UTMS 2009–17 July 29, 2009

A new unicity theorem and Erdös' problem for polarized semi-abelian varieties

by

Pietro CORVAJA and Junjiro NOGUCHI

UNIVERSITY OF TOKYO
GRADUATE SCHOOL OF MATHEMATICAL SCIENCES
KOMABA, TOKYO, JAPAN

A New Unicity Theorem and Erdös' Problem for Polarized Semi-Abelian Varieties *

Pietro Corvaja and Junjiro Noguchi

Abstract

In 1988 P. Erdös asked if the prime divisors of $x^n - 1$ for all $n = 1, 2, \ldots$ determine the given integer x; the problem was affirmatively answered by Corrales-Rodorigáñez and R. Schoof [2] in 1997 together with its elliptic version. Analogously, K. Yamanoi [14] proved in 2004 that the support of the pull-backed divisor f^*D of an ample divisor on an abelian variety A by an algebraically non-degenerate entire holomorphic curve $f : \mathbf{C} \to A$ essentially determines the pair (A, D).

By making use of the main theorem of [10] we here deal with this problem for semi-abelian varieties: namely, given two polarized semi-abelian varieties (A_1, D_1), (A_2, D_2) and entire non-degenerate holomorphic curves $f_i : \mathbf{C} \to A_i$, $i = 1, 2$, we classify the cases when the inclusion $\mathrm{Supp} f_1^* D_1 \subset \mathrm{Supp} f_2^* D_2$ holds. We also apply the main result of [4] to prove an arithmetic counterpart.

1 Introduction and main results.

The purpose of this paper is to study a kind of unicity problem for semi-abelian varieties with given divisors (polarization) in terms of entire holomorphic curves and of arithmetic recurrences.

Let A_i $(i = 1, 2)$ be semi-abelian varieties and let D_i be reduced divisors on A_i. Just for the sake of simplicity, we assume here that D_i $(i = 1, 2)$ are irreducible and have trivial stabilizers

$$\mathrm{Stab}(D_i) = \{0\}, \quad i = 1, 2,$$

(see §2 for the notation). Note that these assumptions are not restrictive by a reduction argument: Cf. §4 for the general case.

Our first result is as follows (cf. §2 for the notation):

*Research supported in part by Grant-in-Aid for Scientific Research (S) 17104001.

Theorem 1.1. *Let $f_i : \mathbf{C} \to A_i$ $(i = 1, 2)$ be non-degenerate holomorphic curves.*

(i) *Assume that*

$$\underline{\mathrm{Supp} f_1^* D_1}_\infty \subset \underline{\mathrm{Supp} f_2^* D_2}_\infty, \tag{1.2}$$

$$N_1(r, f_1^* D_1) \sim N_1(r, f_2^* D_2)||. \tag{1.3}$$

Then there is a (finite) étale morphism $\phi : A_1 \to A_2$ such that $\phi \circ f_1 = f_2$ and $D_1 \subset \phi^* D_2$.

(ii) *If equality holds in* (1.2)*, then $\phi : A_1 \to A_2$ of* (i) *is an isomorphism and $D_1 = \phi^* D_2$.*

(iii) *If A_i $(i = 1, 2)$ are abelian varieties and D_2 is smooth or more generally locally irreducible at every point of D_2, then ϕ of* (i) *is an isomorphism and $D_1 = \phi^* D_2$.*

N.B. (i) When A_i are abelian varieties $(i = 1, 2)$, the above (ii) is K. Yamanoi's Unicity Theorem ([14], Theorem 3.2.1).

(ii) Because of the proof, assumption (1.2) can be replaced by the following estimate:

$$N_1(r, f_1^* D_1) - N_1(r, f_1^* D_1 \cap f_2^* D_2) = o(N_1(r, f_1^* D_1))||.$$

(iii) Assumption (1.3) is necessary (cf. Example 3.6).

The following corollary follows immediately from Theorem 1.1.

Corollary 1.4. (i) *Let $f_1 : \mathbf{C} \to \mathbf{C}^*$ and $f_2 : \mathbf{C} \to A_2$ with an elliptic curve A_2 be holomorphic and non-constant. Then $\underline{f_1^{-1}\{1\}}_\infty \neq \underline{f_2^{-1}\{0\}}_\infty$, where 0 is the zero element of A_2.*

(ii) *If $\dim A_1 \neq \dim A_2$ in Theorem 1.1, then $\underline{f_1^{-1} D_1}_\infty \neq \underline{f_2^{-1} D_2}_\infty$.*

N.B. (i) The first statement means that the difference of the value distribution property caused by the quotient $\mathbf{C}^* \to \mathbf{C}^*/\langle\tau\rangle = A_2$ can not be recovered by the choice of $f_1 : \mathbf{C} \to \mathbf{C}^*$ and $f_2 : \mathbf{C} \to A_2$, even though f_i are allowed to be arbitrarily transcendental.

(ii) The second statement implies that the distribution of $f_i^{-1} D_i$ about ∞ contains the topological information of $\dim A_i$; it is interesting to observe that this works even for one parameter subgroups with Zariski dense image.

Due to the well-known correspondence between Number Theory and Nevanlinna Theory, it is tempting to give a number-theoretic analogue of Theorem 1.1. In 1988, Pál Erdös raised the following problem:

Erdös' Problem. *Let x, y be positive integers with the property that for all positive integers n the set of prime numbers dividing $x^n - 1$ is equal to the set of prime numbers dividing $y^n - 1$. Is then $x = y$?*

In [2] Corrales-Rodorigáñez and R. Schoof solved this problem, by proving that if, for every n, each prime dividing $x^n - 1$ divides also $y^n - 1$, then y is a power of x. They also solved the natural elliptic analogue, for two (elliptic) recurrences in a same elliptic curve.

A related problem asks to classify the cases where $x^n - 1$ divides $y^n - 1$; in [3], it was proved that the same conclusion holds (i.e. y is a power of x) under the hypothesis that $x^n - 1$ divides $y^n - 1$ for infinitely many positive integers n. Note that the conclusion that y is a power of x can be translated in geometric terms by saying that there exists an isogeny $\phi : \mathbf{G}_m \to \mathbf{G}_m$ such that $\phi(x) = y$ and $\phi^*\{1\} \supset \{1\}$.

The natural generalization to several variables is represented by Pisot's problem, asking to characterize the pairs of linear recurrent sequences $(n \mapsto \mathbf{f}_1(n)), (n \mapsto \mathbf{f}_2(n))$ such that $\mathbf{f}_1(n)$ divides $\mathbf{f}_2(n)$ for every integer n (or for infinitely many integers n). We shall explain in §2 why these problems can be viewed as some analogue of the unicity problem for holomorphic maps to semi-abelian varieties. A first case (where it is assumed that the divisibility holds for every integer n) was solved by van der Poorten in [11] (see also [12]), while in [4] this was generalized to the case when the divisibility is assumed to hold just for infinitely many integers n.

We would like to deal with the case of a semi-abelian variety with a given divisor, i.e., a polarized semi-abelian variety. As it often happens, the complex-analytic theory is more advanced, and we dispose only of partial results in the arithmetic case. In the present situation, we can prove an analogue of Theorem 1.1 in the toric case, but not in the general case of semi-abelian varieties, that is left to be a *Conjecture*.

Here is our result in the arithmetic case; again, the notation is explained in §2:

Theorem 1.5. *Let $\mathcal{O}_S$ be a ring of S-integers in a number field k. Let $\mathbf{G}_1$ and $\mathbf{G}_2$ be linear tori, and let $g_i \in \mathbf{G}_i(\mathcal{O}_S)$ be elements generating Zariski-dense subgroups in $\mathbf{G}_i$ $(i = 1, 2)$. Let D_i be reduced divisors defined over k, with defining ideals $\mathcal{I}(D_i)$, such that each irreducible component has a finite stabilizer and the stabilizer of D_2 is trivial. Suppose that for infinitely many natural numbers $n \in \mathbf{N}$, the inclusion of ideals*

$$(g_1^n)^*\mathcal{I}(D_1) \supset (g_2^n)^*\mathcal{I}(D_2) \tag{1.6}$$

holds. Then there exists an étale morphism $\phi : \mathbf{G}_1 \to \mathbf{G}_2$, defined over k, and a positive integer h such that $\phi(g_1^h) = g_2^h$ and $D_1 \subset \phi^*(D_2)$.

(Note that inclusion of ideals (1.6) is a stronger version of the inclusion (1.2), although it apparently goes in the opposite direction!)

N.B. (i) Theorem 1.5 is essentially due to the main results of [4]. We shall derive it from Corollary 1 of [4].

(ii) Example 5.7 will show that we cannot take $h = 1$ in general.

(iii) By Example 5.8, the condition on the finiteness of the stabilizer of each component of D_1 and D_2 cannot be omitted, nor can the condition on the triviality of the stabilizer of D_2. However, it is easy to derive from our proof a nontrivial consequence even in the general case, see Theorem 5.10.

(iv) We assume that inequality (inclusion) (1.6) of ideals holds only for an infinite sequence of n, not necessarily for all large n. On the contrary, we need the inequality of ideals, not only of their *supports*, i.e. of the primes containing the corresponding ideals.

One might ask for a similar conclusion assuming only the inequality of supports,

$$\mathrm{Supp}\,(g_1^n)^*\mathcal{I}(D_1) \subset \mathrm{Supp}\,(g_2^n)^*\mathcal{I}(D_2), \tag{1.7}$$

but we then need (1.7) to hold for *every* exponent n (this is in analogy with Erdös' original problem). In this case, using a work of Barski, Bézivin and Schinzel [1], we will show some result which is slightly weaker than the analogue of the above theorems (see Proposition 6.1).

Acknowledgement. The present joint research was initiated when the authors shared a time at the Fields Institute, Toronto, participating in the Thematic Program on Arithmetic Geometry, Hyperbolic Geometry and Related Topics, July-December, 2008. The second author had a useful discussion on the difference of divisors on abelian and semi-abelian varieties with our colleague, Professor Jörg Winkelmann. We would like to express our sincere gratitude to the Fields Institute, and Professor J. Winkelmann.

2 Notation.

In this paragraph we introduce the necessary notation, both in the analytic setting (a), and in the arithmetic one (b).

(a) A subset $Z \subset \mathbf{C}$ determines a germ, denoted by $\underline{Z}_\infty$, of subsets at the infinity of the Riemann sphere $\mathbf{C} \cup \{\infty\}$. For two subsets $Z_i \subset \mathbf{C}, i = 1, 2$, the relation

$$\underline{Z_1}_\infty \supset \underline{Z_2}_\infty$$

makes sense: it means there is a number $r_0 > 0$ such that

$$Z_1 \cap \{z \in \mathbf{C}; |z| > r_0\} \supset Z_2 \cap \{z \in \mathbf{C}; |z| > r_0\}.$$

For two functions $\phi_i(r) \geq 0, r > 0, i = 1, 2$, we write

$$\phi_1(r) \leq \phi_2(r)||,$$

if there is an exceptional (Borel) subset $E \subset (0, \infty)$ with a finite measure $m(E) < \infty$, satisfying

$$\phi_1(r) \leq \phi_2(r), \quad r \in (0, \infty) \setminus E.$$

If there is a positive constant $C > 0$ satisfying $\phi_1(r) \leq C\phi_2(r)||$, we write

$$\phi_1(r) = O(\phi_2(r))||.$$

If $\phi_1(r) \leq \epsilon\phi_2(r)||$ for an arbitrary $\epsilon > 0$, we write

$$\phi_1(r) = o(\phi_2(r))||.$$

Here, it is noted that the exceptional subset in $\phi_1(r) \leq \epsilon\phi_2(r)||$ may depend on $\epsilon > 0$. We write

$$\phi_1(r) \sim \phi_2(r)||, \tag{2.1}$$

if $\phi_1(r) = O(\phi_2(r))||$ and $\phi_2(r) = O(\phi_1(r))||$. If the exceptional subset of $(0, \infty)$ is empty in the above expressions, we will simply drop the symbol "$||$".

A complex algebraic group A admitting a representation

$$0 \to \mathbf{G}_m^t \to A \to A_0 \to 0,$$

where $\mathbf{G}_m = \mathbf{C} \setminus \{0\}$ is the multiplicative group and A_0 is an abelian variety, is called a *semi-abelian variety*. Let A be a semi-abelian variety and let B be an algebraic subset of A. Set

$$\mathrm{Stab}(B) = \{x \in A; x + B = B\},$$

which is called the *stabilizer* of B (in A). We denote by $\mathrm{Stab}(B)^0$ the identity component of $\mathrm{Stab}(B)$. For a given B there is an equivariant compactification $\bar{A}$ (smooth) of A such that the closure $\bar{B}$ of B in $\bar{A}$ contains no A-orbit (cf. [10]). If B is a reduced divisor D on A, then

2.2. *such $\bar{D}$ is big on $\bar{A}$ if* $\mathrm{Stab}(D)$ *is finite* (cf. [10]).

Let $f : \mathbf{C} \to A$ be a holomorphic curve. If the image $f(\mathbf{C})$ is (resp. is not) Zariski dense in A, f is said to be algebraically non-degenerate (resp. degenerate); from now on, we simply say that f is *non-degenerate* (resp. degenerate). We use $T(r, \omega_{B,f})$ and $T_f(r, c_1(\bar{D}))$

for the order functions, $N(r, f^*D)$ and $N_k(r, f^*D)$ for the counting functions, as defined as in [8], [9] and [10].

Let $f : \mathbf{C} \to M$ be a non-constant holomorphic curve into a projective algebraic manifold M, and let $T_f(r)$ denote the order function of f with respect to an ample divisor on M. Note that there are several ways to define the order function of f, cf. [8]; they are equivalent in the sense of (2.1). Then f is rational if and only if $\overline{\lim}\ (\log r)/T_f(r) > 0$ (cf. [8]). Now, let M be a compactification of a semi-abelian variety A, and suppose that $f(\mathbf{C}) \subset A$. Since there is no non-constant rational map from $\mathbf{C}$ into A, we have

$$\log r = o(T_f(r)).$$

Let Z_i $(i = 1, 2)$ be effective divisors on $\mathbf{C}$ such that $\underline{Z_1}_\infty = \underline{Z_2}_\infty$. Then,

$$N_k(r, Z_1) = N_k(r, Z_2) + O(\log r) = N_k(r, Z_2) + o(T_f(r)) \quad (1 \leq k \leq \infty). \tag{2.3}$$

(b) We now switch to the arithmetic setting. Let k be a number field, let S be a finite set of places of k, containing the archimedean ones, and let $\mathcal{O}_S$ be the corresponding ring of S-integers. Let $\mathbf{G} \cong \mathbf{G}_m^n$ be a (split) linear torus; since the algebraic variety $\mathbf{G}_m$ is canonically defined over the ring of rational integers, we can tacitly considered it as a scheme $\mathbf{G}_{\mathcal{O}_S} \to \operatorname{Spec}(\mathcal{O}_S)$. Every integral point $g \in \mathbf{G}(\mathcal{O}_S)$ can be viewed as a morphism $g : \operatorname{Spec}(\mathcal{O}_S) \to \mathbf{G}_{\mathcal{O}_S}$. A divisor D in $\mathbf{G}$, defined over $\mathcal{O}_S$, corresponds to an ideal $\mathcal{I}(D)$ of $\mathcal{O}_S[\mathbf{G}]$; its pull-back $g^*\mathcal{I}(D)$ is naturally defined, and is an ideal of $\mathcal{O}_S$. In fact, if $F(X_1, \ldots, X_n) = 0$ is an equation for D, where $F(X_1, \ldots, X_n) \in \mathcal{O}_S[X_1, \ldots, X_n]$ is a polynomial, then $g^*\mathcal{I}(D)$ is the ideal of $\mathcal{O}_S$ generated by $F(g)$. The support of this ideal, denoted by $\operatorname{Supp} g^*(D)$, is the set of maximal ideals containing $g^*\mathcal{I}(D)$.

3 Proof of Theorem 1.1.

Before beginning with the proof, we give a key lemma. In general, let $f_i : \mathbf{C} \to A_i$ be non-degenerate holomorphic curves into semi-abelian varieties A_i $(i = 1, 2)$. Put $g = (f_1, f_2) : \mathbf{C} \to A_1 \times A_2$. Let A_0 be the Zariski closure of $g(\mathbf{C})$. Then, by Log Bloch-Ochiai's Theorem [6] A_0 is a translate of a semi-abelian subvariety of $A_1 \times A_2$, so that we have a non-degenerate holomorphic curve and the natural projections,

$$\begin{aligned} &g : \mathbf{C} \to A_0, \\ &p_i : A_0 \to A_i \quad (i = 1, 2). \end{aligned} \tag{3.1}$$

Since f_i are non-degenerate, $p_i(A_0) = A_i$ $(i = 1, 2)$.

Key Lemma 3.2. *Let the notation be as above. Let D_i be reduced divisors on A_i for $i=1,2$. Assume that*

(i) D_1 *is irreducible,*

(ii) $\underline{\mathrm{Supp} f_1^* D_1}_\infty \subset \underline{\mathrm{Supp} f_2^* D_2}_\infty$,

(iii) $N_1(r, f_1^* D_1) \sim N_1(r, f_2^* D_2)||$.

Then, $p_1^* D_1 \subset p_2^* D_2$ *on* A_0.

Proof. We reduce the case to the one where

$$\mathrm{Stab}(D_i) = \{0\}, \qquad i = 1, 2. \tag{3.3}$$

For the reduction we set

$$
\begin{aligned}
& q_i : A_i \to A_i / \mathrm{Stab}(D_i) = A_i', \\
& q_i(D_i) = D_i / \mathrm{Stab}(D_i) = D_i', \\
& f_i' = q_i \circ f_i, \\
& g' = (f_1', f_2') : \mathbf{C} \to A_0' \ (\subset A_1' \times A_2') \quad (A_0' \text{ is the Zariski closure of } g'(\mathbf{C})), \\
& p_i' : A_0' \to A_i' \quad (\text{the natural projections}), \\
& \tilde{q} = (q_1, q_2) : A_1 \times A_2 \to A_1' \times A_2', \\
& \tilde{q}_0 = \tilde{q}|_{A_0} : A_0 \to A_0'.
\end{aligned}
$$

Then we see that $\mathrm{Stab}(D_i') = \{0\}$ and assumptions (i)∼(iii) are satisfied for f_i' and D_i'. Suppose that the present lemma was proved in this case. Then we would have that ${p_1'}^* D_1' \subset {p_2'}^* D_2'$. It follows that $\tilde{q}_0^*({p_1'}^* D_1') \subset \tilde{q}_0^*({p_2'}^* D_2')$, and hence that $p_1^*(q_1^* D_1') \subset p_2^*(q_2^* D_2')$. Therefore, $p_1^* D_1 \subset p_2^* D_2$.

Now we assume (3.3). By virtue of the second main theorem established by [10] there exists for each $i = 1, 2$ an equivariant compactification $\bar{A}_i$ of A_i such that

$$N_1(r, f_i^* D_i) = (1 + o(1)) T_{f_i}(r, c_1(\bar{D}_i))||,$$

where $\bar{D}_i$ is the closure of D_i. Since $\bar{D}_i$ is big on $\bar{A}_i$ (cf. 2.2), we may take the order function of f_i by

$$T_{f_i}(r) = T_{f_i}(r, c_1(\bar{D}_i)) \quad (i = 1, 2),$$

and for $g = (f_1, f_2)$ by

$$T_g(r) = T_{f_1}(r) + T_{f_2}(r).$$

It follows that

$$T_g(r) \sim T_{f_i}(r)|| \quad (i = 1, 2).$$

Let F be an arbitrary irreducible component of $p_1^* D_1$. We are going to show

Claim **3.4.** $F \subset p_2^* D_2$.

There are finitely many elements $a_\nu \in \operatorname{Ker} p_1, 1 \le \nu \le \nu_0$, such that

$$p_1^* D_1 = \sum_{\nu=1}^{\nu_0} (F + a_\nu).$$

Let $\bar{A}_0$ be an equivariant compactification of A_0 such that the closure $\overline{p_1^* D}_1$ of $p_1^* D_1$ in $\bar{A}_0$ contains no A_0-orbit, and p_1 extends to a morphism $\bar{p}_1 : \bar{A}_0 \to \bar{A}_1$. Then $c_1(\bar{F} + a_\nu) = c_1(\bar{F})$, and

$$\begin{aligned} T_{f_1}(r, c_1(\bar{D}_1)) = T_g(r, c_1(\overline{p_1^* D}_1)) &= \sum_{\nu=1}^{\nu_0} T_g(r, c_1(\bar{F} + a_\nu)) \\ &= \nu_0 T_g(r, c_1(\bar{F})). \end{aligned}$$

Again, by the Second Main Theorem [10] and the above we have

$$N_1(r, g^* F) = (1 + o(1)) T_g(r, c_1(\bar{F}))|| \sim T_{f_1}(r)||.$$

It follows from assumption (ii) and (2.3) that

$$N_1(r, g^* F) = N_1(r, g^*(F \cap p_2^* D_2)) + O(\log r).$$

Suppose that Claim 3.4 is not true. Then we would have

$$\operatorname{codim} (F \cap \operatorname{Supp} p_2^* D_2) \ge 2.$$

The Main Theorem (ii) of [10] implies that

$$N_1(r, g^*(F \cap p_2^* D_2)) = o(T_g(r))||.$$

Thus we obtain a contradictory estimate,

$$T_{f_1}(r) = o(T_{f_1}(r))||.$$

Q.E.D.

Proof of Theorem 1.1. Firstly by [7] there are points $\zeta_i \in \mathbf{C}$ $(i = 1, 2)$ such that $f_i(\zeta_i) \in D_i$. Then, considering the composites with the translations

$$z \in \mathbf{C} \to f_i(z + \zeta_i) - f(\zeta_i), \quad D_i - f(\zeta_i),$$

we may assume that $f_i(0) = 0 \in D_i$ $(i = 1, 2)$. In what follows, we keep this setup.

(i) Let $g = (f_1, f_2) : \mathbf{C} \to A_0 (\subset A_1 \times A_2)$ and $p_i : A_0 \to A_i$ be as above. Note that A_0 is a semi-abelian subvariety of $A_1 \times A_2$. Set $E_i = p_i^* D_i$. Let F_i be an irreducible component of E_i containing 0. Since D_i are assumed to be irreducible, there are finitely many points $a_{i\nu} \in \operatorname{Ker} p_i, 1 \leq \nu \leq \nu_i$ $(a_{i1} = 0)$ such that

$$E_i = \sum_{\nu=1}^{\nu_i} (a_{i\nu} + F_i) \quad (i = 1, 2),$$

and $a_{i\nu} + F_i, 1 \leq \nu \leq \nu_i$, are mutually distinct for each i. By the Key Lemma 3.2 we have in fact that

$$F_1 = F_2, \qquad E_1 \subset E_2.$$

Note that $\operatorname{Stab}(E_1)^0 = \operatorname{Stab}(E_2)^0 = \operatorname{Stab}(F_1)^0$ $(= \operatorname{Stab}(F_2)^0)$. If $\dim \operatorname{Stab}(F_1) > 0$, there should be a non-zero holomorphic vector field v on A_0 that is tangent to E_1 and E_2. Therefore, the push-forward $p_{i*}v$ are tangent to D_i $(i = 1, 2)$. Since $\operatorname{Stab}(D_i) = \{0\}$, $p_{i*}v = 0$ $(i = 1, 2)$, and hence $v = 0$; this is absurd. Therefore we see that $\operatorname{Ker} p_i$ are finite, and that

$$\operatorname{Ker} p_i = \{a_{i\nu}\}_{\nu=1}^{\nu_i} = \operatorname{Stab}(E_i).$$

Since $F_1 + a_{1\nu}$ is an irreducible component of $p_2^* D_2$ and D_2 is irreducible, $p_2(F_1 + a_{1\nu}) = p_2(F_1) + p_2(a_{1\nu}) = D_2 + p_2(a_{1\nu}) = D_2$. Since $\operatorname{Stab}(D_2) = \{0\}$, $p_2(a_{1\nu}) = 0$, so that $a_{1\nu} \in \operatorname{Ker} p_2$. Therefore, $\operatorname{Ker} p_1 \subset \operatorname{Ker} p_2$, and we have an isogeny $\phi : A_1 \to A_2$ by the composition of the sequence of morphisms,

$$A_1 \cong A_0 / \operatorname{Ker} p_1 \to A_0 / \operatorname{Ker} p_2 \cong A_2. \tag{3.5}$$

It is immediate that $D_1 \subset \phi^* D_2$.

(ii) Assume that equality holds in (1.2). The above proof implies that $E_1 = E_2$ and $\operatorname{Ker} p_1 = \operatorname{Ker} p_2$. It follows from (3.5) that ϕ is an isomorphism.

(iii) It follows from the proof of (i) that $p_i : A_0 \to A_i$ $(i = 1, 2)$ are isogenies. Since $\operatorname{Stab}(E_i)$ are finite, every irreducible component of E_i is ample on A_0. If there were two irreducible components F' and F'' in E_2, $F' \cap F'' \neq \emptyset$. Since $p_2 : A_0 \to A_2$ is étale and $p_2(F') = p_2(F'') = D_2$, D_2 is not locally irreducible; this is a contradiction. Thus, E_2 is irreducible; since $E_1 \subset E_2$, $E_1 = E_2$. Hence, $\phi : A_1 \to A_2$ is an isomorphism. *Q.E.D.*

Example 3.6. Set $A_1 = \mathbf{C}/\mathbf{Z}$ $(\cong \mathbf{G}_m)$ and let $D_1 = 0$ be the zero element of A_1. Let $f_1 : \mathbf{C} \to A_1$ be the covering map. We take a number $\tau \in \mathbf{C}$ with the imaginary part $\Im\tau \neq 0$. Set $A_2 = \mathbf{C}/(\mathbf{Z} + \mathbf{Z}\tau)$ which is an elliptic curve, and let $D_2 = 0$ be the zero element of A_2. Let $f_2 : \mathbf{C} \to A_2$ be the covering map.

Then $f_1^{-1}D_1 = \mathbf{Z} \subset \mathbf{Z} + \tau\mathbf{Z} = f_2^{-1}D_2$: assumption (1.2) of Theorem 1.1 is satisfied. There is, however, no non-constant morphism $\phi : A_1 \to A_2$. Note that

$$N_1(r, f_1^*D_1) \sim r, \quad N_1(r, f_2^*D_2) \sim r^2.$$

Thus, $N_1(r, f_1^*D_1) \not\sim N_1(r, f_2^*D_2)||$: assumption (1.3) is failing.

4 The case of general D_i.

Here we deal with the case where D_i may be reducible in Theorem 1.1. The following example suggests that there must be some restrictions on the given divisors D_i.

Example 4.1. Let $\mathbf{Z}[i]$ denote the lattice of Gauss integers, and set

$$\begin{array}{ll} A_1 = \mathbf{C}/\mathbf{Z}[i] \times \mathbf{C}^*, & D_1 = \{0\} \times \mathbf{C}^* + \mathbf{C}/\mathbf{Z}[i] \times \{1\}, \\ A_2 = \mathbf{C}/\mathbf{Z}[i] \times \mathbf{C}/\mathbf{Z}[i], & D_2 = \{0\} \times \mathbf{C}/\mathbf{Z}[i] + \mathbf{C}/\mathbf{Z}[i] \times \{0\}. \end{array}$$

Taking an irrational number $\alpha \in \mathbf{R}$, we set

$$f_1 : z \in \mathbf{C} \to ([z], e^{2\pi\alpha z}) \in A_1,$$
$$f_2 : z \in \mathbf{C} \to ([z], [\alpha z]) \in A_2,$$

where $[z]$ denotes the point of $\mathbf{C}/\mathbf{Z}[i]$ represented by z. Then f_i are non-degenerate, $f_1^*D_1 \subset f_2^*D_2$, and by calculation $N_1(r, f_i^*D_i) \sim r^2$ $(i = 1, 2)$. There is, however, no morphism $\phi : A_1 \to A_2$ such that $D_1 \subset \phi^*D_2$.

The above example suggests that the stabilizers of the irreducible components of D_i should be restricted, while the assumption of the triviality of $\mathrm{Stab}(D_i)$ is just a matter of reduction to state the result as seen in (3.3).

Let A_i $(i = 1, 2)$ be a semi-abelian variety and let D_i be reduced divisor on A_i, which may be reducible. In the sequel we again assume that

(4.2) $$\mathrm{Stab}(D_i) = \{0\}, \qquad i = 1, 2.$$

Theorem 4.3. *Let $f_i : \mathbf{C} \to A_i$ $(i = 1, 2)$ be a non-degenerate holomorphic curve.*

(a) *Assume that*

(i) *every irreducible component of D_1 has a finite stabilizer;*

(ii) $\underline{\mathrm{Supp} f_1^*D_1}_{\infty} \subset \underline{\mathrm{Supp} f_2^*D_2}_{\infty}$,

(iii) $N_1(r, f_1^*D_1) \sim N_1(r, f_2^*D_2)||$.

Then there exist a semi-abelian variety A_0, a non-degenerate holomorphic curve $g : \mathbf{C} \to A_0$, reduced divisors E_i on A_i ($i = 1, 2$), and morphisms $\phi_i : A_0 \to A_i$ such that

- $E_1 \subset E_2$,
- $\phi_i^* D_i = E_i$ ($i = 1, 2$),
- $f_i = \phi_i \circ g \quad (i = 1, 2)$,
- $\phi_2 : A_0 \to A_2$ *is an étale morphism.*

(b) *Moreover, if every irreducible component of D_2 has a finite stabilizer, and equality holds in the above* (ii), *then $\phi_i : A_0 \to A_i$ are isomorphisms, and $E_1 = E_2$.*

Proof. (a) Let $g : \mathbf{C} \to A_0$ and $p_i : A_0 \to A_i$ be as in (3.1). Set $E_i = p_i^* D_i$. We use the order functions $T_{f_i}(r)$ and $T_g(r)$ in the same sense as in the proof of the Key Lemma 3.2.

Let F be an arbitrary irreducible component of E_1. Then $p_1(F)$ is an irreducible component of D_1, and by assumption (i) $\mathrm{Stab}(p_1(F))$ is finite. Hence, the closure $\overline{p_1(F)}$ in a compactification $\bar{A}_1$ of A_1 is big. The second main theorem [10] implies that

$$N_1(r, f_1^*(p_1(F))) \sim T_{f_1}(r) \sim N_1(r, f_1^* D_1) \sim N_1(r, f_2^* D_2)||.$$

We infer from the Key Lemma 3.2 that $F \subset E_2$; henceforth, $E_1 \subset E_2$.

By the same vector field argument as in the proof of Theorem 1.1 (i) we see that $\dim \mathrm{Stab}(E_2) = 0$. Thus $p_2 : A_0 \to A_2$ is an étale morphism. Setting $\phi_i = p_i$, we finish the proof of (a).

(b) In the same way as in (a) it immediately follows from the assumptions that $E_1 = E_2$. Since $A_i \cong A_0/\mathrm{Stab}(E_i)$ ($i = 1, 2$), ϕ_i are isomorphisms. *Q.E.D.*

We have the following by Theorem 4.3.

Corollary 4.4. *Let $f_i : \mathbf{C} \to A_i$ be non-degenerate holomorphic curves and let D_i be reduced divisors on A_i such that all irreducible components of D_i have finite stabilizers ($i = 1, 2$). If A_1 and A_2 are not isogenous, then $\underline{f_1^{-1} D_1}_\infty \neq \underline{f_2^{-1} D_2}_\infty$.*

Proof. By the assumption, $\mathrm{Stab}(D_i)$ are finite ($i = 1, 2$). Then we have isogenies $q_i : A_i \to A_i/\mathrm{Stab}(D_i)$. By setting $D_i' = q_i(D_i)$, the case is reduced to the one where (4.2) is satisfied. If $\underline{f_1^{-1} D_1}_\infty = \underline{f_2^{-1} D_2}_\infty$, one would infer from Theorem 4.3 that $\phi_i : A_0 \to A_i$ are both étale morphisms for $i = 1, 2$, so that A_1 and A_2 are isogenous. *Q.E.D.*

5 Proof of Theorem 1.5.

Theorem 1.5 will be reduced to statements about diophantine equations involving linear recurrence sequences. For an introduction to the general theory see [11] or [13]. We shall actually be interested in recurrence sequences of the form

$$n \mapsto \mathbf{f}(n) = \sum_{i=1}^{k} b_i \alpha_i^n, \tag{5.1}$$

where $k \geq 1$ is a natural number, $b_1, \ldots, b_k$ are nonzero complex numbers and $\alpha_1, \ldots, \alpha_k$ are nonzero pairwise distinct complex numbers. The representation (5.1) is unique: it suffices to note that the right-hand side in (5.1) cannot vanish for k consecutive values of n (e.g. for $n = 0, \ldots, k-1$) since the van der Mond matrix $(\alpha_i^{n-1})_{1 \leq i,n \leq k-1}$ is non-singular. These recurrence sequences as in (5.1) will be called *power sums*. The complex numbers $\alpha_1, \ldots \alpha_k$ will be called the *roots* of the power sum $\mathbf{f}$. Theorem 1.5 will be reduced to the following result, appearing in a slightly more general formulation as Corollary 2 in [4]:

Lemma 5.2. *Let, for $i = 1, 2$, $n \mapsto \mathbf{f}_i(n)$ be two power sums with values in a ring of S-integers $\mathcal{O}_S$. Suppose that the roots of $\mathbf{f}_1, \mathbf{f}_2$ together generate a torsion-free multiplicative group. If the ratio $\mathbf{f}_2(n)/\mathbf{f}_1(n)$ lies in $\mathcal{O}_S$ for infinitely many n, then the function $n \mapsto \mathbf{f}_2(n)/\mathbf{f}_1(n)$ is a power sum.*

Before starting the proof of Theorem 1.5 we recall some basic facts about the algebraic theory of power sums.

Let $\mathcal{U} \subset \mathbf{C}^*$ be a torsion-free finitely generated multiplicative group. As an abstract group, $\mathcal{U}$ is isomorphic to $\mathbf{Z}^r$, where r is the rank of $\mathcal{U}$. We shall be interested in the algebra of power sums with roots in $\mathcal{U}$. Such an algebra is isomorphic to the algebra $\mathbf{C}[\mathcal{U}]$, which in turn is isomorphic to the algebra of Laurent polynomials $\mathbf{C}[X_1, \ldots, X_r, X_1^{-1}, \ldots, X_r^{-1}] = \mathbf{C}[\mathbf{G}_m^r]$. Letting $u_1, \ldots, u_r$ be generators of $\mathcal{U}$, an isomorphism is obtained by sending the function $(n \mapsto u_i^n)$ to the monomial X_i and extending to a ring homomorphism. Note that the units $\mathbf{C}[X_1^{\pm 1}, \ldots, X_r^{\pm 1}]$ are the monomials, so the units in the ring of power sums are of the form $(n \mapsto a\alpha^n)$, for nonzero $a \in \mathbf{C}^*$ and $\alpha \in \mathcal{U}$.

Let now $g \in \mathbf{G}_m^r$ be an element in a torus, not contained in any subtorus. This means that if we write $g = (u_1, \ldots, u_r)$, the non-zero complex numbers $u_1, \ldots, u_r$ are multiplicative independent, i.e. generate a subgroup of maximal rank, which is necessarily torsion-free. The main link between the theory of linear recurrences and linear tori is represented by the following fact: *For every regular function $F \in \mathbf{C}[\mathbf{G}_m^r]$, the function $(n \mapsto F(g^n))$ is a power sum with roots in the group $\mathcal{U} := \langle u_1, \ldots, u_r \rangle$.*

Another fact will be repeatedly used in the sequel:

Let $F \in \mathbf{C}[\mathbf{G}_m^r]$ be a non zero regular function, $D = F^{-1}(0)$ the corresponding divisor in $\mathbf{G}_m^r$. Then the stabilizer of D has positive dimension if and only if F can be written, after applying an automorphism to $\mathbf{G}_m^r$, in the form $X_r^l G(X_1, \ldots, X_{r-1})$, for some integer l (possibly zero) and a polynomial $G(X_1, \ldots, X_{r-1})$ in $r-1$ indeterminates.

Another equivalent formulation is that the recurrence $(n \mapsto F(g^n))$ is of the form $\alpha^n \mathbf{f}(n)$ where the roots of $\mathbf{f}$ generate a multiplicative group of rank $< r$.

Let us now begin the proof of Theorem 1.5. Let d_i, for $i = 1, 2$, be the dimension of the torus G_i. Let $F_i(X_1, \ldots, X_{d_i}) \in \mathbf{C}[X_1, \ldots, X_{d_i}]$ be Laurent polynomials defining the irreducible divisor D_i: $D_i = F_i^{-1}(0)$. We can clearly suppose that they are polynomials in $X_1, \ldots, X_r$, and also that they have no monomial factor of positive degree: both facts follow from the remark that by multiplying F_i by a monomial, the zero locus in $\mathbf{G}_m^{d_i}$ does not change.

Let g_1, g_2 be the elements appearing in the statement of Theorem 1.5 and let $k \geq 1$ be the order of the torsion subgroup of the group generated by the coordinates of g_1, g_2. Considering the partition of the set of natural numbers in classes modulo k, we obtain that for at least one integer $r \in \{0, \ldots, k-1\}$ there will exist infinitely many positive integers m such that $(g_1^{r+km})^* \mathcal{I}(D_1) \supset (g_2^{r+km})^* \mathcal{I}(D_2)$. Replacing g_i by g_i^k and D_i by its image under the map $x \mapsto g_i^{-r} x$, we reduce the case to $k = 1$. Note that the conclusion we want to prove is not affected by this replacement. So, from now on, we shall suppose that the coordinates of g_1, g_2 together generate a torsion-free multiplicative group $\mathcal{U} = \langle u_1, \ldots, u_r \rangle \subset \mathbf{C}^*$. Finally, let $\mathbf{f}_i(n) = F_i(g_i^n)$, so that $\mathbf{f}_1, \mathbf{f}_2$ are power sums with roots in $\mathcal{U}$. Their values belong to the ring $\mathcal{O}_S$, although the roots and coefficients expressing $\mathbf{f}_1, \mathbf{f}_2$ as power sums are not necessarily in the ring $\mathcal{O}_S$.

The hypothesis of Theorem 1.5 can be reformulated by saying that for infinitely many natural numbers n, the ideal generated in $\mathcal{O}_S$ by $\mathbf{f}_1(n)$ contains the ideal generated by $\mathbf{f}_2(n)$, i.e. $\mathbf{f}_1(n)$ divides $\mathbf{f}_2(n)$, in the ring $\mathcal{O}_S$. Applying Lemma 5.2, we obtain that the power sum $\mathbf{f}_1$ divides the power sum $\mathbf{f}_2$ in the ring of power sums. We shall exploit this fact, and see how it leads to the sought conclusion of Theorem 1.5.

Lemma 5.3. *Let $i \in \{1, 2\}$. Suppose that the irreducible divisor $D_i = F_i^{-1}(0)$ has finite stabilizer. Then the roots of $(n \mapsto F_i(g_i^n))$ generate a finite index subgroup of the group generated by the roots of g_i.*

Proof. Let $\alpha_{1,i}, \ldots, \alpha_{d_i,i}$ be the roots of g_i; since they are multiplicatively independent, the algebra generated by the functions $(n \mapsto \alpha_{j,i}^n)$, for $j = 1, \ldots, d_i$, is isomorphic, as described above, to the algebra $\mathbf{C}[X_1^{\pm 1}, \ldots, X_{d_i}^{\pm 1}]$. Writing $F_i(X_1, \ldots, X_{d_i}) =$

$\sum_{(j_1,\ldots,j_{d_i})} a_{i,(j_1,\ldots,j_{d_i})} X_1^{j_1}\cdots X_{d_i}^{j_{d_i}}$, the roots of $(n\mapsto F_i(g_i^n))$ are the numbers $\alpha_{i,1}^{j_1}\cdots\alpha_{d_i}^{j_{d_i}}$ for which the corresponding coefficient $a_{i,(j_1,\ldots,j_{d_i})}$ does not vanish. The rank of the group they generate is then the rank of the lattice generated in $\mathbf{Z}^{d_i}$ by the exponents $(j_1,\ldots,j_{d_i})$ corresponding to non zero coefficients $a_{i,(j_1,\ldots,j_{d_i})}$. Suppose now by contradiction that group generated by the roots of $(n\mapsto F_i(g_i^n))$ has rank $d<d_i$; then the above mentioned lattice has also rank d, so is generated by d vectors $(l_{1,k},\ldots,l_{d_i,k})\in\mathbf{Z}^d$, for $k=1,\ldots,d$. Then we can write $F_i(X_1,\ldots,X_{d_i})$ as

$$F_i(X_1,\ldots,X_{d_i}) = G_i(X_1^{l_{1,1}}\cdots X_{d_i}^{l_{d_i,1}},\ldots,X_1^{l_{1,d}}\cdots X_{d_i}^{l_{d_i,d}})$$

for some Laurent polynomial $G_i(T_1,\ldots,T_d)\in\mathbf{C}[T_1^{\pm1}\ldots,T_d^{\pm1}]$ in $d<d_i$ variables. Let $(b_1,\ldots,b_{d_i})\in\mathbf{Z}^{d_i}\setminus\{0\}$ be a nonzero vector which is orthogonal to all the vectors $(l_{1,k},\ldots,l_{d_i,k})$, for $k=1,\ldots,d$; consider the algebraic subgroup of $\mathbf{G}_m^{d_i}$ formed by the elements of the form $(t^{b_1},\ldots,t^{b_{d_i}})$, with $t\in\mathbf{G}_m$. Translations with respect to this subgroup, *i.e.* maps of the form $(X_1,\ldots,X_{d_i})\mapsto(t^{b_1}X_1,\ldots,t^{b_{d_i}}X_{d_i})$, leave invariant the zero set of F_i, which gives the desired contradiction. *Q.E.D.*

Lemma 5.4. *In the previous notation, suppose that* $\mathbf{f}_1$ *divides* $\mathbf{f}_2$ *in the ring of power series with roots in* $\mathcal{U}$. *Then the roots of* $\mathbf{f}_2$ *generate a finite index subgroup of* $\mathcal{U}$.

Proof. Using as usual the isomorphism between the ring of power series with roots in $\mathcal{U}$ and the ring $\mathbf{C}[X_1^{\pm1},\ldots,X_r^{\pm1}]$, the power sums $\mathbf{f}_i$ will correspond to Laurent polynomials $G_i\in\mathbf{C}[X_1^{\pm1},\ldots,X_r^{\pm1}]$. By assumption, G_1 divides G_2 in the ring of Laurent polynomials. If, by contradiction, the lemma were false, then after applying a change of variables, we could write G_2 as a Laurent polynomial in $(X_1,\ldots,X_{r-1})$, while the variable X_r would appear in G_1. Also, since the zero set of G_1 has finite stabilizer, G_1 could not be of the form $G_1(X_1,\ldots,X_r)=X_r^a\tilde{G}_1(X_1,\ldots,X_{r-1})$, for any polynomial $\tilde{G}_1(X_1,\ldots,X_{r-1})$ independent of X_r. But then a divisibility relation of the form $G_2(X_1,\ldots,X_{r-1}) = G(X_1,\ldots,X_r)\cdot H(X_1,\ldots,X_r)$ could not be valid, and this contradiction finishes the proof. *Q.E.D.*

Lemma 5.5. *Assume that each component of* D_2 *has finite stabilizer, and the hypotheses of the previous Lemma 5.4. Then the roots of* $\mathbf{f}_1$ *generate a finite index subgroup of* $\mathcal{U}$.

Proof. In the notation of the proof of Lemma 5.4, we must prove that it is impossible that G_1 can be written such that one of the variables $X_1,\ldots,X_r$ is omitted. This is due to the fact that otherwise G_2 will be the product of G_1 and a Laurent polynomial containing the omitted variable, so F_2 too would be reducible, and moreover one of its irreducible factors would omit an indeterminate, up to automorphism of $\mathbf{G}_m^{d_2}$. This is in contradiction with the fact that each component of D_2 has trivial stabilizer, so we have proved so far that the roots of $\mathbf{f}_1$ generate a finite index subgroup of $\mathcal{U}$. *Q.E.D.*

Finally, we have obtained so far that both the roots of g_1 and those of g_2 generate finite index subgroups of $\mathcal{U}$. So in particular $d_1 = d_2 = r$, and the two tori $\mathbf{G}_1, \mathbf{G}_2$ are isomorphic, both having dimension r.

Lemma 5.6. *Assume the hypotheses of the two preceding Lemmas and that D_2 has trivial stabilizer. Then the roots of $\mathbf{f}_1$ generate the whole group $\mathcal{U}$.*

Proof. By the previous lemma, the roots of $\mathbf{f}_1$ generate a finite index subgroup of $\mathcal{U}$. Suppose by contradiction that such an index is larger than one. This means that, after changing generators of $\mathcal{U}$, $\mathbf{f}_1$ can be written as a function of $u_1^p, u_2, \ldots, u_r$, for some prime p, while in $\mathbf{f}_2$ cannot. In terms of F_1, F_2, this implies that F_2 can be written as a polynomial in $X_1^p, X_2, \ldots, X_{d_2}$, so its stabilizer would be non-trivial. *Q.E.D.*

Proof of Theorem 1.5. We now finish the proof. Recall that $\mathbf{f}_i(n)$ can be written as $F_i(g_i^n)$, where $F(X_1, \ldots, X_r) = 0$ is a polynomial equation for D_i and also as $G_i(u_1^n, \ldots, u_r^n)$, where G_i is a Laurent polynomial and $u_1, \ldots, u_r$ are multiplicatively independent. Geometrically this means that we have isogenies $\phi_i : \mathbf{G}_m^r \to \mathbf{G}_m^r$ defined by

$$g_i = \phi_i(u_1, \ldots, u_r),$$

such that $G_i = F_i(\phi_i)$. The fact that $\mathbf{f}_1$ divides $\mathbf{f}_2$ in the ring of power sums means that G_1 divides G_2, in the ring of Laurent polynomials. This means that $\phi_1^*\mathcal{I}(D_1) \supset \phi_2^*\mathcal{I}(D_2)$. Also, by the previous Lemma 5.6, $\deg \phi_1 = 1$, so we finish the proof by putting $\phi = \phi_2 \circ \phi_1^{-1}$. *Q.E.D.*

The following example shows that one cannot expect in general that the morphism ϕ sends g_1 to g_2.

Example 5.7. We take $k = \mathbf{Q}, \mathcal{O}_S = \mathbf{Z}$, $\mathbf{G}_1 = \mathbf{G}_2 = \mathbf{G}_m$, $D_1 = D_2 = 1$ and $g_1 = 2, g_2 = -2$. We obtain that for even values of the exponent n, $g_1^n = g_2^n$, so in particular the ideals $(g_1^n)^*\mathcal{I}(D_1)$ and $(g_2^n)^*\mathcal{I}(D_2)$ coincide (both ideals are generated by the integer $2^n - 1$). Nevertheless there exists no morphism $\phi : \mathbf{G}_1 \to \mathbf{G}_2$ sending $2 \mapsto -2$ and satisfying $\phi^*\mathcal{I}(D_2) \subset \mathcal{I}(D_1)$.

Here is an example in which the divisor D_2 is reducible, has trivial stabilizer, but its components have non trivial stabilizers, so that hypothesis of Theorem 1.5 is not satisfied.

Example 5.8. We take $k = \mathbf{Q}, \mathcal{O}_S = \mathbf{Z}$, $\mathbf{G}_1 = \mathbf{G}_m$, $D_1 = \{1\}$; now put $\mathbf{G}_2 = \mathbf{G}_m^2, D_2 = \{1\} \times \mathbf{G}_m + \mathbf{G}_m \times \{1\}$, so that $D_2 = F_2^{-1}(0)$, for the polynomial $F_2(X_1, X_2) = (X_1 - 1)(X_2 - 1)$. Choose $g_1 = 2, g_2 = (2, 3)$. Clearly, condition (1.6) is satisfied for every n, since it amounts to the fact that $2^n - 1$ divides $(2^n - 1)(3^n - 1)$. There exists no dominant map $\mathbf{G}_1 \to \mathbf{G}_2$, so the conclusion of Theorem 1.5 fails.

Also, it may happens that D_2 has a non trivial stabilizer, while the stabilizer of each of its component is trivial. In this case too, the conclusion of Theorem 1.5 can fail, as shown by the following

Example 5.9. Again, let us take $k = \mathbf{Q}, \mathcal{O}_S = \mathbf{Z}$, $\mathbf{G}_1 = \mathbf{G}_m$, $D_1 = \{1\}$ (so $F_1(X) = X - 1$). Then take $\mathbf{G}_2 = \mathbf{G}_m, D_2 = \{1, -1\}$, so $F_2(X) = X^2 - 1$ and $\mathrm{Stab}(D_2) = \{\pm 1\}$. Take $g_1 = 4$, $g_2 = 2$. For every positive integer n, $F_1(n) = F_2(n)$; nevertheless, there is no integer $h > 0$ and morphism ϕ such that $\phi(g_1^h) = g_2^h$.

Here is a general statement (the analogue of Theorem 4.3), which essentially follows from the proof of Theorem 1.5:

Theorem 5.10. *Let $\mathcal{O}_S$ be a ring of S-integers in a number field k. Let $\mathbf{G}_1$ and $\mathbf{G}_2$ be linear tori, and let $g_i \in \mathbf{G}_i(\mathcal{O}_S)$ be elements generating Zariski-dense subgroups in $\mathbf{G}_i$ $(i = 1, 2)$. Let D_i be reduced divisors defined over $\mathcal{O}_S$, with defining ideals $\mathcal{I}(D_i)$. Suppose that for infinitely many natural numbers $n \in \mathbf{N}$, the inclusion of ideals*

$$(g_1^n)^* \mathcal{I}(D_1) \supset (g_2^n)^* \mathcal{I}(D_2)$$

holds. Then there exists a torus $\mathbf{G}_0$, dominant morphisms $\phi : \mathbf{G}_1 \to \mathbf{G}_0$ and $\psi : \mathbf{G}_2 \to \mathbf{G}_0$ and a divisor E on $\mathbf{G}_0$ such that $\phi(g_1) = \psi(g_2)$ and

$$D_1 \subset \phi^* E, \qquad D_2 \supset \psi^* E.$$

Let us show that the Examples 5.8, 5.9 can be treated by Theorem 5.10. In the situation of Example 5.8, take $\mathbf{G}_0 = \mathbf{G}_m$, $\psi_1(X) = X$, $\psi_2(X, Y) = X$; in Example 5.9, take $\mathbf{G}_0 = \mathbf{G}_m, \psi(X) = X^2, \phi(X) = X$.

To prove Theorem 5.10 we need the following lemma, for which we need a definition: we say that a power sum is *reduced* if in the decomposition (5.1) one of the roots α_i is equal to 1. Notice that whenever D is a divisor in a torus $\mathbf{G} = \mathbf{G}_m^r$, and $g \in \mathbf{G}$ is a point generating a Zariski-dense subgroup, an equation for D can always be found of the form $F = 0$, for a Laurent polynomial $F \in k[X_1^{\pm 1}, \ldots, X_r^{\pm 1}]$ such that the power sum $n \mapsto \mathbf{f}(n) := F(g^n)$ is reduced. Actually, we can also take F to be a polynomial in $k[X_1, \ldots, X_r]$.

Lemma 5.11. *Let* $\mathbf{f}_1, \mathbf{f}_2, \mathbf{g}$ *be power sum, such that the group generated by all their roots is torsion free and* $\mathbf{f}_1$ *is reduced. Suppose that* $\mathbf{f}_2$ *factors as* $\mathbf{f}_2 = \mathbf{f}_1 \cdot \mathbf{g}$*. Then every root of* $\mathbf{f}_1$ *has finite index in the group generated by the roots of* $\mathbf{f}_2$*.*

Proof. Suppose by contradiction that one root of $\mathbf{f}_1$, say γ, does not have finite index in the group generated by the roots of $\mathbf{f}_2$. Then the group generated by the roots of $\mathbf{f}_1, \mathbf{f}_2$ and $\mathbf{g}$ admits a basis of the form $\gamma_1, \ldots, \gamma_r$, where $\gamma_r^d = \gamma$, for a suitable $d > 1$, and $\gamma_1, \ldots, \gamma_{r-1}$ generate a group containing the roots of $\mathbf{f}_2$. After the usual identification between power sums with roots in a torsion-free rank r group and Laurent polynomials in r indeterminates, we can write $\mathbf{f}_2(n) = F_2(\gamma_1^n, \ldots, \gamma_{r-1}^n)$, $\mathbf{f}_1(n) = F_1(\gamma_1^n, \ldots, \gamma_r^n)$ and $\mathbf{g}(n) = G(\gamma_1^n, \ldots, \gamma_r^n)$, for Laurent polynomials $F_1, F_2, G \in k[X_1^{\pm 1}, \ldots, X_r^{\pm 1}]$, where F_2 does not depend on X_r, while F_1 does. From the factorization $\mathbf{f}_2 = \mathbf{f}_1 \cdot \mathbf{g}$ follows the corresponding factorization $F_2 = F_1 \cdot G$. Since $\mathbf{f}_1$ is assumed to be reduced, F_1 is not of the form $X_2^k \cdot \tilde{F}_1$, for any $k \neq \mathbf{Z}$ and $\tilde{F}_1 \in k[X_1^{\pm 1}, \ldots, X_{r-1}]$. Hence F_1 is not invertible in $k(X_1, \ldots, X_{r-1})[X_r^{\pm 1}]$, so it is impossible that F_2 omits the indeterminate X_r, finishing the proof. *Q.E.D.*

Proof of Theorem 5.10. Let, as usual, $\mathbf{G}_i = \mathbf{G}_m^{d_i}$, $g_i = (\alpha_{1,i}, \ldots, \alpha_{d_i,i})$. Let $F_i = 0$ be equations for D_i, such that the power sum $n \mapsto F_1(g_1^n) =: \mathbf{f}_1(n)$ is reduced. The inclusion of ideals in our assumption means that $\mathbf{f}_1(n)$ divides $\mathbf{f}_2(n)$ for infinitely many integers n. As in the previous proofs, we can reduce to the case when the roots of the two power sums together generate a torsino-free multiplicative group (this is obtained by considering separately the arithmetic progressions $n \mapsto qn+r$, for a suitable $q > 1$, $r \in \{0, \ldots, q-1\}$, and applying the assumptions to each such arithmetic progression). Hence we can apply Lemma 5.2, which provides a power sum $n \mapsto \mathbf{g}(n)$ such that identically $\mathbf{f}_2 = \mathbf{f}_1 \cdot \mathbf{g}$. By Lemma 5.11, the roots of $\mathbf{f}_1$ have finite index in the group generated by the roots of $\mathbf{f}_2$. Let q be the minimal common multiple of such indices. Let r be the rank of the group Γ generated by the roots of $\mathbf{f}_1$ and put $\mathbf{G}_0 = \mathbf{G}_m^r$. Then the group $\Gamma^d := \{\gamma^d \, : \, \gamma \in \Gamma\}$ embeds both in the group generated by $\alpha_{1,1}, \ldots, \alpha_{d_1,1}$ and in the group generated by $\alpha_{1,2}, \ldots, \alpha_{d_2,2}$. Let $\gamma_1, \ldots, \gamma_r$ be a basis of Γ. The embeddings of Γ in the two mentioned groups correspond to dominant morphisms $\phi : \mathbf{G}_1 \to \mathbf{G}_0$ and $\psi : \mathbf{G}_2 \to \mathbf{G}_0$ with $\phi(g_1) = \psi(g_2) = (\gamma_1, \ldots, \gamma_r)$. Let us write $\mathbf{f}_1(qn) = F_0(\gamma_1, \ldots, \gamma_r)$ for a Laurent polynomial $F_0 \in k[X_1^{\pm 1}, \ldots, X_r^{\pm 1}]$. Then, putting $E = F_0^{-1}(0)$ we have $D_1 \subset \phi^* E$, $D_2 \supset \psi^*(E)$ as wanted. *Q.E.D.*

6 Arithmetic support problem.

In this section we study the unicity problem only with supports in the assumption of Theorem 1.5. We need some technical hypothesis on the divisors D_i. We prove the following.

Proposition 6.1. *Let $\mathbf{G}_1, \mathbf{G}_2, g_1, g_2$ be as above, let D_1, D_2 be irreducible divisors such that D_1 contains the origin of $\mathbf{G}_1$ and D_2 does not contain any translate of a positive dimensional sub-torus. Suppose that for every sufficiently large integer n the inclusion*

$$\operatorname{Supp}(g_1^n)^*\mathcal{I}(D_1) \subset \operatorname{Supp}(g_2^n)^*\mathcal{I}(D_2) \tag{6.2}$$

holds. Then there exists a dominant morphism $\phi : \mathbf{G}_1 \to \mathbf{G}_2$, defined over k, and an integer $h \geq 1$ such that $\phi(g_1) = g_2^h$.

We note that the condition for D_2 to contain no translate of any positive dimensional sub-torus is much stronger than the (necessary) condition that its stabilizer be trivial. We do not know whether the latter suffices. Also, we do not obtain any relation between $\phi^* D_2$ and D_1. In the case we have equality of supports, we obtain the stronger conclusion that ϕ is étale.

The above results will be reduced to a theorem of Barsky, Bézivin and Schinzel [1]. We state as a lemma a particular case of Theorem 1 of [1]:

Lemma 6.3. *Let k be a number field, let $\mathcal{O}_S \subset k$ be a ring of S-integers, let $\alpha_1, \ldots, \alpha_{d_1} \in \mathcal{O}_S^*$ be multiplicatively independent units in $\mathcal{O}_S$, and let $\beta_1, \ldots, \beta_{d_2} \in \mathcal{O}_S^*$ be units of $\mathcal{O}_S$. Let $F_1(X_1, \ldots, X_{d_1})$ and $F_2(Y_1, \ldots, Y_{d_2})$ be polynomials. Assume that $F_1(1, \ldots, 1) = 0$ and that the equation $F_2(y_1, \ldots, y_{d_2}) = 0$ has only finitely many solutions in the roots of unity $y_1, \ldots, y_{d_2}$. If for all large integers n*

$$\operatorname{Supp}\mathcal{I}(F_1(\alpha_1^n, \ldots, \alpha_{d_1}^n)) \subset \operatorname{Supp}\mathcal{I}(F_2(\beta_1^n, \ldots, \beta_{d_2}^n)), \tag{6.4}$$

then there exists a positive integer h such that $\beta_1^h, \ldots, \beta_{d_2}^h$ belongs to the multiplicative group generated by $\alpha_1, \ldots, \alpha_{d_1}$.

Proof of Proposition 6.1. Write $g_1 = (\alpha_1, \ldots, \alpha_{d_1}) \in \mathbf{G}_m^{d_1}$ and $g_2 = (\beta_1, \ldots, \beta_{d_2})$. The divisors $D_i \subset \mathbf{G}_m^{d_i}$ will be defined in $\mathbf{G}_m^{d_i}$ by the equation $F_i(X_1, \ldots, X_{d_i}) = 0$, where F_i are irreducible polynomials. Suppose the hypotheses of Theorem 6.1 are satisfied. In particular, $\alpha_1, \ldots, \alpha_{d_1}$ are multiplicatively independent, $\beta_1, \ldots, \beta_{d_2}$ are also multiplicatively independent and $F_1(1, \ldots, 1) = 0$. The fact that the divisor D_2 contains no translate of positive dimensional sub-tori implies, by a theorem of Laurent ([5], previously a conjecture of Lang), that it contains only finitely many points whose coordinates are roots of unity. So the equation $F_2(y_1, \ldots, y_{d_2}) = 0$ has only finitely many solutions in roots of unity, as required in Lemma 6.3. The hypothesis that $\mathrm{Supp}\,(g_1^n)^*\mathcal{I}(D_1) \supset \mathrm{Supp}\,(g_2^n)^*\mathcal{I}(D_2)$ is equivalent to condition (6.4), so Lemma 6.3 applies and gives the existence of an integer $h \geq 1$ and a $d_2 \times d_1$ matrix $(a_{ij})_{1\leq i\leq d_2; 1\leq j\leq d_1}$ such that

$$\beta_i^h = \prod_{j=1}^{d_1} \alpha_j^{a_{ij}}.$$

Then, defining $\phi : \mathbf{G}_m^{d_1} \to \mathbf{G}_m^{d_2}$ by sending $(x_1, \ldots, x_{d_1}) \mapsto (\prod_j x_j^{a_{1j}}, \ldots, \prod_j x_j^{a_{d_2 j}})$, we obtain the desired conclusion. *Q.E.D.*

References

[1] Barsky D., Bézivin, J-P. and Schinzel, A., Une caractérisation arithmétique de suites récurrentes linéaires, J. Reine Angew. Math. **494** (1998), 73-84.

[2] Corrales-Rodorigáñez, C. and Schoof, R., The support problem and its elliptic analogue, J. Number Theory **64** (1997), 276-290.

[3] Corvaja, P. and Zannier, U., Diophantine equations with power sums and universal Hilbert sets, Indag. Math. **N.S. 9(3)** (1998), 317-332.

[4] Corvaja, P. and Zannier, U., Finiteness of integral values for the ratio of two linear recurrences, Invent. Math. **149** (2002), 431-451.

[5] Laurent, M., Equations diophantiennes exponentielles, Invent. Math. **78** (1984), 299-327.

[6] Noguchi, J., Lemma on logarithmic derivatives and holomorphic curves in algebraic varieties, Nagoya Math. J. **83** (1981), 213-233.

[7] Noguchi, J., On holomorphic curves in semi-Abelian varieties, Math. Z. **228** (1998), 713-721.

[8] Noguchi, J. and Ochiai, T., Geometric Function Theory in Several Complex Variables, Japanese edition, Iwanami, Tokyo, 1984; English Translation, Transl. Math. Mono. **80**, Amer. Math. Soc., Providence, Rhode Island, 1990.

[9] Noguchi, J., Winkelmann, J. and Yamanoi, K., The second main theorem for holomorphic curves into semi-Abelian varieties, Acta Math. **188** no.1 (2002), 129-161.

[10] Noguchi, J., Winkelmann, J. and Yamanoi, K., The second main theorem for holomorphic curves into semi-Abelian varieties II, Forum Math.**20** (2008), 469-503.

[11] van der Poorten, A. J., Solution de la conjecture de Pisot sur le quotient de Hadamard de deux fractions rationnelles, C. R. Acad. Sci. Sér. I Math. **306** (1988), 97–102.

[12] Rumely, R., Notes on van der Poorten's proof of the Hadamard Quotient Theorem: Part I, II, Séminaire de Théorie des Nombres de Paris 1986-87, pp. 349–382, pp. 383–409, Progr. Math. **75**, Birkhäuser, Boston-Basel, 1988.

[13] Schmidt, W. M., Linear recurrence sequences, in Diophantine Approximations, pp. 171-247, Lecture Notes in Math. **1819**, Springer, Berlin, 2003.

[14] Yamanoi, K., Holomorphic curves in abelian varieties and intersection with higher codimensional subvarieties, Forum Math. **16** (2004), 749-788.

Dipartimento di Matematica e Informatica
University of Udine
Via delle Scienze, 206 - 33100 Udine
e-mail: pietro.corvaja@dimi.uniud.it

Graduate School of Mathematical Sciences
The University of Tokyo
Komaba, Meguro,Tokyo 153-8914
e-mail: noguchi@ms.u-tokyo.ac.jp

UTMS

2009–7 Toshio Oshima and Nobukazu Shimeno: *Heckman-Opdam hypergeometric functions and their specializations.*

2009–8 Atsushi Yamashita: *Compactification of the homeomorphism group of a graph.*

2009–9 Jingzhi Li, Masahiro Yamamoto, and Jun Zou: *Conditional stability and numerical reconstruction of initial temperature.*

2009–10 Taku Ishii and Takayuki Oda: *Calculus of principal series Whittaker functions on* $SL(n, \mathbb{R})$.

2009–11 Atsushi Nitanda: *The growth of the Nevanlinna proximity function.*

2009–12 Paola Loreti and Daniela Sforza: *Reachability problems for a class of integro-differential equations.*

2009–13 Masahiro Yamamoto: *Carleman estimates for parabolic equations and applications.*

2009–14 Seiji Nishioka: *Decomposable extensions of difference fields.*

2009–15 Shigeo Kusuoka: *Gaussian K-Scheme.*

2009–16 Shinichiroh Matsuo and Masaki Tsukamoto: *Instanton approximation, periodic ASD connections, and mean dimension.*

2009–17 Pietro Corvaja and Junjiro Noguchi: *A new unicity theorem and Erdös' problem for polarized semi-abelian varieties.*

The Graduate School of Mathematical Sciences was established in the University of Tokyo in April, 1992. Formerly there were two departments of mathematics in the University of Tokyo: one in the Faculty of Science and the other in the College of Arts and Sciences. All faculty members of these two departments have moved to the new graduate school, as well as several members of the Department of Pure and Applied Sciences in the College of Arts and Sciences. In January, 1993, the preprint series of the former two departments of mathematics were unified as the Preprint Series of the Graduate School of Mathematical Sciences, The University of Tokyo. For the information about the preprint series, please write to the preprint series office.

ADDRESS:
Graduate School of Mathematical Sciences, The University of Tokyo
3–8–1 Komaba Meguro-ku, Tokyo 153-8914, JAPAN
TEL +81-3-5465-7001 FAX +81-3-5465-7012