{Z}}$ be the integral lattice and $\mathfrak{g}^*_{\mathbb{Z}}$ the weight lattice of G.

Proposition 5.6. *Non L-acyclic points are contained in $\mathfrak{g}^*_{\mathbb{Z}}$.*

Proof. First let us recall that the moment map associated to the lift of the G-action on (M, ω) to (L, ∇) is defined by the following equality

$$\frac{d}{dt}\Big|_{t=0} \psi_{\exp -t\xi} \circ s \circ \varphi_{\exp t\xi} = \nabla_{X_\xi} s - 2\pi\sqrt{-1}\,\langle \mu_G, \xi\rangle\, s \tag{5.1}$$

for $\xi \in \mathfrak{g}$ and $s \in \Gamma(L)$, where φ_g denotes the G-action on M and ψ_g denotes the lift of φ_g to L for $g \in G$.

Let $\eta^* \in \mathfrak{g}^*$ be a non L-acyclic point and $\mathcal{O}$ a non L-acyclic orbit that is contained in $\mu_G^{-1}(\eta^*)$. Then, there exists a non-trivial parallel section which we denote by $s \in \Gamma(L|_{\mathcal{O}})$. For arbitrary element $\xi \in \mathfrak{g}_{\mathbb{Z}}$ and $x \in \mathcal{O}$, by (5.1) we have

$$\frac{d}{dt}\psi_{\exp -t\xi} \circ s \circ \varphi_{\exp t\xi}(x) = -2\pi\sqrt{-1}\,\langle \eta^*, \xi\rangle\, \psi_{\exp -t\xi} \circ s \circ \varphi_{\exp t\xi}(x).$$

This implies that

$$\psi_{\exp -t\xi} \circ s \circ \varphi_{\exp t\xi}(x) = e^{-2\pi\sqrt{-1}t\langle \eta^*, \xi\rangle} s(x).$$

Since $\xi \in \mathfrak{g}_{\mathbb{Z}}$, by putting $t = 1$, we have

$$s(x) = \psi_{\exp -\xi} \circ s \circ \varphi_{\exp \xi}(x) = e^{-2\pi\sqrt{-1}\langle \eta^*, \xi\rangle} s(x).$$

Then, $\langle \eta^*, \xi\rangle$ must be in $\mathbb{Z}$. □

Remark 5.7. There exist the following three conditions which are related to acyclicity on a G-invariant open subset V' of M.

(1) Every point in $\mu_G(V')$ is L-acyclic.
(2) $\mu_G(V')$ does not contain any integral points.
(3) The compatible system on V' is acyclic.

Neither the condition (1) nor (3) implies (2) because there may exist a lattice point without non-zero parallel section on its fiber. The condition (2) implies (1) by Prop 5.6. The condition (3) implies (1) because if every $H^{\perp}$-orbit does not have any non-zero parallel sections, then every G-orbit does not have. Conversely neither the condition (1) nor (2) implies (3). When $G = S^1$, the conditions (1) and (3) are equivalent because A consists of the single element $\{e\}$. In this case, since the action of G on $H^*(G, \mathbb{C})$ is trivial, two conditions $(1)'(L, G)$-acyclicity and $(3)'G$-acyclicity are also equivalent. Note that only the condition (3) (resp. $(3)'$) is our sufficient condition to define the index (resp. G-invariant index), and we only use it.

Let V be a $G \times K$-invariant open subset of M. We fix a $G \times K$-invariant ω-compatible almost complex structure on V. As in Subsection 3.1 V has a structure of $G \times K$-equivariant compatible fibration with the open covering $\{V_H\}_{H \in A}$ parameterized by the set of subgroups of G which appear as the identity components of the stabilizers of the G-action. Consider the $\mathbb{Z}/2$-graded Clifford module bundle $W_L = \wedge^{0,\bullet} T^*V \otimes L|_V$. Suppose that there is a $G \times K$-invariant open subset V' of V such that the family of Dirac type operators along leaves $\{D_H\}$ constructed as in Subsection 3.2 defines a $G \times K$-equivariant acyclic (resp. G-acyclic) compatible system. Using these data we have the equivariant local index $\mathrm{ind}_{G\times K}(V, V', W_L) \in R(G \times K)$ (resp. $\mathrm{ind}_K^G(V, V', W_L) \in R(K)$).

Definition 5.8 (Equivariant Riemann-Roch number)**.** For the above data we define the *equivariant Riemann-Roch number* $RR_{G\times K}(V, V', L) \in R(G\times K)$ by putting $RR_{G\times K}(V, V', L) = \mathrm{ind}_{G\times K}(V, V', W_L)$. As in the similar way when $\{D_\alpha\}$ is G-acyclic the *G-invariant part of the Riemann-Roch number* $RR_K^G(V, V', L) \in R(K)$ is defined by $RR_K^G(V, V', L) = \mathrm{ind}_K^G(V, V', W_L)$.

5.2. **Proof of Theorem 5.1: S^1-case.** We first show Theorem 5.1 in the case of $\dim G = 1$. We use the results in Section 2.

For each lattice point $k \in \mathfrak{g}^*_{\mathbb{Z}}$ we can take a small $G\times K$-invariant open neighborhood V_k of the compact set $\mu_G^{-1}(k)$ so that the image of the complement $\mu_G(V_k \smallsetminus \mu_G^{-1}(k))$ consists of L-acyclic points. The proof of Proposition 5.6 implies that if k is not equal to 0, then the image of the complement of the fixed point set $\mu_G(V_k \smallsetminus \mu_G^{-1}(k)^G)$ consists of (L, G)-acyclic points.

Lemma 5.9.

$$RR_K^G(M, L) = RR_K^G(V_0, V_0 \smallsetminus \mu_G^{-1}(0), L|_{V_0}) \in R(K).$$

Proof. By Proposition 5.6 the complement $M \smallsetminus \mu_G^{-1}(\mathfrak{g}^*_{\mathbb{Z}})$ has a structure of $G \times K$-equivariant strongly acyclic compatible system and $RR_{G\times K}(M, L)$ is equal to the sum of equivariant local Riemann-Roch numbers $RR_{G\times K}(V_k, V_k \smallsetminus \mu_G^{-1}(k), L|_{V_k})$. Moreover if $k \neq 0$, then $V_k \smallsetminus \mu_G^{-1}(k)^G$ has a structure of K-equivariant G-acyclic compatible system we have the equality for the G-invariant part,

$$RR_K^G(M, L) = RR_K^G(V_0, V_0 \smallsetminus \mu_G^{-1}(0), L|_{V_0}) + \sum_{k\in\mathfrak{g}^*_{\mathbb{Z}}\smallsetminus\{0\}} RR_K^G(V_k, V_k \smallsetminus \mu_G^{-1}(k)^G, L|_{V_k}).$$

On the other hand we have

$$RR_K^G(V_k, V_k \smallsetminus \mu_G^{-1}(k)^G, L|_{V_k}) = 0$$

for $k \neq 0$ by Theorem 4.1, and hence, we complete the proof. $\square$

Now we identify the neighborhood V_0 of $\mu_G^{-1}(0)$. Let T^*G be the cotangent bundle of G with the prequantizing line bundle $L(0)$ as in Subsection 3.3, where $0 \in \mathfrak{g}^*_{\mathbb{Z}}$ corresponds to the one dimensional trivial representation of G.

Lemma 5.10. *There is a $G \times K$-equivariant symplectomorphism between small open neighborhoods of $\mu_G^{-1}(0)$ in M and the zero section in $\mu_G^{-1}(0) \times_G T^*G (\cong \mu_G^{-1}(0) \times \mathfrak{g}^*)$. Moreover this symplectomorphism lifts to a $G \times K$-equivariant isomorphism between prequantizing line bundles over them.*

This lemma follows from the following general proposition, which is a generalization of Darboux's theorem ([8]).

Proposition 5.11. *Let H be a compact Lie group acting on symplectic manifolds (M_i, ω_i) for $i = 0, 1$. We assume each (M_i, ω_i) has a H-equivariant prequantizing line bundle (L_i, ∇_i) . Suppose that there is a compact manifold N with the following properties.*

(1) *There is an embedding $\iota_i : N \hookrightarrow M_i$.*
(2) *Normal bundles $\nu_{\iota_i(N)}$ of $\iota_i(N)$ in M_i are H-equivariantly isomorphic to each other.*
(3) *The collections of data $\iota_i^*(\omega_i, L_i, \nabla_i)$ are H-equivariantly isomorphic.*

Then there is a H-invariant neighborhood U_i of $\iota_i(N)$ and H-equivariant diffeomorphism $\phi : U_1 \to U_2$ such that

(1) *The following diagram commutes.*

$$\begin{tikzcd} U_1 \arrow[rr, "\phi"] & & U_2 \\ & N \arrow[ul, "\iota_1"] \arrow[ur, "\iota_2"'] & \end{tikzcd}$$

(2) *The diffeomorphism ϕ lifts to a H-equivariant isomorphisms between $(\omega_i, L_i, \nabla_i)$.*

Proof. For $i = 0, 1$ consider the associated principal $U(1)$-bundle with connection 1-form (P_i, α_i) of (L_i, ∇_i). Then the proposition follows from $G \times U(1)$-equivariant Morser's argument for contact manifold P_i with contact 1-form α_i for submanifold $\iota_i(N)$. Here the $U(1)$-action comes from that of the structure of a principal bundle. □

Note that $\mu_G^{-1}(0) \to \mu_G^{-1}(0)/G = M_G$ has a structure of a principal G-bundle (in the sense of orbifold), and $\mu_G^{-1}(0) \times_G T^*G$ has the structure of a product of compatible fibrations in the sense of Subsection 4.2. Here the compatible fibration on M_G is the trivial one (all leaves are a point) and the one on T^*G is induced by the induced G-action. Since the symplectomorphism in Lemma 5.10 is G-equivariant and preserves the prequantizing line bundles, we have the following.

Lemma 5.12. *The symplectomorphism in Lemma 5.10 induces an isomorphism between two acyclic compatible systems.*

We take V_0 to be the open neighborhood as in Lemma 5.10. Then by Proposition 3.5 and the product formula (Proposition 4.3), we have

$$RR_K^G(V_0, V_0 \smallsetminus \mu_G^{-1}(0), L|_{V_0}) = RR_K(M_G, L_G)$$

and hence by Lemma 5.9 we complete the proof of Theorem 5.1 in the case of $\dim G = 1$.

Remark 5.13. The argument in this subsection implies that Theorem 5.1 holds in the case that M is not necessarily closed. Let (M, ω) be a prequantized symplectic manifold with prequantizing line bundle (L, ∇). We do not assume that M is closed. Let G be the circle group S^1 and K a compact Lie group. Suppose that $G \times K$ acts effectively on (M, ω) and the $G \times K$-action on M lifts to L preserving ∇ and the Hermitian metric of L. We assume that the corresponding moment map μ_G of the G-action is a proper map and 0 is its regular value. Suppose that there is a G-invariant open subset V of M such that the complement $M \smallsetminus V$ is a compact neighborhood of $\mu_G^{-1}(0)$ and the image $\mu_G(M \smallsetminus V)$ does not contain any non-zero integral point. Under the above assumptions we have the equivariant index $RR_{G\times K}(M, V, L)$. As in the same way in the proof of Lemma 5.9 and by Lemma 5.12, we have

$$RR_K^G(M, V, L) = RR_K(M_G, L_G),$$

where M_G is the symplectic reduction of M at 0 and L_G is the induced prequantizing line bundle on M_G. Note that since μ_G is proper M_G is a closed symplectic orbifold.

On the other hand, when M is not necessarily compact, Vergne [15] formulated a version of quantization conjecture for general compact Lie group G in terms of transversally elliptic operator [1], which is proved by Ma and Zhang [7] and Tian and Zhang [12]. We expect that there is a relation between our G-invariant index $RR_K^G(M, V, L)$ and the index of the transversally elliptic operator for the case of torus action. We, however, do not know the precise relation.

5.3. **Proof of Theorem 5.1: general case.** To show Theorem 5.1 by induction on $\dim G$ we have to choose an appropriate circle subgroup as follows.

Lemma 5.14. *There exists a compact connected 1 dimensional subgroup H of G such that the induced moment map $\mu_H = \iota_H^* \circ \mu_G : M \to \mathfrak{h}^*$ has 0 as its regular value, where $\mathfrak{h}$ is the Lie algebra of H and $\iota_H : \mathfrak{h} \to \mathfrak{g}$ is the natural inclusion.*

Proof. Let A be the set of all subgroups of G which appear as a stabilizer of G-action on M. Since M is compact A is a finite set and the image of the fixed point set $\mu_G(M^G)$ is a finite set which does not contain 0, and we can choose a one-dimensional subspace $\mathfrak{h}$ in $\mathfrak{g}$ which satisfies the following conditions.

(1) $\mathfrak{h}$ is generated by rational vectors.
(2) $\mathfrak{h} \smallsetminus \{0\}$ does not intersect with $\displaystyle\bigcup_{G' \in A \smallsetminus \{G\}} \mathrm{Lie}\, G'$ in $\mathfrak{g}$.
(3) $\mu_G(M^G)$ does not intersect with $\ker \iota_H^*$.

We define H as the one-dimensional connected subgroup of G which is defined as the image of $\mathfrak{h}$ by the exponential map. Note that by the above first condition H is a compact subgroup of G, and we have $M^H = M^G$ from the second condition. Since H is a one-dimensional subgroup $x \in M$ is a critical point of $\mu_H = \iota_H^* \circ \mu_G$ if and only if it is a fixed point $x \in M^H = M^G$. By the last condition for $\mathfrak{h}$, we have that 0 is a regular value of μ_H. □

Proof of Theorem 5.1. We show the theorem by induction on $\dim G$. As in the previous subsection we proved in the case of $\dim G = 1$. We take a one-dimensional connected subgroup H of G as in Lemma 5.14 and a complementary subtorus G'. According to this decomposition we have a decomposition of the moment map μ_G as $\mu_G = \mu_H \oplus \mu_{G'} : M \to \mathfrak{g}^* = \mathfrak{h}^* \oplus (\mathfrak{g}')^*$. Let $\bar{\mu}_{G'} : \mu_H^{-1}(0)/H \to (\mathfrak{g}')^*$ be the induced moment map with respect to the induced G'-action. Since the natural projection $\mu_G^{-1}(0) = (\mu_{G'}|_{\mu_H^{-1}(0)})^{-1}(0) \to \bar{\mu}_{G'}^{-1}(0)$ is a submersion and 0 is a regular value of μ_G and μ_H, 0 is also a regular value of $\bar{\mu}_{G'}$. Then we can prove Theorem 5.1 inductively as follows:

$$\begin{aligned} RR_K^G(M,L) &= (RR_K^H(M,L))^{G'} \\ &= (RR_K(M_H, L_H))^{G'} \\ &= RR_K((M_H)_{G'}, (L_H)_{G'}) \\ &= RR_K(M_G, L_G). \end{aligned}$$

Here the second equality follows from the facts that H is one-dimensional and 0 is a regular value of μ_H, and the third equality follows from the facts that G' is $m-1$-dimensional and 0 is a regular value of $\bar{\mu}_{G'}$. □

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