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On exact dead-core rates for a semilinear heat equation with strong absorption

by

Yukihiro SEKI

UNIVERSITY OF TOKYO
GRADUATE SCHOOL OF MATHEMATICAL SCIENCES
KOMABA, TOKYO, JAPAN

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Abstract

We study a semilinear heat equation with strong absorption $u_t = \Delta u - u^p$ with $0 < p < 1$ in $\mathbf{R}^N$. A solution is known to develop dead-core in finite time for a wide class of initial data. We construct specific solutions with exact dead-core rates faster than the one given by the corresponding ODE. They are constructed formally by means of matched asymptotic expansion technique and rigorously by means of topological fixed-point arguments based on a priori estimates. To obtain the a priori estimates we analyze a certain linearized problem in a new function space $\mathcal{H}'$.

1 Introduction

We discuss the Cauchy problem for a semilinear heat equation with strong absorption

$$u_t = \Delta u - u^p, \quad x \in \mathbf{R}^N,\ t > 0, \tag{1.1a}$$

$$u(x,0) = u_0(x), \quad x \in \mathbf{R}^N, \tag{1.1b}$$

where $p \in (0,1)$, $N \geq 1$ and $u_0 \in L^\infty_{loc}(\mathbf{R}^N)$. The equation (1.1a) arises originally in the Dirichlet problem on a bounded domain in the modeling of an isothermal reaction-diffusion process [4, 33]. It also appears in a description of thermal energy transport in plasmas [23]. It is known that a unique smooth solution u of (1.1) exists globally in time if an initial datum u_0 is positive and has an appropriate bound on growth order as $|x| \to \infty$ (cf. [2, 17]). Once a suitable initial datum is chosen, the solution develops dead-core in finite time. Namely, there exists a finite time T such that the infimum of $u(\cdot,t)$ reaches zero at $t = T$. This is a peculiar phenomenon caused by strongly absorbing effect, which never appears in a solution of the equation with "weak" absorption, i.e., $p \geq 1$. It was investigated in [17] whether a finite-time dead-core occurs in view of the growth order of initial data as $|x| \to \infty$.

For the corresponding Cauchy-Dirichlet problem on a ball with constant boundary condition, it was proven in [14, 16] that dead-core rates are, in general, unexpected ones; they are faster than the self-similar rate, that is,

$$\lim_{t \nearrow T} (T-t)^{-\frac{1}{1-p}} u(0,t) = 0 \tag{1.2}$$

if u_0 is a positive radial nondecreasing function. In other words, they are faster than the dead-core rate of the ODE obtained by dropping Δu from (1.1a). Note, however, that exact dead-core rates are still remained unrevealed there. Our purpose is to construct specific solutions of (1.1) which develop dead-core in finite time and exhibit exact dead-core rates faster than the self-similar rate.

Concerning the case of “extinction time” T_e, the first time at which a nonnegative solution u of (1.1) vanishes identically, it is proven in [7,19] that

$$\|u(t)\|_{L^\infty} \sim (T_e - t)^{\frac{1}{1-p}} \qquad \text{as } t \to T_e$$

under suitable assumptions on initial data. Here and henceforth we write $f(t) \sim g(t)$ as $t \to T$ for some $T > 0$ and (real-valued) functions f, g on $(0, T)$ if there are constants $C_1, C_2 > 0$ and $t_1 \in (0, T)$ such that $C_1 g(t) \le f(t) \le C_2 g(t)$ for all $t \in (t_1, T)$.

Similar problems have been studied for a semilinear heat equation with source

$$u_t = \Delta u + |u|^{q-1}u, \qquad q > 1. \tag{1.3}$$

The equation (1.3) has finite-time blow-up solutions and their blow-up rates have been investigated for past decades. Let q_s denote the Sobolev ciritical exponent: $q_s = \infty$ for $N = 1, 2$ and $q_s = (N+2)/(N-2)$ for $N \ge 3$. For $1 < q < q_s$, every blow-up solution of (1.3) exhibits the self-similar rate, often referred as the type I blow-up rate [10,12,13]:

$$\|u(t)\|_{L^\infty} \sim (T - t)^{-\frac{1}{q-1}}. \tag{1.4}$$

It is known that (1.4) holds also for $N \ge 3$ and $q \ge q_s$ under certain additional assumptions on initial data [8,27,28,37]. The blow-up rate estimate (1.4) is useful to investigate local structures of blow-up sets (cf. for example, [9,11,18,34,35] and the references cited there). When $N \ge 3$ and $q \ge q_s$, (1.4) fails to hold in general. Herrero and Velázquez [20,21] first discovered nonnegative blow-up solutions of (1.3) such that

$$\limsup_{t \to T} (T - t)^{\frac{1}{q-1}} \|u(t)\|_{L^\infty} = +\infty,$$

often referred as type II blow-up or fast blow-up, when q and N are large enough (see also [29]). Sign-changing type II blow-up solutions were formally constructed in [6] for the critical case $q = q_s$ when $N = 3, 4, 5, 6$. Moreover, the exact blow-up rates of these specific solutions are also revealed in the articles. In [20,21] such solutions are constructed formally by means of matched asymptotic expansion techniques and rigorously by a topological fixed-point argument. Related arguments were used also in different contexts, e.g., [22, 32,36]. It is worth pointing out that the analysis of general type II blow-up solutions with radial symmetry has recently advanced on the basis of the specific solutions constructed in [20,21]. The reader is referred to the review [26] and references therein for this topic.

We shall again focus our attention to the dead-core problems. In the following, α denotes the positive constant defined by

$$\alpha = \frac{1}{1-p}. \tag{1.5}$$

For $N = 1$, Guo and Wu [15] recently studied the equation (1.1a) and discovered that for every odd integer $\ell \ge 1$, there exists a solution u of (1.1) which develops dead-core in a finite time T and behaves as

$$u(0, t) = \min_{x \in \mathbf{R}} u(x, t) \sim (T - t)^{\alpha + 2\alpha(\ell - \frac{1}{2})} \qquad \text{as } t \to T^-. \tag{1.6}$$

To show this result, they applied the method of [21] to the dead-core problem. They demand, however, that the parameter ℓ in (1.6) should be odd integers, which need not be assumed at least in the formal level of the matching process. It is therefore natural to ask whether or not the restrictive assumption on ℓ would be essentially requisite to prove the existence of solutions with the property (1.6). In addition, one would expect that this kind of solutions could exist also in arbitrary dimensions $N \geq 2$. In the present article we improve the method of [15], thus giving affirmative answers to these problems: we are able to remove the assumption on ℓ and provide radial solutions with analogous properties to (1.6) in arbitrary dimensions $N \geq 1$.

Theorem 1.1. *Let $N \geq 1$, $T > 0$ and $p \in (0,1)$. Then for every positive integer ℓ, there exists a radial solution u_ℓ of* (1.1) *which develops dead-core at $t = T$ such that $u_\ell(0,t) = \min_{x\in \mathbf{R}^N} u_\ell(x,t)$ and*

$$\eta_1(T-t)^{\frac{2\ell\alpha}{2\alpha-\gamma}} \leq u_\ell(0,t) \leq \eta_2(T-t)^{\frac{2\ell\alpha}{2\alpha-\gamma}} \qquad \text{for } 0 < t < T \tag{1.7}$$

with some constants $\eta_1, \eta_2 > 0$ depending only on p, N and ℓ, where α is the positive constant in (1.5) *and γ is the constant given by*

$$\gamma = \frac{-(N-2) + \sqrt{(N-2)^2 + 8(\alpha-1)(2\alpha+N-2)}}{2}. \tag{1.8}$$

The author expects that the solutions u_ℓ constructed in Theorem 1.1 will be useful, for example, to determine the exact dead-core rates of some solutions of (1.1) with general initial data, as the solutions of (1.3) constructed in [20, 21] play a crucial role in determining the blow-up rates of type II blow-up solutions of (1.3) in a supercritical case.

Remark 1.2. (i) Theorem 1.1 shows that infinite kinds of rates do exist. For every $N \geq 1$ and $p \in (0,1)$ we have $2(\alpha-1) < \gamma < 2\alpha$, whence $2\ell\alpha/(2\alpha-\gamma) > \alpha$ for $\ell \geq 1$. Namely, each dead-core rate specified in (1.7) is precisely faster than the self-similar rate $(T-t)^\alpha$.
(ii) The constant γ in (1.8) comes from the quadratic equation related to the asymptotic expansions of stationary solutions U_η of (1.1a) introduced in §§2.1. It is also related to eigenvalues of the linearized operator $\mathcal{A}$ given in (2.11) below (cf. Lemma 2.2).
(iii) Further information about η_1, η_2 in (1.7) is given in Remark 3.3.

Remark 1.3. In Theorems 3.2 and 6.1 we give detailed descriptions on the asymptotics of the solutions by virtue of the rescaled solutions w_ℓ with the self-similar variables (y, s) introduced in §2. With the help of a fixed-point theorem, we are able to construct a particular initial datum $u_{0,\ell}$ for each positive integer ℓ so that the corresponding solution u_ℓ of the Cauchy problem (1.1) fulfills the properties described in Theorem 1.1.

Remark 1.4. (i) Theorem 1.1 includes the result in [15] mentioned before as a special consequence when $N = 1$. Moreover, our result is still new even in this case in view of the generality of possible dead-core rates. Indeed, we have $\gamma = 2\alpha - 1$ for $N = 1$ and thus the dead-core rates in Theorem 1.1 are $(T-t)^{2\ell\alpha}$ with $\ell = 1, 2, 3, ...$, agreeing with the rates of the solutions constructed in [15] (cf. (1.6)) for the particular cases $\ell = 1, 3, 5, ...$
(ii) For $N \geq 3$ the result in Theorem 1.1 was obtained in a part of the author's thesis [31]. A technical difficulty arises in the proof for $N = 1, 2$ as is discussed below.

Our basic strategy to prove Theorem 1.1 is the same as that of [15]. We first introduce the similarity variables to derive a rescaled equation and then linearize the equation around the singular stationary solution U given in (2.2) below. It is shown that an eigenfunction expansion provides a good approximation for a desired solution of the linearized equation in a region away from the origin. The behavior of the solution near the origin is described by rescaling the regular stationary solutions U_η introduced in §2. Asymptotic expansions of U_η and ϕ_ℓ make the matching possible near the origin and moreover suggest the position at which it takes place. The existence of the solution with these behaviors may be rigorously proved by topological arguments coupled with a priori estimate. The solution enjoys the properties that have been expected by the formal matching argument. Going back to the original variables, we obtain the solution u_ℓ as stated in Theorem 1.1 for each positive integer ℓ.

In order to derive the key a priori estimate, we need to estimate an upper bound of growth order for the possible solutions in a region near infinity. We are able to prove, in the same way as [15], that under certain assumptions on initial data the solutions are below the singular stationary solution U in that region provided that the integer ℓ is odd. This estimate allows us to implement the topological arguments. In the present article, however, we do not impose the restriction that the integer ℓ is odd. We therefore have to show another a priori estimate in that region so as to execute the topological arguments without the restriction on ℓ. This task is accomplished by a comparison argument which works for every positive integer ℓ. It guarantees that the solutions exhibit the same growth order as that of U at infinity.

As was noted in Remark 1.4, this result was obtained in the author's thesis [31] for $N \geq 3$. In that situation, a Hardy type inequality is available (cf. (2.15)) and accordingly the potential term $r^{-2}v$ in the elliptic part $\mathcal{A}v$ of the linearized equation is easily handled in the function space H^1_ρ (cf. (2.11), (2.13b)). To deal with the inverse square potential in arbitrary dimensions, we introduce another function space $\mathcal{H}$ (cf. (2.13c)) in the spectral analysis and make use of its dual space $\mathcal{H}'$, where we prove that a variation of constants formula is valid for a nonhomogeneous problem related to the linearized equation. We explain in more detail in Remark 2.5 why these spaces have to be introduced.

We conclude this introduction by describing the plan of the present article. In the next §2, we briefly recall some preliminary results on stationary solutions of (1.1a) and investigate both spectral properties of the linearized operator and the validity of the variation of constants formula mentioned before. The former part of §3 is concerned with a formal construction of the solution described in Theorem 1.1. The topological arguments, together with the set of functions where they should be applied, are described in the latter part of §3. The proof of Theorem 1.1 is given there as a consequence of the arguments. §4 and §5 are devoted to proving key a priori estimates. Finally, we show further properties on the rescaled solution in §6.

2 Stationary solutions and linearization

In this section we recall and establish some fundamental results. §§2.1 is concerned with stationary solutions of (1.1a), where we briefly introduce some properties of stationary solutions obtained in [17]. In particular, the asymptotic expansion (2.3) plays an important role in the formal matching argument to be presented in §3. In §§2.2 we investigate some spectral properties of the linearized operator $\mathcal{A}$ formally given in (2.11) below.

2.1 Stationary problems

In this subsection we consider the radially symmetric stationary problem of (1.1a),

$$u'' + \frac{N-1}{\xi}u' = u^p \quad \text{for } \xi > 0, \quad u(0) = \eta, \; u'(0) = 0. \tag{2.1}$$

Let α be the constant in (1.5). A simple computation reveals that

$$U(\xi) := c_{p,N}\xi^{2\alpha} \quad \text{with} \quad c_{p,N} = \{2\alpha(2\alpha + N - 2)\}^{-\alpha} \tag{2.2}$$

is a solution of (2.1) such that $U(0) = U'(0) = 0$. It is the unique solution of (2.1) with $\eta = 0$ satisfying $U(\xi) > 0$ for $\xi > 0$ and is referred as the singular steady solution. It is proven in [17] that for each $\eta > 0$, a unique solution U_η of (2.1) exists and behaves as

$$U_\eta(\xi) = U(\xi) + h(\eta)\xi^\gamma(1 + o(1)) \quad \text{as } \xi \to \infty, \tag{2.3}$$

where γ is the constant in (1.8) and $h(\eta) = a\eta^\mu$ with $\mu = 1 - (1-p)\gamma/2 > 0$ and some constant $a > 0$ depending only on p and N. Note that U_η is monotone increasing in $\xi > 0$ for every $\eta > 0$, since $r^{N-1}U_\eta' = \int_0^r z^{N-1}U_\eta^p(z)dz > 0$. It is readily seen that if $0 < \eta_1 < \eta_2$, then $U(r) < U_{\eta_1}(r) < U_{\eta_2}(r)$ for all $r \in [0, \infty)$.

Remark 2.1. Although (2.3) was proven in [17, Proposition 3.1], we shall show its formal derivation for reader's convenience. Set $U_1(\xi) = U(\xi) + W(\xi)$ and observe that

$$W'' + \frac{N-1}{\xi}W' = pc_{p,N}^{p-1}\xi^{-2}W(\xi) + \cdots \quad \text{as } \xi \to \infty. \tag{2.4}$$

Suppose that W grows with algebraic order, say $W(\xi) \approx a\xi^\lambda$ with some $a \neq 0$ and $\lambda > 0$, as $\xi \to \infty$. Formally substituting $W(\xi) = a\xi^\lambda$ to (2.4) and letting $\xi \to \infty$, we get the quadratic equation for λ,

$$\lambda^2 + (N-2)\lambda - 2(\alpha - 1)(2\alpha + N - 2) = 0. \tag{2.5}$$

We then obtain the constant γ in (1.8) as a larger root of (2.5) and get (2.3) with $\eta = 1$. The expansion (2.3) with $\eta > 0$ is reduced to the one with $\eta = 1$ by the scaling property $U_\eta(\xi) = \eta U_1(\eta^{\frac{p-1}{2}}\xi)$.

2.2 Spectral analysis for the linearized operator

For $T > 0$, we introduce the similarity variables

$$u(x,t) = (T-t)^\alpha w(y,s), \quad y = (T-t)^{-1/2}x, \quad s = -\log(T-t). \tag{2.6}$$

A function u satisfies (1.1) in $\mathbf{R}^N \times (0,T)$ if and only if w satisfies

$$w_s = \Delta w - \frac{1}{2}y \cdot \nabla w + \alpha w - w^p \quad \text{in } \mathbf{R}^N \times (s_1, \infty), \tag{2.7}$$

$$w(y, s_1) = w_0(y) \quad \text{in } \mathbf{R}^N, \tag{2.8}$$

where $s_1 = -\log T$ and $w_0(y) = T^{-\alpha} u_0(\sqrt{T} y)$. Since we discuss only radial solutions, we may write (2.7), (2.8) as

$$w_s = w_{rr} + \Big(\frac{N-1}{r} - \frac{r}{2}\Big) w_r + \alpha w - w^p \quad \text{in } (0,\infty)\times(s_1,\infty), \tag{2.9a}$$
$$w_r(0,s) = 0 \quad \text{for } s \in (s_1,\infty), \tag{2.9b}$$
$$w(r,s_1) = w_0(r) \quad \text{in } [0,\infty). \tag{2.9c}$$

Here and henceforth we write $r = |y|$. Notice that $U = U(r)$ is a stationary solution of (2.9). We are discussing the existence of a solution $w(r,s)$ of (2.9) which converges to the singular stationary solution U as $s \to \infty$ in an appropriate way. To this end, we linearize the equation (2.9a) around U, setting

$$v(r,s) = w(r,s) - U(r)$$

for a solution w of (2.9). It then satisfies

$$v_s = -\mathcal{A} v + f(v) \quad \text{for } r > 0 \text{ and } s > s_1, \tag{2.10a}$$
$$v(r,s_1) = v_0(r) \quad \text{for } r > 0, \tag{2.10b}$$

where $v_0 = w_0 - U$ and $-\mathcal{A}$ is the linear differential operator formally given by

$$\begin{aligned} -\mathcal{A} v &= v'' + \Big(\frac{N-1}{r} - \frac{r}{2}\Big) v' + \alpha v - \frac{p c_{p,N}^{p-1}}{r^2} v \\ &= \frac{1}{r^{N-1}\rho(r)} \frac{d}{dr}\Big(r^{N-1}\rho(r)\frac{dv}{dr}\Big) + \alpha v - \frac{p c_{p,N}^{p-1}}{r^2} v \end{aligned} \tag{2.11}$$

with $\rho(r) = \exp(-r^2/4)$ and where

$$f(v) = U(r)^p - \{U(r) + v\}^p + \frac{p c_{p,N}^{p-1}}{r^2} v. \tag{2.12}$$

We work in the following weighted Hilbert spaces;

$$L^2_\rho = \Big\{ h \in L^2_{loc}((0,\infty)) \;\Big|\; \int_0^\infty h(r)^2 r^{N-1} \rho dr < \infty \Big\}, \tag{2.13a}$$
$$H^1_\rho = \Big\{ h \in H^1_{loc}((0,\infty)) \;\Big|\; h, h' \in L^2_\rho \Big\}, \tag{2.13b}$$
$$\mathcal{H} = \Big\{ \phi \in L^2_\rho \;\Big|\; \phi \in H^1_\rho,\ \int_0^\infty \frac{|\phi(r)|^2}{r^2} r^{N-1} \rho dr < \infty \Big\}, \tag{2.13c}$$

equipped with inner products

$$\langle g, h\rangle_{L^2_\rho} = \int_0^\infty g(r) h(r) r^{N-1} \rho dr, \tag{2.14a}$$
$$\langle g, h\rangle_{H^1_\rho} = \langle g, h\rangle_{L^2_\rho} + \langle g', h'\rangle_{L^2_\rho}, \tag{2.14b}$$
$$(\phi, \psi)_{\mathcal{H}} = \langle \phi, \psi\rangle_{H^1_\rho} + p c_{p,N}^{p-1} \int_0^\infty \frac{\phi(r)\psi(r)}{r^2} r^{N-1} \rho dr, \tag{2.14c}$$

respectively, where ρ is as above. Clearly, the space $\mathcal{H}$ is continuously embedded in H^1_ρ as a Banach space. When $N \geq 3$, we have a version of Hardy inequality [32, Lemma 2.2]:

$$\int_0^\infty \frac{1}{r^2}|g(r)|^2 r^{N-1}\rho(r)dr \leq \frac{4\|g'\|^2_{L^2_\rho}}{(N-2)^2} + \frac{\|g\|^2_{L^2_\rho}}{N-2} \tag{2.15}$$

for any $g \in H^1_\rho$. Therefore we have the converse embedding, so that $\mathcal{H} = H^1_\rho$ if $N \geq 3$ thanks to (2.15), whereas $\mathcal{H} \neq H^1_\rho$ for $N = 1, 2$.

We are then led to the spectral analysis for a realization of $\mathcal{A}$ in L^2_ρ so as to investigate the asymptotic behavior of v as $s \to \infty$. The analysis to be discussed below reveals that the realization can be uniquely extended to a self-adjoint operator and its spectrum consists only of eigenvalues.

Lemma 2.2. *The operator $A : L^2_\rho \to L^2_\rho$ defined by $A\psi = \mathcal{A}\psi$ for $\psi \in \mathcal{D}(A)$ with domain $\mathcal{D}(A) = \{\psi \in L^2_\rho \mid \psi \in \mathcal{H},\ \mathcal{A}\psi \in L^2_\rho\}$ may be extended to a unique self-adjoint operator, still denoted by A, which has the following properties:*

$$\mathcal{D}(A) \subset \mathcal{H}; \tag{2.16}$$

$$-\alpha\|\varphi\|^2_{L^2_\rho} \leq \langle A\varphi, \varphi\rangle_{L^2_\rho}, \quad \forall \varphi \in \mathcal{D}(A). \tag{2.17}$$

Moreover, the spectrum of A consists only of the eigenvalues $\{\mu_j\}_{j=0}^\infty$ given by

$$\mu_j = j + \frac{\gamma}{2} - \alpha, \quad j = 0, 1, 2, ... \tag{2.18}$$

and the corresponding eigenfunctions are explicitly represented as

$$\phi_0 = c_0 r^\gamma,\ \phi_j(r) = c_j r^\gamma M\Big(-j; \gamma + \frac{N}{2}, \frac{r^2}{4}\Big), \quad j = 1, 2, ..., \tag{2.19}$$

where $c_j > 0$ are the normalizing constants so that $\|\phi_j\|_{2,\rho} = 1$, $M(a; b, \eta)$ denotes the standard Kummer function, and γ is the constant in (1.8). *Furthermore,*

$$\phi_j(r) = c_j r^\gamma(1 + o(1)) \quad \text{as } r \to 0; \tag{2.20a}$$

$$\phi_j(r) = \tilde{c}_j r^{2(\mu_j+\alpha)}(1 + o(1)) \quad \text{as } r \to \infty, \tag{2.20b}$$

where $\tilde{c}_j$ are constants such that $(-1)^j \tilde{c}_j > 0$ for $j = 1, 2, ...$

Proof. We begin with remarking that

$$\langle \phi, A\psi\rangle_{L^2_\rho} = \int_0^\infty \phi'\psi' r^{N-1}\rho dr - \alpha \int_0^\infty \phi\psi r^{N-1}\rho dr + pc^{p-1}_{p,N} \int_0^\infty \frac{\phi\psi}{r^2} r^{N-1}\rho dr \tag{2.21}$$

for any $\phi, \psi \in \mathcal{H}$ such that $\mathcal{A}\psi, \mathcal{A}\phi \in L^2_\rho$. This is readily seen by an integration by parts and a limiting procedure. In particular, the lower bound (2.17) holds for the original operator A. Thus A is a semi-bounded symmetric operator. Friedrichs' theorem admits then it extending a unique self-adjoint operator such that (2.17) holds (the Friedrichs extension), still denoted by A, whose domain $\mathcal{D}(A)$ is contained in the form domain of $\hat{q}$. Here $\hat{q}$ is the closure of the quadratic form $q(\phi, \psi) = \langle \phi, A\psi\rangle_{L^2_\rho}$ under the norm

$\|\phi\|_{+1} := (q(\phi,\phi) + (\alpha+1)\|\phi\|^2_{L^2_\rho})^{1/2} = \|\phi\|_{\mathcal{H}}$ (cf. for example, [30]). The last assertion simultaneously implies that the form domain of $\hat{q}$ is continuously embedded in $\mathcal{H}$, whence (2.16) follows. For any ϕ and ψ in the form domain of $\hat{q}$, there are Cauchy sequences $\{\phi_n\}$ and $\{\psi_n\}$, in the form domain of q (the original domain of A), which tend to ϕ and to ψ as $n \to \infty$, respectively, in the sense of the norm $\|\cdot\|_{+1}$. We then observe that

$$\langle \phi, A\psi \rangle_{L^2_\rho} = \hat{q}(\phi,\psi) = \lim_{n\to\infty} q(\phi_n,\psi_n). \tag{2.22}$$

Consequently, the representation (2.21) holds for every ϕ, ψ in the domain $\mathcal{D}(A)$ of the Friedrichs extension A.

Consider now the equation $\big(A + (\alpha+1)I\big)\phi = \psi$ for $\psi \in L^2_\rho$. By (2.21) we have $\|\phi\|^2_{H^1_\rho} \le \langle \phi, \psi \rangle_{L^2_\rho} \le \|\phi\|_{L^2_\rho}\|\psi\|_{L^2_\rho}$ and thus

$$\|\phi\|_{H^1_\rho} \le \|\psi\|_{L^2_\rho},$$

which implies that $\big(A + (\alpha+1)I\big)^{-1}$ is a compact operator. Therefore the spectrum of A consists only of a countable number of real eigenvalues.

Let us compute the concrete values of the eigenvalues $\{\mu_j\}$ and the corresponding eigenfunctions $\{\phi_j\}$ to see (2.18) and (2.19), respectively. We begin with remarking that every eigenfunction ϕ of A is smooth and satisfies the differential equation $\mathcal{A}\phi = \mu\phi$ in $(0,\infty)$ for the corresponding eigenvalue μ. Indeed, the self-adjointness of A implies that $\mu\langle\phi,\psi\rangle = \langle \phi, \mathcal{A}\psi\rangle_{L^2_\rho}$ for any $\psi \in C_0^\infty((0,\infty))$. We may exploit Weyl's lemma to observe that $\phi(r)$ is twice differentiable for $r > 0$. An integration by parts shows then that $\langle\phi,\mathcal{A}\psi\rangle_{L^2_\rho} = \langle\mathcal{A}\phi,\psi\rangle_{L^2_\rho}$ and hence $\mathcal{A}\phi = \mu\phi$ a. e. $r > 0$. Since $\phi_j \not\equiv 0$, we may assume, without loss of generality, that each ϕ_j is positive where $r > 0$ is small enough. In order to prove (2.18)-(2.20b), we set $\phi_j(r) = c_j r^\gamma H_j(\eta)$ with $\eta = r^2/4$. A straightforward calculation reveals then that the function ϕ_j solves the equation $\mathcal{A}\phi = \mu_j\phi$ if and only if H_j satisfies Kummer's equation

$$\eta H''(\eta) + (\hat{b} - \eta)H'(\eta) - \hat{a}H(\eta) = 0 \tag{2.23}$$

with $\hat{a} = \gamma/2 - (\mu_j + \alpha)$ and $\hat{b} = \gamma + N/2$. The general solution of (2.23) is given by

$$C_1 M(\hat{a};\hat{b},\eta) + C_2 U(\hat{a};\hat{b},\eta)$$

with arbitrary constants C_1 and C_2 (cf. [1]), where $M(\hat{a};\hat{b},\eta)$ is the Kummer function;

$$M(a;b,z) = 1 + \frac{az}{b} + \frac{(a)_2 z^2}{(b)_2 2!} + \cdots + \frac{(a)_n z^n}{(b)_n n!} + \cdots \tag{2.24}$$

with $(a)_j = a(a+1)(a+2)\cdots(a+j-1)$, $j = 1, 2, ...$, and

$$U(a;b,z) = \frac{\pi}{\sin b\pi}\left[\frac{M(a;b,z)}{\Gamma(1+a-b)\Gamma(b)} - z^{1-b}\frac{M(1+a-b;2-b,z)}{\Gamma(a)\Gamma(2-b)}\right].$$

Here Γ denotes the standard gamma function. Suppose now that $C_2 \neq 0$. Then ϕ_j grows as $r^{\gamma+2(1-\hat{b})}(= r^{-\gamma-N+2})$ as $r \to 0$, but it then contradicts the fact that ϕ_j is in H^1_ρ since

$2(-\gamma - N + 1) + N \le 0$ and accordingly $|\phi_j'(r)|^2 r^{N-1}$ exhibits non-integrable growth as $r \to 0$. Consequently the constant C_2 must vanish. Finally we shall show that $\hat{a}$ must be a nonpositive integer. If it were not so, then

$$M\Big(\hat{a}; \hat{b}, \frac{r^2}{4}\Big) \sim \frac{\Gamma(\hat{b})}{\Gamma(\hat{a})}\Big(\frac{r^2}{4}\Big)^{-(\hat{b}-\hat{a})} \exp\Big(\frac{r^2}{4}\Big) \quad \text{as } r \to \infty.$$

Not belonging to H^1_ρ, it should be excluded, whence $\hat{a} = -n$ for some $n \in \mathbf{N} \cup \{0\}$. We thus conclude (2.18) and (2.19). Moreover, for each n, we have

$$M\Big(-n; \gamma + \frac{N}{2}, \frac{r^2}{4}\Big) = (-1)^n \tilde{c}_n r^{2n} + o(r^{2n}) \quad \text{as } r \to \infty \tag{2.25}$$

with some constant $\tilde{c}_n$ such that $(-1)^n \tilde{c}_n > 0$. Now (2.20b) is readily seen from (2.25), (2.18) and (2.19), whereas (2.20a) follows from (2.19) and (2.24). □

An important fact is that, as is proven below, the eigenfunctions ϕ_j yield a basis of $\mathcal{H}$ for any spatial dimension $N \ge 1$, which suggests us to work with the space $\mathcal{H}$ rather than H^1_ρ. Notice that

$$-1 < \mu_0 < 0 < \mu_1 < \dots \tag{2.26}$$

Corollary 2.3. *Assume the same hypotheses as in Lemma 2.2. Then the sequence* $\{\hat{\phi}_j\}_{j=0}^\infty$ *defined by*

$$\hat{\phi}_j = \frac{\phi_j}{\sqrt{\mu_j + \alpha + 1}}, \quad j = 0, 1, \dots, \tag{2.27}$$

is a complete orthonormal system in $\mathcal{H}$.

Proof. From the proof of Lemma 2.2 (cf. (2.14c), (2.21) and (2.22)) we have

$$(\psi, \phi)_{\mathcal{H}} = \langle A\psi, \phi\rangle_{L^2_\rho} + (\alpha + 1)\langle \psi, \phi\rangle_{L^2_\rho}, \quad \forall \psi \in \mathcal{D}(A), \forall \phi \in \mathcal{H}. \tag{2.28}$$

Substituting $\psi = \phi_j$ to (2.28), one has that

$$(\phi_j, \phi)_{\mathcal{H}} = (\mu_j + \alpha + 1)\langle \phi_j, \phi\rangle_{L^2_\rho}, \quad \forall \phi \in \mathcal{H};\ j = 0, 1, \dots \tag{2.29}$$

and, in particular,

$$(\phi_j, \phi_k)_{\mathcal{H}} = (\mu_j + \alpha + 1)\delta_{jk}, \quad j, k = 0, 1, \dots, \tag{2.30}$$

where the symbol δ_{jk} denotes Kronecker's delta. Since the system $\{\phi_j\}_{j=0}^\infty$ is complete in L^2_ρ by Lemma 2.2, it follows from (2.29) and (2.30) that the system (2.27) is a complete orthonormal system in $\mathcal{H}$. □

We state here some nodal structures of the eigenfunctions ϕ_j on $(0, \infty)$, though we do not require them except for §6. Let J be an interval of $[0, \infty)$. For a function $\Psi : J \to \mathbf{R}$, the zero number $\mathcal{Z}_J[\Psi]$ of Ψ on J is defined by

$$\mathcal{Z}_J[\Psi] := \sharp\{r \in J;\ \Psi(r) = 0\}, \tag{2.31}$$

where $\sharp$ stands for cardinal numbers. As is well-known, if Ψ is of $C^1(J)$, then $\mathcal{Z}_J(\Psi) + 1$ coincides with the number of sign-changes of Ψ in J.

Proposition 2.4. *One has* $\mathcal{Z}_{(0,\infty)}[\phi_j] = j$ *for each* $j = 0, 1, \ldots$ *Moreover, if one denotes by* $r_{j,n}$ *the* n*-th zero of* ϕ_j *in* $(0, \infty)$ *enumerated near the origin for* $j = 1, 2, \ldots$ *and* $n = 1, 2, \ldots, j$*, then* $r_{1,1} = \sqrt{2(2\gamma + N)}$ *and* $r_{j+1,1} \in (0, r_{j,1})$*,* $r_{j+1,2} \in (r_{j,1}, r_{j,2}), \ldots, r_{j+1,n} \in (r_{j,n-1}, r_{j,n}), \ldots, r_{j+1,j+1} \in (r_{j,j}, \infty)$ *for each* $j \geq 1$*.*

Proof. These are fundamental consequences of the well-known Sturm-Liouville theory. We appeal to an induction on j. The assertions are clear for $j = 0, 1$ by the representation formula of ϕ_j in (2.19), (2.24). Assume that they hold for some integer $j \geq 1$. Consider two eigenpairs (μ_m, ϕ_m), (μ_n, ϕ_n) of A with $\mu_m < \mu_n$. Suppose that $\phi_m(r_i) = 0$, $\phi_m(r_{i+1}) = 0$ and that ϕ_m has a constant sign in (r_i, r_{i+1}) for some $r_i, r_{i+1} \in (0, \infty)$. We claim that there exists a zero of ϕ_n in (r_i, r_{i+1}) at which ϕ_n changes its sign. We may assume that $\phi_m(r) > 0$ in (r_i, r_{i+1}) without loss of generality. Note that $\phi_m'(r_i) \geq 0$ and $\phi_m'(r_{i+1}) \leq 0$. An integration by parts reveals then that

$$(\mu_n - \mu_m) \int_{r_i}^{r_{i+1}} \phi_m \phi_n r^{N-1} \rho dr = \phi_n(r_{i+1}) \phi_m'(r_{i+1}) r_{i+1}^{N-1} \rho(r_{i+1}) - \phi_n(r_i) \phi_m'(r_i) r_i^{N-1} \rho(r_i).$$

This identity implies that ϕ_n cannot have a constant sign in (r_i, r_{i+1}). A similar argument shows that there exists a zero of ϕ_n in (r_{i+1}, ∞) at which ϕ_n changes its sign. Thus $\mathcal{Z}_{(0,\infty)}[\phi_{j+1}] \geq \mathcal{Z}_{(0,\infty)}[\phi_j] + 1 = j+1$. On the other hand, the representation formula of ϕ_j in (2.19), (2.24) guarantees that $\mathcal{Z}_{(0,\infty)}[\phi_\ell] \leq \ell$ for every $\ell \geq 1$. Therefore $\mathcal{Z}_{(0,\infty)}[\phi_{j+1}] = j+1$. The claim on the positions of zeros is obvious by virtue of the above argument. □

Our next task is to show that the differential operator $\mathcal{A}$ may be understood to be an operator in $\mathcal{H}'$. Moreover, we prove that a solution of the equation (2.10) may be considered as an element of $\mathcal{H}'$ and enjoys the integral equation in $\mathcal{H}'$ corresponding to the problem (2.10). Here and henceforth, $\mathcal{H}'$ stands for the dual space of $\mathcal{H}$.

Remark 2.5. It should be noticed that the space $\mathcal{H}$ was already used in [15] for $N = 1$ to produce eigenvalues of $\mathcal{A}$, but its dual space $\mathcal{H}'$ was not used explicitly there. The use of $\mathcal{H}'$ should be essential because a solution of (1.1) is positive everywhere before its dead-core appears and thus does not belong to $\mathcal{H}$ unless $N \geq 3$. To show this issue, let us consider a positive function $\Phi \in L^2_\rho$ and the integral

$$\int_0^\infty \frac{\Phi(r)\psi(r)}{r^2} r^{N-1} \rho dr \quad \text{with} \quad \psi \in H^1_\rho.$$

This integral is not finite for $N = 1, 2$ unless ψ decays at the origin. This fact suggests us to restrict the set of ψ's to $\mathcal{H}$ and thus regard Φ as a bounded linear functional on $\mathcal{H}$.

Suppose that a function $w_0 \in L^\infty_{loc}(\mathbf{R}^N)$ has at most algebraic growth as $|y| \to \infty$. Let w be a solution of (2.7) with initial data $w(s_1) = w_0$. Then by a standard argument of parabolic equations (cf. [2, 18, 34]), w, ∇w and $\nabla^2 w$ are locally bounded in $\mathbf{R}^N$ and have at most algebraic growth as $|y| \to \infty$ for each $s \in (s_1, \infty)$. Namely, for each $0 < s_1 < s_2 < \infty$, there are constants $C_0, C_1, C_2 > 0$ such that

$$|w(y, s)| \leq C_0(|y| + 1)^\beta, \ |\nabla w(y, s)| \leq C_1(|y| + 1)^\beta, \ |\nabla^2 w(y, s)| \leq C_2(|y| + 1)^\beta \quad (2.32)$$

with some $\beta > 1$. Here the constant C_0 does not depend on s_1. Let us consider a function $v_0 \in L^\infty_{loc}([0,\infty))$ satisfying

$$|v_0(r)| \le C(r+1)^\beta, \quad r > 0 \tag{2.33}$$

with some constant $C > 0$ and $\beta > \max\{1, 2\alpha - 2\}$. We then observe that a solution v of (2.10) with initial datum v_0 satisfying (2.33) fulfills

$$|v(r,s)| \le C_0'(r+1)^\beta,\ |v_r(r,s)| \le C_1'(r+1)^\beta, \quad r > 0,\ s_1 < s \le s_2; \tag{2.34a}$$

$$|v_{rr}(r,s)| \le C_2'\Big(1 + \frac{N-1}{r}\Big)(r+1)^\beta, \quad r > 0,\ s_1 < s \le s_2 \tag{2.34b}$$

with some constants $C_0', C_1', C_2' > 0$, since $w = v + U$.

The following lemma plays a crucial role to deal with a solution of (2.10) in $\mathcal{H}'$. Throughout the present article the topology of $\mathcal{H}'$ is understood to be that given by the norm $\|\cdot\|_{\mathcal{H}'} = \sup\{\langle \cdot, \psi\rangle\ ;\ \|\psi\|_{\mathcal{H}} = 1\}$.

Lemma 2.6. *(i) Let g be a measurable function on $[0,\infty)$ satisfying*

$$|g(r)| \le \Big(1 + \frac{1}{r^q}\Big)\varphi(r) \qquad \text{a.e. } r > 0 \tag{2.35}$$

with some constant $q \ge 0$ and nonnegative function $\varphi \in L^\infty_{loc}([0,\infty)) \cap H^1_\rho$. If

$$q < \gamma + \min\Big\{N, 1 + \frac{N}{2}\Big\}, \tag{2.36}$$

then g may be regarded as an element of $\mathcal{H}'$ in the sense that one may associate it with $\tilde{g} \in \mathcal{H}'$ defined by

$$\langle \tilde{g}, \psi\rangle_{\mathcal{H}'\times\mathcal{H}} := \int_0^\infty g(r)\psi(r) r^{N-1}\rho dr \qquad \text{for } \psi \in \mathcal{H} \tag{2.37}$$

and, moreover, $\|\tilde{g}\|_{\mathcal{H}'} \le C\|\varphi\|_{H^1_\rho}$ with some constant $C > 0$.
(ii) If v is a solution of (2.10) in $(0,\infty)\times[s_1, s_2]$ for some $s_2 \in (s_1,\infty)$ such that $|v(r,s)| \le \varphi(r)$ with some $\varphi \in H^1_\rho$, then $f(v)$ is in the class $L^1(s_1, s_2; \mathcal{H}') \cap C((s_1, s_2]; \mathcal{H}')$. In particular, $f(v)$ belongs to this class provided that the initial datum v_0 satisfies (2.33).

Proof. (i) Note that $\psi = \sum_{j=0}^\infty (\psi, \hat{\phi}_j)_{\mathcal{H}}\hat{\phi}_j$ in $\mathcal{H}$ for every $\psi \in \mathcal{H}$ by Corollary 2.3. We first prove that for any $R > 0$ there exists a constant $C(R) > 0$ such that

$$|\phi_j(r)| \le C(R) r^\gamma j^{-\frac{1}{4}} \tag{2.38}$$

for all $0 < r \le R$ and $j \ge 1$. Using (2.19) and the formula

$$\int_0^\infty x^{\beta-1}e^{-x}M(-j,\beta;x)^2 dx = \frac{\Gamma(\beta)^2\Gamma(j+1)}{\Gamma(\beta+j)},$$

we get

$$c_j^2 = \frac{\Gamma(j + \frac{N}{2} + \gamma)}{\Gamma(\frac{N}{2}+\gamma)\Gamma(j+1)}.$$

Stirling's formula then yields a positive constant C such that

$$c_j = C(j+1)^{\frac{1}{2}(\frac{N}{2}+\gamma-1)} \tag{2.39}$$

for $j \gg 1$. By a classical asymptotic formula

$$M(-j,\beta;x) \sim \pi^{-\frac{1}{2}}\Gamma(\beta)e^x\Big\{\Big(\frac{\beta}{2}+j\Big)x\Big\}^{\frac{1}{4}-\frac{\beta}{2}}\cos\Big\{\sqrt{2(\beta+2j)x}-\frac{\beta\pi}{2}+\frac{\pi}{4}\Big\}$$

as $j \to \infty$ (cf. [1]), we observe

$$\Big|M\Big(-j,\gamma+\frac{N}{2};\frac{r^2}{4}\Big)\Big| \le C(R)j^{\frac{1-2\gamma-N}{4}}, \quad j \ge j_0 \tag{2.40}$$

for some $j_0 \gg 1$ and $C(R) > 0$. The claim (2.38) then follows from (2.19), (2.39) and (2.40) for $j \ge j_0$. Clearly, (2.38) holds for $1 \le j < j_0$ due to the expressions of eigenfunctions (2.19), (2.24). We use the estimate (2.38) with $R = 1$ to obtain

$$\int_0^1 \Big(1+\frac{1}{r^q}\Big)\varphi(r)|\hat{\phi}_j(r)|r^{N-1}\rho dr \le C(1)j^{-\frac{3}{4}}\int_0^1 \varphi(r)r^{\gamma+N-1-q}\rho dr \tag{2.41}$$

for each $j \ge 1$. We get, by an integration by parts and the hypothesis on q in (2.36),

$$\begin{aligned}\int_0^\infty \varphi(r)r^{\gamma+N-1-q}\rho dr &= \frac{1}{\gamma+N-q}\int_0^\infty r^{\gamma+N-q}\Big\{-\varphi'(r)+\frac{r}{2}\varphi(r)\Big\}\rho dr\\ &\le C\|\varphi\|_{H^1_\rho}\Big(\int_0^\infty r^{2(\gamma+1-q)+N-1}(1+r^2)\rho dr\Big)^{\frac{1}{2}}\\ &\le C\|\varphi\|_{H^1_\rho}.\end{aligned} \tag{2.42}$$

By (2.35), (2.41) and (2.42), we get

$$\sum_{j=0}^\infty \Big(\int_0^\infty g(r)\hat{\phi}_j(r)r^{N-1}\rho dr\Big)^2 \le C\|\varphi\|^2_{H^1_\rho}. \tag{2.43}$$

We then proceed to prove that

$$\int_0^1 g(r)\psi(r)r^{N-1}\rho dr = \sum_{j=0}^\infty (\psi,\hat{\phi}_j)_{\mathcal{H}}\int_0^1 g(r)\hat{\phi}_j(r)r^{N-1}\rho dr. \tag{2.44}$$

Notice that the completeness of $\{\hat{\phi}_j\}_{j=0}^\infty$ in $\mathcal{H}$ and (2.43) guarantee the convergence of the series in (2.44). To prove (2.44), we set $S_n(r) = \sum_{j=0}^n (\psi,\hat{\phi}_j)_{\mathcal{H}}\hat{\phi}_j(r)$. Since $S_n \to \psi$ in $\mathcal{H}$, it is possible to find a subsequence $\{S_{n_k}\}$ such that $S_{n_k}(r) \to \psi(r)$ a. e. $r \in [0,1]$ as $k \to \infty$. Our assumption (2.35), (2.41) and (2.42) imply that $|g(r)S_n(r)r^{N-1}\rho|$ is estimated above by an integrable function in $[0,\infty)$ independently of n, which provides that

$$\lim_{k\to\infty}\int_0^1 g(r)S_{n_k}(r)r^{N-1}\rho dr = \int_0^1 g(r)\psi(r)r^{N-1}\rho dr.$$

Therefore (2.44) holds. It then follows from (2.44) and (2.43) that

$$|\langle \tilde{g}, \psi \rangle_{\mathcal{H}' \times \mathcal{H}}| \leq C \|\psi\|_{\mathcal{H}} \|\varphi\|_{H^1_\rho},$$

which completes the proof of (i).

(ii) It is readily seen that

$$\int_0^\infty \Big(U(r)^p + \{U(r) + v\}^p \Big) |\psi| r^{N-1} \rho dr \leq C(1 + \|\varphi\|^2_{L^2_\rho})^{\frac{p}{2}} \|\psi\|_{L^2_\rho}.$$

Consider the integral $\int_0^\infty r^{-2} v \psi r^{N-1} \rho dr$. Since $2 < \gamma + \min\{N, 1 + N/2\}$, we may apply the result of (i) with $q = 2$ to this integrand to observe that $f(v) \in L^\infty(s_1, s_2; \mathcal{H}')$ and $\|f(v)\|_{L^\infty(s_1,s_2;\mathcal{H}')} \leq L$ with a constant $L > 0$ depending only on p, N, s_1, s_2 and $\|\varphi\|_{H^1_\rho}$. Let us prove the continuity of $f(v)$. Since $\Lambda := \inf\{w(r, s); (r, s) \in (0, \infty) \times [s_1, s_2]\} > 0$, there is a constant $C > 0$ depending only on p, N, s_1, s_2 and Λ such that

$$|f(v(s)) - f(v(\bar{s}))| \leq C\Big(1 + \frac{1}{r}\Big) |v(r, s) - v(r, \bar{s})|, \quad r > 0,\ s_1 < s, \bar{s} \leq s_2.$$

Hence $\|f(v(s)) - f(v(\bar{s}))\|_{\mathcal{H}'} \leq C \|v(s) - v(\bar{s})\|_{H^1_\rho}$ by (i), which completes the proof. □

Based on Corollary 2.3 and Lemma 2.6, we define an operator $\tilde{A} : \mathcal{H}' \to \mathcal{H}'$. Let v be an element of H^1_ρ. By Lemma 2.6 there is a constant $C > 0$ depending only on p and N such that

$$\Big| \int_0^\infty (\mathcal{A}v)(r) \psi(r) r^{N-1} \rho dr \Big| \leq C \|v\|_{H^1_\rho} \|\psi\|_{\mathcal{H}}$$

for every $\psi \in \mathcal{H}$. Hence $\mathcal{A}v \in \mathcal{H}'$ and

$$\|\mathcal{A}v\|_{\mathcal{H}'} \leq C \|v\|_{H^1_\rho}. \tag{2.45}$$

Consequently we may define a linear operator $\tilde{A} : \mathcal{H}' \to \mathcal{H}'$ with domain $\mathcal{D}(\tilde{A})$ by

$$\mathcal{D}(\tilde{A}) = H^1_\rho, \tag{2.46a}$$
$$\tilde{A}v = \mathcal{A}v \ \text{ in } \mathcal{H}' \quad \text{ for } v \in \mathcal{D}(\tilde{A}). \tag{2.46b}$$

In particular, $\phi_j \in \mathcal{D}(\tilde{A})$ and $\tilde{A}\phi_j = \mu_j \phi_j$ for each $j = 0, 1, \ldots$

Lemma 2.7. *Let v be a solution of (2.10) in $(0, \infty) \times [s_1, s_2]$ for some $s_2 \in (s_1, \infty)$ satisfying the growth condition (2.34). Denote by $v(s)$ the corresponding element of $\mathcal{H}'$ in the sense of (2.37). Then v is in $C^1((s_1, s_2]; \mathcal{H}') \cap C((s_1, s_2]; \mathcal{D}(\tilde{A}))$ and solves the evolution equation*

$$\frac{dv}{ds} = -\tilde{A}v + f(v) \quad \text{in } \mathcal{H}'. \tag{2.47}$$

Proof. For any $s \in (s_1, s_2]$, $s + h \in (s_1, s_2)$ and $r > 0$, there is $\theta \in (0, 1)$ such that

$$\Big\| \frac{v(s+h) - v(s)}{h} - v_s \Big\|^2_{\mathcal{H}'} \leq \int_0^\infty \big\{ v_s(r, s + \theta h) - v_s(r, s) \big\}^2 r^{N-1} \rho dr \tag{2.48}$$

Since v_s is bounded and continuous in $(0,1]\times[s^*,s_2]$ and is estimated above by $C(1+r)^{2\beta+2}$ in $[1,\infty)\times[s^*,s_2]$ for each $s^*\in(s_1,s_2)$ by virtue of (2.34), the right hand side of (2.48) tends to zero as $h\to 0$. It shows that $v(s)$ is differentiable in $(s_1,s_2]$ and $dv(s)/ds=v_s(\cdot,s)$ in $\mathcal{H}'$. A similar argument shows that $v_s(\cdot,s)$ is continuous in $s\in(s_1,s_2]$ with values in $\mathcal{H}'$. Namely $v\in C^1((s_1,s_2];\mathcal{H}')$. We next observe by (2.45) that

$$\|\tilde{A}v(s)-\tilde{A}v(\bar{s})\|_{\mathcal{H}'}\le\|v(s)-v(\bar{s})\|_{H^1_\rho}\qquad\text{for } s,\bar{s}\in(s_1,s_2].$$

This implies that v is in $C((s_1,s_2];\mathcal{D}(\tilde{A}))$. Since $dv(s)/ds=v_s(\cdot,s)$, the function $v(s)$ solves the equation (2.47) for $s\in(s_1,s_2]$. □

We define a family of linear operators $\{e^{-(s-s_1)\tilde{A}}\}_{s\ge s_1}$ acting on $\mathcal{H}'$ by $e^{0\tilde{A}}=I$, i.e., the identity map in $\mathcal{H}'$ and by

$$e^{-(s-s_1)\tilde{A}}\Phi:=\sum_{j=0}^{\infty}e^{-\mu_j(s-s_1)}\langle\Phi,\phi_j\rangle_{\mathcal{H}'\times\mathcal{H}}\phi_j \tag{2.49}$$

for $\Phi\in\mathcal{H}'$ and $s>s_1$.

Proposition 2.8. *(i) For each $\Phi\in\mathcal{H}'$, $e^{-(s-s_1)\tilde{A}}\Phi$ is an element of $\mathcal{H}'\cap\mathcal{H}$ and*

$$\|e^{-(s-s_1)\tilde{A}}\Phi\|_{\mathcal{H}'}\le\|\Phi\|_{\mathcal{H}'}. \tag{2.50}$$

The family $\{e^{-(s-s_1)\tilde{A}}\}_{s\ge s_1}$ is a semigroup on $\mathcal{H}'$. Namely, $e^{0\tilde{A}}=I$ and

$$e^{-(s-s_1)\tilde{A}}e^{-(\bar{s}-s_1)\tilde{A}}=e^{-(s+\bar{s}-2s_1)\tilde{A}}\qquad\text{for each } s,\bar{s}\ge s_1. \tag{2.51}$$

(ii) The function $s\mapsto e^{-(s-s_1)\tilde{A}}\Phi$ is continuous in $[s_1,\infty)$ with values in $\mathcal{H}'$ if and only if Φ belongs to the closure $\overline{\mathcal{D}(\tilde{A})}$ of $\mathcal{D}(\tilde{A})$ in $\mathcal{H}'$.
(iii) Let $\Phi\in\mathcal{H}'$. If $s>s_1$, then $e^{-(s-s_1)\tilde{A}}\Phi\in\mathcal{D}(\tilde{A})$ and

$$\tilde{A}e^{-(s-s_1)\tilde{A}}\Phi=\sum_{j=0}^{\infty}\mu_je^{-\mu_j(s-s_1)}\langle\Phi,\phi_j\rangle_{\mathcal{H}'\times\mathcal{H}}\phi_j. \tag{2.52}$$

Moreover, the function $s\mapsto e^{-(s-s_1)\tilde{A}}\Phi$ is differentiable for each $s>s_1$ and

$$\frac{d}{ds}e^{-(s-s_1)\tilde{A}}\Phi=\sum_{j=0}^{\infty}(-\mu_j)e^{-\mu_j(s-s_1)}\langle\Phi,\phi_j\rangle_{\mathcal{H}'\times\mathcal{H}}\phi_j=-\tilde{A}e^{-(s-s_1)\tilde{A}}\Phi. \tag{2.53}$$

Proof. (i) We begin with remarking that for each $\Phi\in\mathcal{H}'$ there is $F_\Phi\in\mathcal{H}$ with $\|\Phi\|_{\mathcal{H}'}=\|F_\Phi\|_{\mathcal{H}}$ such that $\langle\Phi,\psi\rangle_{\mathcal{H}'\times\mathcal{H}}=(F_\Phi,\psi)_{\mathcal{H}}$ for every $\psi\in\mathcal{H}$. We observe with (2.27) that

$$\sum_{j=n}^{\infty}|e^{-\mu_j(s-s_1)}||\langle\Phi,\phi_j\rangle_{\mathcal{H}'\times\mathcal{H}}|\|\phi_j\|_{\mathcal{H}'}\le M_s\Big(\sum_{j=n}^{\infty}|(F_\Phi,\hat{\phi}_j)_{\mathcal{H}}|^2\Big)^{\frac{1}{2}},\quad s>s_1,$$

where $M_s^2=\sum_{j=0}^{\infty}|(\mu_j+\alpha+1)e^{-2\mu_j(s-s_1)}|<\infty$. By the completeness of the system $\{\hat{\phi}_j\}_{j=0}^{\infty}$ in $\mathcal{H}$ the right hand side tends to zero as $n\to\infty$, which implies the convergence of the series (2.49) in $\mathcal{H}'$. A similar argument shows that the series is convergent also in the norm of $\mathcal{H}$. We next claim that

$$\Phi = \sum_{j=0}^{\infty} \langle \Phi, \phi_j \rangle_{\mathcal{H}' \times \mathcal{H}} \phi_j \quad \text{ in } \mathcal{H}'; \tag{2.54a}$$

$$\|\Phi\|_{\mathcal{H}'}^2 = \sum_{j=0}^{\infty} |\langle \Phi, \hat{\phi}_j \rangle_{\mathcal{H}' \times \mathcal{H}}|^2. \tag{2.54b}$$

To show (2.54a) we use Corollary 2.3 to see that

$$F_\Phi = \sum_{j=0}^{\infty} \langle \Phi, \hat{\phi}_j \rangle_{\mathcal{H}' \times \mathcal{H}} \hat{\phi}_j. \tag{2.55}$$

Since $(\hat{\phi}_j, \psi)_{\mathcal{H}} = \sqrt{\mu_j + \alpha + 1} \langle \phi_j, \psi \rangle_{L^2_\rho}$ for any $\psi \in \mathcal{H}$ and j by (2.27) and (2.29), we get $(F_\Phi, \psi)_{\mathcal{H}} = \sum_{j=0}^{\infty} \langle \Phi, \phi_j \rangle_{\mathcal{H}' \times \mathcal{H}} \langle \phi_j, \psi \rangle_{L^2_\rho}$, which implies (2.54a). Since $\|\Phi\|_{\mathcal{H}'} = \|F_\Phi\|_{\mathcal{H}}$, the claim (2.54b) follows from (2.55). Replacing Φ by $e^{-(s-s_1)\tilde{A}}\Phi$ in (2.54b), we obtain

$$\|e^{-(s-s_1)\tilde{A}}\Phi\|_{\mathcal{H}'}^2 = \sum_{j=0}^{\infty} |e^{-\mu_j(s-s_1)} \langle \Phi, \hat{\phi}_j \rangle_{\mathcal{H}' \times \mathcal{H}}|^2, \tag{2.56}$$

and get (2.50) by (2.54b) and (2.56). The semigroup property (2.51) is now obvious.

(ii) It is enough to prove the continuity of $e^{-(s-s_1)\tilde{A}}\Phi$ at $s = s_1$. By (2.54b) we have

$$\|e^{-(s-s_1)\tilde{A}}\Phi - \Phi\|_{\mathcal{H}'}^2 = \sum_{j=0}^{\infty} |e^{-\mu_j(s-s_1)} - 1|^2 |\langle \Phi, \hat{\phi}_j \rangle_{\mathcal{H}' \times \mathcal{H}}|^2. \tag{2.57}$$

Suppose that Φ belongs to H^1_ρ. Then $|\langle \Phi, \hat{\phi}_j \rangle_{\mathcal{H}' \times \mathcal{H}}| \le |\langle \Phi, \phi_j \rangle_{L^2_\rho}|$ (cf. (2.27)) and

$$\limsup_{n \to \infty} \sum_{j=n}^{\infty} |\langle \Phi, \hat{\phi}_j \rangle_{\mathcal{H}' \times \mathcal{H}}|^2 \le \limsup_{n \to \infty} \sum_{j=n}^{\infty} |\langle \Phi, \phi_j \rangle_{L^2_\rho}|^2 = 0 \tag{2.58}$$

by the completeness of the system $\{\phi_j\}_{j=0}^{\infty}$ in L^2_ρ. It follows from (2.57) and (2.58) that

$$\lim_{s \to s_1} \|e^{-(s-s_1)\tilde{A}}\Phi - \Phi\|_{\mathcal{H}'}^2 = 0. \tag{2.59}$$

The convergence (2.59) for general $\Phi \in \overline{\mathcal{D}(\tilde{A})}$ is proven by a standard limiting procedure.

(iii) Let $s > s_1$. An argument similar to the one used in the proof of (i) shows that the series $\sum_{j=0}^{\infty} \mu_j e^{-\mu_j(s-s_1)} \langle \Phi, \phi_j \rangle_{\mathcal{H}' \times \mathcal{H}} \phi_j$ is convergent in $\mathcal{H}$ and in $\mathcal{H}'$. Substituting $\Phi = \tilde{A}\psi$ with $\psi = e^{-(s-s_1)\tilde{A}}\Phi$ to (2.54a), we get $\tilde{A}e^{-(s-s_1)\tilde{A}}\Phi = \sum_{j=0}^{\infty} \mu_j e^{-\mu_j(s-s_1)} \langle \Phi, \phi_j \rangle_{\mathcal{H}' \times \mathcal{H}} \phi_j$, which yields (2.52). The differentiability of $s \mapsto e^{-(s-s_1)\tilde{A}}\Phi$ and (2.53) are proven by an argument similar to that of (ii). The remaining proof is thus left for the reader. □

Now, let us look at the initial value problem for an evolution equation

$$\frac{dv}{ds} = -\tilde{A}v + F, \quad \text{in } \mathcal{H}' \ \text{ for } s \in (s_1, s_2], \tag{2.60a}$$

$$v(s_1) = v_0. \tag{2.60b}$$

with $F \in L^1([s_1, s_2]; \mathcal{H}') \cap C((s_1, s_2]; \mathcal{H}')$ and $v_0 \in \mathcal{H}'$. A function $v \in C([s_1, s_2]; \mathcal{H}') \cap C^1((s_1, s_2]; \mathcal{H}') \cap C((s_1, s_2]; \mathcal{D}(\tilde{A}))$ is understood to be a classical solution of (2.60) if it solves the equation (2.60a) for all $s \in (s_1, s_2]$ and satisfies the initial condition (2.60b).

Proposition 2.9. *If v is a classical solution of the initial value problem* (2.60), *then*

$$v(s) = e^{-(s-s_1)\tilde{A}}v_0 + \int_{s_1}^{s} e^{-(s-\tau)\tilde{A}}F(\tau)d\tau \quad \text{in } \mathcal{H}' \tag{2.61}$$

for $s_1 \le s \le s_2$.

Proof. This is readily obtained by Proposition 2.8 and the semigroup theory (cf. [25]). □

Corollary 2.10. *Let $v_0 \in L^\infty_{loc}([0, \infty))$ satisfy the growth condition* (2.33). *Suppose that $v(r, s)$ be a solution of* (2.10) *in $(0, \infty) \times [s_1, s_2]$ for some $s_2 > s_1$ with initial data $v(r, s_1) = v_0(r)$. Then the solution $v(s) = v(\cdot, s)$ may be regarded as an element of $\mathcal{H}'$ in the sense of* (2.37) *and*

$$v(s) = e^{-(s-s_1)\tilde{A}}v_0 + \int_{s_1}^{s} e^{-(s-\tau)\tilde{A}}f(v(\tau))d\tau \quad \text{in } \mathcal{H}' \tag{2.62}$$

for $s_1 \le s \le s_2$.

Proof. Since $v_0 \in L^2_\rho$, it is approximated by a sequence $\{v_{0,n}\} \subset C_0^\infty([0, \infty))$ in the norm of L^2_ρ and thus belongs to the closure of $\mathcal{D}(\tilde{A})$ in $\mathcal{H}'$. We then see that v is a classical solution of the problem (2.7) with $F = f(v)$ due to Lemma 2.7. Since $f(v) \in L^1([s_1, s_2]; \mathcal{H}') \cap C((s_1, s_2]; \mathcal{H}')$ by Lemma 2.6(ii), the formula (2.62) follows from Proposition 2.9. □

3 Dead-core rates and topological arguments

In this section we show the existence of a solution u_ℓ of (1.1) that satisfies the bound (1.7). At first the solution w_ℓ of (2.9) corresponding to u_ℓ is formally constructed in §§3.1 by means of a matched asymptotic expansion technique. Suggested by this formal construction, we prove rigorously that such a solution does exist by a priori estimates and topological arguments in §§3.2.

3.1 A formal matching argument

We begin with splitting the half line $\{r > 0\}$ into three regions: inner, intermediate and outer regions. The inner region is a very narrow layer disappearing as $s \to \infty$. The outer region lies very far away from the origin and moves to infinity as $s \to \infty$. The intermediate region is their complement part, which lies between the inner and outer regions, and expands to the half line as $s \to \infty$. At first their interfaces are unknown. Let $r_m(s)$ denote the boundary between the inner and intermediate regions. Let $\delta(s) > 0$ be such that $\delta(s) \ll r_m(s)$. Suppose that a solution $w_\ell(r, s)$ of (2.9) behaves as

$$w_\ell(r,s) \approx \begin{cases} \delta(s)^{2\alpha} U_\eta(\delta(s)^{-1} r), & \text{for } 0 \le r < r_m(s), \\ U(r) + \chi(s)\phi_\ell(r), & \text{for } r_m(s) < r < R_m(s), \\ U(r) + \nu(s_1) r^{2\alpha}, & \text{for } R_m(s) < r < \infty, \end{cases} \tag{3.1}$$

and for $s_1 \le s \le s_2$, where $R_m(s)$ is the frontier between the intermediate and the outer regions, and $\nu(s_1) = o(s_1)$ as $s_1 \to \infty$, that is, $\nu(s_1)/s_1 \to 0$ as $s_1 \to \infty$. Here and hereafter we loosely use the notation "$\approx$" to express approximation. The number η is a positive constant to be determined later. Notice that the first representation in (3.1) is a rescaling of stationary solution and that the second representation in (3.1) claims that the eigenfunction expansion yields an approximation of $w_\ell - U$ $(= v)$. Assuming that they are comparable at $r = r_m(s)$, we have

$$\delta(s)^{2\alpha} \left[U(\delta(s)^{-1} r_m(s)) + h(\eta) \{\delta(s)^{-1} r_m(s)\}^\gamma \right] \approx U(r_m(s)) + \chi(s) \cdot c_\ell (r_m(s))^\gamma \tag{3.2}$$

by (2.3) and (2.20a). We now select $\eta = \eta^* > 0$ so that

$$h(\eta^*) = c_\ell. \tag{3.3}$$

It then follows from (3.2) and (3.3) that

$$\chi(s) \approx \delta(s)^{2\alpha - \gamma}. \tag{3.4}$$

Taking the duality product in the equation (2.47), we see

$$\chi'(s) \approx -\mu_\ell \chi(s) + \langle f(v(s)), \phi_\ell \rangle_{\mathcal{H}' \times \mathcal{H}}. \tag{3.5}$$

At this formal level, it is convenient to assume that $\langle f(v(s)), \phi_\ell \rangle_{\mathcal{H}' \times \mathcal{H}} = o(e^{-\mu_\ell s})$ as $s \to \infty$ (cf. (4.9) below), so that the first term in the right hand side of (3.5) is dominant and

$$\chi(s) \approx e^{-\mu_\ell (s - s_1)} \chi(s_1) \tag{3.6}$$

as far as $s \gg 1$. Therefore if the initial data is chosen so that $\chi(s_1) \approx e^{-\mu_\ell s_1}$, then $\chi(s) \approx e^{-\mu_\ell s}$ and $\delta(s) \approx e^{-\omega_\ell s}$ with $\omega_\ell = \mu_\ell/(2\alpha - \gamma)$ by (3.4). Here we may choose the matching point $r_m(s)$ as $r_m(s) = K\delta(s) = Ke^{-\omega_\ell s}$ with $K \gg 1$ being a constant. A similar argument shows that $R_m(s) \approx e^{\sigma s}$ with some $\sigma \in (0, 1/2)$. Substituting these values to (3.1), we deduce that

$$w_\ell(r,s) \approx \begin{cases} e^{-2\alpha\omega_\ell s} U_{\eta^*}(e^{\omega_\ell s} r), & \text{for } 0 \le r < Ke^{-\omega_\ell s}, \\ U(r) + e^{-\mu_\ell s}\phi_\ell(r), & \text{for } Ke^{-\omega_\ell s} < r < e^{\sigma s}, \\ U(r) + \nu(s_1) r^{2\alpha}, & \text{for } e^{\sigma s} < r < \infty, \end{cases} \tag{3.7}$$

and for $s_1 \le s \le s_2$. Therefore $\inf_{r \ge 0} w(r, s) = w(0, s) \approx \eta^* e^{-2\alpha\omega_\ell s}$. We then go back to the original variables (x, t) through (2.6) to see that the corresponding solution u_ℓ of (1.1a) satisfies

$$u_\ell(0, t) \approx \eta^* (T - t)^{(2\omega_\ell + 1)\alpha}. \tag{3.8}$$

The dead-core rate (3.8) is precisely faster than the self-similar rate $(T - t)^\alpha$.

Remark 3.1. We have selected the particular constant $\eta = \eta^*$ as in (3.3). In fact, arbitrary choices of $\eta > 0$ are available. We only have to replace the values of $\delta(s), w_\ell(0,s)$ and $u_\ell(0,t)$ by $M(\eta,\ell)e^{-\omega_\ell s}, \eta M(\eta,\ell)^{2\alpha}e^{-2\alpha\omega_\ell s}$ and $\eta M(\eta,\ell)^{2\alpha}(T-t)^{(2\omega_\ell+1)\alpha}$ with $M(\eta,\ell) = \{c_\ell/h(\eta)\}^{1/(2\alpha-\gamma)}$, respectively. No changes appearing in the possible dead-core rates, we have chosen the simplest case where $M(\eta^*,\ell) = 1$ in the above argument.

We expect from (3.7) that a desired solution w_ℓ would converge to U for $\varepsilon_0 < r < 1/\varepsilon_0$ with every $\varepsilon_0 > 0$ as $s \to \infty$. However, the presence of unstable eigenvalue μ_0 makes the situation involved (cf. (2.26)). We have to perturb the initial datum of w so as to get rid of unstable effects caused by μ_0. Topological aspects of mapping degree, the subjects of the next section, provide a suitable perturbation.

3.2 Topological arguments

As is noted in Remark 1.2, the constant γ in (1.8) admits a useful inequality

$$2(\alpha-1) < \gamma < 2\alpha \tag{3.9}$$

with constant α in (1.5) for every $N \geq 1$ and $p \in (0,1)$. For each $\ell = 1, 2, ...$, we set

$$\omega_\ell = \frac{\mu_\ell}{2\alpha-\gamma} = \frac{\ell}{2\alpha-\gamma} - \frac{1}{2} > 0. \tag{3.10}$$

We select positive constants K and σ respectively as

$$K = e^{k\omega_\ell s_1} \quad \text{with} \quad \max\left\{\frac{1}{2\alpha+1-\gamma}, \frac{1}{2}\right\} < k < 1; \tag{3.11}$$

$$\frac{\mu_\ell}{2\ell} < \sigma < \min\left\{\frac{1}{2}, k\mu_\ell\right\}. \tag{3.12}$$

Let η_1, η_2 be positive constants such that

$$h(\eta_1) < c_\ell < h(\eta_2), \tag{3.13}$$

where $h(\eta)$ is as in §2.1 (cf. (2.3)) and c_ℓ is the constant appearing in (2.20a).

Theorem 3.2. *Let $k, \sigma, \eta_1, \eta_2$ be as above and let $\varepsilon > 0$ be a constant such that $h(\eta_1) < c_\ell(1-3\varepsilon)$ and $c_\ell(1+3\varepsilon) < h(\eta_2)$. Then for any $G > 0$ and for each positive integer ℓ, there exists a radial solution w_ℓ of (2.7) with the following properties: there exists a positive constant s_1 depending only on $p, N, \ell, G, k, \sigma, \eta_1, \eta_2$ and ε such that*

$$e^{-2\alpha\omega_\ell s}U_{\eta_1}(e^{\omega_\ell s}r) < w_\ell(r,s) < e^{-2\alpha\omega_\ell s}U_{\eta_2}(e^{\omega_\ell s}r) \tag{3.14}$$

for $0 \leq r \leq Ke^{-\omega_\ell s}$ and $s \geq s_1$;

$$|w_\ell(r,s) - U(r) - e^{-\mu_\ell s}\phi_\ell(r)| \leq \varepsilon e^{-\mu_\ell s}(r^\gamma + r^{2(\mu_\ell+\alpha)}) \tag{3.15}$$

for $Ke^{-\omega_\ell s} \leq r \leq e^{\sigma s}$ and $s \geq s_1$;

$$|w_\ell(r,s) - U(r)| < Gr^{2\alpha} \tag{3.16}$$

for $e^{\sigma s} \leq r$ and $s \geq s_1$.

Remark 3.3. (i) In §6 we give further properties on the solution w_ℓ, such as the convergence $\lim_{s\to\infty} e^{\mu_\ell s}\{w_\ell(r,s)-U(r)\}$ in compact sets of $(0,\infty)$ and the zeros of $w_\ell(s)-U$.
(ii) By fixed-point arguments we construct a suitable initial datum for each positive integer ℓ so that the corresponding solution w_ℓ of the problem (2.9) fulfills the above properties.
(iii) One may choose the constants η_1,η_2 arbitrarily close to each other if they satisfy (3.13), though ε must be selected so small accordingly (cf. (3.22) below).

In the sequel let $\varepsilon\in(0,1)$ be a constant and we denote by $w(r,s;d)$ the solution of (2.9) with initial datum $w(r,s_1;d)$.

Definition 3.4. Let $d=(d_0,d_1,...,d_{\ell-1})\in\mathbf{R}^\ell$ be such that

$$\sum_{n=0}^{\ell-1}|d_n|<\varepsilon e^{-\mu_\ell s_1}. \tag{D}$$

A solution $w(r,s;d)$ of (2.9) is said to be in a *class* $\mathcal{W}^\theta_{s_1,s_2}$, $\theta\in(0,1]$, and written as $w(r,s;d)\in\mathcal{W}^\theta_{s_1,s_2}$ if

$$|w(r,s;d)-U(r)-e^{-\mu_\ell s}\phi_\ell(r)|\le\theta\varepsilon e^{-\mu_\ell s}(r^\gamma+r^{2(\mu_\ell+\alpha)})$$

for $Ke^{-\omega_\ell s}\le r\le e^{\sigma s}$ and $s_1\le s\le s_2$. We set

$$U_{s_1,s_2}=\{d\in\mathbf{R}^\ell;\ d\text{ satisfies the property }(D)\text{ and }w(r,s;d)\in\mathcal{W}^1_{s_1,s_2}\}. \tag{3.17}$$

For the constants k and σ respectively in (3.11) and (3.12), we choose positive constants $\tilde K$ and $\tilde\sigma$ as

$$\sigma<\tilde\sigma<\min\Big\{\frac12,k\mu_\ell\Big\}, \tag{3.18}$$

$$\tilde K=e^{\tilde k\omega_\ell s_1}\quad\text{with}\quad 0<\tilde k<k. \tag{3.19}$$

Let η^* be the constant in (3.3). We select a function $\tilde\phi_\ell\in L^\infty_{loc}([0,\infty))$ satisfying

$$\tilde\phi_\ell(r)=-e^{\mu_\ell s_1}\Big\{U(r)-e^{-2\alpha\omega_\ell s_1}U_{\eta^*}(e^{\omega_\ell s_1}r)+\sum_{n=0}^{\ell-1}d_n\phi_n(r)\Big\}\ \text{ in }[0,\tilde Ke^{-\omega_\ell s_1}); \tag{$V1$}$$

$$\tilde\phi_\ell(r)=\phi_\ell(r)\ \text{ in }[\tilde Ke^{-\omega_\ell s_1},e^{\tilde\sigma s_1}); \tag{$V2$}$$

$$\Big|\sum_{n=0}^{\ell-1}d_n\phi_n(r)+e^{-\mu_\ell s_1}\tilde\phi_\ell(r)\Big|\le\frac12 Gr^{2\alpha}\ \text{ for some }G>0\text{ in }[e^{\tilde\sigma s_1},\infty); \tag{$V3$}$$

$$U(r)+\sum_{n=0}^{\ell-1}d_n\phi_n(r)+e^{-\mu_\ell s_1}\tilde\phi_\ell(r)>0\ \text{ in }[0,\infty). \tag{$V4$}$$

For this function $\tilde\phi_\ell(r)$, we take an initial function as

$$w(r,s_1;d)=U(r)+v_{0,\ell}(r;d) \tag{3.20a}$$

with $d \in \mathbf{R}^\ell$ satisfying the property (D) and

$$v_{0,\ell}(r;d) := \sum_{n=0}^{\ell-1} d_n \phi_n(r) + e^{-\mu_\ell s_1} \tilde{\phi}_\ell(r). \tag{3.20b}$$

The initial data $w(r, s_1; d)$ may have jumps in $[\tilde{K}e^{-\omega s_1}, \infty)$ and thus can be discontinuous, but this thing is not a serious drawback since the corresponding mild solutions become smooth immediately for $s > s_1$ in view of parabolic regularizing effects. Further remarks on the choice of initial data are given after the proof of Proposition 3.5 below.

For the future references, we note that for each $j = 0, 1, ...,$ there is a constant $C_j > 0$ such that

$$|\langle \phi_\ell - \tilde{\phi}_\ell, \phi_j \rangle| \leq C_j e^{-(\gamma+N)(1-\tilde{k})\omega_\ell s_1}, \tag{3.21}$$

where $\tilde{k} \in (0, 1)$ is the constant in (3.19). The estimate (3.21) is readily obtained by splitting the integral defining the inner product according to $(V1)$-$(V3)$ and using (2.20a) as well as (3.19) and the exponentially decaying factor of the weight function ρ as $r \to \infty$.

In the following, for a constant $\varepsilon > 0$, we write $s_1 \gg 1$ if $s_1 \geq s_0$ with some sufficiently large $s_0 \geq 1$ depending only on $p, N, \ell, k, \sigma, G, \eta_1$ and η_2 as well as on ε. We now choose $\varepsilon > 0$ so small to fulfill

$$h(\eta_1) < c_\ell(1 - 3\varepsilon) < c_\ell < c_\ell(1 + 3\varepsilon) < h(\eta_2). \tag{3.22}$$

Such a constant ε does exist due to our choices of η_1 and η_2 in (3.13).

Proposition 3.5. *Let η_1, η_2 and ε be as above. Assume that $d \in \overline{U}_{s_1,s_2}$ for some $s_2 > s_1$. Then there exist $\delta \in (0, 1)$ and $s_1 \gg 1$such that*

$$(1 + \delta)e^{-2\alpha\omega_\ell s} U_{\eta_1}(e^{\omega_\ell s} r) < w(r, s) < (1 - \delta)e^{-2\alpha\omega_\ell s} U_{\eta_2}(e^{\omega_\ell s} r) \tag{3.23}$$

for $0 \leq r \leq Ke^{-\omega_\ell s}$ and $s_1 \leq s \leq s_2$. In particular, there holds

$$\eta_1 e^{-2\alpha\omega_\ell s} < w(r, s) - U(r). \tag{3.24}$$

Proof. Since $d \in \overline{U}_{s_1,s_2}$, we have

$$w(r, s) - U(r) \leq e^{-\mu_\ell s}\{\phi_\ell(r) + \varepsilon r^\gamma(1 + r^{2\ell})\} \tag{3.25}$$

for $Ke^{-\omega_\ell s} \leq r \leq e^{\sigma s}$ and $s_1 \leq s \leq s_2$. Take a constant $\delta_0 \in (0, 1)$ small enough so that $c_\ell(1+3\varepsilon) < h(\eta_2)(1-\delta_0)$. We set $\delta = \delta(s_1) = e^{-\mu_\ell s_1}$. Then $c_\ell(1+3\varepsilon) < h(\eta_2)(1-\delta_0)(1-\delta)$ as far as $s_1 \gg 1$. Recalling (2.20a) and (2.3), we then get from (3.25) that

$$\begin{aligned} w(Ke^{-\omega_\ell s}, s) &\leq e^{-2\alpha\omega_\ell s}U(K) + h(\eta_2)(1 - \delta_0)(1 - \delta)e^{-\mu_\ell s}(Ke^{-\omega_\ell s})^\gamma - \frac{\varepsilon}{2}e^{-\mu_\ell s}(Ke^{-\omega_\ell s})^\gamma \\ &= (1 - \delta)e^{-2\alpha\omega_\ell s}\{U(K) + h(\eta_2)(1 - \delta_0)K^\gamma\} + \delta e^{-2\alpha\omega_\ell s}U(K) - \frac{\varepsilon}{2}e^{-2\alpha\omega_\ell s}K^\gamma \\ &\leq (1 - \delta)e^{-2\alpha\omega_\ell s}U_{\eta_2}(K) \end{aligned}$$

if $s_1 \gg 1$, because $\delta = o(K^{-2\alpha+\gamma})$ and thus $\delta U(K) = o(K^\gamma)$ as $s_1 \to \infty$ (cf. (3.10) and (3.11)). A similar argument shows that

$$w(Ke^{-\omega_\ell s}, s) \geq (1 + \delta)e^{-2\alpha\omega_\ell s}U_{\eta_1}(K)$$

if $s_1 \gg 1$. Consequently (3.23) holds for $r = Ke^{-\omega_\ell s}$ and $s_1 \le s \le s_2$.

We then proceed to observe, by our choice of the initial data, that

$$w(r, s_1) = e^{-2\alpha\omega_\ell s_1} U_{\eta^*}(e^{\omega_\ell s_1} r) \tag{3.26}$$

for $0 \le r \le \tilde{K}e^{-\omega_\ell s_1}$, and

$$\begin{aligned} |w(r, s_1) - U(r)| &= \Big| \sum_{n=0}^{\ell-1} d_n \phi_n(r) + e^{-\mu_\ell s_1} \phi_\ell(r) \Big| \\ &\le c_\ell e^{-\mu_\ell s_1} r^\gamma (\varepsilon + 1)(1 + o(1)) \end{aligned} \tag{3.27}$$

for $\tilde{K}e^{-\omega_\ell s_1} \le r \le Ke^{-\omega_\ell s_1}$ as $s_1 \to \infty$. On the other hand, there holds

$$e^{-2\alpha\omega_\ell s_1} U_{\eta_i}(e^{\omega_\ell s_1} r) = U(r) + h(\eta_i) e^{-\mu_\ell s_1} r^\gamma (1 + o(1)) \tag{3.28}$$

as $s_1 \to \infty$ for $i = 1, 2$ by (2.3). Arguing as above, we obtain

$$(1+\delta) e^{-2\alpha\omega_\ell s_1} U_{\eta_1}(e^{-\omega_\ell s_1} r) \le w(r, s_1) \le (1-\delta) e^{-2\alpha\omega_\ell s_1} U_{\eta_2}(e^{-\omega_\ell s_1} r) \tag{3.29}$$

for $\tilde{K}e^{-\omega_\ell s_1} \le r \le Ke^{-\omega_\ell s_1}$, if $s_1 \gg 1$. It follows from (3.26) and (3.29) that (3.23) holds for $0 \le r \le Ke^{-\omega_\ell s_1}$ and $s = s_1$. Thus (3.23) valids in $\{(r, s_1);\ 0 \le r \le Ke^{-\omega_\ell s_1}\} \cup \{(r, s)|\ r = Ke^{-\omega_\ell s},\ s_1 \le s \le s_2\}$. We shall show that this estimate keeps to hold for $0 \le r \le Ke^{-\omega_\ell s},\ s_1 \le s \le s_2$ by a comparison argument. We set

$$\begin{aligned} \underline{w}(r, s) &= (1+\delta) e^{-2\alpha\omega_\ell s} U_{\eta_1}(e^{\omega_\ell s} r); \\ \overline{w}(r, s) &= (1-\delta) e^{-2\alpha\omega_\ell s} U_{\eta_2}(e^{\omega_\ell s} r). \end{aligned}$$

A direct calculation yields then that

$$\begin{aligned} &\underline{w}_s - \underline{w}_{rr} - \Big(\frac{N-1}{r} - \frac{r}{2}\Big)\underline{w}_r - \alpha\underline{w} + \underline{w}^p \\ =&(1+\delta) e^{-2(\alpha-1)\omega_\ell s}\Big[- \big\{1 - (1+\delta)^{-(1-p)}\big\} U_{\eta_1}^p(e^{\omega_\ell s} r) + e^{-2\omega_\ell s} B(e^{\omega_\ell s} r)\Big] \end{aligned} \tag{3.30}$$

with

$$B(\xi) := (2\omega_\ell + 1)\Big\{\frac{1}{2}\xi U_{\eta_1}'(\xi) - \alpha U_{\eta_1}(\xi)\Big\}.$$

Notice that the function B is negative near $\xi = 0$ and $\xi = \infty$ due to (2.3) and (3.9). Because $U_{\eta_1}(\xi) > 0$, $B(\xi)$ is bounded and $e^{-2\omega_\ell s_1} \lll \delta$ when $s_1 \gg 1$ (cf. (3.9)), the right hand side of (3.30) is negative for $0 < r < Ke^{-\omega_\ell s}$ and $s_1 < s < s_2$ if $s_1 \gg 1$, that is, $\underline{w}$ is a subsolution of (2.9a). A similar argument shows that $\overline{w}$ is a supersolution of (2.9a). Note that $U_{\eta_1}'(0) = U_{\eta_2}'(0) = w_r(0, s) = 0$ for $s_1 < s < s_2$. The desired estimate (3.23) is then obtained by the comparison theorem. Since $U_{\eta_1}' - U' \ge 0$, we get (3.24). □

We have remarked in §2 that under the general hypothesis on initial data of having algebraic growth bound as $|y| \to \infty$, the corresponding solution w of (2.7) grows with at most algebraic order for every $s \in (s_1, s_2)$. The estimate, however, can depend on s_2. We next prove that for our particular initial data, the corresponding solutions have growth bound as $r \to \infty$ that is independent of $s \in [s_1, s_2]$.

Proposition 3.6. *Assume that $d \in \overline{U}_{s_1,s_2}$ for some $s_2 > s_1$. Then there exists $s_1 \gg 1$ such that*

$$|w(r,s) - U(r)| < Gr^{2\alpha} \quad \text{for } r \geq e^{\sigma s} \text{ and } s_1 \leq s \leq s_2,$$

where G is the positive constant in the choice of initial data described in $(V3)$.

Proof. Since we are assuming that d belongs to $\overline{U}_{s_1,s_2}$, we certainly have that

$$|w(r,s) - U(r)| \leq Ce^{-\mu_\ell s}(r^\gamma + r^{2(\mu_\ell+\alpha)}) < \frac{1}{2}Gr^{2\alpha}$$

for $1 \leq r \leq e^{\sigma s}$ and $s_1 \leq s \leq s_2$, provided that $s_1 \gg 1$. We then go back to the original variables $(u; x, t)$ to see that

$$|u(x,t) - U(|x|)| < \frac{1}{2}G|x|^{2\alpha} \quad \text{for } |x| \leq (T-t)^{\frac{1}{2}-\sigma},\ t_1 \leq t \leq T_1, \tag{3.31}$$

where $t_1 := T - e^{-s_1}$ and $T_1 := T - e^{-s_2}$. Define

$$\varphi(x,t) := u(x,t) - U(|x|)$$

and observe that

$$\partial_t\varphi = \Delta\varphi - \frac{u^p - U^p}{u - U}\varphi \equiv \Delta\varphi - \Psi(x,t)\varphi.$$

We then multiple $\mathrm{sgn}\varphi$ and make use of Kato's inequality to get

$$\partial_t(|\varphi|) \leq \Delta(|\varphi|) - \Psi(x,t)|\varphi| \leq \Delta(|\varphi|). \tag{3.32}$$

Namely, $|\varphi|$ is a subsolution of the heat equation. On the other hand, we have

$$0 = \Delta U(|x|) - U(|x|)^{-(1-p)}U(|x|) \geq \Delta U(|x|) - c_{p,N}^{-(1-p)}(T-t)^{-(1-2\sigma)}U(|x|).$$

for $|x| \geq (T-t)^{\frac{1}{2}-\sigma}$. A simple computation shows then that the function

$$Z(x,t) := \exp\Big(c_{p,N}^{-(1-p)}\int_{t_1}^{t}(T-\tau)^{-(1-2\sigma)}d\tau\Big)U(|x|)$$

is a supersolution of the heat equation, that is,

$$Z_t \geq \Delta Z \quad \text{for } \ |x| \geq (T-t)^{\frac{1}{2}-\sigma},\ t_1 < t \leq T_1 \tag{3.33}$$

(and so is $CZ(x,t)$ for each constant $C > 0$). Moreover, our choice of initial data (cf. $(V2)$, $(V3)$ and (3.20b)) implies that if $s_1 \gg 1$, then

$$|u(x,t_1) - U(|x|)| \leq \frac{1}{2}G|x|^{2\alpha} \quad \text{for } |x| \geq (T-t_1)^{\frac{1}{2}-\sigma}, \tag{3.34}$$

where the same argument as the derivation of (3.31) has been used to obtain (3.34) for $(T-t_1)^{\frac{1}{2}-\sigma} \leq |x| \leq (T-t_1)^{\frac{1}{2}-\tilde{\sigma}}$. It then follows from (3.31)-(3.34) that

$$|u(x,t) - U(|x|)| \leq C_0Z(x,t) \quad \text{for } |x| \geq (T-t)^{\frac{1}{2}-\sigma},\ t_1 \leq t \leq T_1$$

by the comparison theorem, where $C_0 = (2c_{p,N})^{-1}G$. Since

$$\int_{t_1}^{t} (T-\tau)^{-(1-2\sigma)} d\tau = \frac{1}{2\sigma}\big((T-t_1)^{2\sigma} - (T-t)^{2\sigma}\big),$$

we conclude

$$|u(x,t) - U(|x|)| < G|x|^{2\alpha} \quad \text{for } |x| \geq (T-t)^{\frac{1}{2}-\sigma},\ t_1 \leq t \leq T_1,$$

if $T - t_1 = e^{-s_1} \ll 1$. Changing the variables to $(w; y, s)$, we get the desired estimate. □

We may define a map $P: \overline{U}_{s_1,s_2} \to \mathbf{R}^\ell$ by

$$P(d; s_1, s_2) = (p_0, p_1, ..., p_{\ell-1}), \quad p_n = \langle w(\cdot, s_2; d) - U, \phi_n \rangle_{L^2_\rho},\ n = 0, 1, ..., \ell - 1. \tag{3.35}$$

The following proposition plays a key role in the proof of Theorem 3.2.

Proposition 3.7. *Let $s_1 \gg 1$. If there exists $d \in \overline{U}_{s_1,s_2}$ for some $s_2 > s_1$ such that $P(d; s_1, s_2) = 0$, then $w(r, s; d) \in \mathcal{W}^{\theta}_{s_1,s_2}$ for some $\theta \in (0, 1)$.*

The proof of Proposition 3.7 requiring quite heavy analysis, we postpone the proof to §5. Once Proposition 3.7 is proven, a topological fixed-point argument by mapping degree guarantees the existence of the solution as stated in Theorem 3.2. This step is a purely topological argument and is therefore essentially the same as the corresponding parts of [15, 21], but we present highlights for readers' convenience.

Proposition 3.8. *Let $s_1 \gg 1$. If $\overline{U}_{s_1,s_2} \neq \emptyset$ for some $s_2 > s_1$, then there exists $d \in \overline{U}_{s_1,s_2}$ such that $P(d; s_1, s_2) = 0$.*

Proof. Note that $p_n(d; s_1, s_1) = d_n + e^{-\mu_\ell s_1}\langle \tilde{\phi}_\ell, \phi_n \rangle$, $n = 0, 1, ..., \ell - 1$, and

$$U_{s_1,s_1} = \Big\{ d \in \mathbf{R}^\ell\ ;\ \sum_{n=0}^{\ell-1} |d_n| < \varepsilon e^{-\mu_\ell s_1},\ \Big|\sum_{n=0}^{\ell-1} d_n \phi_n(r)\Big| \leq e^{-\mu_\ell s_1}(r^\gamma + r^{2(\mu_\ell+\alpha)}) \\ \text{for } K e^{-\omega_\ell s_1} \leq r \leq e^{\sigma s_1} \Big\}.$$

Lemma 4.2 below implies that if $P(d; s_1, s_1) = 0$ for some $d \in \overline{U}_{s_1,s_2}$, then d lies in the interior of U_{s_1,s_2}. Since the map P is continuous in d and is homotopic with the identity map I and there is no $d \in \partial U_{s_1,s_1}$ such that $P(d; s_1, s_1) = 0$, homotopy invariance of mapping degree yields

$$\deg(P(\cdot; s_1, s_1), U_{s_1,s_1}, 0) = \deg(I, U_{s_1,s_1}, 0) = 1. \tag{3.36}$$

Admitting Proposition 3.7, we have

$$0 \notin P(\partial U_{s_1,s}; s_1, s) \quad \text{for } s_1 \leq s \leq s_2.$$

It then follows from (3.36) and the homotopy invariance [24, Theorem 2.2.4] that

$$\deg(P(\cdot; s_1, s_2), U_{s_1,s_2}, 0) = \deg(P(\cdot; s_1, s_1), U_{s_1,s_1}, 0) = 1.$$

Hence there exists $d \in U_{s_1,s_2}$ such that $P(d; s_1, s_2) = 0$, which completes the proof. □

Proposition 3.9. *If $s_1 \gg 1$ then $\overline{U}_{s_1,s_2} \neq \emptyset$ for every $s_2 > s_1$.*

Proof. By our choice of initial data w, we have $w(r, s_1; 0) \in \mathcal{W}^{1/2}_{s_1,s_1}$. Then there is a constant $\eta > 0$ such that $w(r, s; 0) \in \mathcal{W}^{1/2}_{s_1,s_1+\eta}$, whence $0 \in U_{s_1,s_1+\eta}$. We set

$$s^* := \sup\{s > s_1;\ U_{s_1,s} \neq \emptyset\}$$

and prove $s^* = +\infty$ by contradiction. Suppose that s^* were finite. Then there is a sequence $\{s_j\} \subset (s_1, s^*)$ such that $s_j \to s^*$ as $j \to \infty$ and $U_{s_1,s_j} \neq \emptyset$ for every j. It follows from Proposition 3.8 that for each j, there exists $d_j \in U_{s_1,s_j}$ such that $P(d_j; s_1, s_j) = 0$. We then use Proposition 3.7 to see that $w(r, s; d_j) \in \mathcal{W}^{\theta_j}_{s_1,s_j}$ for some $\theta_j \in (0, 1)$. Taking a subsequence which converges to some $d^* \in \mathbf{R}^N$, we obtain $P(d^*; s_1, s^*) = 0$ and $w(r, s; d^*) \in \mathcal{W}^1_{s_1,s^*}$. Namely, $d^* \in U_{s_1,s^*}$. We again use Proposition 3.7 to observe that $w(r, s; d^*) \in \mathcal{W}^{\theta^*}_{s_1,s^*}$ for some $\theta^* \in (0, 1)$, whence $d^* \in U_{s^*,s^*+\eta}$ with some $\eta > 0$, contradicting the definition of s^*. □

We have now arrived at a position to prove Theorems 3.2 and 1.1.

Proof of Theorem 3.2. For a monotone increasing unbounded sequence $\{s_j\}$, we may take $d^{(j)} \in \overline{U}_{s_1,s_j}$ such that $P(d^{(j)}; s_1, s_j) = 0$ for each j by Propositions 3.8 and 3.9. We then apply Proposition 3.7 to deduce that $w(r, s; d^{(j)}) \in \mathcal{W}^{\theta_j}_{s_1,s_j}$ for some $\theta_j \in (0, 1)$. Taking a subsequence converging to some $d^* \in \cap_{j\geq 1}\overline{U}_{s_1,s_j}$, we see $w(r, s; d^*) \in \mathcal{W}^1_{s_1,\infty} = \cap_{s_2\geq s_1}\mathcal{W}^1_{s_1,s_2}$, which is the desired solution $w_\ell(r, s)$. The properties (3.14), (3.15) and (3.16) are guaranteed by Propositions 3.5, 3.7 and 3.6, respectively. □

Proof of Theorem 1.1. Let d^* be as in the proof of Theorem 3.2. For $T > 0$ small enough, the function $u_\ell(x, t) := (T-t)^\alpha w_\ell((T-t)^{-1/2}x, -\log(T-t); d^*)$ is the desired solution of (1.1) with initial data $u_{0,\ell}(x) = T^\alpha w_\ell(T^{-1/2}x, -\log T; d^*)$. The existence for each $T > 0$ is shown by rescaling $u_\ell \mapsto u^{(\lambda)}_\ell(x, t) = \lambda^{-\alpha}u_\ell(\lambda x, \lambda^2 t)$ and selecting a suitable $\lambda > 0$. □

4 Fundamental estimates

In this section we derive some auxiliary estimates leading to the proof of the key a priori estimate of Proposition 3.7. As is previously noted in §3, these estimates are essential to execute our topological argument within $\mathcal{W}^\theta_{s_1,s_2}$, where $\mathcal{W}^\theta_{s_1,s_2}$ is the set defined in Definition 3.4 together with U_{s_1,s_2}. To this end we just recall here some notations introduced in the previous sections. Let f be as in (2.12), i.e.,

$$f(\psi) = U(r)^p - \{U(r) + \psi\}^p + \frac{pc^{p-1}_{p,N}}{r^2}\psi, \qquad r > 0,$$

where $r = |y|$ and $U(r)$ is the singular stationary solution given in (2.2). It may be regarded as an element of $\mathcal{H}'$ by Lemma 2.6. We have denoted the duality product between $\mathcal{H}'$ and $\mathcal{H}$ by $\langle\cdot,\cdot\rangle_{\mathcal{H}'\times\mathcal{H}}$, but we shall henceforth write it, together with the inner product in L^2_ρ, simply as $\langle\cdot,\cdot\rangle$. We denote by $w(r, s)$ a solution of (2.9) with initial data $w(r, s_1; d) = v_0(r; d) + U(r)$, where v_0 is the function defined in (3.20b), and set

$v(r,s) := w(r,s) - U(r)$. Then v is a solution of the linearized equation (2.10) and admits the integral equation described in Corollary 2.10. The linearized operator is denoted by A as well as its Friedrichs extension in Lemma 2.2, whose eigenvalues $\{\mu_j\}_{j=0}^{\infty}$ and eigenfunctions $\{\phi_j\}_{j=0}^{\infty}$ are given in (2.18) and (2.19), respectively.

In the following, we shall denote by C a generic positive constant which varies from line to line. As is used in the previous sections, for a positive constant ν, we write $s_1 \gg 1$ if $s_1 \geq s_0$ with some sufficiently large $s_0 \geq 1$ which may depend only on $p, N, \ell, k, \sigma, G, \eta_1$ and η_2 as well as on ν. Here k, σ and η_i are the positive constants given in (3.11), (3.12) and (3.13), respectively, while G is the positive constant appearing in $(V3)$ to define the class of initial data $w(r, s_1; d)$ given in $(V1)$-$(V4)$ and (3.20).

Lemma 4.1. *Assume $d \in \overline{U}_{s_1,s_2}$. Let K be the constant as in* (3.11)*. If $s_1 \gg 1$, then*

$$0 \leq f(v) \leq CK^{\gamma} e^{-2\alpha\omega_\ell s} r^{-2} \quad \text{for } 0 < r \leq K e^{-\omega_\ell s},\ s_1 \leq s \leq s_2; \tag{4.1}$$
$$0 \leq f(v) \leq C e^{-2\mu_\ell s} r^{2\gamma - 2\alpha - 2} \quad \text{for } K e^{-\omega_\ell s} < r \leq 1,\ s_1 \leq s \leq s_2; \tag{4.2}$$
$$0 \leq f(v) \leq C e^{-2\mu_\ell s} r^{4\gamma - 2\alpha - 2} \quad \text{for } 1 < r \leq e^{\sigma s},\ s_1 \leq s \leq s_2; \tag{4.3}$$
$$0 \leq f(v) \leq C r^{2\alpha - 2} \quad \text{for } e^{\sigma s} \leq r,\ s_1 \leq s \leq s_2. \tag{4.4}$$

Proof. Notice that if $s_1 \gg 1$, then

$$\begin{aligned} 0 \leq v(r,s) &< e^{-2\alpha\omega_\ell s} U_{\eta_2}(e^{\omega_\ell s} r) - U(r) \\ &\leq 2h(\eta_2) K^{\gamma} e^{-2\alpha\omega_\ell s} \end{aligned}$$

for $0 \leq r \leq K e^{-\omega_\ell s}$ and $s_1 \leq s \leq s_2$ by Proposition 3.5 and (2.3). Then there holds

$$\begin{aligned} f(v) &\leq C U^{-(1-p)} v \\ &\leq C K^{\gamma} e^{-2\alpha\omega_\ell s} r^{-2}, \end{aligned}$$

which yields (4.1).

We next note that Taylor's theorem yields

$$0 \leq f(v) \leq C U^{-(2-p)} v^2 \quad \text{for any } r > 0. \tag{4.5}$$

By the assumption $d \in \overline{U}_{s_1,s_2}$, we have

$$|w(r,s;d) - U(r) - e^{-\mu_\ell s}| \leq \varepsilon e^{-\mu_\ell s} (r^{\gamma} + r^{2(\mu_\ell + \alpha)})$$

for $K e^{-\omega_\ell s} \leq r \leq e^{\sigma s}$ and $s_1 \leq s \leq s_2$, whence

$$|v| \leq C e^{-\mu_\ell s} (r^{\gamma} + r^{2(\mu_\ell + \alpha)}) \tag{4.6}$$

there. It then follows from (4.5) and (4.6) that

$$f(v) \leq \begin{cases} C e^{-2\mu_\ell s} r^{2\gamma - 2\alpha - 2} & \text{for } K e^{-\omega_\ell s} \leq r \leq 1,\ s_1 \leq s \leq s_2; \\ C e^{-2\mu_\ell s} r^{4\mu_\ell + 2\alpha - 2} & \text{for } 1 \leq r \leq e^{\sigma s},\ s_1 \leq s \leq s_2, \end{cases}$$

which shows (4.2) and (4.3).

Finally, (4.4) immediately follows from (4.5) and Proposition 3.6. □

Lemma 4.2. *Let $s_1 \gg 1$. Suppose that for any $\nu > 0$, there exists $d = (d_0, d_1, ..., d_{\ell-1}) \in \overline{U}_{s_1,s_2}$ such that $P(d; s_1, s_2) = 0$. Then*

$$\sum_{n=0}^{\ell-1} |d_n| < \nu e^{-\mu_\ell s_1}. \tag{4.7}$$

Proof. Taking the duality product with ϕ_n in (2.62) at $s = s_2$, we have

$$0 = e^{-\mu_n(s_2-s_1)}\langle v_0, \phi_n\rangle + \int_{s_1}^{s_2} e^{-\mu_n(s_2-\tau)}\langle f(v(\tau)), \phi_n\rangle d\tau \tag{4.8}$$

for $n = 0, 1, ..., \ell - 1$. In order to prove (4.8), we show

$$\|f(v(\tau))\|_{\mathcal{H}'} \le Ce^{-(1+\kappa)\mu_\ell\tau}, \quad s_1 \le \tau \le s \tag{4.9}$$

with some constants $C > 0$ and $\kappa > 0$ for $s_1 \gg 1$. To show (4.9) we take $\phi \in \mathcal{H}$ and use the estimates of $f(v(r, \tau))$ in Lemma 4.1 and (2.29) to observe that

$$\begin{aligned}
&\big|\langle f(v(\tau)), \phi\rangle\big| \\
&\le C\sum_{j=0}^{\infty}\langle\phi, \phi_j\rangle\left(K^\gamma e^{-2\alpha\omega_\ell\tau}\int_0^{Ke^{-\omega_\ell\tau}} |\phi_j(r)|r^{N-3}\rho dr + e^{-2\mu_\ell\tau}\int_{Ke^{-\omega_\ell\tau}}^1 |\phi_j(r)|r^{2\gamma-2\alpha+N-3}\rho dr\right) \\
&\qquad + Ce^{-2\mu_\ell\tau}\int_1^{e^{\sigma\tau}} r^{4\gamma-2\alpha+N-3}|\phi(r)|\rho dr + C\int_{e^{\sigma\tau}}^\infty r^{2\alpha+N-3}|\phi(r)|\rho dr \\
&=: \sum_{j=0}^{\infty}\frac{(\phi, \phi_j)_{\mathcal{H}}}{\mu_j + \alpha + 1}\big(L_{1,j} + L_{2,j}\big) + L_3 + L_4.
\end{aligned} \tag{4.10}$$

Since $\gamma + N - 2 > 0$, we may estimate $L_{1,j}$ by (2.38) as

$$\begin{aligned}
|L_{1,j}| &\le CK^\gamma e^{-2\alpha\omega_\ell\tau} j^{-\frac{1}{4}}\int_0^{Ke^{-\omega_\ell\tau}} r^{\gamma+N-3}\rho dr \\
&\le Cj^{-\frac{1}{4}}e^{-(1+\kappa)\mu_\ell\tau} \qquad \text{for } j \ge 1
\end{aligned}$$

and get similarly $|L_{1,0}| \le Ce^{-(1+\kappa)\mu_\ell\tau}$, where $\kappa = (1-k)(2\gamma + N - 2)/(2\alpha - \gamma) > 0$ with $k \in (0, 1)$ being the constant as in (3.11). To estimate $L_{2,j}$ we take a positive constant $a := 2\alpha - \gamma - \varepsilon_0$ with $0 < \varepsilon_0 < \min\{2\alpha - \gamma, 2\gamma + N - 2\}$ and then obtain, by (2.38), that

$$\begin{aligned}
|L_{2,j}| &\le Ce^{-2\mu_\ell\tau}\int_{Ke^{-\omega_\ell\tau}}^1 r^{2\gamma-2\alpha-3+N}|\phi_j(r)|\rho dr \\
&\le Cj^{-\frac{1}{4}}e^{-2\mu_\ell\tau}(Ke^{-\omega_\ell\tau})^{-a}\int_0^1 r^{2\gamma+N-3-\varepsilon_0}dr \\
&\le Cj^{-\frac{1}{4}}K^{-a}e^{-\mu_\ell\tau}e^{-\varepsilon_0\omega_\ell\tau} \qquad \text{for } j \ge 1
\end{aligned}$$

and get analogously $|L_{2,0}| \le K^{-a} e^{-\mu_\ell \tau} e^{-\varepsilon_0 \omega_\ell \tau}$. Using these estimates in (4.10), we have

$$\begin{aligned}
&\sum_{j=0}^{\infty} \frac{(\phi, \phi_j)_{\mathcal{H}}}{\mu_j + \alpha + 1} \big(L_{1,j} + L_{2,j}\big) \\
\le & C\Big(\sum_{j=0}^{\infty} (\phi, \hat{\phi}_j)_{\mathcal{H}}^2\Big)^{\frac{1}{2}} \Big(j_0 + \sum_{j=j_0}^{\infty} \frac{j^{-\frac{1}{2}}}{\mu_j + \alpha + 1}\Big)^{\frac{1}{2}} \big(e^{-(1+\kappa)\mu_\ell \tau} + e^{-\mu_\ell \tau} e^{-\varepsilon_0 \omega_\ell \tau}\big) \\
\le & C\|\phi\|_{\mathcal{H}} \big(e^{-(1+\kappa)\mu_\ell \tau} + e^{-\mu_\ell \tau} e^{-\varepsilon_0 \omega_\ell \tau}\big).
\end{aligned} \tag{4.11}$$

As for L_3 and L_4, it is readily seen that

$$|L_3| + |L_4| \le C\Big\{e^{-2\mu_\ell \tau} + C \exp\Big(-\frac{e^{2\sigma\tau}}{8}\Big)\Big\} \|\phi\|_{\mathcal{H}}. \tag{4.12}$$

We substitute (4.11) and (4.12) to (4.10), so that the claim (4.9) holds.

Since $\langle v_{0,\ell}, \phi_n\rangle = d_n + e^{-\mu_\ell s_1} \langle \tilde{\phi}_\ell, \phi_n\rangle$ by our choice of $v_{0,\ell}$ and $P(d; s_1, s_2) = 0$, we have

$$d_n = -e^{-\mu_\ell s_1} \langle \tilde{\phi}_\ell, \phi_n\rangle - \int_{s_1}^{s_2} e^{\mu_n(\tau - s_1)} \langle f(v(\tau)), \phi_n\rangle d\tau \tag{4.13}$$

for $n = 0, 1, ..., \ell - 1$ by (4.8) and hence, using (3.21) and (4.9),

$$\begin{aligned}
|d_n| &\le |\langle \tilde{\phi}_\ell, \phi_n\rangle| e^{-\mu_\ell s_1} + C\sqrt{\mu_n + \alpha + 1} e^{-\mu_\ell s_1} \int_{s_1}^{s_2} e^{-(\mu_\ell - \mu_n)(\tau - s_1)} e^{-\kappa \mu_\ell \tau} d\tau \\
&< C e^{-(1+q_0)s_1} e^{-\mu_\ell s_1}
\end{aligned} \tag{4.14}$$

where $q_0 = \min\{\kappa, (\gamma + N)(1 - \tilde{k})(2\alpha - \gamma)\} > 0$. Summing up (4.14) for $n = 0, 1, ..., \ell - 1$, we obtain (4.7) for $s_1 \gg 1$. □

5 Proofs of a priori estimates.

This section is devoted to proving the key a priori estimate described in Proposition 3.7. We continue to use the notations having been used in the previous sections and partly recalled at the beginning of §4. Throughout this section, we always assume that there is $d \in \overline{U}_{s_1,s_2}$ such that $P(d; s_1, s_2) = 0$ for some $s_2 > s_1$. The proof is divided into two parts; short-time $s_1 \le s \le s_1 + 1$ and long-time $s_1 + 1 \le s \le s_2$. The former part is discussed in §§5.1, while the latter is argued in §§5.2. The proof of Proposition 3.7 is concluded at the end of this section.

5.1 Short-time estimates

In this subsection we prove Proposition 3.7 for the short-time interval. We set

$$\Sigma_{m,s}^{s_1} := \big\{(r, s)|\ K e^{-\omega_\ell s} \le r \le e^{\sigma s},\ s_1 \le s \le s_1 + 1\big\}.$$

The formula (2.62) in Corollary 2.10 provides that

$$v(s) = \sum_{j=0}^{\infty} e^{-\mu_j(s-s_1)} \langle v_{0,\ell}, \phi_j\rangle \phi_j + \int_{s_1}^{s} \sum_{j=0}^{\infty} e^{-\mu_j(s-\tau)} \langle f(v(\tau)), \phi_j\rangle \phi_j d\tau \tag{5.1}$$
$$= S_1(\cdot, s) + S_2(\cdot, s) + S_3(\cdot, s),$$

where

$$S_1(\cdot, s) := e^{-\mu_\ell s} \langle \tilde{\phi}_\ell, \phi_\ell\rangle \phi_\ell, \tag{5.2a}$$

$$S_2(\cdot, s) := \sum_{j\neq \ell} e^{-\mu_j(s-s_1)} e^{-\mu_\ell s_1} \langle \tilde{\phi}_\ell, \phi_j\rangle \phi_j + \sum_{n=0}^{\ell-1} d_n e^{-\mu_n(s-s_1)} \phi_n(r), \tag{5.2b}$$

$$S_3(\cdot, s) := \sum_{j=0}^{\infty} \int_{s_1}^{s} e^{-\mu_j(s-\tau)} \langle f(v(\tau)), \phi_j\rangle \phi_j d\tau. \tag{5.2c}$$

Lemma 5.1. *For any $\nu > 0$, there exists $s_1 \gg 1$ such that*

$$S_2 < \nu e^{-\mu_\ell s_1}(r^\gamma + r^{2(\mu_\ell+\alpha)}) \quad \textit{in } \Sigma_{m,s}^{s_1}. \tag{5.3}$$

Proof. Since S_2 satisfies the equation $S_s = S_{rr} + ((N-1)/r - r/2)S_r + \alpha S - (pc_{p,N}^{p-1}/r^2)S$, it may be represented as $S_2(r, s) = r^\gamma V(r, s)$ with a solution V of the equation

$$V_s = V_{rr} + \Big(\frac{2\gamma + N - 1}{r} - \frac{r}{2}\Big)V_r + \Big(\alpha - \frac{\gamma}{2}\Big)V \tag{5.4}$$

for $r > 0$ and $s_1 \le s \le s_2$. The equation (5.4) is further reduced to

$$W_s = W_{rr} + \Big(\frac{2\gamma + N - 1}{r} - \frac{r}{2}\Big)W_r$$

by setting $W = V \exp(-(\alpha - \gamma/2)(s - s_1))$. Hence we have

$$W(r, s) = \frac{Ce^{(\alpha+\frac{N-2}{4})(s-s_1)}}{1 - e^{-(s-s_1)}} \int_0^\infty I_{\frac{2\gamma+N-2}{2}}\left(\frac{e^{-\frac{s-s_1}{2}}\xi r}{2(1 - e^{-(s-s_1)})}\right)$$
$$\cdot \exp\left(-\frac{r^2e^{-(s-s_1)} + \xi^2}{4(1 - e^{-(s-s_1)})}\right) r^{-\gamma+1-\frac{N}{2}} \xi^{\gamma+\frac{N}{2}} W(\xi, s_1) d\xi, \tag{5.5}$$

where I_μ denotes the modified Bessel function of order μ (cf. [29, Proposition 6.1]). We recast (5.5) for S_2 to get

$$S_2 = Cr^\gamma \frac{e^{(\alpha+\frac{N-2}{4})(s-s_1)}}{1 - e^{-(s-s_1)}} \int_0^\infty H(\xi, r; s - s_1) \xi^{\frac{N}{2}} S_2(\xi, s_1) d\xi \tag{5.6}$$

with

$$H(\xi, r; t) = I_{\frac{2\gamma+N-2}{2}}\left(\frac{e^{-\frac{t}{2}}\xi r}{2(1 - e^{-t})}\right) \exp\left(-\frac{r^2e^{-t} + \xi^2}{4(1 - e^{-t})}\right) r^{-\gamma+1-\frac{N}{2}}. \tag{5.7}$$

Since

$$|I_\mu(z)| \le \frac{Cz^\mu e^z}{(1+z)^{\mu+1/2}}, \qquad z \in \mathbf{R},$$

for any $\mu > 0$ (cf. [1]), we have

$$\begin{aligned} H(\xi, r; s-\tau) \le\ & C\Big\{\frac{e^{-\frac{s-\tau}{2}}\xi r}{2(1-e^{-(s-\tau)})}\Big\}^{\frac{2\gamma+N-2}{2}} \left\{1+\frac{e^{-\frac{s-\tau}{2}}\xi r}{2(1-e^{-(s-\tau)})}\right\}^{-\frac{2\gamma+N-1}{2}} \\ & \cdot \exp\left(-\frac{r^2 e^{-(s-\tau)}+\xi^2}{4(1-e^{-(s-\tau)})}\right) r^{-\gamma-1+\frac{N}{2}} \\ =:\ & T(\xi, r; s-\tau). \end{aligned} \tag{5.8}$$

Therefore S_2 is estimated as

$$\begin{aligned} |S_2| \le\ & Cr^\gamma \frac{\exp\Big(\big(\alpha+\frac{N-2}{4}\big)(s-s_1)\Big)}{1-e^{-(s-s_1)}} \int_0^\infty \Big(\frac{e^{-\frac{s-s_1}{2}}\xi r}{2(1-e^{-(s-s_1)})}\Big)^{\frac{2\gamma+N-2}{2}} \\ & \cdot \Big(1+\frac{e^{-\frac{s-s_1}{2}}\xi r}{2(1-e^{-(s-s_1)})}\Big)^{-\frac{2\gamma+N-1}{2}} \exp\left(-\frac{(re^{-\frac{s-s_1}{2}}-\xi)^2}{4(1-e^{-(s-s_1)})}\right) r^{-\gamma+1-\frac{N}{2}} \xi^{\frac{N}{2}} |S_2(\xi, s_1)| d\xi \\ \le\ & \frac{Cr^\gamma}{1-e^{-(s-s_1)}} \left(\int_0^{\tilde K e^{-\omega_\ell s}} + \int_{\tilde K e^{-\omega_\ell s}}^\infty\right) T(\xi, r; s-s_1) \xi^{\frac{N}{2}} |S_2(\xi, s_1)| d\xi \\ =:\ & S_{2,1} + S_{2,2}. \end{aligned}$$

We first consider $S_{2,1}$. Since

$$\begin{aligned} S_2(r, s_1) &= e^{-2\alpha\omega_\ell s_1}\Big\{U_{\eta^*}(e^{\omega_\ell s_1} r) - U(e^{\omega_\ell s_1} r)\Big\} - e^{-\mu_\ell s_1}\langle\tilde\phi_\ell, \phi_\ell\rangle \phi_\ell(r) \\ &= O(e^{-2\alpha\omega_\ell s_1}\tilde K^\gamma) \end{aligned} \tag{5.9}$$

for $0 \le r \le \tilde K e^{-\omega_\ell s_1}$ as $s_1 \to \infty$, $S_{2,1}$ may be estimated as

$$\begin{aligned} S_{2,1}(r, s) &\le \frac{Ce^{-2\alpha\omega_\ell s_1}\tilde K^\gamma r^\gamma}{1-e^{-(s-s_1)}} \int_0^{\tilde K e^{-\omega_\ell s}} T(\xi, r; s-s_1) \xi^{\frac{N}{2}} d\xi \\ &\le \frac{Ce^{-2\alpha\omega_\ell s_1}\tilde K^\gamma r^{-\frac{N-1}{2}}}{\sqrt{1-e^{-(s-s_1)}}} \int_0^{\tilde K e^{-\omega_\ell s}} \exp\Big(-\frac{\xi^2}{2(1-e^{-(s-s_1)})}\Big) \xi^{\frac{N-1}{2}} d\xi, \end{aligned}$$

where the fact that $re^{-(s-s_1)/2} - \xi \ge \xi$ for $r \ge Ke^{-\omega_\ell s}$ and $0 \le \xi \le \tilde K e^{-\omega_\ell s_1}$ has been used. By a change of variable $\xi \mapsto t = \xi/\sqrt{1-e^{-(s-s_1)}}$, we obtain

$$\begin{aligned} S_{2,1}(r, s) &\le Ce^{-2\alpha\omega_\ell s_1}\tilde K^\gamma \int_0^\infty \exp\Big(-\frac{t^2}{2}\Big) dt \\ &< Ce^{-(k-\tilde k)\gamma\omega_\ell s_1} e^{-\mu_\ell s} r^\gamma \quad \text{ in } \Sigma_{m,s}^{s_1}. \end{aligned} \tag{5.10}$$

We then proceed to estimate $S_{2,2}$. We know

$$
\begin{aligned}
|S_2(r,s_1)| &= \Big|\sum_{n=0}^{\ell-1} d_n\phi_n(r) + e^{-\mu_\ell s_1}\tilde{\phi}_\ell(r) - e^{-\mu_\ell s_1}\langle\tilde{\phi}_\ell,\phi_\ell\rangle\phi_\ell(r)\Big| \\
&\le \begin{cases} Ce^{-(1+q_1)\mu_\ell s_1}e^{-\mu_\ell s_1}(r^\gamma + r^{2(\mu_\ell+\alpha)}) & \text{for } \tilde{K}e^{-\omega_\ell s_1} \le r \le e^{\tilde{\sigma} s_1}, \\ Cr^{2(\mu_\ell+\alpha)} & \text{for } e^{\tilde{\sigma}s_1} \le r \le \infty, \end{cases}
\end{aligned}
\tag{5.11}
$$

where $q_1 = (\gamma+N)(1-\tilde{k})(2\alpha-\gamma) > 0$ (cf. (3.21)). It then follows that

$$
\begin{aligned}
S_{2,2}(r,s) &\le \frac{Cr^\gamma}{1-e^{-(s-s_1)}}\int_{e^{\tilde{\sigma}s_1}}^\infty T(\xi,r;s-s_1)\xi^{2(\mu_\ell+\alpha)+\frac{N}{2}}d\xi \\
&\quad + \frac{Ce^{-(1+q_1)\mu_\ell s_1}r^\gamma}{1-e^{-(s-s_1)}}\bigg(\int_{2re^{-\frac{s-s_1}{2}}}^{e^{\tilde{\sigma}s_1}} + \int_{\tilde{K}e^{-\omega_\ell s_1}}^{2re^{-\frac{s-s_1}{2}}}\bigg)T(\xi,r;s-s_1)\xi^{\frac{N}{2}+\gamma}(1+\xi^{2\ell})d\xi \\
&=: S_{2,2}^1 + S_{2,2}^2 + S_{2,2}^3.
\end{aligned}
$$

Consider $S_{2,2}^1$. Since the conditions $\xi \ge e^{\tilde{\sigma}s_1}$ and $r \le e^{\sigma s}$ imply $\xi - re^{-\frac{s-s_1}{2}} \ge \xi(1-e^{-(\tilde{\sigma}-\sigma)s_1}) \ge \xi/2$, as long as $s_1 \le s \le s_1+1$ and $s_1 \gg 1$, we have

$$
T(\xi,r;s-s_1) \le \Big(\frac{\xi}{2(1-e^{-(s-s_1)})}\Big)^{\frac{2\gamma+N-2}{2}}\exp\Big(-\frac{\xi^2}{4(1-e^{-(s-s_1)})}\Big),
$$

whence

$$
\begin{aligned}
S_{2,2}^1(r,s) \le \frac{Cr^\gamma}{1-e^{-(s-s_1)}}\exp\Big(-\frac{e^{\tilde{\sigma}s_1}}{32}\Big)\int_{e^{\tilde{\sigma}s_1}}^\infty \Big(\frac{\xi}{2(1-e^{-(s-s_1)})}\Big)^{\frac{2\gamma+N-2}{2}} \\
\cdot\exp\Big(-\frac{\xi^2}{32(1-e^{-(s-s_1)})}\Big)\xi^{2(\mu_\ell+\alpha)+\frac{N}{2}}d\xi.
\end{aligned}
$$

We then change the integral variable by setting $t = \xi(1-e^{-(s-s_1)})^{-1/2}$ to deduce that

$$
\begin{aligned}
S_{2,2}^1(r,s) &\le \frac{Cr^\gamma}{\sqrt{1-e^{-(s-s_1)}}}\exp\Big(-\frac{e^{\tilde{\sigma}s_1}}{32}\Big)\int_0^\infty\Big(\frac{t}{2\sqrt{1-e^{-(s-s_1)}}}\Big)^{\frac{2\gamma+N-2}{2}} \\
&\quad\cdot\exp\Big(-\frac{t^2}{32}\Big)(1-e^{-(s-s_1)})^{\mu_\ell+\alpha+\frac{N}{4}}t^{2(\mu_\ell+\alpha)+\frac{N}{2}}dt. \\
&\le Cr^\gamma\exp\Big(-\frac{e^{\tilde{\sigma}s_1}}{32}\Big)\int_0^\infty t^{\gamma+N+2(\mu_\ell+\alpha)-1}\exp\Big(-\frac{t^2}{32}\Big)dt.
\end{aligned}
$$

Therefore we obtain

$$
S_{2,2}^1(r,s) < Ce^{-2\mu_\ell s_1}r^\gamma \quad \text{in } \Sigma_{m,s}^{s_1}. \tag{5.12}
$$

Next consider $S_{2,2}^2$. Since $\xi - re^{-(s-s_1)/2} \ge \xi^2/2$ when $\xi \ge 2re^{-(s-s_1)/2}$, we see

$$
\begin{aligned}
S_{2,2}^2 < \frac{Ce^{-(1+q_1)\mu_\ell s_1}r^\gamma}{1-e^{-(s-s_1)}}\int_{2re^{-\frac{s-s_1}{2}}}^{\tilde{\sigma}s_1}\Big(\frac{\xi}{2(1-e^{-(s-s_1)})}\Big)^{\frac{2\gamma+N-2}{2}} \\
\cdot\exp\Big(-\frac{\xi^2}{16(1-e^{-(s-s_1)})}\Big)\xi^{\gamma+\frac{N}{2}}(1+\xi^{2\ell})d\xi.
\end{aligned}
$$

We then use the same transformation of the integral variable as above, so that the integral is estimated by a constant multiple of $1-e^{-(s-s_1)}$, and thus

$$S_{2,2}^2 < Ce^{-(1+q_1)\mu_\ell s_1} r^\gamma \quad \text{in} \ \ \Sigma_{m,s}^{s_1}. \tag{5.13}$$

We finally estimate $S_{2,2}^3$. Since $e^{-(s-s_1)/2}\xi r \geq \xi^2/2$ for $\xi \leq 2re^{-(s-s_1)/2}$, we have

$$\begin{aligned} S_{2,2}^3 < & \frac{Ce^{-(1+q_1)\mu_\ell s_1} r^\gamma}{1-e^{-(s-s_1)}} \int_0^{2re^{-\frac{s-s_1}{2}}} \Big\{ \frac{\xi^2}{2(1-e^{-(s-s_1)})} \Big\}^{\frac{2\gamma+N-2}{2}} \\ & \cdot \Big\{ \frac{\xi^2}{4(1-e^{-(s-s_1)})} \Big\}^{-\frac{2\gamma+N-1}{2}} \exp\Big(- \frac{|\xi - re^{-\frac{s-s_1}{2}}|^2}{4(1-e^{-(s-s_1)})} \Big) \xi^{\gamma+\frac{N}{2}} (1+\xi^{2\ell}) d\xi \\ \leq & \frac{Ce^{-(1+q_1)\mu_\ell s_1} r^\gamma}{\sqrt{1-e^{-(s-s_1)}}} (1+r^{2\ell}) \int_0^{2re^{-\frac{s-s_1}{2}}} \exp\Big(- \frac{|\xi - re^{-\frac{s-s_1}{2}}|^2}{4(1-e^{-(s-s_1)})} \Big) d\xi. \end{aligned}$$

Since

$$\begin{aligned} & \int_0^\infty \exp\Big(- \frac{|\xi - re^{-\frac{s-s_1}{2}}|^2}{4(1-e^{-(s-s_1)})} \Big) d\xi \\ = & \int_0^{re^{-\frac{s-s_1}{2}}} \exp\Big(- \frac{|\xi - re^{-\frac{s-s_1}{2}}|^2}{4(1-e^{-(s-s_1)})} \Big) d\xi + \int_0^\infty \exp\Big(- \frac{z^2}{4(1-e^{-(s-s_1)})} \Big) dz \\ \leq & 4\pi^{\frac{N}{2}} \sqrt{1-e^{-(s-s_1)}}, \end{aligned}$$

we obtain

$$S_{2,2}^3 < Ce^{-(1+q_1)\mu_\ell s_1} (r^\gamma + r^{2(\mu_\ell+\alpha)}) \quad \text{in} \ \ \Sigma_{m,s}^{s_1}. \tag{5.14}$$

Summing up (5.12) to (5.14), we have

$$S_{2,2} < Ce^{-(1+q_1)\mu_\ell s_1} (r^\gamma + r^{2(\mu_\ell+\alpha)}) \quad \text{in} \ \ \Sigma_{m,s}^{s_1}. \tag{5.15}$$

The desired estimate (5.3) for $s_1 \gg 1$ then follows from (5.10) and (5.15). □

We shall show that a similar bound holds also for S_3. The proof requires a number of steps. Note that

$$S_3(r,s) = \int_{s_1}^s Z(\tau; r, s) d\tau \quad \text{with} \quad Z(\tau; r, s) = e^{-(s-\tau)\tilde{A}} f(v(\tau)). \tag{5.16}$$

Since the function $Z(\tau; r, s)$ satisfies

$$Z_s = Z_{rr} + \Big(\frac{N-1}{r} - \frac{N}{2} \Big) Z_r + \alpha Z - \frac{pc_{p,N}^{p-1}}{r^2} Z, \quad r > 0, \ s > \tau,$$

we may write (5.16) as

$$S_3(r,s) = Cr^\gamma \int_{s_1}^s \frac{e^{(\alpha+\frac{N-2}{4})(s-\tau)}}{1-e^{-(s-\tau)}} \int_0^\infty H(\xi, r; s-\tau) \xi^{\frac{N}{2}} f(v(\xi,\tau)) d\xi d\tau,$$

where H is the function defined in (5.7). It then follows from (5.8) that

$$\begin{aligned}
S_3(r,s) &\le Cr^{1-\frac{N}{2}} \int_{s_1}^{s} \frac{\exp\Big(\big(\alpha+\frac{N-2}{4}\big)(s-\tau)\Big)}{1-e^{-(s-\tau)}} \int_0^\infty \Big\{\frac{e^{-\frac{s-\tau}{2}}\xi r}{2(1-e^{-(s-\tau)})}\Big\}^{\frac{2\gamma+N-2}{2}} \\
&\quad \cdot \Big\{1+\frac{e^{-\frac{s-\tau}{2}}\xi r}{2(1-e^{-(s-\tau)})}\Big\}^{-\frac{2\gamma+N-1}{2}} \exp\Big(-\frac{r^2 e^{-(s-\tau)}+\xi^2}{4(1-e^{-(s-\tau)})}\Big)\xi^{\frac{N}{2}} f(v(\xi,\tau))d\xi d\tau \\
&\le \int_{s_1}^{s}\bigg(\int_0^{Le^{\omega_\ell \tau}} + \int_{Le^{\omega_\ell\tau}}^{e^{\sigma\tau}} + \int_{e^{\sigma\tau}}^{\infty}\bigg) I(\xi,\tau;r,s)d\xi d\tau \\
&=: S_{3,1}+S_{3,2}+S_{3,3},
\end{aligned} \tag{5.17}$$

where $L = e^{\vartheta\omega_\ell s_1}$ with $\max\{(2\alpha+1-\gamma)^{-1}, 1/2\} < \vartheta < k$ and

$$\begin{aligned}
I(\xi,\tau;r,s) := Cr^{\gamma}(s-\tau)^{-\gamma-\frac{N}{2}} &\left\{1+\frac{e^{-\frac{s-\tau}{2}}\xi r}{2(1-e^{-(s-\tau)})}\right\}^{-\frac{2\gamma+N-1}{2}} \\
&\cdot \exp\left(-\frac{|re^{-\frac{s-\tau}{2}}-\xi|^2}{4(1-e^{-(s-\tau)})}\right)\xi^{\gamma+N-1} f(v(\xi,\tau)).
\end{aligned} \tag{5.18}$$

Lemma 5.2. *Let $S_{3,1}$ be as in (5.17). For any $\nu>0$, there exists $s_1 \gg 1$ such that*

$$S_{3,1} < \nu e^{-\mu_\ell s_1} r^{\gamma} \quad \text{in } \Sigma_{m,s}^{s_1}. \tag{5.19}$$

Proof. When $\xi \le Le^{-\omega_\ell\tau}$, $r \ge Ke^{-\omega_\ell s}$ and $s_1 \le \tau \le s \le s_1+1$, one may readily check that $2\xi r e^{-(s-\tau)/2} < e^{-(s-\tau)} r^2$ if $L \ll K$. We then have

$$\exp\left(-\frac{|re^{-(s-\tau)/2}-\xi|^2}{4(1-e^{-(s-\tau)})}\right) \le \exp\left(-\frac{(2e)^{-1}r^2+\xi^2}{4(1-e^{-(s-\tau)})}\right)$$

there and obtain, by (4.1),

$$\begin{aligned}
S_{3,1} &\le C(Lr)^{\gamma}\int_{s_1}^{s}(s-\tau)^{-\gamma-\frac{N}{2}} e^{-2\alpha\omega_\ell\tau}\int_0^{Le^{\omega_\ell\tau}}\left\{1+\frac{e^{-\frac{s-\tau}{2}}\xi r}{2(1-e^{-(s-\tau)})}\right\}^{-\frac{2\gamma+N-1}{2}} \\
&\quad \cdot \exp\Big(-\frac{(2e)^{-1}r^2+\xi^2}{4(1-e^{-(s-\tau)})}\Big)\xi^{\gamma+N-3}d\xi d\tau \\
&\le C(Lr)^{\gamma}e^{-2\alpha\omega_\ell s_1}\int_{s_1}^{s}(s-\tau)^{-\frac{\gamma}{2}-1}\int_0^\infty \exp\Big(-\frac{Cr^2}{s-\tau}\Big)e^{-Cz^2}z^{\gamma+N-3}dzd\tau,
\end{aligned}$$

where we have used the change of variable $\xi \mapsto z = \xi/\sqrt{s-\tau}$ and the fact that $\gamma+N-2>0$. We again change the variable $\tau \mapsto \zeta = r/\sqrt{s-\tau}$ to observe that

$$\begin{aligned}
S_{3,1} &\le Cr^2(Lr)^{\gamma}e^{-2\alpha\omega_\ell s_1}\int_0^\infty \Big(\frac{\zeta}{r}\Big)^{\gamma+2} e^{-C\zeta^2}\zeta^{-3}\int_0^\infty e^{-Cz^2}z^{\gamma+N-3}dzd\zeta \\
&\le CL^{\gamma}e^{-2\alpha\omega_\ell s_1} = C\Big(\frac{L}{K}\Big)^{\gamma}(Ke^{-\omega_\ell s})^{\gamma}e^{\omega_\ell\gamma s-2\alpha\omega_\ell s_1} \\
&< Ce^{-(1+q_2)\mu_\ell s_1}r^{\gamma}
\end{aligned}$$

in $\Sigma_{m,s}^{s_1}$, where $q_2 = \gamma(k-\vartheta)/(2\alpha-\gamma) > 0$. This yields (5.19) for $s_1 \gg 1$ for $s_1 \gg 1$. □

Lemma 5.3. *Let $S_{3,2}$ be as in (5.17). For any $\nu > 0$, there exists $s_1 \gg 1$ such that*

$$S_{3,2} < \nu e^{-\mu_\ell s_1}(r^\gamma + r^{2(\mu_\ell+\alpha)}) \quad \text{in } \Sigma_{m,s}^{s_1}. \tag{5.20}$$

Proof. By the fundamental estimates of $f(v)$ in (4.2) and (4.3), we have

$$\begin{aligned} S_{3,2} \le{} & Cr^\gamma \int_{s_1}^{s} (s-\tau)^{-\gamma-\frac{N}{2}} e^{-2\mu_\ell \tau} \bigg(\int_{Le^{-\omega_\ell \tau}}^{1} \Big(1 + \frac{C\xi r}{s-\tau}\Big)^{-\frac{2\gamma+N-1}{2}} \\ & \cdot \exp\Big(- \frac{(re^{-\frac{s-\tau}{2}} - \xi)^2}{4(1-e^{-(s-\tau)})}\Big) \xi^{3\gamma-2\alpha+N-3} d\xi \\ & + \int_{1}^{e^{\sigma\tau}} \Big(1 + \frac{C\xi r}{s-\tau}\Big)^{-\frac{2\gamma+N-1}{2}} \exp\Big(- \frac{(re^{-\frac{s-\tau}{2}} - \xi)^2}{4(1-e^{-(s-\tau)})}\Big) \xi^{\gamma+4\mu_\ell+2\alpha+N-3} d\xi \bigg) d\tau \\ =:{} & S_{3,2}^1 + S_{3,2}^2. \end{aligned}$$

We first consider $S_{3,2}^1$. It may be estimated and split as

$$\begin{aligned} S_{3,2}^1 \le{} & Cr^\gamma e^{-2\mu_\ell s_1} \int_{s_1}^{s} (s-\tau)^{-\gamma-\frac{N}{2}} \bigg(\int_{Le^{-\omega_\ell\tau}}^{4r} + \int_{4r}^{1} \bigg) \Big(1 + \frac{C\xi r}{s-\tau}\Big)^{-\frac{2\gamma+N-1}{2}} \\ & \cdot \exp\Big(- \frac{(re^{-\frac{s-\tau}{2}} - \xi)^2}{4(1-e^{-(s-\tau)})}\Big) \xi^{3\gamma-2\alpha+N-3} d\xi d\tau \\ =:{} & S_{3,2}^{1,1} + S_{3,2}^{1,2}. \end{aligned}$$

Note that $\xi r \ge \xi^2/4$ when $Le^{-\omega_\ell \tau} \le \xi \le 4r$. Changing the variable $\xi \mapsto z = \xi/\sqrt{s-\tau}$, we see

$$\begin{aligned} S_{3,2}^{1,1} \le{} & Cr^\gamma e^{-2\mu_\ell s_1} \int_{s_1}^{s} (s-\tau)^{\frac{\gamma}{2}-\alpha-1} \int_{Le^{-\omega_\ell\tau}/\sqrt{s-\tau}}^{4r/\sqrt{s-\tau}} (1+Cz^2)^{-\frac{2\gamma+N-1}{2}} z^{3\gamma-2\alpha+N-3} \\ & \cdot \exp\Big(- C\Big| \frac{re^{-\frac{s-\tau}{2}}}{\sqrt{s-\tau}} - z\Big|^2\Big) dz d\tau. \end{aligned}$$

We split the region where the integral with respect to z is carried out into

$$D_1 = \Big\{ \Big| \frac{re^{-\frac{s-\tau}{2}}}{\sqrt{s-\tau}} - z\Big| \ge \frac{re^{-\frac{s-\tau}{2}}}{2\sqrt{s-\tau}} \Big\} \text{ and } D_2 = \Big\{ \Big| \frac{re^{-\frac{s-\tau}{2}}}{\sqrt{s-\tau}} - z\Big| < \frac{re^{-\frac{s-\tau}{2}}}{2\sqrt{s-\tau}} \Big\},$$

and denote the corresponding integrals in z as $A_i(r,s;\tau)$, $i=1,2$. It is readily seen that

$$A_1(r,s;\tau) \le \exp\big(-Cr^2\big) \int_{D_1} (1+Cz^2)^{-\frac{2\gamma+N-1}{2}} z^{3\gamma-2\alpha+N-3} dz.$$

We shall argue in the following, dividing two cases: $3\gamma - 2\alpha + N - 3 \le -1$ and $3\gamma - 2\alpha + N - 3 > -1$. In the first case, one may readily check that

$$A_1(r,s;\tau) \le C\big(Le^{-\omega_\ell(s_1+1)}\big)^{3\gamma-2\alpha+N-3} \int_{Le^{-\omega_\ell\tau}}^{4r} \zeta^{-(2\gamma+N-1)} d\zeta.$$

We set

$$S_{3,2}^{1,1,1} := Cr^\gamma e^{-2\mu_\ell s_1} \int_{s_1}^{s} (s-\tau)^{\frac{\gamma}{2}-\alpha-1} A_1(r,s;\tau)d\tau.$$

It follows then that

$$\begin{aligned} S_{3,2}^{1,1,1} &\le Cr^\gamma e^{-2\mu_\ell s_1} e^{-Cr^2} \big(Le^{-\omega_\ell(s_1+1)}\big)^{\gamma-2\alpha-1} \\ &< Ce^{-(1+q_3)\mu_\ell s_1} r^{2(\mu_\ell+\alpha)} \quad \text{ in } \Sigma_{m,s}^{s_1}, \end{aligned}$$

where $q_3 = \{(2\alpha-\gamma+1)\vartheta-1\}/(2\alpha-\gamma) > 0$. In the latter case, setting $t = r^2/(s-\tau)$, we observe that

$$\begin{aligned} S_{3,2}^{1,1,1} &\le Cr^{\gamma+2} e^{-2\mu_\ell s_1} \int_0^\infty e^{-ct} \Big(\frac{r}{\sqrt{t}}\Big)^{\gamma-2\alpha-2} (\sqrt{t})^{3\gamma-2\alpha+N-4} dt \\ &\le Ce^{-2\mu_\ell s_1} r^{2\gamma-2\alpha} \\ &< Ce^{-(1+k)\mu_\ell s_1} r^\gamma \quad \text{ in } \Sigma_{m,s}^{s_1}. \end{aligned}$$

We thus obtain, in the both cases,

$$S_{3,2}^{1,1,1} < Ce^{-(1+q_4)\mu_\ell s_1} \big(r^\gamma + r^{2(\mu_\ell+\alpha)}\big) \quad \text{ in } \Sigma_{m,s}^{s_1}, \tag{5.21}$$

where $q_4 = \min\{q_3, k\}$. We then proceed to estimate A_2 and its integral in τ. Note that $z \in D_2$ if and only if

$$\frac{re^{-\frac{s-\tau}{2}}}{2\sqrt{s-\tau}} < z < \frac{3re^{-\frac{s-\tau}{2}}}{2\sqrt{s-\tau}}.$$

It allows us to estimate A_2 as

$$\begin{aligned} A_2(r,s;\tau) &\le \int_{D_2} (1+Cz^2)^{-\frac{2\gamma+N-1}{2}} z^{3\gamma-2\alpha+N-3} dz \\ &\le C \int_{D_2} z^{\gamma-2\alpha-2} dz \\ &\le C(Ke^{-\omega_\ell s})^{-(1+2\alpha-\gamma)} (s-\tau)^{\frac{1+2\alpha-\gamma}{2}} \end{aligned}$$

for $Ke^{-\omega_\ell s} \le r$ and $s_1 \le \tau \le s \le s_1+1$. It then follows that

$$\begin{aligned} S_{3,2}^{1,1,2} &:= Cr^\gamma e^{-2\mu_\ell s_1} \int_{s_1}^{s} (s-\tau)^{\frac{\gamma}{2}-\alpha-1} A_2(s,\tau)d\tau \\ &\le Cr^\gamma e^{-2\mu_\ell s_1} (Ke^{-\omega_\ell s})^{-(1+2\alpha-\gamma)} \\ &< Ce^{-(1+q_5)\mu_\ell s_1} r^\gamma \end{aligned} \tag{5.22}$$

for $Ke^{-\omega_\ell s} \le r$ and $s_1 \le s \le s_1+1$, where $q_5 = \{(2\alpha-\gamma+1)k-1\}/(2\alpha-\gamma) > 0$. Summing up (5.21) and (5.22), we get

$$S_{3,2}^{1,1} < Ce^{-(1+q_4)\mu_\ell s_1} \big(r^\gamma + r^{2(\mu_\ell+\alpha)}\big) \quad \text{ in } \Sigma_{m,s}^{s_1}. \tag{5.23}$$

We shall show that a similar bound holds also for $S_{3,2}^{1,2}$. Since

$$\exp\Big(-\frac{(re^{-\frac{s-\tau}{2}}-\xi)^2}{4(1-e^{-(s-\tau)})}\Big) \le \exp\Big(-\frac{r^2+\xi^2}{12(1-e^{-(s-\tau)})}\Big)$$

when $\xi \geq 4r$ and $s_1 \leq \tau < s \leq s_1 + 1$, we have

$$\begin{aligned}
S_{3,2}^{1,2} &\leq Cr^\gamma e^{-2\mu_\ell s_1} \int_{s_1}^s (s-\tau)^{-\gamma-\frac{N}{2}} \int_{4r}^1 \exp\Big(-\frac{r^2+\xi^2}{12(1-e^{-(s-\tau)})}\Big)\xi^{3\gamma-2\alpha+N-3} d\xi d\tau \\
&\leq Cr^\gamma e^{-2\mu_\ell s_1} \int_{s_1}^s (s-\tau)^{\frac{\gamma}{2}-\alpha-1} \int_{4r/\sqrt{s-\tau}}^\infty \exp\Big(-\frac{Cr^2}{s-\tau}\Big) e^{-Cz^2} z^{3\gamma-2\alpha+N-3} dz d\tau \\
&\leq Cr^{2\gamma-2\alpha} e^{-2\mu_\ell s_1} \int_{r^2/(s-s_1)}^\infty t^{-(\frac{\gamma}{2}-\alpha-1)} e^{-Ct} \int_{4\sqrt{t}}^\infty e^{-Cz^2} z^{3\gamma-2\alpha+N-3} dz dt,
\end{aligned}$$

where we have used the change of variables $\xi \mapsto z = \xi/\sqrt{s-\tau}$ and $\tau \mapsto t = r^2/(s-\tau)$. To proceed further, we divide the argument into the two cases: $3\gamma - 2\alpha + N - 3 > -1$ and $3\gamma - 2\alpha + N - 3 \leq -1$. For the first case, the integrand $e^{-Cz^2} z^{3\gamma-2\alpha+N-3}$ is integrable in z up to the origin, which yields

$$\begin{aligned}
S_{3,2}^{1,2} &\leq Cr^{2\gamma-2\alpha} e^{-2\mu_\ell s_1} \int_0^\infty t^{\alpha-\frac{\gamma}{2}-1} e^{-Ct} dt \\
&< Ce^{-(1+k)\mu_\ell s_1} r^\gamma \quad \text{ in } \Sigma_{m,s}^{s_1}.
\end{aligned} \tag{5.24}$$

Consider the second case. By the definition of γ in (1.8), there is a constant $a \in (0,1)$ such that $\gamma + (N+a-3)/2 > 0$. We then have

$$\begin{aligned}
S_{3,2}^{1,2} &\leq Cr^{2\gamma-2\alpha} e^{-2\mu_\ell s_1} \int_0^\infty t^{\gamma+\frac{N+a-5}{2}} e^{-Ct} dt \\
&< Ce^{-(1+k)\mu_\ell s_1} r^\gamma \quad \text{ in } \Sigma_{m,s}^{s_1}.
\end{aligned} \tag{5.25}$$

It follows from (5.23), (5.24) and (5.25) that

$$S_{3,2}^1 < Ce^{-(1+q_4)\mu_\ell s_1} (r^\gamma + r^{2(\mu_\ell+\alpha)}) \quad \text{ in } \Sigma_{m,s}^{s_1}. \tag{5.26}$$

We then proceed to estimate $S_{3,2}^2$. It may be estimated as

$$\begin{aligned}
S_{3,2}^2 \leq\ & Cr^\gamma e^{-2\mu_\ell s_1} \int_{s_1}^s (s-\tau)^{-\gamma-\frac{N}{2}} \Bigg(\int_1^{4r} + \int_{4r}^{e^{\sigma\tau}}\Bigg) \Big(1 + \frac{C\xi r}{s-\tau}\Big)^{-\frac{2\gamma+N-1}{2}} \\
&\cdot \exp\Big(-\frac{(re^{-\frac{s-\tau}{2}} - \xi)^2}{4(1-e^{-(s-\tau)})}\Big) \xi^{\gamma+4\mu_\ell+2\alpha+N-3} d\xi d\tau \\
=:\ & S_{3,2}^{2,1} + S_{3,2}^{2,2}.
\end{aligned}$$

Consider $S_{3,2}^{2,1}$. Since $4\xi r \geq \xi^2$ in the interval under consideration, it is readily seen that

$$S_{3,2}^{2,1} \leq Cr^\gamma e^{-2\mu_\ell s_1} \int_{s_1}^s (s-\tau)^{-\gamma-\frac{N}{2}} \int_1^{4r} \Big(\frac{C\xi^2}{s-\tau}\Big)^{-\frac{2\gamma+N-1}{2}} \xi^{\gamma+4\mu_\ell+2\alpha+N-3} d\xi d\tau.$$

We then change the integral variable $\xi \mapsto \theta = \xi/\sqrt{s-\tau}$ to get

$$\begin{aligned}
S_{3,2}^{2,1} &\leq Cr^\gamma e^{-2\mu_\ell s_1} \int_{s_1}^s (s-\tau)^{\frac{\gamma}{2}+2\ell-\alpha-1} \int_{1/\sqrt{s-\tau}}^{4r/\sqrt{s-\tau}} \theta^{-(2\gamma+N-1)} \theta^{\gamma+4\mu_\ell+2\alpha+N-3} d\theta d\tau \\
&\leq Cr^{2\gamma+4\ell-2\alpha-1} e^{-2\mu_\ell s_1} \\
&< Ce^{-(1+q_6)\mu_\ell s_1} r^{2(\mu_\ell+\alpha)} \quad \text{ in } \Sigma_{m,s}^{s_1},
\end{aligned} \tag{5.27}$$

where $q_6 = k + (1-k)(2\ell-1)/(2\alpha-\gamma) > 0$. As for $S_{3,2}^{2,2}$, arguing as in the estimate of $S_{3,2}^{1,2}$, we obtain

$$\begin{aligned}
S_{3,2}^{2,2} &\le Cr^\gamma e^{-2\mu_\ell s_1} \int_{s_1}^{s} (s-\tau)^{-\gamma-\frac{N}{2}} \int_{4r}^{e^{\sigma\tau}} \exp\Big(-\frac{C\xi^2}{s-\tau}\Big)\xi^{\gamma+4\mu_\ell+2\alpha+N-3} d\xi d\tau \\
&\le Cr^\gamma e^{-2\mu_\ell s_1} \int_{s_1}^{s} (s-\tau)^{\frac{\gamma}{2}-\alpha+2\ell-1} \int_{4r/\sqrt{s-\tau}}^{\infty} e^{-Cz^2} z^{\gamma+4\mu_\ell+2\alpha+N-3} dz d\tau \\
&< Ce^{-2\mu_\ell s_1} r^\gamma \quad \text{ in } \Sigma_{m,s}^{s_1}. \qquad (5.28)
\end{aligned}$$

The estimates (5.26)-(5.28) yield then the desired estimate (5.20) for $s_1 \gg 1$. □

Lemma 5.4. *Let $S_{3,3}$ be as in* (5.17)*. For any $\nu > 0$, there exists $s_1 \gg 1$ such that*

$$S_{3,3} < \nu e^{-\mu_\ell s_1} r^{2(\mu_\ell+\alpha)} \quad \text{in } \ \Sigma_{m,s}^{s_1}. \qquad (5.29)$$

Proof. We first use (4.4) to get

$$S_{3,3} \le Cr^\gamma \int_{s_1}^{s} (s-\tau)^{-\gamma-\frac{N}{2}} \int_{e^{\sigma\tau}}^{\infty} \Big\{1+\frac{C\xi r}{s-\tau}\Big\}^{-\frac{2\gamma+N-1}{2}} \exp\Big(-\frac{|re^{-\frac{s-\tau}{2}}-\xi|^2}{s-\tau}\Big)\xi^{\gamma+2\alpha+N-3} d\xi d\tau.$$

Consider the case $r \le e^{\sigma s}/4$. In this case we have

$$\begin{aligned}
S_{3,3} &\le Cr^\gamma \int_{s_1}^{s} (s-\tau)^{-\gamma-\frac{N}{2}} \int_{e^{\sigma\tau}}^{\infty} \exp\Big(-\frac{C(r^2+\xi^2)}{s-\tau}\Big)\xi^{\gamma+2\alpha+N-3} d\xi d\tau \\
&\le C\exp\big(-Ce^{2\sigma s_1}\big) r^\gamma \int_{s_1}^{s} (s-\tau)^{-\frac{\gamma}{2}-1+\alpha} \int_{e^{\sigma\tau}/\sqrt{s-\tau}}^{\infty} e^{-C\theta^2}\theta^{\gamma+2\alpha+N-3} d\theta d\tau \\
&< C\exp\big(-Ce^{2\sigma s_1}\big) r^\gamma \quad \text{ in } \Sigma_{m,s}^{s_1} \cap \{r \le e^{\sigma s}/4\}, \qquad (5.30)
\end{aligned}$$

where we have used the change of variable $\xi \mapsto \theta = \xi/\sqrt{s-\tau}$.

In the case where $r > e^{\sigma s}/4$, we further split the integral as

$$\begin{aligned}
S_{3,3} &\le Cr^\gamma \int_{s_1}^{s} (s-\tau)^{-\gamma-\frac{N}{2}} \bigg(\int_{e^{\sigma\tau}}^{4r} + \int_{4r}^{\infty}\bigg)\Big(1+\frac{C\xi r}{s-\tau}\Big)^{-\frac{2\gamma+N-1}{2}} \\
&\qquad \cdot \exp\Big(-\frac{|re^{-\frac{s-\tau}{2}}-\xi|^2}{s-\tau}\Big)\xi^{\gamma+2\alpha+N-3} d\xi d\tau \\
&=: S_{3,3}^1 + S_{3,3}^2.
\end{aligned}$$

Consider $S_{3,3}^1$. Since $\xi r \ge \xi^2/4$ in the interval under consideration, we have

$$\begin{aligned}
S_{3,3}^1 &\le Cr^\gamma \int_{s_1}^{s} (s-\tau)^{-\gamma-\frac{N}{2}} \int_{e^{\sigma\tau}}^{4r} \Big(\frac{C\xi^2}{s-\tau}\Big)^{-\frac{2\gamma+N-1}{2}} \xi^{\gamma+2\alpha+N-3} d\xi d\tau \\
&\le Cr^\gamma \int_{s_1}^{s} (s-\tau)^{-\frac{\gamma}{2}+\alpha-1} \int_{e^{\sigma\tau}/\sqrt{s-\tau}}^{4r/\sqrt{s-\tau}} \theta^{-\gamma+2\alpha-2} d\theta d\tau,
\end{aligned}$$

where we have used the same change of variables as above. Suppose that $-\gamma+2\alpha-1 \le 0$. We then have

$$\begin{aligned} S_{3,3}^1 &\le Cr^{\gamma+1}\int_{s_1}^{s}(s-\tau)^{-\frac{\gamma}{2}+\alpha-\frac{3}{2}}\Big(\frac{e^{\sigma\tau}}{\sqrt{s-\tau}}\Big)^{-\gamma+2\alpha-2}d\tau \\ &\le Cr^{\gamma+1}e^{-(\gamma-2\alpha+2)\sigma s_1} \\ &< Ce^{-(1+q_7)\mu_\ell s_1}r^{2(\mu_\ell+\alpha)} \end{aligned} \tag{5.31}$$

for $r > e^{\sigma s}/4$ and $s_1 \gg 1$, where $q_7 = 2\ell\sigma/\mu_\ell - 1 > 0$ (cf. (3.12)). On the other hand, if $-\gamma + 2\alpha - 1 > 0$, then

$$\begin{aligned} S_{3,3}^1 &\le Cr^{\gamma}\int_{s_1}^{s}(s-\tau)^{-\frac{\gamma}{2}+\alpha-1}\Big(\frac{4r}{\sqrt{s-\tau}}\Big)^{-\gamma+2\alpha-1}d\tau \\ &\le Cr^{2(\mu_\ell+\alpha)+2\alpha-\gamma-2\mu_\ell-1}K^{-(2\alpha-\gamma)}e^{-\mu_\ell s} \\ &\le Ce^{-(1+q_8)\mu_\ell s_1}r^{2(\mu_\ell+\alpha)} \qquad \text{in } \Sigma_{m,s}^{s_1} \end{aligned} \tag{5.32}$$

by the definition of K in (3.11), where $q_8 = k+\sigma(\gamma-2\alpha+2\mu_\ell+1)/\mu_\ell > 0$. We thus have

$$S_{3,3}^1 < Ce^{-(1+q_9)\mu_\ell s_1}r^{2(\mu_\ell+\alpha)} \qquad \text{in } \Sigma_{m,s}^{s_1}\cap\{r > e^{\sigma s}/4\}, \tag{5.33}$$

where $q_9 = \min\{q_7, q_8\}$. Next consider $S_{3,3}^2$. Arguing as in the estimate for $S_{3,2}^{2,2}$, we get

$$\begin{aligned} S_{3,3}^2 &\le Cr^{\gamma}\int_{s_1}^{s}(s-\tau)^{-\gamma-\frac{N}{2}}\int_0^\infty\Big(\frac{Cr^2}{s-\tau}\Big)^{-\frac{2\gamma+N-1}{2}}\exp\Big(-\frac{C\xi^2}{s-\tau}\Big)\xi^{\gamma+N+2\alpha-3}d\xi d\tau \\ &\le Cr^{-\gamma-N+1}\int_{s_1}^{s}(s-\tau)^{\frac{\gamma+N-3}{2}+\alpha}\int_0^\infty e^{-C\theta^2}\theta^{\gamma+N+2\alpha-3}d\theta d\tau \\ &< Ce^{-(1+q_{10})\mu_\ell s}r^{2(\mu_\ell+\alpha)} \qquad \text{in } \Sigma_{m,s}^{s_1}\cap\{r > e^{\sigma s}/4\}, \end{aligned} \tag{5.34}$$

where $q_{10} = 2\ell\sigma/\mu_\ell > 0$. Summing up (5.33) and (5.34), we see that if $s_1 \gg 1$, then (5.29) holds in $\Sigma_{m,s}^{s_1}\cap\{r > e^{\sigma s}/4\}$ as well as in $\Sigma_{m,s}^{s_1}\cap\{r \le e^{\sigma s}/4\}$ due to (5.30). □

We shall summarize the estimates having been obtained in this subsection.

Corollary 5.5. *For any $\nu > 0$, there exists $s_1 \gg 1$ such that*

$$|v(r,s) - e^{-\mu_\ell s}\langle\tilde{\phi}_\ell,\phi_\ell\rangle\phi_\ell| < \nu e^{-\mu_\ell s_1}(r^\gamma + r^{2(\mu_\ell+\alpha)}) \qquad \text{in } \Sigma_{m,s}^{s_1}. \tag{5.35}$$

5.2 Long-time estimates

Our aim in this subsection is to extend the estimates obtained for the short time interval $s_1 \le s \le s_1+1$ in §§5.1 to the long time interval $s_1+1 \le s \le s_2$. Let $R \ge 1$ be a fixed constant. The following notations will be used throughout this section:

$$\begin{aligned} \Sigma_{R,\ell}^{s_1+1} &:= \big\{(r,s) \mid Ke^{-\omega_\ell s} \le r \le R,\ s_1+1 < s \le s_2\big\}, \\ \Sigma_{m,\ell}^{s_1+1} &:= \big\{(r,s) \mid Ke^{-\omega_\ell s} \le r \le e^{\sigma s},\ s_1+1 < s \le s_2\big\}. \end{aligned}$$

Substituting (4.13) to (5.1) with (5.2), we observe, for $n = 0, 1, 2, ..., \ell-1$, that

$$\begin{aligned} v(r,s) &= e^{-\mu_\ell s}\langle\tilde{\phi}_\ell,\phi_\ell\rangle\phi_\ell(r) + \sum_{j=\ell+1}^{\infty} e^{-\mu_j(s-s_1)-\mu_\ell s_1}\langle\tilde{\phi}_\ell,\phi_j\rangle\phi_j(r) \\ &\quad + \sum_{j=\ell}^{\infty}\int_{s_1}^{s} e^{-\mu_j(s-\tau)}\langle f(v(\tau)),\phi_j\rangle\phi_j(r)d\tau - \sum_{n=0}^{\ell-1}\int_{s}^{s_2} e^{-\mu_n(s-\tau)}\langle f(v(\tau)),\phi_n\rangle\phi_n(r)d\tau \\ &=: I_1 + I_2 + I_3 + I_4. \end{aligned}$$

Note that I_1, I_2, I_3, I_4 are expressed by S_1, S_2, S_3 in (5.2) as

$$I_1 = S_1, \tag{5.36a}$$

$$I_3 = S_3 - \sum_{j=0}^{\ell-1}\int_{s_1}^{s} e^{-\mu_j(s-\tau)}\langle f(v(\tau)),\phi_j\rangle\phi_j(r)d\tau. \tag{5.36b}$$

$$I_2 + I_4 = S_2 + \sum_{j=0}^{\ell-1}\int_{s_1}^{s} e^{-\mu_j(s-\tau)}\langle f(v(\tau)),\phi_j\rangle\phi_j(r)d\tau, \tag{5.36c}$$

Lemma 5.6. *For any $\nu > 0$, there exists $s_1 \gg 1$ such that*

$$|I_4| < \nu e^{-\mu_\ell s}(r^\gamma + r^{2(\mu_\ell+\alpha)}) \quad \textit{for all } r > 0 \textit{ and } s_1 + 1 \le s \le s_2. \tag{5.37}$$

Proof. By (4.9) we obtain

$$\begin{aligned} |I_4| &\le C\sum_{n=0}^{\ell-1}|\phi_n(r)|e^{-\mu_n s}\int_s^{s_2} e^{-(\mu_\ell-\mu_n+\kappa\mu_\ell)\tau}d\tau \\ &\le Ce^{-\kappa\mu_\ell s}e^{-\mu_\ell s}|\phi_n(r)| \end{aligned} \tag{5.38}$$

in $\Sigma_{R,\ell}^{s_1+1}$. The claim (5.37) follows from (5.38), (2.38) and (2.20). □

Lemma 5.7. *For any $\nu > 0$, there exists $s_1 \gg 1$ such that*

$$|I_2| < \nu e^{-\mu_\ell s}r^\gamma \quad \textit{in } \Sigma_{R,\ell}^{s_1+1}. \tag{5.39}$$

Proof. Parseval's identity yields that $\|\tilde{\phi}_\ell - \phi_\ell\|_{2,\rho}^2 \ge \sum_{j=\ell+1}^\infty \left|\langle\tilde{\phi}_\ell,\phi_j\rangle\right|^2$. Recalling (3.21), we obtain

$$|I_2| < e^{-(\mu_\ell+\varepsilon_1)s_1}\sum_{j=\ell+1}^{\infty} e^{-\mu_j(s-s_1)}|\phi_j(r)|, \tag{5.40}$$

where $\varepsilon_1 = \min\{(\gamma+N)(1-\tilde{k})\omega_\ell, 1\}$. By (2.38) we then conclude

$$\begin{aligned} |I_2| &< e^{-\varepsilon_1 s_1}e^{-\mu_\ell s}C(R)e^{-\frac{1}{2}\varepsilon_1(s-s_1)}\sum_{j=\ell+1}^{\infty} e^{-(\mu_j-\mu_\ell-\frac{1}{2}\varepsilon_1)(s-s_1)}j^{-\frac{1}{4}}r^\gamma \\ &< C(R)e^{-\frac{1}{2}\varepsilon_1 s_1}e^{-(\mu_\ell+\frac{1}{2}\varepsilon_1)s}r^\gamma \quad \text{in } \Sigma_{R,\ell}^{s_1+1} \end{aligned} \tag{5.41}$$

if $s_1 \gg 1$, since $\mu_j - \mu_\ell = j - \ell \ge 1$ for every $j \ge \ell+1$. □

Lemma 5.8. *For any $\nu > 0$, there exists $s_1 \gg 1$ such that*

$$|I_3| < \nu e^{-\mu_\ell s} r^\gamma \quad \text{in } \Sigma_{R,\ell}^{s_1+1}. \tag{5.42}$$

Proof. We first split I_3 as

$$I_3 = \sum_{j=\ell}^{\infty} \Big(\int_{s-1}^{s} + \int_{s_1}^{s-1} \Big) e^{-\mu_j(s-\tau)} \langle f(v(\tau)), \phi_j \rangle \phi_j(r) d\tau =: I_{3,1} + I_{3,2}.$$

Making use of the estimate of S_3 in $\Sigma_{m,s}^{s_1}$ (cf. Lemmata 5.2-5.4), we have $|I_{3,1}| < \nu e^{-\mu_\ell(s-1)} r^\gamma$ in $\Sigma_{R,\ell}^{s_1+1}$ for $s_1 \gg 1$. It thus suffices to show a similar bound for $I_{3,2}$ in $\Sigma_{R,\ell}^{s_1+1}$.

An argument similar to the one used in the proof of Lemma 5.7 (cf. (5.41)) yields that

$$\begin{aligned} |I_{3,2}| &\le \int_{s_1}^{s-1} e^{-\mu_\ell(s-\tau)} \left(\sum_{j=\ell}^{\infty} (\mu_j + \alpha + 1) e^{-2(j-\ell)(s-\tau)} |\phi_j|^2 \right)^{\frac{1}{2}} \left(\sum_{j=\ell}^{\infty} \frac{|\langle f(v(\tau)), \phi_j \rangle|^2}{\mu_j + \alpha + 1} \right)^{\frac{1}{2}} d\tau \\ &\le C(R) r^\gamma \int_{s_1}^{s-1} e^{-\mu_\ell(s-\tau)} \|f(v(\tau))\|_{\mathcal{H}} d\tau \quad \text{in } \Sigma_{R,\ell}^{s_1+1}. \end{aligned}$$

By (4.9) we have

$$|I_{3,2}| < C(R) e^{-\kappa\mu_\ell s_1} e^{-\mu_\ell s} r^\gamma \quad \text{in } \Sigma_{R,\ell}^{s_1+1},$$

whence get the desired estimate (5.42) for $s_1 \gg 1$. □

Lemma 5.9. *For any $\nu > 0$, there exists $s_1 \gg 1$ such that*

$$|S_2| < \nu e^{-\mu_\ell s} r^{2(\mu_\ell+\alpha)} \quad \text{in } \Sigma_{m,\ell}^{s_1+1} \cap \big\{ (r,s) \mid r \ge e^{\frac{s-s_1}{2}} \big\}. \tag{5.43}$$

Proof. Recalling (5.9), (5.6) and (5.11), we have

$$\begin{aligned} |S_2| &\le C r^\gamma e^{(\alpha + \frac{N-2}{4})(s-s_1)} \int_0^\infty T(\xi, r; s-s_1) \xi^{\frac{N}{2}} |S_2(\xi, s_1)| d\xi \\ &\le \; S_2^1 + S_2^2 + S_2^3 \quad \text{in } \Sigma_{m,\ell}^{s_1+1} \end{aligned}$$

with the function T in (5.8) and

$$\begin{aligned} S_2^1 &:= C K^\gamma e^{-2\alpha\omega_\ell s_1} e^{(\alpha+\frac{N-2}{4})(s-s_1)} r^{1-\frac{N}{2}} \int_0^{\tilde{K} e^{-\omega_\ell s_1}} \xi^{\frac{N}{2}} d\xi, \\ S_2^2 &:= C\nu e^{-\mu_\ell s_1} r^{1-\frac{N}{2}} \int_{\tilde{K} e^{-\omega_\ell s_1}}^{e^{\tilde{\sigma} s_1}} (e^{-\frac{s-s_1}{2}} \xi r)^{\frac{2\gamma+N-2}{2}} \big(1 + C e^{-\frac{s-s_1}{2}} \xi r\big)^{-\frac{2\gamma+N-1}{2}} \\ &\qquad\qquad \cdot \exp\big(-C\big|re^{\frac{s-s_1}{2}} - \xi\big|^2\big) \xi^{\frac{N}{2}} \big(\xi^\gamma + \xi^{2(\mu_\ell+\alpha)}\big) d\xi, \\ S_2^3 &:= C r^\gamma \int_{e^{\tilde{\sigma} s_1}}^\infty \xi^{\gamma+N-1+2(\mu_\ell+\alpha)} \exp\big(-C\big|\xi - re^{-\frac{s-s_1}{2}}\big|^2\big) d\xi \cdot e^{(\alpha-\frac{N}{2})(s-s_1)}. \end{aligned}$$

We then deduce that, for $r \ge \exp\{(s-s_1)/2\}$ and $s \ge s_1 + 1$,

$$\begin{aligned} S_2^1 &\le C r^{2(\mu_\ell+\alpha)} K^\gamma (e^{\frac{s-s_1}{2}})^{-2(\mu_\ell+\alpha)+1-\frac{N}{2}} e^{(\alpha+\frac{N-2}{4})(s-s_1)-2\alpha\omega_\ell s_1} (\tilde{K} e^{-\omega_\ell s_1})^{1+\frac{N}{2}} \\ &= C e^{-\mu_\ell s_1} r^{2(\mu_\ell+\alpha)} e^{-((\gamma+\frac{N}{2}+1)-(\gamma k+(\frac{N}{2}+1)\tilde{k}))\omega_\ell s_1} e^{-(\mu_\ell-\frac{1}{4})(s-s_1)} \\ &< \nu e^{-\mu_\ell s_1} r^{2(\mu_\ell+\alpha)}, \end{aligned} \tag{5.44}$$

if $s_1 \gg 1$, where we have used the definitions of K and $\tilde{K}$ in (3.11) and (3.19), respectively.

To estimate S_2^2, we split the interval under consideration into

$$D_1 = \Big\{\xi;\ \big|re^{-\frac{s-s_1}{2}} - \xi\big| < \frac{\xi}{2}\Big\} \ \text{ and } \ D_2 = \Big\{\xi;\ \big|re^{-\frac{s-s_1}{2}} - \xi\big| \geq \frac{\xi}{2}\Big\}$$

and denote the corresponding integrals by $S_2^{2,1}$ and $S_2^{2,2}$, respectively. If $\xi \in D_1$, then $\xi \sim re^{-(s-s_1)/2}$ and hence

$$\begin{aligned}
S_2^{2,1} &\leq C\nu e^{-\mu_\ell s_1} r^{1-\frac{N}{2}} \int_{D_1} \xi^{\frac{N}{2}-1} \exp\Big(-C\big|re^{\frac{s-s_1}{2}} - \xi\big|^2\Big)\big(\xi^\gamma + \xi^{2(\mu_\ell+\alpha)}\big) d\xi \\
&\leq C\nu e^{-\mu_\ell s_1} (e^{-\frac{s-s_1}{2}})^{\frac{N}{2}-1} \{(re^{-\frac{s-s_1}{2}})^{-2\ell} + 1\} (re^{-\frac{s-s_1}{2}})^{2(\mu_\ell+\alpha)} \int_0^\infty \exp\big(-Cz^2\big) dz \\
&\leq C\nu e^{-\mu_\ell s} r^{2(\mu_\ell+\alpha)} \quad \text{ in } \Sigma_{m,\ell}^{s_1+1} \cap \big\{(r,s) \mid r \geq e^{\frac{s-s_1}{2}}\big\}.
\end{aligned} \tag{5.45}$$

As to $S_2^{2,2}$, the integrand shows exponentially decay, which gives

$$\begin{aligned}
S_2^{2,2} &\leq C\nu e^{-\mu_\ell s_1} e^{(\alpha - \frac{\gamma}{2})(s-s_1)} r^\gamma \int_{D_2} e^{-C\xi^2} \xi^{\gamma+N-1} \big(\xi^\gamma + \xi^{2(\mu_\ell+\alpha)}\big) d\xi \\
&\leq C\nu e^{-\mu_\ell s} r^{2(\mu_\ell+\alpha)} e^{\ell(s-s_1)} \big(e^{\frac{s-s_1}{2}}\big)^{-2\ell} \int_0^\infty e^{-C\xi^2} \xi^{\gamma+N-1} \big(\xi^\gamma + \xi^{2(\mu_\ell+\alpha)}\big) d\xi \\
&\leq C\nu e^{-\mu_\ell s} r^{2(\mu_\ell+\alpha)} \quad \text{ in } \Sigma_{m,\ell}^{s_1+1} \cap \big\{(r,s) \mid r \geq e^{\frac{s-s_1}{2}}\big\}.
\end{aligned} \tag{5.46}$$

We finally estimate S_2^3. Note that the conditions $r \leq e^{\sigma s}$ and $\xi \geq e^{\tilde{\sigma} s_1}$ imply $\xi - re^{-\frac{s-s_1}{2}} \geq \xi/2$ if $s_1 \gg 1$. An argument similar to that for the estimate of $S_2^{2,2}$ yields that

$$S_2^3 \leq \exp\Big(-\frac{C}{2} e^{2\tilde{\sigma} s_1}\Big) e^{-\mu_\ell s} r^{2(\mu_\ell+\alpha)} \quad \text{ in } \Sigma_{m,\ell}^{s_1+1} \cap \big\{(r,s) \mid r \geq e^{\frac{s-s_1}{2}}\big\}. \tag{5.47}$$

The desired estimate (5.43) follows from (5.44) to (5.47). □

Lemma 5.10. *For any $\nu > 0$, there exists $s_1 \gg 1$ such that*

$$|S_3| < \nu e^{-\mu_\ell s} r^{2(\mu_\ell+\alpha)} \quad \text{ in } \Sigma_{m,\ell}^{s_1+1} \cap \big\{(r,s) \mid r \geq e^{\frac{s-s_1}{2}}\big\}. \tag{5.48}$$

Proof. We begin with splitting S_3 as

$$S_3 = \Big(\int_{s-1}^s + \int_{s_1}^{s-1}\Big) e^{-(s-\tau)\tilde{A}} f(v(\tau)) d\tau =: A + B.$$

Since A can be estimated by the procedure adapted in the previous subsection, we only have to estimate B. With the function T in (5.8) we have

$$\begin{aligned}
|B| &\leq Cr^\gamma \int_{s_1}^{s-1} \frac{e^{(\alpha + \frac{N-2}{4})(s-\tau)}}{1 - e^{-(s-\tau)}} \int_0^\infty T(\xi, r; s-\tau) \xi^{\frac{N}{2}} f(v(\xi,\tau)) d\xi d\tau \\
&\leq Cr^\gamma \int_{s_1}^{s-1} e^{(\alpha - \frac{\gamma}{2})(s-\tau)} \Big(\int_0^{Ke^{-\omega_\ell s}} + \int_{Ke^{-\omega_\ell s}}^1 + \int_1^{e^{\sigma s}} + \int_{e^{\sigma s}}^\infty\Big) \xi^{\gamma+N-1} \\
&\qquad \cdot \big(1 + Ce^{-\frac{s-\tau}{2}} \xi r\big)^{-\frac{2\gamma+N-1}{2}} \exp\big(-C\big|\xi - re^{-\frac{s-\tau}{2}}\big|^2\big) f(v(\xi,\tau)) d\xi d\tau \\
&=: B_1 + B_2 + B_3 + B_4.
\end{aligned}$$

Consider B_1. Using (4.1) and the definition of K in (3.11), we obtain

$$
\begin{aligned}
B_1 &\leq CK^\gamma r^\gamma \int_{s_1}^{s-1} e^{(\alpha-\frac{\gamma}{2})(s-\tau)-2\alpha\omega_\ell\tau} \int_0^{Ke^{-\omega_\ell s}} \xi^{\gamma+N-3} d\xi d\tau \\
&\leq CK^{2\gamma+N-2} r^\gamma e^{(\alpha-\frac{\gamma}{2})(s-s_1)} e^{-2\alpha\omega_\ell s_1} e^{-(\gamma+N-2)\omega_\ell s_1} \\
&\leq Ce^{-\mu_\ell s} r^{2(\mu_\ell+\alpha)} e^{-(1-k)(2\gamma+N-2)\omega_\ell s_1} \quad \text{in } \Sigma_{m,\ell}^{s_1+1} \cap \big\{(r,s) \mid r \geq e^{\frac{s-s_1}{2}}\big\}.
\end{aligned} \tag{5.49}
$$

As for B_2, it follows from (4.2) that

$$
\begin{aligned}
B_2 &\leq Cr^\gamma \int_{s_1}^{s-1} e^{(\alpha-\frac{\gamma}{2})(s-\tau)-2\mu_\ell\tau} \int_{Ke^{-\omega_\ell s}}^1 \xi^{\gamma-2\alpha-1} d\xi d\tau \\
&\leq Cr^\gamma K^{-(2\alpha-\gamma)} e^{(\alpha-\frac{\gamma}{2})(s-s_1)-\mu_\ell s_1} e^{\ell(s-s_1)} \\
&< \nu e^{-\mu_\ell s} r^{2(\mu_\ell+\alpha)} \quad \text{in } \Sigma_{m,\ell}^{s_1+1} \cap \big\{(r,s) \mid r \geq e^{\frac{s-s_1}{2}}\big\},
\end{aligned} \tag{5.50}
$$

if $s_1 \gg 1$. As for B_3, we have

$$
\begin{aligned}
B_3 \leq Cr^\gamma \int_{s_1}^{s-1} & e^{(\alpha-\frac{\gamma}{2})(s-\tau)-2\mu_\ell\tau} \int_1^{e^{\sigma\tau}} \xi^{N+\gamma+4\mu_\ell+2\alpha-3} \big(1 + Ce^{-\frac{s-\tau}{2}}\xi r\big)^{-\frac{2\gamma+N-1}{2}} \\
&\cdot \exp\big(-C\big|\xi - re^{-\frac{s-\tau}{2}}\big|^2\big) d\xi d\tau \\
=:& B_{3,1} + B_{3,2},
\end{aligned}
$$

where we have split the interval of ξ under consideration into

$$
D_1 = \Big\{\xi \;;\; |\xi - re^{-\frac{s-\tau}{2}}| \leq \frac{\xi}{2}\Big\} \quad \text{and} \quad D_2 = \Big\{\xi \;;\; |\xi - re^{-\frac{s-\tau}{2}}| > \frac{\xi}{2}\Big\}
$$

and denoted the corresponding integrals by $B_{3,1}$ and $B_{3,2}$, respectively. Consider firstly $B_{3,1}$. Notice that if $\xi \in D_1$, then $\xi \sim re^{-(s-\tau)/2}$ and thus the measure of D_1 is equal to $(4/3)re^{-(s-\tau)/2}$. We then obtain

$$
\begin{aligned}
B_{3,1} &\leq Cr^\gamma \int_{s_1}^{s-1} e^{(\alpha-\frac{\gamma}{2})(s-\tau)-2\mu_\ell\tau} \int_{D_1} \xi^{4\mu_\ell-\gamma+2\alpha-2} d\xi d\tau \\
&\leq Cr^{\gamma+4\mu_\ell} e^{-2\mu_\ell s} \int_{s_1}^{s-1} e^{(\alpha-\frac{\gamma}{2})(s-\tau)} \big(re^{-\frac{s-\tau}{2}}\big)^{2\alpha-1-\gamma} d\tau \\
&< \nu e^{-\mu_\ell s} r^{2(\mu_\ell+\alpha)} \quad \text{in } \Sigma_{m,\ell}^{s_1+1} \cap \big\{(r,s) \mid r \geq e^{\frac{s-s_1}{2}}\big\},
\end{aligned} \tag{5.51}
$$

if $s_1 \gg 1$. As to $B_{2,2}$, there is an exponentially decaying factor in the integrand in ξ. Hence the procedure used to bound B_2 above yields that

$$
\begin{aligned}
B_{3,2} &\leq Cr^\gamma \int_{s_1}^{s-1} e^{(\alpha-\frac{\gamma}{2})(s-\tau)-2\mu_\ell\tau} \int_1^\infty \xi^{N+\gamma+4\mu_\ell+2\alpha-3} \exp\big(-C\xi^2\big) d\xi d\tau \\
&< \nu e^{-\mu_\ell s} r^{2(\mu_\ell+\alpha)} \quad \text{in } \Sigma_{m,\ell}^{s_1+1} \cap \big\{(r,s) \mid r \geq e^{\frac{s-s_1}{2}}\big\},
\end{aligned} \tag{5.52}
$$

if $s_1 \gg 1$. The above procedure also provides that

$$\begin{aligned}
B_4 &\le Cr^\gamma \int_{s_1}^{s-1} e^{(\alpha-\frac{\gamma}{2})(s-\tau)-2\mu_\ell\tau} \int_{e^{\sigma\tau}}^{\infty} \xi^{N+\gamma+2\alpha-3} \exp\big(-C\xi^2\big) d\xi d\tau \\
&\le e^{-\mu_\ell s} r^{2(\mu_\ell+\alpha)} e^{\ell s} r^{-2\ell} e^{-(\alpha-\frac{\gamma}{2})s_1} \exp\big(-Ce^{2\sigma s_1}\big) \\
&< \nu e^{-\mu_\ell s} r^{2(\mu_\ell+\alpha)} \quad \text{in } \Sigma_{m,\ell}^{s_1+1} \cap \big\{(r,s) \mid r \ge e^{\frac{s-s_1}{2}}\big\},
\end{aligned} \tag{5.53}$$

if $s_1 \gg 1$. Summing up (5.49)-(5.53), we obtain the desired estimate. □

Summarizing Lemmata 5.6 to 5.10, we have the following estimates.

Corollary 5.11. *For any $\nu > 0$, there exists $s_1 \gg 1$ such that*

$$|v(r,s) - e^{-\mu_\ell s}\langle \tilde{\phi}_\ell, \phi_\ell\rangle \phi_\ell(r)| < \nu e^{-\mu_\ell s} r^\gamma \quad \text{in } \Sigma_{R,\ell}^{s_1+1};$$

$$|v(r,s) - e^{-\mu_\ell s}\langle \tilde{\phi}_\ell, \phi_\ell\rangle \phi_\ell(r)| < \nu e^{-\mu_\ell s}(r^\gamma + r^{2(\mu_\ell+\alpha)}) \quad \text{in } \Sigma_{m,\ell}^{s_1+1} \cap \big\{(r,s) \mid r \ge e^{\frac{s-s_1}{2}}\big\}.$$

Proof of Proposition 3.7. We prove that

$$|v(r,s) - e^{-\mu_\ell s}\langle \tilde{\phi}_\ell, \phi_\ell\rangle \phi_\ell(r)| < \nu e^{-\mu_\ell s}(r^\gamma + r^{2(\mu_\ell+\alpha)}) \tag{5.54}$$

in $\Sigma_{m,\ell}^{s_1+1}$. It suffices to show (5.54) for $\Sigma_{m,\ell}^{s_1+1} \cap \{R \le r\}$ by virtue of Corollary 5.11. To this aim we set $\tilde{s}_1 = s_1 + 1$, $\tilde{s}_2 = \tilde{s}_1 + 1$ and $R = \sqrt{e}$. Notice that $r \ge e^{(s-\tilde{s}_1)/2}$ whenever $R \le r$ and $\tilde{s}_1 \le s \le \tilde{s}_2$. Corollary 5.11 implies then that (5.54) holds for $R \le r \le e^{\sigma s}$ and $\tilde{s}_1 \le s \le \min\{\tilde{s}_2, s_2\}$. We have thus proven that (5.54) holds in the all of $\Sigma_{m,\ell}^{s_1+1}$ if $s_2 \le \tilde{s}_2$. When $s_2 > \tilde{s}_2$, we set $\tilde{s}_3 = \tilde{s}_2 + 1$ and argue as above to observe that (5.54) holds in $\Sigma_{m,\ell}^{s_1+1} \cap \{\tilde{s}_2 \le s \le \tilde{s}_3\}$, whence in $\Sigma_{m,\ell}^{s_1+1} \cap \{\tilde{s}_1 \le s \le \tilde{s}_3\}$. Repeating these arguments finitely many times, we observe that (5.54) holds in the all of $\Sigma_{m,\ell}^{s_1+1}$. The desired estimate in Proposition 3.7 follows from (2.20), (3.21), Corollary 5.5 and (5.54). □

6 Further properties on the solution w_ℓ

Having proven our main results, we investigate further properties on the solution w_ℓ obtained in Theorem 3.2 in this final section.

Theorem 6.1. *Let $w_\ell(r,s)$ be the solution of (2.9) obtained in Theorem 3.2. Then there exists a constant $\beta_\ell = \beta_\ell(s_1) > 0$ such that $|\beta_\ell - 1| = O(e^{-\varepsilon_0\omega_\ell s_1})$ with $\varepsilon_0 = \min\{(\gamma+N)(1-\tilde{k}), \kappa(2\alpha-\gamma)\}$ as $s_1 \to \infty$ and*

$$\lim_{s\to\infty} e^{\mu_\ell s}\big\{w_\ell(r,s) - U(r)\big\} = \beta_\ell \phi_\ell(r) \tag{6.1}$$

uniformly in each compact set of $(0,\infty)$. More precisely, there exists a constant $\lambda > 0$ such that, for each $R > 0$,

$$\Big|e^{\mu_\ell s}\big(w_\ell(r,s) - U(r)\big) - \beta_\ell \phi_\ell(r)\Big| \le C_R e^{-\lambda\mu_\ell s} r^\gamma \tag{6.2}$$

with some constant $C_R > 0$ for $Ke^{-\omega_\ell s} \le r \le R$ and $s \ge s_1$.

Proof. We continue to use the notations in the previous sections. We first consider the case $s \ge s_1 + 1$. Note that, by (5.36),

$$\begin{aligned} &w_\ell(r,s) - U(r) - I_1 - \int_{s_1}^{s} e^{-\mu_\ell(s-\tau)}\langle f(v(\tau)), \phi_\ell\rangle \phi_\ell d\tau \\ &= I_2 + I_4 - \int_{s-1}^{s} e^{-(s-\tau)\tilde{A}} f(v(\tau)) d\tau - \sum_{j=\ell+1}^{\infty} \int_{s_1}^{s-1} e^{-\mu_j(s-\tau)}\langle f(v(\tau)), \phi_j\rangle \phi_j d\tau. \end{aligned} \tag{6.3}$$

Let $R > 0$ be a constant. According to the proofs of Lemmata 5.2-5.4, it is possible to find a constant $\lambda_0 > 0$ such that

$$\Big| \int_{s-1}^{s} e^{-(s-\tau)\tilde{A}} f(v(\tau)) d\tau \Big| \le C_R e^{-(1+\lambda_0)\mu_\ell s} r^\gamma \tag{6.4}$$

for $Ke^{-\omega_\ell s} \le r \le R$ and $s \ge s_1 + 1$. Arguing as in the proof of Lemma 5.8, we obtain

$$\Big| \sum_{j=\ell+1}^{\infty} \int_{s_1}^{s-1} e^{-\mu_j(s-\tau)}\langle f(v(\tau)), \phi_j\rangle \phi_j d\tau \Big| \le C_R e^{-(1+\lambda_1)\mu_\ell s} r^\gamma \tag{6.5}$$

for $Ke^{-\omega_\ell s} \le r \le R$ and $s \ge s_1 + 1$, where $\lambda_1 = \kappa/2$ with $\kappa \in (0,1)$ being the constant in (4.9). The proofs of Lemmata 5.6 and 5.7 imply the existence of a constant $\lambda_2 > 0$ such that

$$|I_2| + |I_4| \le C_R e^{-(1+\lambda_2)\mu_\ell s} r^\gamma \tag{6.6}$$

for $Ke^{-\omega_\ell s} \le r \le R$ and $s \ge s_1 + 1$. It then follows from (6.3)-(6.6) that

$$\Big| w_\ell(r,s) - U(r) - I_1 - \int_{s_1}^{s} e^{-\mu_\ell(s-\tau)}\langle f(v(\tau)), \phi_\ell\rangle \phi_\ell d\tau \Big| \le 3C_R e^{-(1+\lambda)\mu_\ell s} r^\gamma$$

for $Ke^{-\omega_\ell s} \le r \le R$ and $s \ge s_1 + 1$, where $\lambda = \min\{\lambda_0, \lambda_1, \lambda_2\}$. By (4.9) one may readily observe that $\chi := \int_{s_1}^{\infty} e^{\mu_\ell \tau}\langle f(v(\tau)), \phi_\ell\rangle d\tau$ exists and satisfies

$$\Big| \int_{s_1}^{s} e^{\mu_\ell \tau}\langle f(v(\tau)), \phi_\ell\rangle d\tau - \chi \Big| \le Ce^{-\kappa\mu_\ell s}$$

for $s \ge s_1 + 1$. Since $I_1 = e^{-\mu_\ell s}\langle \tilde{\phi}_\ell, \phi_\ell\rangle \phi_\ell(r)$, we conclude (6.2) with $\beta_\ell = \langle \tilde{\phi}_\ell, \phi_\ell\rangle + \chi$. By (3.21) and (4.9) we have $|\beta_\ell - 1| \le Ce^{-\varepsilon_0 \omega_\ell s_1}$ with $\varepsilon_0 = \min\{(\gamma + N)(1 - \tilde{k}), \kappa(2\alpha - \gamma)\}$.

Consider the case $s_1 \le s \le s_1 + 1$. According to the proofs of Lemmata 5.1-5.4, there exists a constant $q > 0$ such that

$$\begin{aligned} |w_\ell(r,s) - U(r) - S_1| &\le |S_2| + |S_3| \\ &\le Ce^{-(1+q)\mu_\ell s}(r^\gamma + r^{2(\mu_\ell+\alpha)}) \quad \text{in } \Sigma_{m,s}^{s_1}. \end{aligned} \tag{6.7}$$

Since $|\chi| \le Ce^{-k\mu_\ell s_1}$, we obtain (6.2) for $s_1 \le s \le s_1 + 1$ by taking $\lambda > 0$ smaller than what we have shown above. □

As an immediate consequence of Theorem 6.1, we get information about the number of intersections between the graph of the solution $w_\ell(\cdot, s)$ and that of the singular stationary solution U if we add the following additional condition on the class of initial data $w_{0,\ell}$:

$$\text{The function } w_{0,\ell} - U \text{ has just } \ell \text{ simple zeros in } (0, \infty). \tag{V5}$$

Here a zero $r^* \in (0, \infty)$ of a function $\Psi \in C^1((0, \infty))$ is said to be simple unless $\Phi'(r^*) = 0$.

Corollary 6.2. *Let $w_{0,\ell}$ be a function satisfying the condition $(V5)$ as well as $(V1)$-$(V4)$ in §3 and let $w_\ell(r, s)$ be the solution of* (2.9) *obtained by Theorem 3.2. Then the graph of $w_\ell(\cdot, s)$ has ℓ-intersections with the graph of U in $[0, \infty)$, that is, $\mathcal{Z}_{[0,\infty)}[w_\ell(\cdot, s) - U] = \ell$ for every $s \in [s_1, \infty)$, where $\mathcal{Z}_J[\cdot]$ is the zero number defined in* (2.31)*. Every zero of $w_\ell(\cdot, s) - U$ is simple for every $s \in [s_1, \infty)$ and tends to some zero of ϕ_ℓ as $s \to \infty$.*

Proof. Theorem 6.1 and Lemma 2.4 imply that there exists $s^* \geq s_1$ such that

$$\mathcal{Z}_{[0,\infty)}[w_\ell(\cdot, s) - U] \geq \mathcal{Z}_{[0,\infty)}[\phi_\ell] = \ell \quad \text{for } s \geq s^*. \tag{6.8}$$

On the other hand, the zero number diminishing property (cf. [3, 5, 27]) assures that

$$\mathcal{Z}_{[0,\infty)}[w_\ell(\cdot, s) - U] \leq \mathcal{Z}_{[0,\infty)}[w_\ell(\cdot, s_1) - U] \quad \text{for all } s \geq s_1.$$

Therefore $\mathcal{Z}_{[0,\infty)}[w_\ell(\cdot, s) - U] = \ell$ under the condition $(V5)$. If there is a zero of $w_\ell(\cdot, s^*) - U$ that is not simple in $[0, \infty)$ at some $s^* \in (s_1, \infty)$, then $\mathcal{Z}_{[0,\infty)}[w_\ell(\cdot, s) - U] < \ell$ for $s > s^*$, violating (6.8). Thus every zero $r_{\ell,j}(s)$, $j = 1, 2, ..., \ell$, of $w_\ell(\cdot, s) - U$ is simple for every $s \in [s_1, \infty)$, which simultaneously implies that $r_{\ell,j}(s)$ is a smooth function of s. The last assertion is then readily seen by (6.1) and the continuity of ϕ_ℓ. □

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Yukihiro Seki
Graduate School of Mathematical Sciences
University of Tokyo
3-8-1 Komaba, Meguro-ku Tokyo 153-8914, Japan
E-mail: `seki@ms.u-tokyo.ac.jp`

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semilinear heat equation; absorption; dead-core; matched asymptotic expansion.
2000 Mathematics subject classification: 35B40, 35K15, 35K55.

UTMS

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ADDRESS:
Graduate School of Mathematical Sciences, The University of Tokyo
3–8–1 Komaba Meguro-ku, Tokyo 153-8914, JAPAN
TEL +81-3-5465-7001 FAX +81-3-5465-7012