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Recovering a Lamé Kernel in a viscoelastic equation by a single boundary measurement

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Recovering a Lamé kernel in a viscoelastic equation by a single boundary measurement

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Abstract. We consider the viscoelasticity equation and the problem of recovering the spatial part $p(x)$ of the Lamé kernel depending on two spatial variables. We obtain a stability estimate of the solution to this problem.

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1 Introduction

For a function $u(x,t)$ in the domain $Q = \Omega \times (0,T)$, where $T > 0$ and $\Omega \subset \mathbb{R}^2$ is an open bounded domain with smooth boundary $\partial\Omega$, we consider the equation

$$u_{tt} = \operatorname{div}\Big[\mu_0(x)\nabla u + \int_0^t \mu(x,t-s)\nabla u(x,s)\,ds\Big], \quad (x,t) \in Q. \tag{1.1}$$

The above integro-differential equation occurs in the theory of viscoelastic bodies with a constant density and the Lamé coefficients independent of the variable x_3. As for the physical backgrounds on the viscoelasticity, see Pipkin [22] for example. Then the corresponding third component of the displacement vector satisfies equation (1.1) (e.g., [21]).

In this article, following paper [21], we assume that $\mu_0 \in \mathbf{C}^3(\bar{\Omega})$, $\bar{\Omega} = \Omega \cup \partial\Omega$, is a given positive function, $\mu_0(x) \geq \mu_{00} > 0$, and $\mu(x,t)$ admits the representation

$$\mu(x,t) = k(t)p(x)$$

where $k \in \mathbf{C}^4[0,T]$ is a given function and $p(x)$ is an unknown function.

In order to recover $p(x)$ we prescribe the following initial data

$$u|_{t=0} = \phi(x), \quad u_t|_{t=0} = \psi(x), \quad x \in \Omega, \tag{1.2}$$

and the boundary data

$$u|_\Gamma = g(x,t), \quad \frac{\partial u}{\partial n}\Big|_\Gamma = h(x,t), \tag{1.3}$$

where Γ is a smooth part of the lateral boundary $\partial\Omega \times (0,T)$ of the domain Q. A more precise description of Γ will be given later.

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Our result is concerned with the stability of the solution to problem (1.1)-(1.3) and is based on Carleman estimates. We recall that this method was first adapted for studying inverse problems in the article [3] and then widely used for this goal by many researchers. In [4], the authors consider (1.1) with source term and a Hölder stability estimate is proved in determining a spatially varying factor in a source term. In the paper [21] a Hölder type stability result for the problem of recovering the coefficient $p(x)$ was established by using boundary conditions similar to (1.3) for three different initial data, i.e. using three measurements. In the contrast to this, we obtain here a similar result by using a single measurement. This is the main difference in our paper from the previous paper. Moreover, we use here Carleman estimates with a different weight function. The latter allows to give a more precise restriction of the observation set Γ in (1.3) in comparison with [21] (see relations (1.4), (1.5)).

We note some related works on inverse problems for hyperbolic equations that make use of Carleman estimates: Bellassoued [1], Bukhgeim [2], Imanuvilov and Yamamoto [8] - [11], Isakov [12], [14], [15], Isakov and Yamamoto [16], Klibanov [17], Klibanov and Timonov [18], Klibanov and Yamamoto [19], Romanov [24], Yamamoto [25] and the references therein. As for available Carleman estimates, see Fursikov and Imanuvilov [5], Hörmander [6], Imanuvilov [7], Isakov [13], Lavrent'ev, V.G. Romanov and S.P. Shishat'skiĭ [20], Romanov [23].

Now we introduce some basic assumptions and notations. We assume that $\mu_0(x)$ admits a continuation in the some open domain Ω' that strongly contains $\bar\Omega$ and it is still a positive function in Ω' of $\mathbf{C}^2(\Omega')$ class. Moreover, we assume that the Riemannian metric $d\tau = |dx|/\sqrt{\mu_0(x)}$ is *simple*, i.e., each couple of points x and y can be jointed in Ω' by a single geodesic line, and has non-positive section curvatures in Ω'. The sufficient condition for this is:

$$\sum_{i,j=1}^{2} \frac{\partial^2 \ln \mu_0(x)}{\partial x_i \partial x_j}\, \xi_i \xi_j \leq 0$$

for all ξ_1, ξ_2.

Let y be a fixed point, $y \in \Omega' \setminus \Omega$. Denote by $\tau(x, y)$ the Riemannian distance between y and $x \in \Omega$. Let $\theta(x, t) = \tau^2(x, y) - \beta t^2$, where $\beta \in (0, 1)$. Let the following inequalities hold

$$\inf_{x\in\bar\Omega} \tau^2(x, y) < \varepsilon_0 < \varepsilon^0 = \sup_{x\in\bar\Omega} \tau^2(x, y).$$

For arbitrary $\varepsilon \in (\varepsilon_0, \varepsilon^0)$, following [21], we introduce the sets

$$Q(\varepsilon) = \{(x, t)|\, x \in \bar\Omega,\ \theta(x, t) > \varepsilon\}, \quad \Omega(\varepsilon) = \{x \in \bar\Omega|\, \theta(x, 0) > \varepsilon\}.$$

Note that

$$Q(\varepsilon_2) \subset Q(\varepsilon_1), \quad \Omega(\varepsilon_2) \subset \Omega(\varepsilon_1), \quad \text{if } \varepsilon_0 < \varepsilon_1 < \varepsilon_2 < \varepsilon^0.$$

We suppose also that

$$Q(\varepsilon_0) \subset (\overline{\Omega} \times [0, T]), \quad Q(\varepsilon_0) \cap (\partial\Omega \times [0, T]) \equiv \Gamma_0 \subset \Gamma. \tag{1.4}$$

The former is possible only if the condition

$$T > T^*, \quad T^* = \sqrt{\frac{\varepsilon^0 - \varepsilon_0}{\beta}}, \tag{1.5}$$

holds. It means that the time size of the observation set Γ should be sufficiently large.

Moreover, we assume that $p \in \mathcal{P}$, where

$$\mathcal{P} = \{p \in \mathbf{H}^3(\Omega) |\, \operatorname{supp} p \subset \Omega, \|p\|_{\mathbf{H}^3(\Omega)} \leq p_0\},$$

and $u \in \mathcal{U}$, where

$$\mathcal{U} = \{u, D_t^4 u \in \mathbf{H}^2(\Omega \times [0,T]) |\; \|D_t^4 u\|_{\mathbf{H}^1(\Omega\times[0,T])} \leq u_0\}.$$

The main result of this article is stated in the following stability theorem.

Theorem. *Let $\mu_0 \in \mathbf{C}^2(\Omega')$ and the Riemannian metric $d\tau = |dx|/\sqrt{\mu_0(x)}$ be simple and have non-positive curvature in Ω'. We assume that $\phi \in \mathbf{H}^4(\Omega)$, $k(0) = 1$ and $\psi(x) = -k'(0)\phi(x)$, where $k'(0) = D_t k(0)$. Let functions $p_j \in \mathcal{P}$, $u_j \in \mathcal{U}$, $g_j(x,t)$, $h_j(x,t)$, $j = 1,2$, satisfy relations (1.1)-(1.3) with the same functions $\phi(x)$ and $\psi(x)$. Suppose that condition (1.4) holds and the inequality*

$$\nabla\theta(x,0) \cdot \nabla\phi(x) \geq \lambda > 0, \quad x \in \Omega(\varepsilon_0), \tag{1.6}$$

is valid for some positive constant λ. Set

$$\epsilon^2 = \|D_t^{(4)}(g_1 - g_2)\|^2_{\mathbf{H}^1(\Gamma_0)} + \|D_t^{(4)}(h_1 - h_2)\|^2_{\mathbf{L}^2(\Gamma_0)}. \tag{1.7}$$

Then there exist two constants C and ϵ_0 such that for any $\varepsilon \in (\varepsilon_0, \varepsilon^0)$ and $\epsilon \in (0, \epsilon_0)$ the following estimate holds

$$\|p_1 - p_2\|^2_{\mathbf{H}^1(\Omega(\varepsilon))} \leq C(s^*)^2(p_0^2 + u_0^2)^\gamma \epsilon^{2(1-\gamma)}, \tag{1.8}$$

where

$$s^* = \frac{5}{2(5\varepsilon^0 - 4\varepsilon - \varepsilon_0)} \ln \frac{p_0^2 + u_0^2}{\epsilon^2}, \quad \gamma = \frac{5(\varepsilon^0 - \varepsilon)}{5\varepsilon^0 - 4\varepsilon - \varepsilon_0} \in (0,1).$$

The condition $\psi(x) = -k'(0)\phi(x)$ for initial data is restrictive but thanks to it, a single measurement yields the stability as well as the uniqueness for our inverse problem.

2 Proof of the stability Theorem

Calculating the derivatives of $u(x,t)$ with respect to t, since $\psi(x) = -k'(0)\phi(x)$, we find

$$\begin{aligned}
D_t^{(2)} u|_{t=0} &= \operatorname{div}[\mu_0(x)\nabla\phi], && (2.1)\\
D_t^{(3)} u|_{t=0} &= \operatorname{div}[\mu_0(x)\nabla\psi + p(x)\nabla\phi] \equiv A(x), && (2.2)\\
D_t^{(4)} u|_{t=0} &= \operatorname{div}[\mu_0(x)\nabla\operatorname{div}(\mu_0(x)\nabla\phi)] + p(x)\nabla[\psi(x) + k'(0)\phi(x)] && \\
&= \operatorname{div}[\mu_0(x)\nabla\operatorname{div}(\mu_0(x)\nabla\phi)], && (2.3)\\
D_t^{(5)} u|_{t=0} &= \operatorname{div}\big[\mu_0(x)\nabla A(x) + p(x)\nabla a(x)\big], && (2.4)
\end{aligned}$$

where

$$a(x) = \operatorname{div}(\mu_0(x)\nabla\phi(x)) + k'(0)\psi(x) + \phi(x)D_t^{(2)}k(0)$$
$$= \operatorname{div}(\mu_0(x)\nabla\phi(x)) + [-(k'(0))^2 + D_t^{(2)}k(0)]\phi(x). \tag{2.5}$$

Let us now fix the function $\phi(x)$ and consider relations (1.1)-(1.3), (2.1)-(2.4) for $p_j \in \mathcal{P}$, $u_j \in \mathcal{U}$, g_j, h_j, A_j, $j = 1, 2$. Introduce the differences

$$\tilde{p} = p_1 - p_2, \quad \tilde{u} = u_1 - u_2, \quad \tilde{g} = g_1 - g_2, \quad \tilde{h} = h_1 - h_2, \quad \tilde{A} = A_1 - A_2.$$

Then these functions satisfy the relations

$$\tilde{u}_{tt} = \operatorname{div}\Big[\mu_0(x)\nabla\tilde{u} + p_1(x)\nabla\int_0^t k(s)\tilde{u}(x, t-s)\,ds$$
$$+\tilde{p}(x)\nabla\int_0^t k(s)u_2(x, t-s)\,ds\Big], \quad (x,t) \in \Omega \times (0,T), \tag{2.6}$$

$$\tilde{u}|_{t=0} = 0, \quad \tilde{u}_t|_{t=0} = 0, \quad x \in \Omega, \tag{2.7}$$

$$\tilde{u}|_\Gamma = \tilde{g}(x,t), \quad \frac{\partial\tilde{u}}{\partial n}\Big|_\Gamma = \tilde{h}(x,t). \tag{2.8}$$

$$D_t^{(2)}\tilde{u}|_{t=0} = 0, \tag{2.9}$$

$$D_t^{(3)}\tilde{u}|_{t=0} = \operatorname{div}[\tilde{p}(x)\nabla\phi] \equiv \tilde{A}(x), \tag{2.10}$$

$$D_t^{(4)}\tilde{u}|_{t=0} = 0, \tag{2.11}$$

$$D_t^{(5)}\tilde{u}|_{t=0} = \operatorname{div}\big[\mu_0(x)\nabla\tilde{A}(x) + \tilde{p}(x)a(x)\big] \equiv \tilde{q}(x). \tag{2.12}$$

Introduce the new function $\tilde{v}(x,t) = D_t^{(4)}\tilde{u}(x,t)$. This function satisfies the relations

$$\tilde{v}_{tt} = \operatorname{div}\Big[\mu_0(x)\nabla\tilde{v} + p_1(x)\nabla\int_0^t k(t-s)\tilde{v}(x,s)\,ds$$
$$+p_1(x)k(t)\nabla\tilde{A}(x) + \tilde{p}(x)\nabla b(x,t)\Big], \quad (x,t) \in Q(\varepsilon_0), \tag{2.13}$$

$$\tilde{v}|_{t=0} = 0, \quad \tilde{v}_t|_{t=0} = \tilde{q}(x), \quad x \in \Omega(\varepsilon_0), \tag{2.14}$$

$$\tilde{v}|_{\Gamma_0} = D_t^{(4)}\tilde{g}(x,t), \quad \frac{\partial\tilde{v}}{\partial n}\Big|_{\Gamma_0} = D_t^{(4)}\tilde{h}(x,t), \tag{2.15}$$

where

$$b(x,t) = D_t^{(4)}\int_0^t k(t-s)u_2(x,s)\,ds. \tag{2.16}$$

Introduce the function

$$\tilde{w}(x,t) = \tilde{v}(x,t) - \int_0^t \mu_1(x, t-s)\tilde{v}(x,s)\,ds, \tag{2.17}$$

where $\mu_1(x,t) = -k(t)p_1(x)/\mu_0(x)$. It is well known that function $\tilde{v}(x,t)$ can be expressed via function $\tilde{w}(x,t)$ in a similar way, namely

$$\tilde{v}(x,t) = \tilde{w}(x,t) + \int_0^t R(x,t-s)\tilde{w}(x,s)\,ds, \tag{2.18}$$

where

$$R(x,t) = \sum_{n=0}^{\infty} R_n(x,t), \tag{2.19}$$

and the function $R_n(x,t)$ are defined recursively by the formulae

$$R_0(x,t) = \mu_1(x,t), \quad R_n(x,t) = \int_0^t \mu_1(x,t-s)R_{n-1}(x,s)\,ds, \quad n = 1,2,\dots. \tag{2.20}$$

Performing calculations, we find the following equations for function $\tilde{w}(x,t)$:

$$\begin{aligned}
&\tilde{w}_{tt} + R(x,0)\tilde{w}_t + R_t(x,0)\tilde{w} + \int_0^t R_{tt}(x,t-s)\tilde{w}(x,s)\,ds \\
&= \operatorname{div}\Big\{\mu_0(x)\nabla\tilde{w} + \mu_0(x)\int_0^t \Big[\tilde{w}(x,s) + \int_0^s R(x,s-s_1)\tilde{w}(x,s_1)\,ds_1\Big]\nabla\mu_1(x,t-s)\,ds \\
&\quad + p_1(x)k(t)\nabla\tilde{A}(x) + \tilde{p}(x)\nabla b(x,t)\Big\}, \quad (x,t) \in Q(\varepsilon_0), \qquad (2.21)\\
&\tilde{w}|_{t=0} = 0, \quad \tilde{w}_t|_{t=0} = \tilde{q}(x), \quad x \in \Omega(\varepsilon_0), \qquad (2.22)
\end{aligned}$$

$$\tilde{w} = D_t^{(4)}\tilde{g}(x,t) - \int_0^t \mu_1(x,t-s)D_s^{(4)}\tilde{g}(x,s)\,ds \equiv \tilde{G}(x,t), \quad (x,t) \in \Gamma_0, \tag{2.23}$$

$$\begin{aligned}
\frac{\partial\tilde{w}}{\partial n} &= D_t^{(4)}\tilde{h}(x,t) - \int_0^t \Big[\mu_1(x,t-s)D_s^{(4)}\tilde{h}(x,s) + D_s^{(4)}\tilde{g}(x,s)\frac{\partial\mu_1(x,t-s)}{\partial n}\Big]\,ds \\
&\equiv \tilde{H}(x,t), \quad (x,t) \in \Gamma_0.
\end{aligned} \tag{2.24}$$

Rewrite then equation (2.21) in the form

$$\begin{aligned}
&\tilde{w}_{tt} - \mu_0(x)\Delta\tilde{w} + R(x,0)\tilde{w}_t + R_t(x,0)\tilde{w} - \nabla\mu_0(x)\cdot\nabla\tilde{w} \\
&\quad + \int_0^t [R_1(x,t-s)\tilde{w}(x,s) + R_2(x,t-s)\cdot\nabla\tilde{w}(x,s)]\,ds \\
&= \operatorname{div}\big[p_1(x)k(t)\nabla\tilde{A}(x) + \tilde{p}(x)\nabla b(x,t)\big], \quad (x,t) \in Q(\varepsilon_0),
\end{aligned} \tag{2.25}$$

where

$$R_1(x,t) = R_{tt}(x,t) - \operatorname{div}\left[\mu_0(x)\left(\nabla\mu_1(x,t) + \int\limits_0^t R(x,z)\nabla\mu_1(x,t-z)\,dz\right)\right],$$

$$R_2(x,t) = -\mu_0(x)\left(\nabla\mu_1(x,t) + \int\limits_0^t R(x,z)\nabla\mu_1(x,t-z)\,dz\right).$$

Now we make use of the following lemma:

Lemma 1. *Let $\omega \in \mathbf{H}^2(Q(\varepsilon_0))$, $\mu_0 \in \mathbf{C}^2(\Omega')$ and the Riemannian metric $d\tau = |dx|/\sqrt{\mu_0(x)}$ be simple and have non-positive curvature in Ω'. Then there exist two constants $C > 0$ and $s_0 \geq 1$ such that*

$$\begin{aligned}[s(\omega_t^2 + |\nabla\omega|^2) + s^3\omega^2]e^{2s\theta(x,t)} \;&\leq\; D_t P(x,t) + \operatorname{div} Q(x,t) \\ &\quad + C[\omega_{tt} - \mu_0(x)\Delta\omega]^2 e^{2s\theta(x,t)}, \quad (x,t) \in Q(\varepsilon_0), \end{aligned} \tag{2.26}$$

for all $s \geq s_0 \geq 1$. Here (P,Q) is a vector-valued function satisfying the following estimate

$$|P(x,t)| + |Q(x,t)| \leq C[s(\omega_t^2 + |\nabla\omega|^2) + s^3\omega^2]e^{2s\theta(x,t)}.$$

Moreover, $P(x,0) = 0$, if $\omega(x,0) = 0$ or $\omega_t(x,0) = 0$, $x \in \Omega(\varepsilon_0)$.

The proof of this lemma directly follows from Theorem 3.2 in paper [24], where the pseudo-convexity of the function $\theta(x,t) = \tau^2(x,y) - \beta\, t^2$ with respect to operator $D_t^2 - \mu_0(x)\Delta$ is proved under the conditions of Lemma 1. Explicit expressions for functions $P(x,t)$ and $Q(x,t)$ are given in [23] by formulae (2.7), (2.8).

From Lemma 1, we can easily derive

Corollary. *Let $\omega' \equiv \omega(\cdot,0) \in \mathbf{H}^2(\Omega(\varepsilon_0))$ and $\mu_0(x)$ satisfy the conditions of Lemma 1. Then there exist two constants $C > 0$ and $s_0 \geq 1$ such that*

$$[s|\nabla\omega'|^2 + s^3\omega'^2]e^{2s\theta(x,0)} \leq \operatorname{div} Q'(x) + C[\Delta\omega']^2 e^{2s\theta(x,0)}, \quad x \in \Omega(\varepsilon_0), \tag{2.27}$$

for all $s \geq s_0 \geq 1$. Here $Q'(x) = Q(x,0)$ is a function satisfying the following estimate

$$|Q'(x)| \leq C[s|\nabla\omega'|^2 + s^3\omega'^2]e^{2s\theta(x,0)}.$$

Lemma 2. *Let $\mu_0(x)$ satisfy the conditions of Lemma 1. Then there exist two constants $C > 0$ and $s_0 \geq 1$ such that*

$$\begin{aligned}\int\limits_{Q(\varepsilon)} [s(\tilde{w}_t^2 + |\nabla\tilde{w}|^2) + s^3\tilde{w}^2]e^{2s\theta(x,t)}\,dx\,dt &\leq Cs^3e^{2s\varepsilon}\|\tilde{w}\|^2_{\mathbf{H}^1(Q(\varepsilon_0))} \\ &\quad + \frac{C}{\sqrt{s}}\int\limits_{\Omega(\varepsilon_0)} \left(|\Delta\tilde{A}(x)|^2 + |\nabla\tilde{A}(x)|^2 + \tilde{p}^2(x) + |\nabla\tilde{p}(x)|^2\right)e^{2s\theta(x,0)}\,dx \\ &\quad + C\int\limits_{\Gamma_0} [s(\tilde{w}_t^2 + |\nabla\tilde{w}|^2) + s^3\tilde{w}^2]e^{2s\theta(x,t)}\,d\Sigma \end{aligned} \tag{2.28}$$

for all $\varepsilon \in (\varepsilon_0, \varepsilon^0)$ and $s \geq s_0 \geq 1$.

Proof. Applying Lemma 1 with $\omega = \tilde{w}(x,t)$ and using the usual technique for deriving Carleman estimates, we obtain the inequality

$$\int\limits_{Q(\varepsilon)} [s(\tilde{w}_t^2 + |\nabla \tilde{w}|^2) + s^3\tilde{w}^2]e^{2s\theta(x,t)}\,dx\,dt \leq Cs^3e^{2s\varepsilon}\|\tilde{w}\|^2_{\mathbf{H}^1(Q(\varepsilon_0))}$$

$$+C\int\limits_{Q(\varepsilon_0)} (|\Delta\tilde{A}(x)|^2 + |\nabla\tilde{A}(x)|^2 + |\nabla\tilde{p}(x)|^2 + \tilde{p}(x)^2)e^{2s\theta(x,t)}\,dx\,dt$$

$$+C\int\limits_{\Gamma_0} [s(\tilde{w}_t^2 + |\nabla \tilde{w}|^2) + s^3\tilde{w}^2]e^{2s\theta(x,t)}\,d\Sigma \qquad (2.29)$$

for all $\varepsilon \in (\varepsilon_0, \varepsilon^0)$ and $s \geq s_0 \geq 1$. Here $d\Sigma$ denotes the surface measure element. Note that

$$\int\limits_{Q(\varepsilon_0)} \left(|\Delta\tilde{A}(x)|^2 + |\nabla\tilde{A}(x)|^2 + \tilde{p}^2(x) + |\nabla\tilde{p}(x)|^2\right)e^{2s\theta(x,t)}\,dx\,dt$$

$$\leq \frac{C}{\sqrt{s}}\int\limits_{\Omega(\varepsilon_0)} \left(|\Delta\tilde{A}(x)|^2 + |\nabla\tilde{A}(x)|^2 + \tilde{p}^2(x) + |\nabla\tilde{p}(x)|^2\right)e^{2s\theta(x,0)}\,dx.$$

Using the latter inequality we can rewrite (2.29) in the form (2.28). $\Box$

Fix $\varepsilon \in (\varepsilon_0, \varepsilon^0)$ and define the numbers ε_j, $j = 1, 2, \ldots, 5$, by the formulae

$$\varepsilon_j = \varepsilon_0 + j\delta, \quad j = 1, 2, \ldots, 5, \quad \varepsilon_5 = \varepsilon, \quad \delta = (\varepsilon - \varepsilon_0)/5.$$

Now we can state the following lemma.

Lemma 3. *Let the conditions of Theorem hold. Then for each $\varepsilon \in (\varepsilon_0, \varepsilon^0)$ there exist two constants $C > 0$ and $s_0 \geq 1$ such that*

$$\int\limits_{\Omega(\varepsilon_2)} |\Delta\tilde{A}(x)|^2 e^{2s\theta(x,0)}\,dx \leq Cs^3e^{2s\varepsilon_1}\|\tilde{w}\|^2_{\mathbf{H}^1(Q(\varepsilon_0))} + Ce^{2s\varepsilon_2}\|\Delta\tilde{A}\|^2_{\mathbf{L}^2(\Omega(\varepsilon_0))}$$

$$+C\int\limits_{\Gamma_0} [s(\tilde{w}_t^2 + |\nabla \tilde{w}|^2) + s^3\tilde{w}^2]e^{2s\theta(x,t)}\,d\Sigma$$

$$+C\int\limits_{\Omega(\varepsilon_0)} \left(|\nabla\tilde{A}(x)|^2 + \tilde{p}^2(x) + |\nabla\tilde{p}(x)|^2\right)e^{2s\theta(x,0)}\,dx \qquad (2.30)$$

for all $s \geq s_0$.

Proof. Let a function $\chi \in \mathbf{C}^\infty(Q(\varepsilon_0))$ satisfy the following requirements

$$0 \leq \chi(x,t) \leq 1, \quad \chi(x,t) = \begin{cases} 1, & (x,t) \in Q(\varepsilon_2), \\ 0, & (x,t) \in Q(\varepsilon_0) \setminus Q(\varepsilon_1), \end{cases} \qquad (2.31)$$

and $z(x,t)=\chi(x,t)\tilde{w}(x,t)e^{s\theta(x,t)}$. We can write an equation for $z(x,t)$ in the form

$$z_{tt}-\mu_0(x)\Delta z=F(x,t), \tag{2.32}$$

where

$$\begin{aligned}F(x,t)=e^{s\theta}\Big\{&2\tilde{w}_t(\chi_t+s\chi\theta_t)-2\mu_0\nabla\tilde{w}\cdot(\nabla\chi+s\chi\nabla\theta)\\&+\tilde{w}[\chi_{tt}-\mu_0\Delta\chi+s\chi(\theta_{tt}-\mu_0\Delta\theta)\\&+2s(\chi_t\theta_t-\mu_0\nabla\chi\cdot\nabla\theta)+s^2\chi(\theta_t^2-\mu_0|\nabla\theta|^2)]\\&-\chi\Big[R(x,0)\tilde{w}_t+R_t(x,0)\tilde{w}-\nabla\mu_0(x)\cdot\nabla\tilde{w}\\&+\int\limits_0^t[R_1(x,t-s)\tilde{w}(x,s)+R_2(x,t-s)\cdot\nabla\tilde{w}(x,s)]\,ds\\&-\operatorname{div}\big(p_1(x)k(t)\nabla\tilde{A}(x)+\tilde{p}(x)\nabla b(x,t)\big)\Big]\Big\}.\end{aligned} \tag{2.33}$$

Using the identity

$$-2z_t\big(z_{tt}-\mu_0(x)\Delta z\big)=-\frac{\partial}{\partial t}\big(z_t^2+\mu_0(x)|\nabla z|^2\big)+\operatorname{div}\big(2\mu_0(x)z_t\nabla z\big)-2z_t\nabla\mu_0(x)\cdot\nabla z, \tag{2.34}$$

and equation (2.32), we can find a constant C such that

$$-\frac{\partial}{\partial t}\big(z_t^2+\mu_0(x)|\nabla z|^2\big)+\operatorname{div}\big(2\mu_0(x)z_t\nabla z\big)\le C(s^{-1}F^2+sz_t^2+s|\nabla z|^2) \tag{2.35}$$

for any $s\ge 1$. Integrating the latter relation over $Q(\varepsilon_0)$ and recalling (2.22), we obtain the inequality

$$\begin{aligned}\int\limits_{\Omega(\varepsilon_0)}[\chi(x,0)\tilde{q}(x)]^2e^{2s\theta(x,0)}\,dx\le C\int\limits_{Q(\varepsilon_0)}(s^{-1}F^2+sz_t^2+s|\nabla z|^2)\,dx\,dt\\+C\int\limits_{\Gamma_0}(z_t^2+|\nabla z|^2)\,d\Sigma.\end{aligned} \tag{2.36}$$

Note that for $s\ge 1$ we have

$$\begin{aligned}\int\limits_{Q(\varepsilon_0)}(s^{-1}F^2+sz_t^2+s|\nabla z|^2)\,dx\,dt\le C\int\limits_{Q(\varepsilon_1)}[s(\tilde{w}_t^2+|\nabla\tilde{w}|^2)+s^3\tilde{w}^2]e^{2s\theta(x,t)}\,dx\,dt\\+\frac{C}{\sqrt{s^3}}\int\limits_{\Omega(\varepsilon_0)}\big(|\Delta\tilde{A}(x)|^2+|\nabla\tilde{A}(x)|^2+\tilde{p}^2(x)+|\nabla\tilde{p}(x)|^2\big)e^{2s\theta(x,0)}\,dx,\end{aligned} \tag{2.37}$$

$$\int\limits_{\Gamma_0}(z_t^2+|\nabla z|^2)\,d\Sigma\le C\int\limits_{\Gamma_0}(s\tilde{w}_t^2+s|\nabla\tilde{w}|^2+s^3\tilde{w}^2)e^{2s\theta(x,t)}\,d\Sigma. \tag{2.38}$$

Using inequality (2.28) with $\varepsilon = \varepsilon_1$ for an estimation of the first term on the right-hand side of (2.37), we obtain

$$\int\limits_{\Omega(\varepsilon_0)} [\chi(x,0)\tilde{q}(x)]^2 e^{2s\theta(x,0)}\,dx \le Cs^3 e^{2s\varepsilon_1}\|\tilde{w}\|^2_{\mathbf{H}^1(Q(\varepsilon_0))}$$
$$+\frac{C}{\sqrt{s}}\int\limits_{\Omega(\varepsilon_0)} \big(|\Delta\tilde{A}(x)|^2 + |\nabla\tilde{A}(x)|^2 + \tilde{p}^2(x) + |\nabla\tilde{p}(x)|^2\big)e^{2s\theta(x,0)}\,dx$$
$$+C\int\limits_{\Gamma_0}[s(\tilde{w}_t^2 + |\nabla\tilde{w}|^2) + s^3\tilde{w}^2]e^{2s\theta(x,t)}\,d\Sigma \qquad (2.39)$$

for all $s \ge s_0 \ge 1$. On the other hand, recalling that

$$\tilde{q}(x) = \mu_0(x)\Delta\tilde{A}(x) + \nabla\mu_0(x)\cdot\nabla\tilde{A}(x) + \tilde{p}(x)\cdot a(x) + \nabla\tilde{p}(x)\cdot\nabla a(x)$$

(see (2.12)), we have

$$\int\limits_{\Omega(\varepsilon_0)} [\chi(x,0)\tilde{q}(x)]^2 e^{2s\theta(x,0)}\,dx \ge \int\limits_{\Omega(\varepsilon_2)} \tilde{q}^2(x)e^{2s\theta(x,0)}\,dx \ge \mu_{00}\int\limits_{\Omega(\varepsilon_2)} |\Delta\tilde{A}(x)|^2 e^{2s\theta(x,0)}\,dx$$
$$-C\int\limits_{\Omega(\varepsilon_0)} (|\nabla\tilde{A}(x)|^2 + |\nabla\tilde{p}(x)|^2 + |\tilde{p}(x)|^2)e^{2s\theta(x,0)}\,dx. \qquad (2.40)$$

Hence, formula (2.39) leads to the following estimate

$$\int\limits_{\Omega(\varepsilon_2)} |\Delta\tilde{A}(x)|^2 e^{2s\theta(x,0)}\,dx \le Cs^3 e^{2s\varepsilon_1}\|\tilde{w}\|^2_{\mathbf{H}^1(Q(\varepsilon_0))} + \frac{C}{\sqrt{s}}\int\limits_{\Omega(\varepsilon_0)} |\Delta\tilde{A}(x)|^2 e^{2s\theta(x,0)}\,dx$$
$$+C\int\limits_{\Omega(\varepsilon_0)} \big(|\nabla\tilde{A}(x)|^2 + \tilde{p}^2(x) + |\nabla\tilde{p}(x)|^2\big)e^{2s\theta(x,0)}\,dx$$
$$+C\int\limits_{\Gamma_0}[s(\tilde{w}_t^2 + |\nabla\tilde{w}|^2) + s^3\tilde{w}^2]e^{2s\theta(x,t)}\,d\Sigma. \qquad (2.41)$$

Since

$$\int\limits_{\Omega(\varepsilon_0)} |\Delta\tilde{A}(x)|^2 e^{2s\theta(x,0)}\,dx = \int\limits_{\Omega(\varepsilon_2)} |\Delta\tilde{A}(x)|^2 e^{2s\theta(x,0)}\,dx + \int\limits_{\Omega(\varepsilon_0)\setminus\Omega(\varepsilon_2)} |\Delta\tilde{A}(x)|^2 e^{2s\theta(x,0)}\,dx$$
$$\le \int\limits_{\Omega(\varepsilon_2)} |\Delta\tilde{A}(x)|^2 e^{2s\theta(x,0)}\,dx + e^{2s\varepsilon_2}\int\limits_{\Omega(\varepsilon_0)\setminus\Omega(\varepsilon_2)} |\Delta\tilde{A}(x)|^2\,dx,$$

from formula (2.37) follows (2.30) for all sufficiently large s. $\square$

Now we need the following lemma.

Lemma 4. *Under the conditions of Lemma 1 there exist two constants $C > 0$ and $s_0 \geq 1$ such that*

$$\int\limits_{\Omega(\varepsilon_3)} \Big[s|\nabla\tilde{A}|^2 + s^3\tilde{A}^2\Big] e^{2s\theta(x,0)}\, dx \leq C \int\limits_{\Omega(\varepsilon_2)} |\Delta\tilde{A}(x)|^2 e^{2s\theta(x,0)}\, dx + Ce^{2s\varepsilon_3}\|\tilde{A}\|^2_{\mathbf{H}^1(\Omega(\varepsilon_0))} \tag{2.42}$$

for all $s \geq s_0$.

Proof. Introduce the function $\hat{A}(x) = \chi_1(x)\tilde{A}(x)$, where $\chi_1 \in \mathbf{C}^\infty(\Omega(\varepsilon_0))$ satisfies the following requirements

$$0 \leq \chi_1(x) \leq 1, \quad \chi_1(x) = \begin{cases} 1, & x \in \Omega(\varepsilon_3), \\ 0, & x \in \Omega(\varepsilon_0) \setminus \Omega(\varepsilon_2). \end{cases} \tag{2.43}$$

Note that $\operatorname{supp} \hat{A}(x)$ belongs to the open set $\overline{\Omega}(\varepsilon_2) \cap \Omega$. Applying the corollary of Lemma 1 with $\omega' = \hat{A}(x)$, we conclude that there exist two constants C and $s_0 \geq 1$ such that

$$\big(s|\nabla\hat{A}|^2 + s^3\hat{A}^2\big)e^{2s\theta(x,0)} \leq C[\operatorname{div} Q'(x) + (\Delta\hat{A})^2 e^{2s\theta(x,0)}], \tag{2.44}$$

where the function $Q'(x)$ satisfies the estimate

$$|Q'(x)| \leq C[s|\nabla\hat{A}|^2 + s^3\hat{A}^2]e^{2s\theta(x,0)}$$

and $\operatorname{supp} Q'(x) \subset \overline{\Omega}(\varepsilon_2) \cap \Omega$. Integrating (2.44) over $\Omega(\varepsilon_0)$, we obtain

$$\int\limits_{\Omega(\varepsilon_3)} \Big[s|\nabla\tilde{A}|^2 + s^3\tilde{A}^2\Big] e^{2s\theta(x,0)}\, dx \leq C \int\limits_{\Omega(\varepsilon_2)} (\Delta\hat{A})^2 e^{2s\theta(x,0)}\, dx. \tag{2.45}$$

On the other hand,

$$\Delta\hat{A} \;=\; \chi_1\Delta\tilde{A} + 2\nabla\chi_1\cdot\nabla\tilde{A} + \tilde{A}\Delta\chi_1, \quad x \in \Omega(\varepsilon_2). \tag{2.46}$$

Hence,

$$\begin{aligned} \int\limits_{\Omega(\varepsilon_3)} [s|\nabla\tilde{A}|^2 + s^3\tilde{A}^2] e^{2s\theta(x,0)}\, dx \;&\leq\; C \int\limits_{\Omega(\varepsilon_2)} |\Delta\tilde{A}|^2 e^{2s\theta(x,0)}\, dx \\ &\quad + C \int\limits_{\Omega(\varepsilon_2)\setminus\Omega(\varepsilon_3)} (|\nabla\tilde{A}|^2 + \tilde{A}^2)e^{2s\theta(x,0)}\, dx. \end{aligned} \tag{2.47}$$

Therefore estimate (2.42) follows. $\square$

An estimate for $\tilde{p}$ via $\tilde{A}$ is given by the following lemma.

Lemma 5. *Let the condition (1.6) of Theorem hold. Then there exist two constants $C > 0$ and $s_0 \geq 1$ such that*

$$\begin{aligned} \int\limits_{\Omega(\varepsilon_4)} \Big(s^2|\nabla\tilde{p}|^2 + s^4\tilde{p}^2\Big) e^{2s\theta(x,0)}\, dx \;&\leq\; Cs^3 e^{2s\varepsilon_4}\|\tilde{p}\|^2_{\mathbf{H}^1(\Omega(\varepsilon_0))} \\ &\quad + C \int\limits_{\Omega(\varepsilon_3)} \Big(s|\nabla\tilde{A}|^2 + s^3\tilde{A}^2 + s^3\tilde{p}^2\Big) e^{2s\theta(x,0)}\, dx \end{aligned} \tag{2.48}$$

for all $s \geq s_0$.

Proof. From (2.2) we have

$$\nabla \tilde{p} \cdot \nabla \phi + \tilde{p} \Delta \phi = \tilde{A}. \tag{2.49}$$

Set $\chi_2(x)\tilde{p}(x) = \bar{p}(x)$, $\chi_2(x)\tilde{A}(x) = \bar{A}(x)$, $\theta(x,0) = \theta_0(x)$, where $\chi_2 \in \mathbf{C}^\infty(\Omega(\varepsilon_0))$ satisfy the following requirements

$$0 \leq \chi_2(x) \leq 1, \quad \chi_2(x) = \begin{cases} 1, & x \in \Omega(\varepsilon_4), \\ 0, & x \in \Omega(\varepsilon_0) \setminus \Omega(\varepsilon_3). \end{cases} \tag{2.50}$$

Then $\bar{p}$ satisfies the equation

$$\nabla \bar{p} \cdot \nabla \phi + \bar{p} \Delta \phi = (\bar{A} + \tilde{p} \nabla \chi_2 \cdot \nabla \phi). \tag{2.51}$$

Multiplying both sides of the above equation by $-2e^{2s\theta_0(x)}\bar{p}/s$, one has

$$-\frac{1}{s}\mathrm{div}(\bar{p}^2 e^{2s\theta_0} \nabla \phi) + \bar{p}^2 e^{2s\theta_0}(2\nabla\theta_0 \cdot \nabla \phi - \frac{1}{s}\Delta\phi) = -\frac{2}{s}\bar{p}(\bar{A} + \tilde{p}\nabla\chi_2 \cdot \nabla\phi)e^{2s\theta_0}. \tag{2.52}$$

According to condition (1.6), we have

$$\nabla\theta_0(x) \cdot \nabla\phi(x) \geq \gamma > 0, \quad x \in \Omega(\varepsilon_0).$$

Then for sufficiently large s one has

$$\bar{p}^2 e^{2s\theta_0} \leq \frac{C}{s}\mathrm{div}(\bar{p}^2 e^{2s\theta_0}\nabla\phi) + \frac{C}{s}\Big(\bar{A}^2 e^{2s\theta_0} + \tilde{p}^2 e^{2s\varepsilon_4} + \bar{p}^2 e^{2s\theta_0}\Big). \tag{2.53}$$

Assuming that $s > C$, we absorb the last summand in the above equation to obtain

$$\bar{p}^2 e^{2s\theta_0} \leq \frac{C}{s}\mathrm{div}(\bar{p}^2 e^{2s\theta_0}\nabla\phi) + \frac{C}{s}\Big[\bar{A}^2 e^{2s\theta_0} + \tilde{p}^2 e^{2s\varepsilon_4}\Big]. \tag{2.54}$$

Similarly, we have

$$\nabla\bar{p}_{x_i} \cdot \nabla\phi + \bar{p}_{x_i}\Delta\phi = (\bar{A} + \tilde{p}\nabla\chi_2 \cdot \nabla\phi)_{x_i} - \nabla\bar{p} \cdot \nabla\phi_{x_i} - \bar{p}\Delta\phi_{x_i}, \quad i = 1, 2, \tag{2.55}$$

$$\begin{aligned} &-\frac{1}{s}\mathrm{div}(\bar{p}_{x_i}^2 e^{2s\theta_0}\nabla\phi) + \bar{p}_{x_i}^2 e^{2s\theta_0}(2\nabla\theta_0 \cdot \nabla\phi - \frac{1}{s}\Delta\phi) \\ &\qquad = -\frac{2}{s}\bar{p}_{x_i}\Big[(\bar{A} + \tilde{p}\nabla\chi_2 \cdot \nabla\phi)_{x_i} - \nabla\bar{p} \cdot \nabla\phi_{x_i} - \bar{p}\Delta\phi_{x_i}\Big]e^{2s\theta_0}, \ i = 1, 2. \end{aligned} \tag{2.56}$$

From (2.56), for sufficiently large s, one finds the estimate

$$|\nabla\bar{p}|^2 e^{2s\theta_0} \leq \frac{C}{s}\mathrm{div}\big(|\nabla\bar{p}|^2 e^{2s\theta_0}\nabla\phi\big) + \frac{C}{s}\big[(|\nabla\bar{A}|^2 + \bar{p}^2)e^{2s\theta_0(x)} + (|\nabla\tilde{p}|^2 + \tilde{p}^2)e^{2s\varepsilon_4}\big]. \tag{2.57}$$

Integrating inequalities (2.54), (2.57) over $\Omega(\varepsilon_0)$, we find that all the divergence terms vanish and we obtain the following estimates

$$\begin{aligned}\int\limits_{\Omega(\varepsilon_4)} \tilde{p}^2 e^{2s\theta_0(x)}\,dx \ &\leq \int\limits_{\Omega(\varepsilon_0)} \bar{p}^2 e^{2s\theta_0(x)}\,dx \leq \frac{C}{s} \int\limits_{\Omega(\varepsilon_3)} \left[\bar{A}^2 e^{2s\theta_0(x)} + \tilde{p}^2 e^{2s\varepsilon_4}\right] dx \\ &\leq \frac{C}{s} \int\limits_{\Omega(\varepsilon_3)} \left[\tilde{A}^2 e^{2s\theta_0(x)} + \tilde{p}^2 e^{2s\varepsilon_4}\right] dx, \end{aligned} \tag{2.58}$$

$$\begin{aligned}\int\limits_{\Omega(\varepsilon_4)} |\nabla\tilde{p}|^2 e^{2s\theta_0(x)}\,dx &\leq \int\limits_{\Omega(\varepsilon_0)} |\nabla\bar{p}|^2 e^{2s\theta_0(x)}\,dx \\ &\leq \frac{C}{s} \int\limits_{\Omega(\varepsilon_3)} \left[(|\nabla\bar{A}|^2 + \bar{p}^2) e^{2s\theta_0(x)} + (|\nabla\tilde{p}|^2 + \tilde{p}^2) e^{2s\varepsilon_4}\right] dx \\ &\leq \frac{C}{s} \int\limits_{\Omega(\varepsilon_3)} \left[(|\nabla\tilde{A}|^2 + |\tilde{A}|^2 + \tilde{p}^2) e^{2s\theta_0(x)} + (|\nabla\tilde{p}|^2 + \tilde{p}^2) e^{2s\varepsilon_4}\right] dx. \end{aligned} \tag{2.59}$$

Therefore the inequality (2.48) follows. □

Now we derive other estimate of the function $\tilde{p}(x)$:

Lemma 6. *Let the conditions of Theorem hold. Then there exist two constants $C > 0$ and $s_0 \geq 1$ such that*

$$\begin{aligned}\int\limits_{\Omega(\varepsilon)} \left(s^2|\nabla\tilde{p}|^2 + s^4\tilde{p}^2\right) e^{2s\theta(x,0)}\,dx &\leq C s^3 e^{2s\varepsilon_4}\left(\|\tilde{w}\|^2_{\mathbf{H}^1(Q(\varepsilon_0))} + \|\tilde{p}\|^2_{\mathbf{H}^3(Q(\varepsilon_0))}\right) \\ &+ C\int\limits_{\Gamma_0} [s(\tilde{w}_t^2 + |\nabla\tilde{w}|^2) + s^3\tilde{w}^2] e^{2s\theta(x,t)}\,d\Sigma \end{aligned} \tag{2.60}$$

for all $s \geq s_0$.

Proof. Recall that $\varepsilon = \varepsilon_5 = \varepsilon_4 + \delta$, and use estimates given by Lemma 4 and Lemma 5. From inequalities (2.48), (2.42) we obtain

$$\begin{aligned}\int\limits_{\Omega(\varepsilon_4)} \left(s^2|\nabla\tilde{p}|^2 + s^4\tilde{p}^2\right) e^{2s\theta(x,0)}\,dx \ &\leq \ C \int\limits_{\Omega(\varepsilon_2)} (|\Delta\tilde{A}(x)|^2 + s^3\tilde{p}^2) e^{2s\theta(x,0)}\,dx \\ &+ Ce^{2s\varepsilon_3}\|\tilde{A}\|^2_{\mathbf{H}^1(\Omega(\varepsilon_0))} + Ce^{2s\varepsilon_4}\|\tilde{p}\|^2_{\mathbf{H}^1(\Omega(\varepsilon_0))}. \end{aligned} \tag{2.61}$$

Combining (2.42) and (2.61), we obtain

$$\begin{aligned}&\int\limits_{\Omega(\varepsilon_4)} \left[s(|\nabla\tilde{A}|^2 + s|\nabla\tilde{p}|^2) + s^3(\tilde{A}^2 + s\tilde{p}^2)\right] e^{2s\theta(x,0)}\,dx \\ &\leq C \int\limits_{\Omega(\varepsilon_2)} (|\Delta\tilde{A}(x)|^2 + s^3\tilde{p}^2) e^{2s\theta(x,0)}\,dx + Ce^{2s\varepsilon_4}(\|\tilde{A}\|^2_{\mathbf{H}^1(\Omega(\varepsilon_0))} + \|\tilde{p}\|^2_{\mathbf{H}^1(\Omega(\varepsilon_0))}). \end{aligned} \tag{2.62}$$

Applying now Lemma 3, we obtain

$$\int_{\Omega(\varepsilon_4)} \Big[s(|\nabla\tilde{A}|^2 + s|\nabla\tilde{p}|^2) + s^3(\tilde{A}^2 + s\tilde{p}^2)\Big] e^{2s\theta(x,0)}\,dx$$
$$\leq C \int_{\Omega(\varepsilon_0)} \big(|\nabla\tilde{A}(x)|^2 + s^3\tilde{p}^2(x) + |\nabla\tilde{p}(x)|^2\big) e^{2s\theta(x,0)}\,dx$$
$$+C\int_{\Gamma_0} [s(\tilde{w}_t^2 + |\nabla\tilde{w}|^2) + s^3\tilde{w}^2] e^{2s\theta(x,t)}\,d\Sigma$$
$$+Ce^{2s\varepsilon_4}(\|\tilde{w}\|^2_{\mathbf{H}^1(\Omega(\varepsilon_0))} + \|\tilde{A}\|^2_{\mathbf{H}^2(\Omega(\varepsilon_0))} + \|\tilde{p}\|^2_{\mathbf{H}^1(\Omega(\varepsilon_0))}). \quad (2.63)$$

Decompose the first integral on the right-hand side into two integrals: the first integral over $\Omega(\varepsilon_4)$ and the second over $\Omega(\varepsilon_0)\setminus\Omega(\varepsilon_4)$. Absorbing for sufficiently large s the integral over $\Omega(\varepsilon_4)$ on the left-hand side of the inequality and noting that $\theta(x,0)\leq\varepsilon_4$ for $x\in\Omega(\varepsilon_0)\setminus\Omega(\varepsilon_4)$, we obtain

$$\int_{\Omega(\varepsilon_4)} \Big[s(|\nabla\tilde{A}|^2 + s|\nabla\tilde{p}|^2) + s^3(\tilde{A}^2 + s\tilde{p}^2)\Big] e^{2s\theta(x,0)}\,dx$$
$$\leq C\int_{\Gamma_0} [s(\tilde{w}_t^2 + |\nabla\tilde{w}|^2) + s^3\tilde{w}^2] e^{2s\theta(x,t)}\,d\Sigma$$
$$+Cs^3e^{2s\varepsilon_4}(\|\tilde{w}\|^2_{\mathbf{H}^1(\Omega(\varepsilon_0))} + \|\tilde{A}\|^2_{\mathbf{H}^2(\Omega(\varepsilon_0))} + \|\tilde{p}\|^2_{\mathbf{H}^1(\Omega(\varepsilon_0))}). \quad (2.64)$$

Noting that

$$\|\tilde{A}\|^2_{\mathbf{H}^2(\Omega(\varepsilon_0))} + \|\tilde{p}\|^2_{\mathbf{H}^1(\Omega(\varepsilon_0))} \leq C\|\tilde{p}\|^2_{\mathbf{H}^3(\Omega(\varepsilon_0))}, \quad (2.65)$$

and $\Omega(\varepsilon)\subset\Omega(\varepsilon_4)$, we obtain (2.60). $\square$

Now we can obtain the final estimate for $\tilde{p}$. Using inequality (2.60) and the boundary data (2.23), (2.24), we have the following inequality

$$e^{2s\varepsilon}\|\tilde{p}\|^2_{\mathbf{H}^1(\Omega(\varepsilon))} \leq Cs^2e^{2s\varepsilon_4}\Big(\|\tilde{p}\|^2_{\mathbf{H}^1(\Omega(\varepsilon_0))} + \|\tilde{w}\|^2_{\mathbf{H}^1(Q(\varepsilon_0))}\Big)$$
$$+Cs^2e^{2s\varepsilon^0}\Big(\|\tilde{G}\|^2_{\mathbf{H}^1(\Gamma_0)} + \|\tilde{H}\|^2_{\mathbf{L}^2(\Gamma_0)}\Big). \quad (2.66)$$

Since $\varepsilon-\varepsilon_4=\delta$,

$$\|\tilde{p}\|^2_{\mathbf{H}^1(\Omega(\varepsilon))} \leq Cs^2e^{-2s\delta}\Big(\|\tilde{p}\|^2_{\mathbf{H}^1(\Omega(\varepsilon_0))} + \|\tilde{w}\|^2_{\mathbf{H}^1(Q(\varepsilon_0))}\Big)$$
$$+Cs^2e^{2s(\varepsilon^0-\varepsilon)}\Big(\|\tilde{G}\|^2_{\mathbf{H}^1(\Gamma_0)} + \|\tilde{H}\|^2_{\mathbf{L}^2(\Gamma_0)}\Big). \quad (2.67)$$

Note that

$$\|\tilde{G}\|^2_{\mathbf{H}^1(\Gamma_0)} + \|\tilde{H}\|^2_{\mathbf{L}^2(\Gamma_0)} \leq C\Big(\|D_t^{(4)}\tilde{g}\|^2_{\mathbf{H}^1(\Gamma_0)} + \|D_t^{(4)}\tilde{h}\|^2_{\mathbf{L}^2(\Gamma_0)}\Big) \equiv C\epsilon^2. \quad (2.68)$$

Using the assumptions in Theorem, we find that

$$\|\tilde{p}\|^2_{\mathbf{H}^1(\Omega(\varepsilon))} \le Cs^2\Big(e^{-2s\delta}(p_0^2+u_0^2)+e^{2s(\varepsilon^0-\varepsilon)}\epsilon^2\Big) \tag{2.69}$$

for all s larger than some $s_0 \ge 1$. Let $\epsilon^2 < (p_0^2+u_0^2)$. Choose s^* as a root of the equation

$$e^{-2s\delta}(p_0^2+u_0^2) = e^{2s(\varepsilon^0-\varepsilon)}\epsilon^2,$$

i.e.,

$$s^* = \frac{1}{2(\varepsilon^0-\varepsilon+\delta)}\ln\frac{p_0^2+u_0^2}{\epsilon^2} = \frac{5}{2(5\varepsilon^0-4\varepsilon-\varepsilon_0)}\ln\frac{p_0^2+u_0^2}{\epsilon^2}.$$

Observe that $s^* \ge s_0$ if ϵ is chosen small enough. Then

$$\|\tilde{p}\|^2_{\mathbf{H}^1(\Omega(\varepsilon))} \le C(s^*)^2 e^{2s^*(\varepsilon^0-\varepsilon)}\epsilon^2 = C(s^*)^2(p_0^2+u_0^2)^\gamma\epsilon^{2(1-\gamma)}, \tag{2.70}$$

where

$$\gamma = \frac{5(\varepsilon^0-\varepsilon)}{5\varepsilon^0-4\varepsilon-\varepsilon_0} \in (0,1).$$

Therefore (2.70) proves estimate (1.8) of Theorem.

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