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Gaussian K-Scheme

by

Shigeo Kusuoka

UNIVERSITY OF TOKYO

GRADUATE SCHOOL OF MATHEMATICAL SCIENCES

KOMABA, TOKYO, JAPAN

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Shigeo KUSUOKA *
Graduate School of Mathematical Sciences
The University of Tokyo
Komaba 3-8-1, Meguro-ku, Tokyo 153-8914, Japan

1 Introduction

Let $W_0 = \{w \in C([0,\infty);\mathbf{R}^d);\ w(0) = 0\}$, $\mathcal{G}$ be the Borel algebra over W_0 and μ be the Wiener measure on $(W_0,\mathcal{G})$. Let $B^i : [0,\infty) \times W_0 \to \mathbf{R}$, $i = 1,\ldots,d$, be given by $B^i(t,w) = w^i(t)$, $(t,w) \in [0,\infty) \times W_0$. Then $\{(B^1(t),\ldots,B^d(t); t \in [0,\infty)\}$ is a d-dimensional Brownian motion. Let $B^0(t) = t$, $t \in [0,\infty)$. Let $\{\mathcal{F}_t\}_{t\geqq 0}$ be the Brownian filtration generated by $\{(B^1(t),\ldots,B^d(t); t \in [0,\infty)\}$. Let $\mathcal{S}$ denote the set of continuous $\{\mathcal{F}_t\}$-semimartingales.

Let $V_0, V_1, \ldots, V_d \in C_b^\infty(\mathbf{R}^N;\mathbf{R}^N)$. Here $C_b^\infty(\mathbf{R}^N;\mathbf{R}^n)$ denotes the space of $\mathbf{R}^n$-valued smooth functions defined in $\mathbf{R}^N$ whose derivatives of any order are bounded. We regard elements in $C_b^\infty(\mathbf{R}^N;\mathbf{R}^N)$ as vector fields on $\mathbf{R}^N$.

Now let $X(t,x)$, $t \in [0,\infty)$, $x \in \mathbf{R}^N$, be the solution to the Stratonovich stochastic integral equation

$$X(t,x) = x + \sum_{i=0}^d \int_0^t V_i(X(s,x)) \circ dB^i(s). \tag{1}$$

Then there is a unique solution to this equation. Moreover we may assume that $X(t,x)$ is continuous in t and smooth in x and $X(t,\cdot) : \mathbf{R}^N \to \mathbf{R}^N$, $t \in [0,\infty)$, is a diffeomorphism with probability one.

Let $A = A_d = \{v_0, v_1, \ldots, v_d\}$, be an alphabet, a set of letters, and A^* be the set of words consisting of A including the empty word which is denoted by 1. For $u = u^1 \cdots u^k \in A^*$, $u^j \in A$, $j = 1,\ldots,k$, $k \geqq 0$, we denote by $n_i(u)$, $i = 0,\ldots,d$, the cardinal of $\{j \in \{1,\ldots,k\}; u^j = v_i\}$. Let $|u| = n_0(u)+\ldots+n_d(u)$, a length of u, and $\| u \| = |u|+n_0(u)$ for $u \in A^*$. Let $\mathbf{R}\langle A\rangle$ be the $\mathbf{R}$-algebra of noncommutative polynomials on A, $\mathbf{R}\langle\langle A\rangle\rangle$ be the $\mathbf{R}$-algebra of noncommutative formal power series on A.

Let $r : A^* \setminus \{1\} \to \mathcal{L}(A)$ denote the right normed bracketing operator inductively given by

$$r(v_i) = v_i, \qquad i = 0,1,\ldots,d,$$

and

$$r(v_i u) = [v_i, r(u)], \qquad i = 0,1,\ldots,d,\ u \in A^* \setminus \{1\}.$$

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For any $w_1 = \sum_{u\in A^*} a_{1u}u \in \mathbf{R}\langle\langle A\rangle\rangle$ and $w_2 = \sum_{u\in A^*} a_{2u}u \in \mathbf{R}\langle A\rangle$, let us define a kind of an inner product $\langle w_1, w_2\rangle$ by

$$\langle w_1, w_2\rangle = \sum_{u\in A^*} a_{1u}a_{2u} \in \mathbf{R}.$$

Also, we denote by $||w||$ $\langle w, w\rangle^{1/2}$ for $w \in \mathbf{R}\langle A\rangle$.

Let $A^*_m = \{u \in A^*;\ \| u \|= m\}$, $m \geqq 0$, and let $\mathbf{R}\langle A\rangle_m = \sum_{u\in A^*_m} \mathbf{R}u$, and $\mathbf{R}\langle A\rangle_{\leqq m} = \sum_{k=0}^m \mathbf{R}\langle A\rangle_k$, $m \geqq 0$. Let $j_m : \mathbf{R}\langle\langle A\rangle\rangle \to \mathbf{R}\langle A\rangle_m$ be natural sujective linear maps such that $j_m(\sum_{u\in A^*} a_u u) = \sum_{u\in A^*_m} a_u u$. Let $j_{\leqq m} : \mathbf{R}\langle\langle A\rangle\rangle \to \mathbf{R}\langle A\rangle_{\leqq m}$ be given by $j_{\leqq m} = \sum_{k=0}^m j_k$.

Let $A^{**} = \bigcup_{i=1}^d \{uv_i \in A^*;\ u \in A^*\}$, $A^{**}_m = \{u \in A^{**};\ \| u \|= m\}$, and $A^{**}_{\leqq m} = \{u \in A^{**};\ \| u \|\leqq m\}$, $m \geqq 1$. Let $\mathbf{R}^{**}\langle A\rangle$ be the $\mathbf{R}$-subalgebra of $\mathbf{R}\langle A\rangle$ generated by 1 and $r(u)$, $u \in A^{**}$. Also, we denote $\mathbf{R}^{**}\langle A\rangle \cap \mathbf{R}\langle A\rangle_m$ by $\mathbf{R}^{**}\langle A\rangle_m$. We can regard vector fields $V_0, V_1, \dots, V_d$ as first differential operators over $\mathbf{R}^N$. Let $\mathcal{DO}(\mathbf{R}^N)$ denotes the set of linear differential operators with smooth coefficients over $\mathbf{R}^N$. Then $\mathcal{DO}(\mathbf{R}^N)$ is a noncommutative algebra over $\mathbf{R}$. Let $\Phi : \mathbf{R}\langle A\rangle \to \mathcal{DO}(\mathbf{R}^N)$ be a homomorphism given by

$$\Phi(1) = Identity, \qquad \Phi(v_{i_1}\cdots v_{i_n}) = V_{i_1}\cdots V_{i_n}, \qquad n \geqq 1,\ i_1, \dots, i_n = 0, 1, \dots, d.$$

Then we see that

$$\Phi(r(v_i u)) = [V_i, \Phi(r(u))], \qquad i = 0, 1, \dots, d,\ u \in A^* \setminus \{1\}.$$

Now we introduce a condition (UFG) on the family of vector field $\{V_0, V_1, \dots, V_d\}$ as follows.

(UFG) There are an integer ℓ_0 and $\tilde\varphi_{u,u'} \in C_b^\infty(\mathbf{R}^N)$, $u \in A^{**}_{\ell_0+1} \cup A^{**}_{\ell_0+2}$, $u' \in A^{**}_{\leqq \ell_0}$, satisfying the following.

$$\Phi(r(u)) = \sum_{u'\in A^{**}_{\leqq\ell_0}} \tilde\varphi_{u,u'}\Phi(r(u')), \qquad u \in A^{**}_{\ell_0+1} \cup A^{**}_{\ell_0+2}$$

For any vector field $W \in C_b^\infty(\mathbf{R}^N; \mathbf{R}^N)$, we can think of an ordinary differential equation on $\mathbf{R}^N$

$$\frac{d}{dt}y(t, x) = W(y(t, x)),$$
$$y(0, x) = x.$$

We denote $y(1, x)$ by $\exp(W)(x)$. Then $\exp(W) : \mathbf{R}^N \to \mathbf{R}^N$ is a diffeomorphism. We define a linear operator $Exp(W)$ in $C^\infty(\mathbf{R}^N)$ by

$$(Exp(W)f)(x) = f(\exp(W)(x)), \qquad x \in \mathbf{R}^N,\ f \in C^\infty(\mathbf{R}^N).$$

Since our main result is rather complicated to present, we will explain our result by using operators introduced by Ninomiya-Victoir [5] in the following. We define a family of Markov operator $Q_{(s)}$, $s > 0$, defined on $C_b^\infty(\mathbf{R}^N; \mathbf{R})$ by

$$(Q_{(s)}f)(x)$$

$$= \frac{1}{2}E[(Exp(\frac{s}{2}V_0)Exp(B^1(s)V_1)\cdots Exp(B^d(s)V_d)Exp(\frac{s}{2}V_0)f)(x)]$$
$$+\frac{1}{2}E[(Exp(\frac{s}{2}V_0)Exp(B^d(s)V_d)\cdots Exp(B^1(s)V_1)Exp(\frac{s}{2}V_0)(x))f)(x)],$$

$f \in C_b^\infty(\mathbf{R}^N;\mathbf{R})$.

Then we can show the following result.

Theorem 1 *For any $T>0$, there are $C>0$ and $w \in \mathbf{R}^{**}\langle A\rangle_6$ such that*

$$||Q^n_{(T/n)}f - P_Tf - (\frac{T}{n})^2\int_0^T P_{T-t}\Phi(w)P_tf dt||_\infty \leqq \frac{C}{n^3}||f||_\infty, \qquad f \in C_b^\infty(\mathbf{R}^N),\ n \geqq T.$$

We see by the result in [1] that for any $T>0$ there is a $C'>0$ such that

$$||\int_0^T P_{T-t}\Phi(w)P_tf dt||_\infty \leqq C'||f||_\infty, \qquad f \in C_b^\infty(\mathbf{R}^N).$$

Therefore we see that the following.

Corollary 2 *For any $T>0$ and any bounded measurable function $f:\mathbf{R}^N \to \mathbf{R}$, there are $c>0$ and $C>0$ such that*

$$||Q^n_{(T/n)}f - P_Tf - \frac{c}{n^2}||_\infty \leqq \frac{C}{n^3}.$$

This corollary allows us to use the Romberg extrapolation in numerical computation. We use the notaion in Shigekawa [6] for Malliavin calculus.

2 Preparations

We say that $Z:[0,\infty)\times W_0 \to \mathbf{R}\langle\langle A\rangle\rangle$ is an $\mathbf{R}\langle\langle A\rangle\rangle$-valued continuous semimartingale, if there are continuous semimartingales Z_u, $u \in A^*$, such that $Z(t) = \sum_{u\in A^*} Z_u(t)u$. For $\mathbf{R}\langle\langle A\rangle\rangle$-valued continuous semimartingale $Z_1(t), Z_2(t)$, we can define $\mathbf{R}\langle\langle A\rangle\rangle$-valued continuous semimartingales $\int_0^t Z_1(s)\circ dZ_2(s)$ and $\int_0^t \circ dZ_1(s)Z_2(s)$ by

$$\int_0^t Z_1(s)\circ dZ_2(s) = \sum_{u,w\in A^*}(\int_0^t Z_{1,u}(s)\circ dZ_{2,w}(s))uw,$$

$$\int_0^t \circ dZ_1(s)Z_2(s) = \sum_{u,w\in A^*}(\int_0^t Z_{1w}(s)\circ dZ_{2,u}(s))uw,$$

where

$$Z_1(t) = \sum_{u\in A^*} Z_{1,u}(t)u, \qquad Z_2(t) = \sum_{w\in A^*} Z_{2,w}(t)w.$$

Then we have

$$Z_1(t)Z_2(t) = Z_1(0)Z_2(0) + \int_0^t Z_1(s)\circ dZ_2(s) + \int_0^t \circ dZ_1(s)Z_2(s).$$

Since $\mathbf{R}$ is regarded a vector subspace in $\mathbf{R}\langle\langle A\rangle\rangle$, we can define $\int_0^t Z(s)\circ dB^i(s)$, $i=0,1,\ldots,d$, naturally.

Let $\mathcal{S}$ be the set of continuous semimartigales. Let us define $S:\mathcal{S}\times A^*\to\mathcal{S}$ and $\hat{S}:\mathcal{S}\times A^*\to\mathcal{S}$ inductively by

$$S(Z;1)(t)=Z(t),\qquad \hat{S}(Z;1)(t)=Z(t),\quad t\geqq 0,\qquad Z\in\mathcal{S},\tag{2}$$

and

$$S(Z;uv_i)(t)=\int_0^t S(Z,u)(r)\circ dB^i(r),$$

$$\hat{S}(Z;v_iu)(t)=-\int_0^t S(Z,u)(r)\circ dB^i(r),\quad t\geqq 0,\tag{3}$$

for any $Z\in\mathcal{S},\ i=0,1,\ldots,d,\ u\in A^*$. Also, we denote $S(1;u)(t)$ by $B(t;u)$, $t\geqq 0$, $u\in A^*$.

We define $I:\mathcal{S}\times A^*\to\mathcal{S}$ inductively by

$$I(Z;1)(t)=Z(t),\quad t\geqq 0,\qquad Z\in\mathcal{S},\tag{4}$$

and

$$I(Z;uv_i)(t)=\int_0^t S(Z,u)(r)dB^i(r),\quad t\geqq 0,\tag{5}$$

for any $Z\in\mathcal{S},\ i=0,1,\ldots,d,\ u\in A^*$.

Let us consider the following SDE on $\mathbf{R}\langle\langle A\rangle\rangle$

$$\hat{X}(t)=1+\sum_{i=0}^d\int_0^t\hat{X}(s)v_i\circ dB^i(s),\qquad t\geq 0.\tag{6}$$

One can easily solve this SDE and obtains

$$\hat{X}(t)=\sum_{u\in A^*}B(t;u)u.$$

Let $(W_0.\mathcal{G},\mu)$ be a Wiener space as in Introduction. Let H denote the associated Cameron-Martin space, $\mathcal{L}$ denote the associated Ornstein-Uhlenbeck operator, and $W^{r,p}(E)$, $r\in\mathbf{R}$, $p\in(1,\infty)$, be Watanabe-Sobolev space, i.e. $W^{r,p}=(I-\mathcal{L})^{-r/2}(L^p(W_0;E,d\mu))$ for any separable real Hilbert space E. Let D denote the gradient operator. Then D is a bounded linear operator from $W^{r,p}(E)$ to $W^{r-1,p}(H\otimes E)$. Let D^* denote the adjoint operator of D. (See Shigekawa [6] for details.) Now let $(\tilde{\Omega},\tilde{\mathcal{B}},\tilde{P})$ be a probability space and let $(\Omega,\mathcal{F},P)=(W_0\times\tilde{\Omega},\mathcal{G}\times\tilde{\mathcal{B}},\mu\otimes\tilde{P})$. Note that we can narurally identify $L^p(\Omega;E,dP)$ with $L^p(\tilde{\Omega};L^p(W_0;E,d\mu),d\tilde{P})$ for any $p\in(1,\infty)$ by the mapping Ψ given by $\Psi(f)(\tilde{\omega})(w)=f(w,,\tilde{\omega})$, for $(w,,\tilde{\omega})\in\Omega$ and $f\in L^p(\Omega;E,dP)$. Since $W^{r,p}(E)$ is a subset of $L^p(W;E,d\mu)$ for any $p\in(1,\infty)$ and $r\geqq 0$, we can define $\hat{W}^{r,p}(E)=\Psi^{-1}(L^p(\Omega;W^{r,p}(E),dP))$ as a subset of $L^p(\tilde{\Omega};E,d\tilde{P})$. We identify $\hat{W}^{r,p}(E)$ with $L^p(\Omega;W^{r,p}(E),dP)$. Then $\hat{W}^{r,p}(E)$ is a Banch space.

We can define $\hat{D}:\hat{W}^{r,p}(E)\to\hat{W}^{r-1,p}(H\otimes E)$ and $\hat{D}^*:\hat{W}^{r,p}(H\otimes E)\to\hat{W}^{r-1,p}(E)$ by $\hat{D}=\Psi^{-1}\circ D\circ\Psi$ and $\hat{D}^*=\Psi^{-1}\circ D^*\circ\Psi$. Then $\hat{D}:\hat{W}^{r,p}(E)\to\hat{W}^{r-1,p}(H\otimes E)$ and $\hat{D}^*:\hat{W}^{r,p}(H\otimes E)\to\hat{W}^{r-1,p}(E)$ are continuous for $r\geqq q$ and $p\in(1,\infty)$.

Also, we define a Frechet space $\hat{W}^{\infty,\infty-}(E)$ by

$$\hat{W}^{\infty,\infty-}(E)=\bigcap_{n=1}^{\infty}\hat{W}^{n,n}(E).$$

3 Gaussian K-Scheme

Let $(\Omega_0, \mathcal{B}_0, P_0)$ be a probability space, and let $(\tilde{\Omega}, \tilde{\mathcal{B}}, \tilde{P}) = (\Omega_0, \mathcal{B}, P_0)^{\mathbf{N}}$. Let $(W_0.\mathcal{G}, \mu)$ be a Wiener space as in Introduction. Now let $(\Omega, \mathcal{F}, P) = (W_0.\mathcal{G}, \mu) \times (\tilde{\Omega}, \tilde{\mathcal{B}}, \tilde{P})$ and we think on this probability space.

Let $B^i : [0, \infty) \times \Omega \to \mathbf{R}^d$, $i = 0, 1, \ldots, d$, and $Z_n : \Omega \to \Omega_0$, $n = 1, 2, \ldots,$ be $B^0(t, (w, \{\tilde{\omega}_k\}_{k=1}^{\infty}) = t$, $B^i(t, (w, \{\tilde{\omega}_k\}_{k=1}^{\infty}) = w^i(t)$, $i = 1, \ldots, d$, $t \in [0, \infty)$, and $Z_n(w, \{\tilde{\omega}_k\}_{k=1}^{\infty}) == \tilde{\omega}_n$, for $(w, \{\tilde{\omega}_k\}_{k=1}^{\infty}) \in \Omega$.

Let $s \in (0, 1]$. Let $\mathcal{F}_n^{(s)}$, $n = 1, 2, \ldots,$ be sub σ-algebras of $\mathcal{F}$ generated by $\{W(t);\ t \in [0, ns]\}$, and $\{Z_k;\ k = 1, 2, \ldots, n\}$. Now let $\tilde{\eta}^i_{(s)} : [0, s) \times \Omega \to \mathbf{R}$, $i = 0, 1, \ldots, d$, be $\mathcal{F}_1^{(s)}$ measurable functions satisfying the following conditions.

(G-1) There exists an $\varepsilon_0 > 0$ such that

$$\sup_{s \in (0,1]} E[\exp(\varepsilon_0(s^{-1} \int_0^s |\tilde{\eta}^0_{(s)}(t)|^2 dt)) + \sum_{i=1}^{d} (\int_0^s |\tilde{\eta}^i_{(s)}(t)|^2 dt))] < \infty.$$

(G-2) For any $i = 0, 1, \ldots, d$,

$$\int_0^s \tilde{\eta}^i_{(s)}(t) dt = B^i(s).$$

(G-3) There is a $C_0 > 0$ such that

$$|E^P[\int_0^s \tilde{\eta}^i_{(s)}(t)(\int_0^t \tilde{\eta}^j_{(s)}(r) dr) dt] - \frac{s}{2}\delta'_{ij}| \leqq C_0 s^2,\ i, j = 0, 1, \ldots, d.$$

Here δ'_{ij}, $i, j = 0, \ldots, d$, be given by

$$\delta'_{ij} == \begin{cases} 1, & \text{if } i = j \text{ and } 1 \leqq i \leqq d\ , \\ 0, & \text{otherwise}\ . \end{cases}$$

(G-4) The map $t \in [0, s)$ to $\tilde{\eta}^i_{(s)}(t)$ is a measurrable map from $[0, s)$ to $\hat{W}^{r,p}(\mathbf{R})$ for any $i = 0, 1, \ldots, d$, and $r \geqq 0$, $p \in (1, \infty)$. Moreover,

$$\hat{D}^2 \tilde{\eta}^i_{(s)}(t) = 0, \qquad t \in [0, s),$$

and

$$\sup_{s \in (0,1]} E^P[(\int_0^s ||\hat{D}\tilde{\eta}^i_{(s)}(t)||_H^2 dt)^p]^{1/p} < \infty, \qquad t \in [0, s)$$

for any $p \in (1, \infty)$.

Let $\theta_{(s)} : \Omega \to \Omega$, $s \in (0, 1]$, be given by

$$\theta_{(s)}(w, \{\tilde{\omega}_k\}_{k=1}^{\infty}) = (w(\cdot + s) - w(s), \{\tilde{\omega}_{k+1}\}_{k=1}^{\infty}), \qquad (w, \{\tilde{\omega}_k\}_{k=1}^{\infty}) \in \Omega.$$

We define $\eta^i_{(s)} : [0, s) \times \Omega \to \mathbf{R}$, $i = 0, 1, \ldots, d$, by

$$\eta^i_{(s)} : (t, \omega) = \tilde{\eta}^i_{(s)}(t - (n-1)s, \theta_{(s)}^{n-1}\omega),\ \text{if } t \in: [(n-1)s, ns),\ n = 1, 2, \ldots.$$

Let $Y_{(s)} : [0, \infty) \times \mathbf{R}^N \times \Omega \to \mathbf{R}^N$, $s \in (0, 1]$, be a solution to the following ordinary differential equation.

$$\frac{d}{dt} Y_{(s)}(t, x) = \sum_{i=0}^{d} V_i(Y(t, x; s)) \eta^i_{(s)}(t)$$

$$Y_{(s)}(0,x) = x \in \mathbf{R}^N.$$

Let $Q_{(s)}$, $s \in (0,1]$, be linear operators in $C_b^\infty(\mathbf{R}^N)$ given by

$$(Q_{(s)}f)(x) = E^P[f(Y_{(s)}(s,x)].$$

Also let $\hat{Y}_{(s)} : [0,1] \times \mathbf{R}^d \times \Omega \to \mathbf{R}\langle\langle A\rangle\rangle$ be a solution to the following ordinary differential equation.

$$\frac{d}{dt}\hat{Y}_{(s)}(t) = \sum_{i=0}^{d} \hat{Y}_{(s)}(t) v_i \eta^i_{(s)}(t)$$

$$\hat{Y}_{(s)}(0) = 1.$$

Theorem 3 *Let $m \geqq 2$ and assume that*

$$j_{\leqq m}(E^P[\tilde{Y}_{(s)}(s)]) = j_{\leqq m}(\exp(s(\frac{1}{2}\sum_{i=1}^{d} v_i^2 + v_0)).$$

Then for any $T > 0$, there is a $C_T > 0$ for which

$$||P_T f - Q^n_{(T/n)} f||_\infty \leqq \frac{C_T}{n^{(m-1)/2}} ||f||_\infty, \qquad f \in C_b^\infty(\mathbf{R}^N),\ n \geqq T.$$

Theorem 4 *Let $m \geqq 2$ and assume that there is a $w_0 \in \mathbf{R}^*\langle A\rangle_{m+1}$ such that*

$$j_{\leqq m+2}(E^P[\tilde{Y}_{(s)}(s)]) = s^{(m+1)/2)} w_0 + j_{\leqq m+2}(\exp(s(\frac{1}{2}\sum_{i=1}^{d} v_i^2 + v_0)).$$

Then $w_0 \in \mathbf{R}^{**}\langle A\rangle_{m+1}$ and for any $T > 0$, there is a $C_T > 0$ for which

$$||P_T f - Q^n_{(T/n)} f + (\frac{T}{n})^{(m-1)/2} \int_0^T P_{T-t}\Phi(w) P_t f dt||_\infty \leqq \frac{C_T}{n^{(m+1)/2}} ||f||_\infty$$

for any $f \in C_b^\infty(\mathbf{R}^N)$, $n \geqq T$.

We give two examples for the above Theorem.

Example 1(Ninomiya-Victoir)

Let $\Omega_0 = \{0,1\}$ and $P_0(\{0\}) = P_0(\{1\}) = 1/2$. Let us define $\tilde{\eta}^i_{(s)} : [0,s) \times \Omega \to \mathbf{R}$, $i = 0, 1, \ldots, d$, by the following.

$$\tilde{\eta}^i_{(s)}(t, (w, \{\tilde{\omega}\}_{k=1}^\infty))$$

$$= \begin{cases} (d+1)s^{-1}B^i(s), & \text{if } t \in [\frac{2i-1}{2d+2}s, \frac{2i+1}{2d+2}s),\ i = 1,\ldots,d, \text{ and } \tilde{\omega}_1 = 0\ , \\ (d+1)s^{-1}B^i(s), & \text{if } t \in [\frac{2d-2i+1}{2d+2}s, \frac{2d-2i+3}{2d+2}s),\ i = 1,\ldots,d, \text{ and } \tilde{\omega}_1 = 1\ , \\ d+1, & \text{if } t \in [0, \frac{1}{2d+2}s) \cup (\frac{2d-1}{2d+2}s, s),\ i = 0, \\ 0 & \text{otherwise}\ . \end{cases}$$

Then the assumption (G-1)-(G-4) are satisfied and the assumption of Theorem 4 for $m = 5$ is satisfied. Moreover, the operaor $Q_{(s)}$ is the same as the one given in Introduction. Therefore Theorem 1 is a corolary to Theorem 4.

Example 2(Ninomiya-Ninomiya)

Let $\Omega_0 = \mathbf{R}^d$, and $P_0(dz) = (2\pi)^{-N/2}\exp(-|z|^2/2)dz$. Let us define $\tilde{\eta}^i_{(s)} : [0,s)\times\Omega \to \mathbf{R}$, $i = 0,1,\dots,d$, by the following.

$$\tilde{\eta}^0_{(s)}(t,(w,\{z_k\}_{k=1}^\infty)) = \begin{cases} 0, & t \in [0,s/2), \\ 2, & t \in [s/2,s), \end{cases}$$

and

$$\tilde{\eta}^i_{(s)}(t,(w,\{z_k\}_{k=1}^\infty)) = \begin{cases} 2s^{-1/2}z_1^i, & t \in [0,s/2), \\ 2s^{-1}B^i(s) - 2s^{-1/2}z^i, & t \in [s/2,s), \end{cases}$$

for $i = 1,\dots,d$.

Then the assumption (G-1)-(G-4) are satisfied and the assumption of Theorem 4 for $m = 5$ is satisfied.

This example has been introduced by Ninomiya-Ninomiya [4]. Actually Theorem 4 applies to all examples given in [4].

4 Approximation of SDE

From now on, we assume that the conditions (G-1)-(G-4) are satisfied.

Let $\delta'_{ij}(s)$, $s \in (0,1]$, $i,j = 0,\dots,d$, be given by

$$\delta'_{ij}(s) = E^P[\int_0^s \tilde{\eta}^i_{(s)}(t)(\int_0^t \tilde{\eta}^j_{(s)}(r)dr)dt] - \frac{s}{2}\delta'_{ij}$$

Then by the condition (G-3)

$$|\delta^{ij}(s)| \leqq C_0 s^2, \qquad s \in (0,1],\ i,j = 0,\dots,d.$$

Also, let $d^{ij}_{(s)}(n) : \tilde{\Omega} \to \mathbf{R}$, $s > 0$, $i,j = 0,\dots,d$, $n = 1,2,\dots$, be given by

$$d^{ij}_{(s)}(n) = \int_{(n-1)s}^{ns} dr_1 \eta^i_{(s)}(r_1)(\int_{(n-1)s}^{r_1} dr_1 \eta^j_{(s)}(r_2)) - \frac{s}{2}\delta'_{ij} - \delta'_{ij}(s).$$

Then from the assumptions (G-1)-(G-3), we see that $d^{ij}_{(s)}$ is $\mathcal{F}^{(s)}_n$-measurable and

$$E[d^{ij}_s(n)|\mathcal{F}^{(s)}_{n-1}] = 0, \qquad i,j = 0,\dots,d,\ n \geq 0.$$

Since

$$|d^{ij}_s(n)| \leqq s(1 + C_0 + \sum_{k=0}^d \int_{(n-1)s}^{ns} |\eta^k_{(s)}(r)|^2 dr),$$

we see from the assumption (G-1) that for any $p \in (1,\infty)$ there is a constant $C_p > 0$ such that

$$E[|d^{ij}_s(n)|^{2p}|\mathcal{F}^{(s)}_{n-1}] \leqq C_p s^{2p}, \qquad s \in (0,1],\ n = 1,2,3,\dots. \tag{7}$$

Proposition 5 *For any $T > 0$*

$$\sup_{x\in\mathbf{R}^N}\sup_{s\in(0,1]} s^{-1/3}E[\max_{t\in[0,T]}|X(t,x)-Y_{(s)}(t,x)|^p]^{1/p} < \infty.$$

Note that

$$f(Y_{(s)}(t,x)) = f(Y_{(s)}((n-1)s,x)) + \sum_{i=0}^{d}\int_{(n-1)s}^{t}(V_i f)(Y_{(s)}(r,x))\eta^i_{(s)}(r)dr$$

for any $f\in C^\infty(\mathbf{R}^N)$. Therefore we see that for $t\in[(n-1)s,ns)$,

$$Y_{(s)}(t,x) = Y_{(s)}((n-1)s,x) + \sum_{i=0}^{d}\int_{(n-1)s}^{t} V_i(Y_{(s)}(r,x))\eta^i_{(s)}(r)dr$$

$$= Y_{(s)}((n-1)s,x) + \sum_{i=0}^{d} V_i(Y_{(s)}((n-1)s,x))\int_{(n-1)s}^{t}\eta^i_{(s)}(r)dr$$

$$+\sum_{i_1,i_2=0}^{d}\int_{(n-1)s}^{t} dr_1\eta^{i_1}_{(s)}(r_1)(\int_{(n-1)s}^{r_1}(V_{i_2}(V_{i_1}))(Y_{(s)}(r_2,x))\eta^{i_2}_{(s)}(r_2)dr_2).$$

Therefore we see that

$$\max_{t\in[(n-1)s,ns)}|Y_{(s)}(t,x)-Y_{(s)}((n-1)s,x)|$$

$$\leqq s^{1/2}(1+d)(\max_{i=0,\dots,d}||V_i||_\infty)(\sum_{i=0}^{d}\int_{(n-1)s}^{ns}|\eta^i_{(s)}(r)|^2dr)^{1/2} \tag{8}$$

and

$$Y_{(s)}(ns,x)$$

$$= Y_{(s)}((n-1)s,x) + \sum_{i=0}^{d} V_i(Y_{(s)}((n-1)s,x))(B^i(ns)-B^i((n-1)s))$$

$$+\frac{1}{2}\sum_{i=1}^{d}(V_i(V_i))(Y_{(s)}((n-1)s,x))s + \sum_{i_1,i_2=0}^{d}(V_{i_2}(V_{i_1}))(Y_{(s)}((n-1)s,x))d^{i_1,i_2}_{(s)} + R_{(s)}(n,x),$$

where

$$R_{(s)}(n,x)$$

$$= \sum_{i_1,i_2=0}^{d}(V_{i_2}(V_{i_1}))(Y_{(s)}((n-1)s,x))d'_{ij}(s)$$

$$+\sum_{i_1,i_2,i_3=0}^{d}\int_{(n-1)s}^{t} dr_1\eta^{i_1}_{(s)}(r_1)(\int_{(n-1)s}^{r_1} dr_2\eta^{i_2}_{(s)}(r_2)(\int_{(n-1)s}^{r_2}(V_{i_3}(V_{i_2}(V_{i_1})))(Y_{(s)}(r_3,x))\eta^{i_3}_{(s)}(r_3)dr_3)).$$

Then we see that

$$|R_{(s)}(n,x)|$$

$$\leqq s^2(d+1)^2C_0(\max_{i=0,\dots,d}||V_i||_{C_b^2}) + s^{3/2}(d+1)^3(\max_{i=0,\dots,d}||V_i||_{C_b^3})(\sum_{i=0}^d \int_{(n-1)s}^{ns} |\eta^i_{(s)}(r)|^2 dr)^{3/2}.$$

Also, we see that

$$X(ns,x)$$
$$= X((n-1)s,x) + \sum_{i=0}^d V_i(X((n-1)s,x))(B^i(ns) - B^i((n-1)s))$$
$$+\frac{1}{2}\sum_{i=1}^d (V_i(V_i))(X((n-1)s,x))s + R(n,x;s),$$

where

$$R(n,x,s)$$
$$= \sum_{i_1,i_2,i_3=0}^d \int_{(n-1)s}^t \circ dB^{i_1}(r_1)(\int_{(n-1)s}^{r_1} \circ dB^{i_2}(r_2)(\int_{(n-1)s}^{r_2} (V_{i_3}(V_{i_2}(V_{i_1})))(X(r_3,x)) \circ dB^{i_3}(r_3))).$$

Then we can easily see that

$$\sup_{s\in(0,1]} \sup_{x\in\mathbf{R}^N} \max_{n=1,\dots,[T/s]} s^{-3/2}(||R_{(s)}(n,x)||_{L^p} + ||R(n;x,s)||_{L^p}) < \infty \tag{9}$$

for any $T > 0$ and $p \in (1,\infty)$.

Note that

$$X(ns,x) - Y_{(s)}(ns,x)$$
$$= X((n-1)s,x) - Y_{(s)}((n-1)s,x) + (M_{0,s}(n,x) - M_{0,s}(n-1,x)) + (M_{1,s}(n,x) - M_{1,s}(n-1,x))$$
$$+\sum_{i=0}^d (V_i(X((n-1)s,x)) - V_i(Y_{(s)}((n-1)s,x)))(B^i(ns) - B^i((n-1)s))$$
$$+R(n;x,s) - R_{(s)}(n,x).$$

Here

$$M_{0,s}(n) = \sum_{k=1}^n \sum_{i=0}^d (V_i(X((k-1)s,x)) - V_i(Y_{(s)}((k-1)s,x)))(B^i(ks) - B^i((k-1)s))$$

and

$$M_{1,s}(n,x) = \sum_{m=1}^n \sum_{i,j=0}^d (V_j(V_i))(Y_{(s)}((n-1)s,x))d^{i,j}_{(s)}.$$

Let

$$A(n;s,x) = \max_{k=1,\dots,n} |X(ns,x) - Y_{(s)}(ns,x)|.$$

Then we have

$$A(n;s,x) \leqq \sum_{j=0}^1 \max_{n=1,\dots,[T/s]} |M_{j,s}(n,x)| + \sum_{n=1}^{[T/s]} (|R(n;x,s)| + |R_{(s)}(n,x)|).$$

Since $\{M_{j,s}(n,x)\}_{n=0}^{\infty}$, $j=0,1$ is an $\mathcal{F}_n^{(s)}$ martingale, by Burkholder-Davis-Gundy's inequality we see that for any $p\in(2,\infty)$ there is a $C_p'>0$ such that

$$E[|\max_{k=1,\ldots,n}|M_{0,s}(k,x)|^p]$$

$$\leqq C_p'E[(\sum_{k=1}^{n}\sum_{i=0}^{d}|(V_i(X((k-1)s,x))-V_i(Y_{(s)}((k-1)s,x)))|^2(B^i(ks)-B^i((k-1)s))^2)^{p/2}]$$

$$\leqq C_p'E[(n(d+1))^{(p-2)/2}(\sum_{k=1}^{n}\sum_{i=0}^{d}|(V_i(X((k-1)s,x))-V_i(Y_{(s)}((k-1)s,x)))|^p \times|B^i(ks)-B^i((k-1)s)|^p)]$$

and

$$E[\max_{n=1,\ldots,[T/s]}|M_{1,s}(n,x)|^p]\leqq C_p'E[(\sum_{n=1}^{[T/s]}\sum_{i,j=0}^{d}|V_j(V_i))(Y_{(s)}((n-1)s,x))|^2|d_{(s)}^{i,j}(n)|^2)^{p/2}].$$

Therefore we see by Equation (7) that

$$\sup_{s\in(0,1]}\sup_{x\in\mathbf{R}^N}s^{-1/3}E[|\max_{n=1,\ldots,[T/s]}|M_{1,s}(n,x)|^p]^{1/p}<\infty,\qquad T>0,\ p\in(2,\infty),$$

and there is a $C>0$ for each $p\in(2,\infty)$ such that

$$E[|\max_{k=1,\ldots,n}|M_{0,s}(k,x)|^p]\leqq C(ns)^{(p-2)/2}s\sum_{k=1}^{n}E[A(k-1;s,x)^p],\quad n\geqq 0,\ s\in(0,1],\ x\in\mathbf{R}^N.$$

Let $p\in(2,\infty)$ and let $b(n;s,x)=\sum_{k=1}^{n}E[A(n;s,x)^p]$, $n\geqq 0$, $s\in(0,1]$, $x\in\mathbf{R}^N$. Then combining with Equation (9), we see that for any $p\in(2,\infty)$, and $T>0$ there is a constant $C>0$ such that

$$b(n;s,x)-b(n-1;s,x)\leqq C(sb(n-1;s,x)+s^{1/3})$$

for any $n=1,2,\ldots,[T/s]$, $s\in(0,1]$, and $x\in\mathbf{R}^N$. Then we have

$$(1+Cs)^{-n}b(n;s,x)\leqq nCs^{1/3}$$

and so

$$E[A(n;s,x)^p]\leqq Cns\exp(Csn)s^{1/3}$$

for any $n=1,2,\ldots,[T/s]$, $s\in(0,1]$, and $x\in\mathbf{R}^N$. This implies

$$\sup_{x\in\mathbf{R}^N}\sup_{s\in(0,1]}s^{-1/3}E[\max_{n=1,\ldots,[T/s]}|X(ns,x)-Y_{(s)}(ns,x)|^p]^{1/p}<\infty,\qquad T>0.\qquad(10)$$

Also, by Equation (8) we have for $T>0$

$$E[\max_{n=1,\ldots,[T/s]}\max_{t\in[(n-1)s,ns)}|Y_{(s)}(t,x)-Y_{(s)}((n-1)s,x)|^{2p}]$$

$$\leqq E[\sum_{n=1}^{[T/s]} \max_{t\in[(n-1)s,ns)} |Y_{(s)}(t,x) - Y_{(s)}((n-1)s,x)|^{2p}]$$

$$\leqq s^p[T/s](d+1)^{2p}(\max_{i=0,\dots,d} ||V_i||_\infty)^{2p} \sum_{i=0}^{d} E[(\int_0^s |\tilde{\eta}^i_{(s)}(r)|^2 dr)^p].$$

Therefore by (G-1) we see that for any $p \in (1,\infty)$ and $T > 0$

$$\sup_{x\in\mathbf{R}^N} \sup_{s\in(0,1]} s^{-1/3} E[\max_{n=1,\dots,[T/s]} \max_{t\in[(n-1)s,ns)} |Y_{(s)}(t,x) - Y_{(s)}((n-1)s,x)|^p]^{1/p} < \infty.$$

Similarly we have

$$\sup_{x\in\mathbf{R}^N} \sup_{s\in(0,1]} s^{-1/3} E[\max_{n=1,\dots,[T/s]} \max_{t\in[(n-1)s,ns)} |X(t,x) - X((n-1)s,x)|^p]^{1/p} < \infty.$$

These and Equation (10) imply our assertion. ∎

5 Approximation of Linear SDE

Let $M \geqq 1$, $a_0 \in \mathbf{R}^M$ and $c_{i,jk} \in C_b^\infty(\mathbf{R}^N;\mathbf{R})$, $i = 0,1,\dots,d$ and $j,k = 1,\dots M$.

Let $A : [0,\infty) \times \mathbf{R}^N \times \Omega \to \mathbf{R}^M$ and Let $Z_{(s)} : [0,\infty) \times \mathbf{R}^N \times \Omega \to \mathbf{R}^M$ be solutions to the following equations.

$$A_j(t;x) = a_0 + \sum_{i=0}^{d} \sum_{k=1}^{M} \int_0^t c_{i,jk}(X(r,x)) A_k(r;x) \circ dB^i(r). \tag{11}$$

$$Z_{(s),j}(t;x) = a_0 + \sum_{i=0}^{d} \sum_{k=1}^{M} \int_0^t c_{i,jk}(Y_{(s)}(r,x)) Z_{(s),k}(r;x) \eta^i_{(s)}(r) dr. \tag{12}$$

Proposition 6 *For any $T > 0$ and $p \in (1,\infty)$*

$$\sup_{x\in\mathbf{R}^N} \sup_{s\in(0,1]} E[\max_{t\in[0,T]} |Z_{(s)}(t,x)|^p] < \infty.$$

Proof. It is easy to see that for $F \in C_b^\infty(\mathbf{R}^N;\mathbf{R}^M)$, $j = 1,\dots m$,

$$\frac{d}{dt}(F(Y_{(s)}(t,x)) \cdot Z_{(s)}(t,x))) = \sum_{i=0}^{d} \tilde{F}_i(Y_{(s)}(t,x);F) \cdot Z_{(s)}(t,x)) \eta^i_{(s)}(t),$$

where

$$\tilde{F}_{i,j}(x;F) = \sum_{k=1}^{M} F_k(x) c_{i,kj}(x) + V_i F_j(x) \quad x \in \mathbf{R}^N,\ j = 1,\dots,M.$$

Note that

$$\frac{d}{dt}\log(1 + |Z_{(s)}(t,x))|^2)$$

$$= (1 + |Z_{(s)}(t,x))|^2)^{-1} \sum_{i=0}^{d} \sum_{j,k=1}^{M} Z_{(s),j}(t,x) c_{i,jk}(Y_{(s)}(r,x)) Z_{(s),k}(t;x) \eta^i_{(s)}(t),$$

and so

$$|\frac{d}{dt}\log(1+|Z_{(s)}(t,x)|^2)| \leqq \sum_{i=0}^{d}\sum_{j,k=1}^{M}||c_{i,jk}||_\infty|\eta^i_{(s)}(t)|.$$

So we have

$$\max_{t\in[(n-1)s,ns)}(1+|Z_{(s)}(t,x))|^2)$$

$$\leqq (1+|Z_{(s)}((n-1)s,x)|^2)\exp(\gamma_0 s^{1/2}(\sum_{i=1}^{d}\int_{(n-1)s}^{ns}|\eta^i_{(s)}(t)|^2dt)^{1/2}) \tag{13}$$

where $\gamma_0=\sum_{i=0}^d\sum_{j,k=1}^M||c_{i,jk}||_\infty$.

Also, we see that there are bounded smooth functions $G_{1,i}:\mathbf{R}^N\times\mathbf{R}^M\to\mathbf{R}$, $i=0,1\ldots d$, and $G_{2,ij}:\mathbf{R}^N\times\mathbf{R}^M\to\mathbf{R}$, $i,j=0,1\ldots d$, such that

$$\log(1+|Z_{(s)}(ns,x)|^2)-\log(1+|Z_{(s)}((n-1)s,x)|^2)$$

$$=\sum_{i=0}^{d}G_{1,i}(Y_{(s)}((n-1)s,x),Z_{(s)}((n-1)s,x))(B^i(ns)-B^i((n-1)s))+\hat{R}_{(s)}(n,x),$$

and

$$\hat{R}_{(s)}(n,x)$$

$$=\sum_{i_1,i_2=0}^{d}\int_{(n-1)s}^{ns}dr_1\eta^{i_1}_{(s)}(r_1)(\int_{(n-1)s}^{r_1}dr_2\eta^{i_2}_{(s)}(r_2)G_{2,i_1.i_2}(Y_{(s)}(r_2,x),Z_{(s)}(r_2,x))$$

Note that

$$|\hat{R}_{(s)}(n,x)|\leqq(\sum_{i_1,i_2=0}^{d}||G_{2,i_1.i_2}||_\infty)s\sum_{j=0}^{d}\int_{(n-1)s}^{ns}\eta^j_{(s)}(r)^2dr$$

Since $e^{sx}\leqq 1+s(e^x-1)$ for any $x\geqq 0$ and $s\in(0,1]$, we see from the assumption (G-1)

$$\sup_{s\in(0,1/\gamma]}E[\exp(s\gamma\sum_{n=1}^{[T/s]}\sum_{j=0}^{d}\int_{(n-1)s}^{ns}\eta^j_{(s)}(r)^2dr)]$$

$$\leqq\sup_{s\in(0,1/\gamma]}(1+\frac{s\gamma}{\varepsilon_0}E[\exp(\varepsilon_0\sum_{i=0}^{d}(\int_0^s|\tilde{\eta}^i_{(s)}(t)|^2dt))])^{[T/s]}<\infty \tag{14}$$

for any $\gamma>1$ and $T>0$. Also we see that

$$\exp(\sum_{k=1}^{n}(\gamma\sum_{i=1}^{d}G_{1,i}(Y_{(s)}((k-1)s,x),Z_{(s)}((k-1)s,x))(B^i(ks)-B^i((k-1)s)))$$

$$-\frac{\gamma^2 s}{2}\sum_{i=1}^{d}G_{1,i}(Y_{(s)}((k-1)s,x),Z_{(s)}((k-1)s,x))^2)$$

is a $\{\mathcal{F}^{(s)}_n\}_{n\geqq0}$-martingale for any $\gamma>0$. Also it is obvious from Equation (13) that

$$\sup_{x\in\mathbf{R}^N}\sup_{s\in(s_0,1]}E[\max_{n=1,\ldots,[T/s]}(1+|Z_{(s)}(ns,x)|^2)^p]<\infty$$

for any $s_0 > 0$ and $T > 0$. So we seefrom Equation (13) and (14) that

$$\sup_{x\in\mathbf{R}^N}\sup_{s\in(0,1]} E[\max_{n=1,\dots,[T/s]}(1+|Z_{(s)}(ns,x)|^2)^p] < \infty.$$

By Equation (13) we see that

$$\sup_{t\in[0,T]}(1+|Z_{(s)}(t,x)|^2)$$

$$\leqq \max_{n=0,\dots,[T/s]}(1+|Z_{(s)}(ns,x)|)^2)\exp(\gamma_0 s^{1/2}\sum_{i=0}^d\int_{ns}^{(n+1)s}|\eta^i_{(s)}(r)|dr).$$

Since $(1+|Z_{(s)}(ns,x)|)^2$ and $\sum_{i=0}^d\int_{ns}^{(n+1)s}|\eta^i_{(s)}(r)|dr$ are independent, we have by Burkholder's inequality

$$E[\sup_{t\in[0,T]}(1+|Z_{(s)}(t,x)|^2)^p]^{1/p}$$

$$\leqq \kappa(s)E[\max_{n=0,\dots,[T/s]}(1+|Z_{(s)}(ns,x)|^2)^p]^{1/p}$$

$$+C_pE[(\sum_{n=0}^{[T/s]}\{(1+|Z_{(s)}(ns,x)|^2)^2(\exp(\gamma_0 s^{1/2}(\sum_{i=0}^d\int_{ns}^{(n+1)s}|\eta^i_{(s)}(r)|^2)^{1/2}dr)-\kappa(s))^2)^{p/2}]^{1/p}$$

$$\leqq E[\max_{n=0,\dots,[T/s]}(1+|Z_{(s)}(ns,x)|^2)^{2p}]^{1/2p}$$

$$\times(\kappa(s)+C_pE[(\sum_{n=0}^{[T/s]}(\exp(\gamma_0 s^{1/2}\sum_{i=0}^d\int_{ns}^{(n+1)s}|\eta^i_{(s)}(r)|dr)-\kappa(s))^2)^p]^{1/2p}$$

where

$$\kappa(s) = E[\exp(\gamma_0 s^{1/2}(\sum_{i=0}^d\int_0^s|\eta^i_{(s)}(r)|^2dr)^{1/2})].$$

This implies our assertion. ∎

By using Proposition 6, we can prove the following similarly to Proposition 5.

Proposition 7 *For any $T > 0$ and $p \in (1,\infty)$*

$$\sup_{x\in\mathbf{R}^N}\sup_{s\in(0,1]} s^{-1/3}E[\max_{t\in[0,T]}||A(t,x)-Z_{(s)}(t,x)||^p]^{1/p} < \infty.$$

Corollary 8 *For any $T > 0$ and $p \in (1,\infty)$*

$$\sup_{x\in\mathbf{R}^N}\sup_{s\in(0,1]} s^{-1/3}E[\max_{t\in[0,T]}||\nabla X(t,x)-\nabla Y_{(s)}(t,x)||^p]^{1/p} < \infty.$$

and

$$\sup_{x\in\mathbf{R}^N}\sup_{s\in(0,1]} s^{-1/3}E[\max_{t\in[0,T]}||(\nabla X(t,x))^{-1}-(\nabla Y_{(s)}(t,x))^{-1}||^p]^{1/p} < \infty.$$

Proof. Since

$$d\nabla X(t,x) = \sum_{i=0}^{d} \nabla V_i(X(t,x)) \circ dB^i(t)$$

and

$$\frac{d}{dt}\nabla Y_{(s)}(t,x) = \sum_{i=0}^{d} \nabla V_i(Y_{(s)}(t,x))\eta^i_{(s)}(t),$$

we have our first assertion from Proposition 7. ∎

Proposition 9 *For any $T > 0$, $m \geqq 0$, $p \in (1,\infty)$ and any multi-index $\alpha \in \mathbf{Z}^N_{\geqq 0}$*

$$\sup_{x\in\mathbf{R}^N} \sup_{s\in(0,1]} s^{-1/3} E[\max_{t\in[0,T]} ||\frac{\partial^\alpha}{\partial x^\alpha}\hat{D}^m(X(t,x) - Y_{(s)}(t,x))||^p_{H^{\otimes m}\otimes\mathbf{R}^N}]^{1/p} < \infty.$$

Proof. Note that for any $h \in H$

$$\frac{d}{dt}\hat{D}Y_{(s)}(t,x)(h)$$

$$= \sum_{i=0}^{d} \nabla V_i(Y_{(s)}(t,x))\hat{D}Y_{(s)}(t,x)(h)\eta^i_{(s)}(t) + \sum_{i=0}^{d} V_i(Y_{(s)}(t,x))\hat{D}\eta^i_{(s)}(t)(h).$$

Therefore we have

$$(\nabla Y_{(s)}(t,x))^{-1}\hat{D}Y_{(s)}(t,x)(h)$$

$$= \sum_{i=0}^{d} \int_0^t (\nabla Y_{(s)}(r,x))^{-1} V_i(Y_{(s)}(r,x))\hat{D}\eta^i_{(s)}(r)(h)dr$$

and so

$$(\nabla Y_{(s)}(ns,x))^{-1}\hat{D}Y_{(s)}(ns,x)(h)$$

$$\sum_{i=1}^{d}\sum_{k=1}^{n}(\nabla Y_{(s)}((k-1)s,x))^{-1}V_i(Y_{(s)}((k-1)s,x))(h^i(ks) - h^i((k-1)s) + R_0(n;s,x)(h),$$

where

$$R_0(n;s,x)(h)$$

$$= \sum_{k=1}^{n}\sum_{i=0}^{d}\int_{(k-1)s}^{ks} ((\nabla Y_{(s)}(r,x))^{-1}V_i(Y_{(s)}(r,x)) - (\nabla Y_{(s)}((k-1)s,x))^{-1}V_i(Y_{(s)}((k-1)s,x))$$

$$\hat{D}\eta^i_{(s)}(r)(h)dr.$$

Since $\eta^i_{(s)}(r)$, $r \in ((k-1)s, ks)$, $i = 0,1,\ldots,d$ is $\sigma\{B(u) - B((k-1)s); u \in ((k-1)s, ks)\} \vee \sigma\{Z_k\}$-measurable, $k = 1,2,\ldots,$ we see that

$$||R_0(n;s,x)||^2_H$$

$$= \sum_{k=1}^{n} ||\sum_{i=1}^{d}\int_{(k-1)s}^{ks} ((\nabla Y_{(s)}(r,x))^{-1}V_i(Y_{(s)}(r,x)) - (\nabla Y_{(s)}((k-1)s,x))^{-1}V_i(Y_{(s)}((k-1)s,x))$$

$$\hat{D}\eta^i_{(s)}(r)dr||^2_H$$

$$\leqq d^2 s \sum_{k=1}^n \sum_{i=1}^d \max_{r\in[(k-1)s,ks]} |(\nabla Y_{(s)}(r,x))^{-1}V_i(Y_{(s)}(r,x)) - (\nabla Y_{(s)}((k-1)s,x))^{-1}V_i(Y_{(s)}((k-1)s,x)|^2$$

$$\times(\int_{(k-1)s}^{ks} ||\hat{D}\eta^i_{(s)}(r)||^2_H dr)$$

and so we see by the assumption (G-4) that

$$\sup_{x\in\mathbf{R}^N} \sup_{s\in(0,1]} s^{-1/3} E[\max_{n=1,2,\dots[T/s]} ||R_0(n;s,x)||^p_H]^{1/p} < \infty$$

for any $p \in (1,\infty)$ and $T > 0$. Note that

$$(\nabla X(t,x))^{-1}\hat{D}X(t,x)(h)$$

$$= \sum_{i=0}^d \int_0^t (\nabla X(r,x))^{-1}V_i(X(r,x))\frac{dh^i}{dr}(r)dr.$$

So by Propositions 5 and 7, we have the assetion for $m = 1$.

Also, we have our assertion inductively in m and α. ∎

Similarly we have the following.

Proposition 10 *Let A and $Z_{(s)}$ be the solutions to Equations* (11) *and* (12). *For any $T > 0$, $m \geqq 0$, $p \in (1,\infty)$ and any multi-index $\alpha \in \mathbf{Z}^N_{\geqq 0}$*

$$\sup_{x\in\mathbf{R}^N} \sup_{s\in(0,1]} s^{-1/3} E[\max_{t\in[0,T]} ||\frac{\partial^\alpha}{\partial x^\alpha}\hat{D}^m(A(t,x) - Z_{(s)}(t,x))||^p_{H^{\otimes m}\otimes\mathbf{R}^M}]^{1/p} < \infty.$$

6 Structure of vector fields

From now on, we assume that that the condition (UFG) and the conditions (G-1)-(G-4) are satisfied.

Let $J^j_i(t,x) = \frac{\partial}{\partial x^i}X^j(t,x)$. Then for any C^∞_b vector field W on $\mathbf{R}^N$, we see that $(X(t)_*W)(X(t,x)) = \sum_{j=1}^N J^i_j(t,x)W^j(x)$, where $X(t)_*$ is a push-forward operator with respect to the diffeomorphism $X(t,\cdot) : \mathbf{R}^N \to \mathbf{R}^N$. Therefore we see that

$$d(X(t)^{-1}_*W)(x) = \sum_{i=0}^d (X(t)^{-1}_*[V_i,W])(x) \circ dB^i(t)$$

for any C^∞_b vector field W on $\mathbf{R}^N$ (cf. [2]). So we have

$$d(X(t)^{-1}_*\Phi(r(u)))(x)$$

$$= \sum_{i=0}^d (X(t)^{-1}_*\Phi(r(v_i u)))(x) \circ dw^i(t),$$

for any $u \in A^* \setminus \{1\}$.

Also, we see that

$$\frac{d}{dt}(Y_{(s)}(t)_*^{-1}\Phi(r(u)))(x)$$
$$= \sum_{i=0}^d (Y_{(s)}(t)_*^{-1}\Phi(r(v_i u)))(x)\eta^i_{(s)}(t),$$

for any $u \in A^* \setminus \{1\}$.

Proposition 11 *There are $\varphi_{u,u'} \in C_b^\infty(\mathbf{R}^N)$, $u \in A^{**}$, $u' \in A^{**}_{\leqq \ell_0}$ such that*

$$\Phi(r(u)) = \sum_{u' \in A^{**}_{\leqq \ell_0}} \varphi_{u,u'}\Phi(r(u')), \qquad u \in A^{**}.$$

Proof. It is obivious that our assertion is valid for $u \in A^{**}_{\leqq \ell_0+2}$. Suppose that our assertion is valid for any $u \in A^{**}_{\leqq m}$, $m \geqq \ell_0$. Then we have for any $i = 0, 1, \ldots, d$ and $u \in A^{**}_{\leqq m}$,

$$\Phi(r(v_i u)) = [V_i, \Phi(r(u))] = \sum_{u' \in A^{**}_{\leqq \ell_0}} [V_i, \varphi_{u,u'}\Phi(r(u'))]$$
$$= \sum_{u' \in A^{**}_{\leqq \ell_0}} (V_i\varphi_{u,u'})\Phi(r(u')) + \sum_{u' \in A^{**}_{\leqq \ell_0}} \varphi_{u,u'}\Phi(r(v_i u'))$$
$$= \sum_{u' \in A^{**}_{\leqq \ell_0}} (V_i\varphi_{u,u'})\Phi(r(u')) + \sum_{u',u'' \in A^{**}_{\leqq \ell_0}} \varphi_{u,u'}\varphi_{u',u''}\Phi(r(u''))$$

So we see that our assertion is valid for any $u \in A^{**}_{\leqq m+1}$. Thus by induction we have our Proposition. ∎

Let $m \geqq \ell_0$. Let $c_i^{(m)}(\cdot, u, u') \in C_b^\infty(\mathbf{R}^N, \mathbf{R})$, $i = 0, 1, \ldots, d$, $u, u' \in A^{**}_{\leqq m}$, be given by

$$c_i^{(m)}(x; u, u') = \begin{cases} 1, & \text{if } ||v_i u|| \leqq m \text{ and } u' = v_i u, \\ \varphi_{v_i u, u'}(x), & \text{if } ||v_i u|| > m \text{ and } ||u'|| \leqq \ell_0, \\ 0, & \text{otherwise.} \end{cases}$$

Here $\varphi_{v_i u, u'}$ is one as in Proposition 11. Then we have

$$d(X(t)_*^{-1}\Phi(r(u)))(x)$$
$$= \sum_{i=0}^d \sum_{u' \in A^{**}_{\leqq m}} (c_i^{(m)}(X(t,x); u, u')(X(t)_*^{-1}\Phi(r(u')))(x) \circ dB^i(t), \quad u \in A^{**}_{\leqq m},$$

and

$$\frac{d}{dt}(Y_{(s)}(t)_*^{-1}\Phi(r(u)))(x)$$
$$= \sum_{i=0}^d \sum_{u' \in A^{**}_{\leqq m}} (c_i^{(m)}(X(t,x); u, u')(Y_{(s)}(t)_*^{-1}\Phi(r(u')))(x)\eta^i_{(s)}(t).$$

Note that $c_i^{(m)}(\cdot;u,u') \in C_b^\infty(\mathbf{R}^N)$. As is shown in [1], there exists a solution $a^{(m)}(t,x;u,u')$, $u,u' \in A^{**}_{\leqq m}$, to the following SDE

$$da^{(m)}(t,x;u,u') = \sum_{i=0}^d \sum_{u'' \in A^{**}_{\leqq m}} (c_i^{(m)}(X(t,x);u,u'')a^{(m)}(t,x;u'',u')) \circ dB^i(t). \tag{15}$$

$$a^{(m)}(0,x;u,u') = \langle u,u' \rangle,$$

such that
(1) $a^{(m)}(t,x;u,u')$ is smooth in x and $\frac{\partial^\alpha}{\partial x^\alpha} a^{(m)}(t,x;u,u')$ is continuous in $(t,x) \in [0,\infty) \times \mathbf{R}^N$ for any multi-indez $\alpha \in \mathbf{Z}^N_{\geqq 0}$ with probability one,
and
(2) for any multi-indez $\alpha \in \mathbf{Z}^N_{\geqq 0}$ and $T > 0$

$$\sup_{x \in \mathbf{R}^N} E[\sup_{t \in [0,T]} |\frac{\partial^\alpha}{\partial x^\alpha} a^{(m)}(t,x;u,u')|^p] < \infty.$$

Then the uniqueness of SDE implies

$$(X(t)_*^{-1}\Phi(r(u)))(x) = \sum_{u' \in A^{**}_{\leqq m}} a^{(m)}(t,x;u,u')\Phi(r(u'))(x), \;\; u \in A^{**}_{\leqq m}. \tag{16}$$

Similarly we see that there exists a unique good solution $b^{(m)}(t,x;u,u')$, $u,u' \in A^{**}_{\leqq m}$, to the SDE

$$b^{(m)}(t,x;u,u') = \langle u,u' \rangle - \sum_{i=0}^d \sum_{u'' \in A^{**}_{\leqq m}} \int_0^t (b^{(m)}(r,x;u,u''))(c_i^{(m)}(X(r,x);u'',u')) \circ dB^i(r). \tag{17}$$

Then we see that

$$\sum_{u'' \in A^{**}_{\leqq m}} a^{(m)}(t,x,u,u'')b^{(m)}(t,x,u'',u) = \langle u,u' \rangle, \qquad u,u' \in A^{**}_{\leqq m},$$

and so we see that

$$\Phi(r(u))(x) = \sum_{u' \in A^{**}_{\leqq m}} b^{(m)}(t,x;u,u')(X(t)_*^{-1}\Phi(r(u')))(x), \;\; u \in A^{**}_{\leqq m}. \tag{18}$$

Also, there exists a solution $a_{(s)}^{(m)}(t,x;u,u')$, $b_{(s)}^{(m)}(t,x;u,u')$ $u,u' \in A^{**}_{\leqq m}$, to the following ODE

$$\frac{d}{dt} a_{(s)}^{(m)}(t,x;u,u') = \sum_{i=0}^d \sum_{u'' \in A^{**}_{\leqq m}} (c_i^{(m)}(Y_{(s)}(t,x);u,u'')a_{(s)}^{(m)}(t,x;u'',u'))\eta^i_{(s)}(t). \tag{19}$$

$$a_{(s)}^{(m)}(0,x;u,u') = \langle u,u' \rangle.$$

and

$$\frac{d}{dt}b^{(m)}_{(s)}(t,x;u,u') = -\sum_{i=0}^{d}\sum_{u''\in A^{**}_{\leqq m}} b^{(m)}_{(s)}(t,x;u',u''))(c^{(m)}_i(Y_{(s)}(t,x);u'',u')\eta^i_{(s)}(t). \tag{20}$$

$$a^{(m)}_{(s)}(0,x;u,u') = \langle u,u'\rangle.$$

Then we see that

$$(Y_{(s)}(t)^{-1}_{*}\Phi(r(u)))(x) = \sum_{u'\in A^{**}_{\leqq m}} a^{(m)}_{(s)}(t,x;u,u')\Phi(r(u'))(x),\ u\in A^{**}_{\leqq m}. \tag{21}$$

$$\Phi(r(u))(x) = \sum_{u'\in A^{**}_{\leqq m}} b^{(m)}_{(s)}(t,x;u,u')(Y_{(s)}(t)^{-1}_{*}\Phi(r(u')))(x),\ u\in A^{**}_{\leqq m}. \tag{22}$$

Then we have the following similarly to proofs of Propositions prop:5 and prop:6.

Proposition 12 *For any $T>0$ $\alpha\in \mathbf{Z}^{\mathbf{N}}_{\geqq 0}$, and $u,u'\in A^{**}_{\leqq m}$*

$$\sup_{x\in\mathbf{R}^N}\sup_{s\in(0,1]} s^{-1/3}E[\max_{t\in[0,T]}||\frac{\partial^\alpha}{\partial x^\alpha}\hat{D}^n(a^{(m)}(t,x;u,u')-a^{(m)}_{(s)}(t,x;u,u'))||^p_{H^{\otimes n}}]^{1/p}<\infty$$

$$\sup_{x\in\mathbf{R}^N}\sup_{s\in(0,1]} s^{-1/3}E[\max_{t\in[0,T]}||\frac{\partial^\alpha}{\partial x^\alpha}\hat{D}^n(b^{(m)}(t,x;u,u')-b^{(m)}_{(s)}(t,x;u,u'))||^p_{H^{\otimes n}}]^{1/p}<\infty$$

Let

$$R^*_{m,0}=A^*_{m-1}\cup A^*_m \qquad R^*_{m,i}=A^*_m,\ i=1,\ldots,d,$$

and

$$R^*_m=\bigcup_{i=0}^{d}\{v_iu; u\in R^*_{m,i}, ||u||=m\}. \tag{23}$$

Then we have the following.

Proposition 13 *For any $m\geqq \ell_0+1$,*

$$\begin{aligned} &a^{(m)}(t,x,u,u')\\ &=\sum_{u_1\in A^*_{\leqq m}}\langle u_1u,u'\rangle B(t;u_1)\\ &+\sum_{u_1\in A^*:u_1u\in R^*_m}\sum_{u_2\in A^{**}_{\leqq \ell_0}} S(\varphi_{u_1u,u_2}(X(\cdot,x))a^{(m)}(\cdot,x,u_2,u'),u_1)(t)\end{aligned}$$

for any $t\in[0,\infty)$, $x\in\mathbf{R}^N$, and $u,u'\in A^{**}_{\leqq m}$.

Proof. Note that for $u, u' \in A^{**}_{\leqq m}$

$$a^{(m)}(t,x;u,u')$$

$$= \langle u,u'\rangle + \sum_{i=0}^{d} \sum_{u_1 \in A^{**}_{\leqq m}} S(c_i^{(m)}(X(\cdot,x);u,u_1)a^{(m)}(\cdot,x;u_1,u'),v_i)(t).$$

So the assertion is obvious from the definition, if $||u|| = m$. If $||u|| = m-1$, we have

$$a^{(m)}(t,x;u,u')$$

$$= \langle u,u'\rangle + \sum_{i=1}^{d} S(\langle v_i u,u'\rangle a^{(m)}(\cdot,x;v_i u,u'),v_i)(t)$$

$$+ \sum_{u_1 \in A^{**}_{\leqq \ell_0}} S(\varphi_{v_0 u,u_1}(X(\cdot,x))a^{(m)}(\cdot,x,u_1,u'),v_0)(t)$$

$$= \langle u,u'\rangle + \sum_{i=1}^{d} \langle v_i u,u'\rangle S(1,v_i)(t)$$

$$+ \sum_{i=1}^{d} \sum_{j=0}^{d} \sum_{u_1 \in A^{**}_{\leqq \ell_0}} S(S(\varphi_{v_j v_i u,u_1}(X(\cdot,x))a^{(m)}(\cdot,x,u_1,u'),v_j),v_i)(t)$$

$$+ \sum_{u_1 \in A^{**}_{\leqq \ell_0}} S(\varphi_{v_0 u,u_1}(X(\cdot,x))a^{(m)}(\cdot,x,u_1,u'),v_0)(t).$$

So we have our assertion. Similarly by induction in $m - ||u||$ we have our assertion. ∎

Corollary 14 *For any $m \geqq \ell_0 + 1$,*

$$a^{(m)}(t,x;u,u')$$

$$= \langle \hat{X}(t)u,u'\rangle$$

$$+ \sum_{u_1 \in A^*: u_1 u \in R^*_m} \sum_{u_2 \in A^*_{\leqq \ell_0}} S(\varphi_{u_1 u,u_2}(X(\cdot,x))a^{(m)}(\cdot,x;u_2,u'),u_1)(t)$$

for any $t \in [0,\infty)$, $x \in \mathbf{R}^N$, and $u,u' \in A^{**}_{\leqq m}$.

Similarly we have the following .

Proposition 15 *For any $m \geqq \ell_0 + 1$,*

$$b^{(m)}(t,x;u,u')$$

$$= \langle \hat{X}(t)^{-1}u,u'\rangle$$

$$+ \sum_{i=0}^{d} \sum_{u_1 \in A^*, u_2 \in A^*_{\leqq \ell_0}: u' = u_1 u_2} \sum_{u_3 \in R^*_{m,i}} \tilde{S}(b^{(m)}(\cdot,x;u,u_3)\varphi_{v_i u_3,u_2}(X(\cdot,x)),u_1)(t)$$

$$= \langle \hat{X}(t)^{-1}u, u' \rangle$$

$$+ \sum_{i=0}^{d} \sum_{u_1 \in A^*, u_2 \in A^*_{\leqq \ell_0} : u' = u_1 u_2} \sum_{u_3 \in R^*_{m,i}} \tilde{S}(\langle X(\cdot)^{-1}u, u_3 \rangle \varphi_{v_i u_3, u_2}(X(\cdot, x)), u_1)(t)$$

$$+ \sum_{i,j=0}^{d} \sum_{u_1 \in A^*, u_2 \in A^*_{\leqq \ell_0} : u' = u_1 u_2} \sum_{u_3 \in R^*_{m,i}} \sum_{u_4 \in A^*, u_5 \in A^*_{\leqq \ell_0} : u_3 = u_4 u_5} \sum_{u_6 \in R^*_{m,j}}$$

$$\tilde{S}(\tilde{S}(b^{(m)}(\cdot, x; u, u_6)\varphi_{v_i u_6, u_5}(X(\cdot, x)), u_4)\varphi_{v_i u_3, u_2}(X(\cdot, x)), u_1)(t)$$

for any $t \in [0, \infty)$, $x \in \mathbf{R}^N$, *and* $u, u' \in A^{**}_{\leqq m}$.

Finally let $y_0 : \mathbf{R} \times \mathbf{R}^N \to \mathbf{R}^N$ be a solution to the following ODE.

$$\frac{d}{dt} y_0(t, x) = V_0(y_0(t, x)), \qquad t \in \mathbf{R}$$

$$y(0, x) = x \in \mathbf{R}^N.$$

Let $c_0(\cdot; u, u') \in C_b^\infty(\mathbf{R}^N)$, $u, u' \in A^*_{\leqq \ell_0}$ be given by

$$c_0(x, u, u') = \begin{cases} 1, & if ||u|| \leqq \ell_0 - 1 \text{ and } u' = v_0 u, \\ \varphi_{v_0 u, u'}(x), & if ||u|| = \ell_0 - 1, \\ 0, & \text{otherwise} . \end{cases}$$

Let $a_0(t, x; u, u')$, $b_0(t, x; u, u')$, $u, u' \in A^*_{\leqq \ell_0}$ be solutions to the following ODE.

$$\frac{d}{dt} a_0(t, x; u, u') = \sum_{u'' \in A^*_{\leqq \ell_0}} c(y_0(t, x), u, u'') a_0(t, x; u'', u')$$

$$\frac{d}{dt} b_0(t, x; u, u') = - \sum_{u'' \in A^*_{\leqq \ell_0}} b_0(t, x; u, u'') c(y_0(t, x), u'', u)$$

$$a_0(0, x, u, u') = b_0(0, x, u, u') = \langle u, u' \rangle$$

Then we see that

$$y_0(t)_*^{-1} r(u) = \sum_{u' \in A^*_{\leqq \ell_0}} a_0(t, x, u, u') r(u') \tag{24}$$

$$r(u) = \sum_{u' \in A^*_{\leqq \ell_0}} b_0(t, x, u, u')(y_0(t)_*^{-1} r(u')) \tag{25}$$

for any $u \in A^*_{\leqq \ell_0}$.

7 A certain class of Wiener functionals

For any separable real Hilbert space E let $\hat{\mathcal{K}}_0(E)$ be the set of $F : (0,\infty) \times \mathbf{R}^N \times \Omega \to E$ such that
(1) $F(t,\cdot,\omega) : \mathbf{R}^N \to E$ is smooth for any $t \in (0,\infty)$ and $\omega \in \Omega$,
(2) $\partial^\alpha F/\partial x^\alpha(\cdot,*,\omega) : (0,\infty) \times \mathbf{R}^N \to E$ is continuous for any $\omega \in \Omega$ and $\alpha \in \mathbf{Z}^N_{\geqq 0}$,
(3) $\partial^\alpha F/\partial x^\alpha(t,x,\cdot) \in \hat{W}^{r,p}$ for any $r,p \in (1,\infty)$, $\alpha \in \mathbf{Z}^N_{\geqq 0}$, $t \in (0,\infty)$ and $x \in \mathbf{R}^N$, and
(4) for any $r,p \in (1,\infty)$, $\alpha \in \mathbf{Z}^N_{\geqq 0}$, and $T > 0$

$$\sup_{t \in (0,T]} \sup_{x \in \mathbf{R}^N} ||\frac{\partial^\alpha}{\partial x^\alpha} F(t,x)||_{\hat{W}^{r,p}} < \infty.$$

Then it is easy to see the following.

Proposition 16 (1) *Let $F \in \hat{\mathcal{K}}_0(E)$ and $\gamma \geqq 0$. Let $\tilde{F}_i : (0,\infty) \times \mathbf{R}^N \times \Omega \to E$, $i = 0,\ldots,d$ be given by*

$$\tilde{F}_0(t,x) = t^{-(\gamma+||v_i||/2)} \int_0^t r^\gamma F(r,x) dB^i(r) \qquad (t,x) \in (0,\infty) \times \mathbf{R}^N.$$

Then $\tilde{F}_i \in \hat{\mathcal{K}}_0(E)$, $i = 0,1\ldots,d$, if we take a good version.
(2) *Let $F_i \in \hat{\mathcal{K}}_0(E)$, $c \in C_b^\infty(\mathbf{R}^N;E)$. Let $\tilde{F} : (0,\infty) \times \mathbf{R}^N \times A^* \times \Omega \to E$, be given by*

$$\tilde{F}(t,x;1) = c(x) + \sum_{i=0}^d \int_0^t F_i(r,x) dB^i(r),$$

and

$$\tilde{F}(t,x;u) = S(\tilde{F}(\cdot,x);u)(t)$$

for $(t,x) \in (0,\infty) \times \mathbf{R}^N$. Then $t^{-||u||/2}\tilde{F}(t,x;u) \in \hat{\mathcal{K}}_0(E)$, if we take a good version.

Let us define $k^{(m)} : [0,\infty) \times \mathbf{R}^N \times A^{**}_{\leqq m} \times \Omega \to H$ by

$$k^{(m)}(t,x;u) = (\int_0^{t\wedge\cdot} a^{(m)}(r,x;v_i,u)dr)_{i=1,\ldots d}.$$

Let $M^{(m)}(t,x) = \{M^{(m)}(t,x;u,u')\}_{u,u' \in A^{**}_{\leqq m}}$ be a matrix-valued random variable given by

$$M^{(m)}(t,x;u,u') = t^{-(||u||+||u'||)/2}(k^{(m)}(t,x;u), k^{(m)}(t,x;u'))_H.$$

Then it has been shown in [1]

$$\sup_{t \in (0,T]} \sup_{x \in \mathbf{R}^N} E^P[|det M^{(m)}(t,x)|^{-p}] < \infty \text{ for any } p \in (1,\infty) \text{ and } T > 0.$$

Let $M^{(m)-1}(t,x) = \{M^{(m)-1}(t,x;u,u')\}_{u,u' \in A^{**}_{\leqq m}}$ be the inverse matrix of $M^{(m)}(t,x)$

Note that

$$||\hat{D}^n \frac{\partial^\alpha}{\partial x^\alpha} k^{(m)}(t,x;u)||^2_{H^{\otimes(n+1)}} = \int_0^t ||\hat{D}^n \frac{\partial^\alpha}{\partial x^\alpha} a^{(m)}(r,x;v_i u)||^2_{H^{\otimes n}} dr$$

Therefore we have the following by Corollary 14, Propositions 15 and 16.

Proposition 17 *Let $m \geqq 2\ell_0 + 1$.*
(1) $a^{(m)}(t,x;u,u')$, $b^{(m)}(t,x;u,u')$, $M^{(m)}(t,x;u,u')$, and $M^{(m)-1}(t,x;u,u')$ belong to $\hat{\mathcal{K}}_0(\mathbf{R})$ for any $u,u' \in A^{**}_{\leqq m}$.
(2) $t^{-(m-||u||)/2}(a^{(m)}(t,x;u,u') - \langle \hat{X}(t)u,u'\rangle)$ and $t^{-(m-||u||)/2}(b^{(m)}(t,x;u,u') - \langle \hat{X}(t)^{-1}u,u'\rangle)$ belong to $\hat{\mathcal{K}}_0(\mathbf{R})$ for any $u,u' \in A^{**}_{\leqq m}$. In particular, $t^{-(||u'||-||u||)/2}a^{(m)}(t,x;u,u')$ and $t^{-(||u'||-||u||)/2}b^{(m)}(t,x;u,u')$ belong to $\hat{\mathcal{K}}_0(\mathbf{R})$ for any $u,u' \in A^{**}_{\leqq m}$.
(3) $t^{-||u||/2}k^{(m)}(t,x;u)$ belongs to $\hat{\mathcal{K}}_0(H)$ for any $u,u' \in A^{**}_{\leqq m}$.

Let us define $k^{(m)}_{(s)} : [0,\infty) \times \mathbf{R}^N \times A^{**}_{\leqq m} \times \Omega \to H$ by

$$k^{(m)}_{(s)}(t,x;u) = \sum_{i=0}^{d} \int_0^t a^{(m)}_{(s)}(r,x;v_i,u) D\eta^i_{(s)}(r)(h)dr.$$

Note that

$$(k^{(m)}_{(s)}(t,x;u),h)_H = \sum_{i=0}^{d} \sum_{k=1}^{[t/s]} a^{(m)}_{(s)}((k-1)s,x;v_i,u)(h^i(ks) - h^i((k-1)s))$$

$$+ \sum_{i=0}^{d} \sum_{k=1}^{[t/s]} \int_{(k-1)s}^{ks} (a^{(m)}_{(s)}(r,x;v_i,u) - a^{(m)}_{(s)}((k-1)s,x;v_i,u)) D\eta^i_{(s)}(r)(h)dr$$

$$+ \sum_{i=0}^{d} \int_{[t/s]s}^{t} a^{(m)}_{(s)}(r,x;v_i,u) D\eta^i_{(s)}(r)(h)dr,$$

and so we see that

$$k^{(m)}_{(s)}(t,x;u) = (\int_0^{s[t/s]\wedge\cdot} a^{(m)}_{(s)}(s[r/s],x;v_i,u)dr)_{i=1,\dots,d}$$

$$+ \int_0^{s[t/s]} (a^{(m)}_{(s)}(r,x;v_i,u) - a^{(m)}_{(s)}(s[r/s],x;v_i,u)) D\eta^i_{(s)}(r)dr$$

$$+ \sum_{i=0}^{d} \int_{[t/s]s}^{t} a^{(m)}_{(s)}(r,x;v_i,u) D\eta^i_{(s)}(r)dr. \tag{26}$$

Therefore we see that

$$||k^{(m)}(t,x;u) - k^{(m)}_{(s)}(t,x;u)||^2_H$$

$$\leqq 5 \sum_{i=1}^{d} \int_0^{s[t/s]} |a^{(m)}(s[r/s],x;v_i,u) - a^{(m)}_{(s)}(s[r/s],x;v_i,u)|^2$$

$$+5 \sum_{i=1}^{d} \int_0^{s[t/s]} |a^{(m)}(r,x;v_i,u) - a^{(m)}(s[r/s],x;v_i,u)|^2 dr$$

$$+5(d+1)s \sum_{i=0}^{d} \sum_{k=1}^{[t/s]} \max_{r\in[(k-1)s,ks]} |a^{(m)}_{(s)}(r,x;v_i,u) - a^{(m)}_{(s)}(s[r/s],x;v_i,u))|^2 \int_{(k-1)s}^{ks} ||D\eta^i_{(s)}(r)||^2_H dr$$

$$+5s\sum_{i=0}^{d}\max_{r\in[s[t/s],t]}|a^{(m)}_{(s)}(r,x;v_i,u)|^2\int_{[t/s]s}^{t}||D\eta^i_{(s)}(r)||^2dr+5s\sum_{i=0}^{d}\max_{r\in[s[t/s],t]}|a^{(m)}(r,x;v_i,u)|^2$$

This implies that

$$\sup_{s\in(0,1]}s^{-1/3}\sup_{x\in\mathbf{R}^N}E^P[\sup_{t\in[0,T]}||k^{(m)}(t,x;u)-k^{(m)}_{(s)}(t,x;u)||^p_H]^{1/p}<\infty \tag{27}$$

for any $u\in A^{**}_{\leqq m}$ and $T>0$.

Also note that

$$\frac{\partial^\alpha}{\partial x^\alpha}\hat{D}^n k^{(m)}_{(s)}(t,x;u)$$

$$=(\int_0^{s[t/s]\wedge\cdot}\frac{\partial^\alpha}{\partial x^\alpha}D^n a^{(m)}_{(s)}(s[r/s],x;v_i,u)dr)_{i=1,\dots,d}$$

$$+\int_0^{s[t/s]}\frac{\partial^\alpha}{\partial x^\alpha}\hat{D}^n(a^{(m)}_{(s)}(r,x;v_i,u)-a^{(m)}_{(s)}(s[r/s],x;v_i,u))D\eta^i_{(s)}(r)dr$$

$$+\sum_{i=0}^{d}\int_{[t/s]s}^{t}\frac{\partial^\alpha}{\partial x^\alpha}\hat{D}^n a^{(m)}_{(s)}(r,x;v_i,u)D\eta^i_{(s)}(r)dr.$$

So by a similar argument we have the following.

Proposition 18 *For any $n\geqq 0$, $\alpha\in\mathbf{Z}^N_{\geqq 0}$, $u\in A^{**}_{\leqq m}$ and $T>0$ we have*

$$\sup_{s\in(0,1]}s^{-1/3}\sup_{x\in\mathbf{R}^N}E^P[\sup_{t\in[0,T]}||\frac{\partial^\alpha}{\partial x^\alpha}\hat{D}^n(k^{(m)}(t,x;u)-k^{(m)}_{(s)}(t,x;u))||^p_{H^{\otimes(n+1)}}]^{1/p}<\infty$$

8 Random linear operators

Let N_k, $k=0,1,\dots,$ be the dimension of $\mathbf{R}$-vector space $\mathbf{R}^{**}\langle A\rangle_{\leqq k}$. Then there are a basis $\{e_n\}_{n=0}^\infty$ of $\mathbf{R}^{**}\langle A\rangle$ such that $e_0=1$, and that $\{e_n\}_{n=N_{k-1}}^{N_k-1}$ is a basis of $\mathbf{R}^{**}\langle A\rangle_k$, $k=1,2,\dots$ For each e_i belongs to $\mathbf{R}^{**}\langle A\rangle_k$ for some $k\geqq 0$. We denote this k by $||e_i||$.

Let us define random linear operators $U(t)$, $U_{(s)}(t)$, and $U_0(t)$ in $C^\infty(\mathbf{R}^N)$ by

$$(U(t)f)(x)=f(X(t,x)),\qquad (U_{(s)}(t)f)(x)=f(Y_{(s)}(t,x)),$$

for $t\in[0,\infty)$ and $f\in C^\infty(\mathbf{R}^N)$, and

$$U_0(t)=Exp(tV_0),\qquad U_{(s),0}(t)=Exp((\int_0^t\eta^0_{(s)}(r)dr)V_0)\qquad t\in[0,\infty).$$

Then we have

$$dU(t)=\sum_{i=0}^{d}U(t)\Phi(v_i)\circ dB^i(t)$$

$$\frac{d}{dt}U_{(s)}(t)=\sum_{i=0}^{d}U_{(s)}(t)\Phi(v_i)\eta^i_{(s)}(t)$$

$$\frac{d}{dt}U_0(t) = U_0(t)\Phi(v_0) = \Phi(v_0)U_0(t)$$

$$\frac{d}{dt}U_0(t)^{-1} = -U_0(t)^{-1}\Phi(v_0) = -\Phi(v_0)U_0(t)^{-1},$$

and

$$\frac{d}{dt}U_{(s),0}(t)^{-1} = -\eta_{(s)}(t)\Phi(v_0)U_0(t)^{-1}.$$

Note that for any $u \in A^{**}_{\leqq m}$

$$(U(t)\Phi(r(u))f)(x) = \langle X(t)^* df, X(t)_*^{-1}\Phi(r(u))\rangle_x$$

$$= \sum_{u' \in A^{**}_{\leqq m}} \langle X(t)^* df, a^{(m)}(t, x; u, u')\Phi(r(u'))\rangle_x = \sum_{u' \in A^{**}_{\leqq m}} a^{(m)}(t, x; u, u')(\Phi(r(u'))U(t)f)(x).$$

Let $a^{(m)}(t; u, u')$, $u, u' \in A^{**}_{\leqq m}$, be multiplier operators in $C^\infty(\mathbf{R}^N)$ defined by

$$(a^{(m)}(t; u, u')f)(x) = a^{(m)}(t, x; u, u')f(x).$$

Then we have

$$U(t)\Phi(r(u)) = \sum_{u' \in A^{**}_{\leqq m}} a^{(m)}(t; u; u')\Phi(r(u'))U(t).$$

So we have the following.

Proposition 19 *For any* $n \geqq 1$ *and* $u_1, \ldots, u_n \in A^{**}_{\leqq m}$,

$$U(t)\Phi(r(u_1)\cdots r(u_n))$$

$$= \sum_{k=1}^{n} \sum_{u'_1, \ldots, u'_k \in A^{**}_{\leqq m}} a^{(m)}(t; u_1, \ldots, u_n; u'_1, \ldots, u'_k)\Phi(r(u'_1))\cdots r(u'_k))U(t),$$

where $a^{(m)}(t; u_1, \ldots, u_{n+1}; u'_1, \ldots, u'_k)$*'s are multiplier operators iductively defined by*

$$a^{(m)}(t; u_1, \ldots, u_{n+1}; u'_1, \ldots, u'_k)$$

$$= a^{(m)}(t, u_1; u'_1)a^{(m)}(t; u_2, \ldots, u_n; u'_2, \ldots, u'_k)$$

$$+ \sum_{\tilde{u} \in A^{**}_{\leqq m}} a^{(m)}(t; u_1, \tilde{u})[\Phi(r(\tilde{u})), a^{(m)}(t; u_2, \ldots, u_n; u'_1, \ldots, u'_k)].$$

In particular, $a^{(m)}(t; u_1, \ldots, u_n; u'_1, \ldots, u'_k)$*'s are multiplier operators multiplying* $a^{(m)}(t, x; u_1, \ldots, u_{n+1}; u'_1, \ldots, u'_k)$ *belonging to* $\hat{\mathcal{K}}_0(\mathbf{R})$.

Similarly we have the following.

Proposition 20 *For any* $n \geqq 1$ *and* $u_1, \ldots, u_n \in A^{**}_{\leqq m}$,

$$U_{(s)}(t)\Phi(r(u_1)\cdots r(u_n))$$

$$= \sum_{k=1}^{n} \sum_{u'_1,\ldots,u'_k \in A^{**}_{\leqq m}} a^{(m)}_{(s)}(t; u_1, \ldots, u_n; u'_1, \ldots, u'_k)\Phi(r(u'_1)) \cdots r(u'_k))U_{(s)}(t),$$

where $a^{(m)}_{(s)}(t, x; u_1, \ldots, u_{n+1}; u'_1, \ldots, u'_k)$*'s are multiplier operators inductively defined by the following.* $a^{(m)}_{(s)}(t; u, u')$, $u, u' \in A^{**}_{\leqq m}$, *are multiplier operators in* $C^\infty(\mathbf{R}^N)$ *defined by*

$$(a^{(m)}_{(s)}(t; u, u')f)(x) = a^{(m)}(t, x; u, u')f(x),$$

and

$$\begin{aligned} &a^{(m)}_{(s)}(t; u_1, \ldots, u_{n+1}; u'_1, \ldots, u'_k) \\ &= a^{(m)}_{(s)}(t; u_1; u'_1)a^{(m)}_{(s)}(t; u_2, \ldots, u_n; u'_2, \ldots, u'_k) \\ &+ \sum_{\tilde{u} \in A^{**}_{\leqq m}} a^{(m)}_{(s)}(t; , u_1, \tilde{u})[\Phi(r(\tilde{u})), a^{(m)}_{(s)}(t; u_1, \ldots, u_n; u'_1, \ldots, u'_k)]. \end{aligned}$$

In particular, $a^{(m)}_{(s)}(t; u_1, \ldots, u_n; u'_1, \ldots, u'_k)$*'s are multiplier operators multiplying* $a^{(m)}_{(s)}(t, x; u_1, \ldots, u_{n+1}; u'_1, \ldots, u'_k)$ *such that*

$$\sup_{s\in(0,1]} \sup_{t\in(0,T]} \sup_{x\in\mathbf{R}^N} || \frac{\partial^\alpha}{\partial x^\alpha} a^{(m)}_{(s)}(t, x; u_1, \ldots, u_{n+1}; u'_1, \ldots, u'_k)||_{\hat{W}^{r,p}} < \infty.$$

for any $r, p \in (1, \infty)$, $\alpha \in \mathbf{Z}^N_{\geqq 0}$, *and* $T > 0$.

By the above two Propositions, we have the following.

Proposition 21 *For any* $i \geqq 0$, *there are* $M_i \geqq 1$ *and* a_{ij}, $a_{(s),ij} \in \hat{\mathcal{K}}_0(\mathbf{R})$, $j = 0, 1, \ldots, M_i$, $s \in (0, 1]$ *satisfying the following.*
(1) *For any* $j = 1, 2, \ldots, M_i$, $T > 0$, $r, p \in (1, \infty)$ *and* $\alpha \in \mathbf{Z}^N_{\geqq 0}$,

$$\sup_{s\in(0,1]} \sup_{t\in(0,T]} \sup_{x\in\mathbf{R}^N} || \frac{\partial^\alpha}{\partial x^\alpha} a_{(s),ij}(t, x)||_{\hat{W}^{r,p}} < \infty.$$

(2)*For any* $t \geqq 0$,

$$U(t)\Phi(e_i) = \sum_{j=0}^{M_i} \Phi(e_j)a_{ij}(t)U(t)$$

and

$$U_{(s)}(t)\Phi(e_i) = \sum_{j=0}^{M_i} \Phi(e_j)a_{(s),ij}(t)U_{(s)}(t).$$

Here $a_{ij}(t)$ *and* $a_{(s),ij}(t)$ *are multiplier operators multiplying* $a_{ij}(t, x)$ *and* $a_{(s),ij}(t, x)$ *respectively.*

Similarly we have the following.

Proposition 22 *For any $i \geqq 0$, there are $M_i \geqq 1$ and b_{ij}, $b_{(s),ij} \in \hat{\mathcal{K}}_0(\mathbf{R})$, $j = 0, 1, \ldots, M_i$, $s \in (0,1]$ satisfying the following.*
(1) *For any $j = 1, 2, \ldots, M_i$, $T > 0$, $r, p \in (1, \infty)$ and $\alpha \in \mathbf{Z}^N_{\geqq 0}$,*

$$\sup_{s\in(0,1]} \sup_{t\in(0,T]} \sup_{x\in\mathbf{R}^N} \left|\left|\frac{\partial^\alpha}{\partial x^\alpha} b_{(s),ij}(t,x)\right|\right|_{\hat{W}^{r,p}} < \infty.$$

(2)*For any $t \geqq 0$,*

$$\Phi(e_i)U(t) = \sum_{j=0}^{M_i} b_{ij}(t)U(t)\Phi(e_j)$$

and

$$\Phi(e_i)U_{(s)}(t) = \sum_{j=0}^{M_i} b_{(s),ij}(t)U_{(s)}(t)\Phi(e_j).$$

Here $b_{ij}(t)$ and $b_{(s),ij}(t)$ are multiplier operators multiplying $b_{ij}(t,x)$ and $b_{(s),ij}(t,x)$ respectively.

Also, by Equations (24) and (25), we have the following.

Proposition 23 *For any $i \geqq 0$, there are $M_i \geqq 1$ and a continuous map $c_{ijk} : [0,\infty) \to C_b^\infty(\mathbf{R}^N)$, $j = 0, 1, \ldots, M_i$, $k = 0, 1$ satisfying the following.*

$$\Phi(e_i)U_0(t) = \sum_{j=0}^{M_i} c_{ij1}(t)U_0(t)\Phi(e_j)$$

and

$$U_0(t)\Phi(e_i) = \sum_{j=0}^{M_i} \Phi(e_j)c_{ij1}(t)U_0(t).$$

As a corollary to Propositions 21 and 22, we have the following.

Proposition 24 *For any $i \geqq 0$, there are $M_i \geqq 1$ and linear operators $R_{ijk}(t)$, in $C_b^\infty(\mathbf{R}^N)$, $t \geqq 0$, $s \in (0,1]$, $j = 1, \ldots, M_i$ $k = 0, 1$, such that*
(1) *For any $T > 0$, there is a $C > 0$ such that*

$$||R_{ik0}(t)f||_\infty + ||R_{ik1}(t)f||_\infty \leqq C||f||_\infty$$

for any $f \in C_b^\infty(\mathbf{R}^N)$, $t \in (0,T]$ $j = 0, \ldots, M_i$.
(2) *For any $t \geqq 0$*

$$P_t\Phi(e_i) = \sum_{j=0}^{M_i} \Phi(e_j)R_{i,k,0}(t),$$

and

$$\Phi(e_i)P_t = \sum_{j=0}^{M} R_{i,j,1}(t)\Phi(e_j).$$

Let $\tilde{a}^{(m)}(t,x;u,u')$, $u,u' \in A^{**}_{\leqq m}$, be given by

$$\tilde{a}^{(m)}(t,x;u,u') = t^{(||u||-||u'||)/2} a^{(m)}(t,x;u,u')$$

and let $\tilde{a}^{(m)}(t;u,u')$, be a corresponding multiplier operators in $C^\infty(\mathbf{R}^N)$. By Proposition 17, we see that $\tilde{a}^{(m)}(\cdot,*;u,u')$ belongs to $\mathcal{K}_0(\mathbf{R})$.

Then we have

$$t^{||u||/2} U(t)\Phi(r(u)) = \sum_{u' \in A^{**}_{\leqq m}} \tilde{a}^{(m)}(t;u;u') t^{||u'||/2} \Phi(r(u'))U(t),$$

where $\tilde{a}^{(m)}(t;u,u')$ is a multiplier given by

$$(\tilde{a}^{(m)}(t;u,u')f)(x) = \tilde{a}^{(m)}(t,x;u,u')f(x).$$

So we have the following.

Proposition 25 *For any $n \geqq 1$ and $u_1,\ldots,u_n \in A^{**}_{\leqq m}$,*

$$t^{(||u_1||+\ldots+||u_n||)/2} U(t)\Phi(r(u_1)\cdots r(u_n))$$

$$= \sum_{k=1}^{n} \sum_{u'_1,\ldots,u'_k \in A^{**}_{\leqq m}} \tilde{a}^{(m)}(t;u_1,\ldots,u_n;u'_1,\ldots,u'_k) t^{(||u'_1||+\ldots+||u'_k||)/2} \Phi(r(u'_1))\cdots r(u'_k))U(t),$$

where $\tilde{a}^{(m)}(t;u_1,\ldots,u_{n+1};u'_1,\ldots,u'_k)$'s are multiplier operators inductively defined by

$$\begin{aligned} &\tilde{a}^{(m)}(t;u_1,\ldots,u_{n+1};u'_1,\ldots,u'_k) \\ &= \tilde{a}^{(m)}(t,u_1;u'_1) a^{(m)}(t;u_2,\ldots,u_n;u'_2,\ldots,u'_k) \\ &+ \sum_{\tilde{u} \in A^{**}_{\leqq m}} \tilde{a}^{(m)}(t;u_1,\tilde{u}) t^{||\tilde{u}||/2} [\Phi(r(\tilde{u})), a^{(m)}(t;u_2,\ldots,u_n;u'_1,\ldots,u'_k)]. \end{aligned}$$

In particular, $\tilde{a}^{(m)}(t;u_1,\ldots,u_n;u'_1,\ldots,u'_k)$'s are multiplier operators multiplying $\tilde{a}^{(m)}(t,x;u_1,\ldots,u_{n+1};u'_1,\ldots,u'_k)$ belonging to $\hat{\mathcal{K}}_0(\mathbf{R})$.

By the above Propositions, we have the following.

Proposition 26 *For any $i \geqq 0$, there are $M_i \geqq 1$ and $\tilde{a}_{ij} \in \hat{\mathcal{K}}_0(\mathbf{R})$, $j = 0,1,\ldots,M_i$, such that*

$$t^{||e_i||/2} U(t)\Phi(e_i) = \sum_{j=0}^{M_i} t^{||e_j||/2} \Phi(e_j) \tilde{a}_{ij}(t) U(t), \qquad t > 0.$$

Here $a_{ij}(t)$ is a multiplier operators multiplying $\tilde{a}_{ij}(t,x)$.

Similarly we have the following.

Proposition 27 *For any $i \geqq 0$, there are $M_i \geqq 1$ and $\tilde{b}_{ij} \in \hat{\mathcal{K}}_0(\mathbf{R})$, $j = 0,1,\ldots,M_i$, such that*

$$t^{||e_i||/2} \Phi(e_i) U(t) = \sum_{j=0}^{M_i} t^{||e_j||/2} \tilde{b}_{ij}(t) U(t) \Phi(e_j).$$

Here $\tilde{b}_{ij}(t)$ is a multiplier operator multiplying $\tilde{b}_{ij}(t,x)$.

Note that
$$X(t)_*(x)^{-1}\hat{D}X(t,x) = (\int_0^{t\wedge\cdot}(X(r)_*^{-1}V_i)(x)dr)_{i=1,\dots d}.$$
Then we see that
$$X(t)_*(x)^{-1}\hat{D}X(t,x) = \sum_{u\in A^{**}_{\leqq m}} k^{(m)}(t,x;u)\Phi(r(u))(x),$$
and so we have
$$\begin{aligned}\hat{D}(f(X(t,x))) &= \langle (X(t)^*df)(x), X(t)_*(x)^{-1}\hat{D}X(t,x)\rangle \\ &= \sum_{u\in A^{**}_{\leqq m}} (\Phi(r(u))U(t)f)(x)k^{(m)}(t,x;u) \qquad (28)\end{aligned}$$
for any $f\in C_b^\infty(\mathbf{R}^N)$.

Then we have the following.

Proposition 28 *For any $u\in A^{**}_{\leqq m}$ and $F\in\tilde{\mathcal{K}}_0(\mathbf{R})$, we have*
$$t^{||u||/2}E^P[F(t,x)(\Phi(r(u))U(t)f)(x))] = E^P[(\mathcal{R}(u)F)(t,x)(U(t)f)(x)],$$
where
$$(\mathcal{R}(u)F)(t,x) = \sum_{u'\in A_{\leqq m}} \hat{D}^*(M^{(m)-1}(t,x;u,u'))F(t,x)t^{-||u'||/2}k^{(m)}(t,x,u))$$
for any $f\in C_b^\infty(\mathbf{R}^N)$, $t>0$ and $x\in\mathbf{R}^N$. Moreover $\mathcal{R}(u)F$ belongs to $\tilde{\mathcal{K}}_0(\mathbf{R})$.

Then we have the following.

Proposition 29 *For any $i,j\geqq 0$ and $F\in\tilde{\mathcal{K}}_0(\mathbf{R})$, there is an $F_{ij}\in\tilde{\mathcal{K}}_0(\mathbf{R})$ such that*
$$t^{(||e_i||+||e_j||)/2}E^P[F(t,x)(\Phi(e_i)U(t)\Phi(e_j)f)(x))] = E^P[F_{ij}(t,x)(U(t)f)(x)].$$

9 Basic lemma

Let $Q_{(s)}(t)$, $t>0, s\in(0,1]$ be linear operators in $C_b^\infty(\mathbf{R}^N)$ given by
$$(Q_{(s)}(t)f)(x) = E^P[f(Y_{(s)}(t,x))], \qquad f\in C_b^\infty(\mathbf{R}^N).$$
In this section, we prove the following lemma

Lemma 30 *There are linear operators $Q_{(s),0}(t)$, and $Q_{(s),1}(t)$, $t>0, s\in(0,1]$, in $C_b^\infty(\mathbf{R}^N)$ satisfying the following.*
(1) $Q_{(s)}(t) = Q_{(s),0}(t) + Q_{(s),1}(t)$.
(2) *For any $w,w'\in\mathbf{R}^{**}\langle A\rangle$ and $T_1>T_0>0$, there is a $C>0$ such that*
$$||\Psi(w)Q_{(s),0}(t)\Psi(w')f||_\infty \leqq C||f||_\infty$$
for any $t\in[T_0,T_1]$, $s\in(0,1]$, and any $f\in C_b^\infty(\mathbf{R}^N)$.
(3) *For any $n\geqq 1$ and $T_1>T_0>0$, there is a $C>0$ such that*
$$||Q_{(s),1}(t)f||_\infty \leqq Cs^{-n}||f||_\infty$$
for any $t\in[T_0,T_1]$, $s\in(0,1]$, and any $f\in C_b^\infty(\mathbf{R}^N)$.

We make some preparations to prove this lemma.

Let $M^{(m)}_{(s)}(t,x) = \{M^{(m)}_{(s)}(t,x;u,u')\}_{u,u'\in A^{**}_{\leqq m}}$ be a matrix-valued random variable given by

$$M^{(m)}_{(s)}(t,x;u,u') = t^{-(||u||+||u'||)/2}(k^{(m)}_{(s)}(t,x;u), k^{(m)}_{(s)}(t,x;u'))_H.$$

Then we have

$$(\hat{D}(f(Y_{(s)}(t,x))), k^{(m)}_{(s)}(t,x))_H = \sum_{u\in A^{**}_{\leqq m}} M^{(m)}_{(s)}(t,x;u,u')(\Phi(r(u))f)(x)k_{(s)}(t,x.u).$$

Let $\delta^{(m)}_{(s)}(t,x)$, $t>0$, $x\in\mathbf{R}^N$, $s\in(0,1]$ be given by

$$\delta^{(m)}_{(s)}(t,x) = \det M^{(m)}(t,x)^{-1}\det M^{(m)}_{(s)}(t,x) - 1$$

Then we see that

$$\sup_{s\in(0,1]} s^{-1/3}\sup_{x\in\mathbf{R}^N}\sup_{t\in(0,T]} t^{\gamma_m}||\frac{\partial^\alpha}{\partial x^\alpha}\delta^{(m)}_{(s)}(t,x)||_{\hat{W}^{r,p}} < \infty \tag{29}$$

for any $T>0$, $r,p\geqq 1$ and $\alpha\in\mathbf{Z}^N_{\geqq 0}$. Here

$$\gamma_m = \sum_{u\in A^{*}_{\leqq m}} ||u||.$$

Let us define $M^{(m)-1}_{(s)}(t,x) = \{M^{(m)-1}_{(s)}(t,x;u,u')\}_{u,u'\in A^{**}_{\leqq m}}$ be a matrix-valued random variable given by

$$M^{(m)-1}_{(s)}(t,x) = \lim_{\varepsilon\downarrow 0} M^{(m)}_{(s)}(t,x)(\varepsilon I_{A^{**}_{\leqq m}} + M^{(m)}_{(s)}(t,x))^{-2}.$$

Then one can easy to see that for any $\varphi\in C^\infty_0((-1/2,1/2))$, $\varphi(\delta^{(m)}_{(s)}(t,x))M^{(m)-1}_{(s)}(t,x;u,u')$ belongs to $\hat{W}^{r,p}$ for all $r,p\in(1,\infty)$,

$$\sum_{u_2\in A^{**}_{\leqq m}} (\varphi(\delta^{(m)}_{(s)}(t,x))M^{-1}_{(s)}(t,x;u_1,u_2)M^{(m)}_{(s)}(t,x;u_2,u_3) = \langle u_1,u_3\rangle, \qquad u_1,u_3\in A^{**}_{\leqq m},$$

and

$$\sup_{s\in(0,1]} s^{-1/3}\sup_{x\in\mathbf{R}^N}\sup_{t\in(0,T]} t^{(r+1+|\alpha|)\gamma_m}||\frac{\partial^\alpha}{\partial x^\alpha}(\varphi(\delta^{(m)}_{(s)}(t,x))M^{(m)-1}_{(s)}(t,x;u_1,u_2))||_{\hat{W}^{r,p}} < \infty \tag{30}$$

for any $T>0$, $r,p\in(1,\infty)$, $\alpha\in\mathbf{Z}^N_{\geqq 0}$ and $u_1,u_3\in A^{**}_{\leqq m}$.

Note that

$$\frac{d}{dt}\hat{D}Y_{(s)}(t,x)(h) = \sum_{i=0}^d V_i(Y_{(s)}(t,x))\hat{D}\eta^i_{(s)}(t)(h) + \sum_{i=0}^d (\nabla V_i)(Y_{(s)}(t,x))\hat{D}Y_{(s)}(t,x)(h)\eta^i_{(s)}(t)$$

Therefore we have

$$Y_{(s)}(t)^{-1}_*\hat{D}Y_{(s)}(t)(h))(x) = \sum_{i=0}^d \int_0^t (Y_{(s)}(r)^{-1}_* V_i)(x)\eta^i_{(s)}(t)(h)$$

Then we see that for any $f \in C_b^\infty(\mathbf{R}^N)$

$$\hat{D}(f(Y_{(s)}(t,x))) = \langle (Y_{(s)}^* df)(x), Y_{(s)}(t)_*(x)^{-1}\hat{D}Y_{(s)}(t,x)\rangle$$

$$= \sum_{u \in A_{\leqq m}^{**}} (\Phi(r(u))U_{(s)}(t)f)(x)k_{(s)}(t,x.u).$$

$$= \sum_{u,u' \in A_{\leqq m}^{**}} b^{(m)}(t,x;u,u')(U_{(s)}(t)\Phi(r(u'))f)(x)k_{(s)}^{(m)}(t,x.u). \tag{31}$$

Then we have the following by using Equation (31).

Proposition 31 *Let $\varphi, \psi \in C_0^\infty((-1/2,1/2))$ and $F : (0,\infty) \times \mathbf{R}^N \to \hat{W}^{\infty,\infty-}$ be a continuous map. We assume that $\psi = 1$ in the neighborhood of the clousure of $\{z \in (-1/2,1/2);\ \varphi(z) > 0\}$. Then we see that for any $u \in A_{\leqq m}^{**}$*

$$E^P[F(t,x)\varphi(\delta_{(s)}^{(m)})(U_{(s)}(t)\Phi(r(u))f)(x))]$$

$$= E^P[(\mathcal{R}_{(s)}F)(t,x;u,\varphi)\psi(\delta_{(s)}^{(m)}(t,x))(U_{(s)}(t)f)(x)],$$

where

$$(\mathcal{R}_{(s)}F)(t,x;u,\varphi)$$

$$= \sum_{u_1,u_2 \in A_{\leqq m}^{**}} \hat{D}^*(\varphi(\delta_{(s)}^{(m)}(t,x))M_{(s)}^{-1}(t,x;u_1,u_2)a^{(m)}(t,x;u,u_2)F(t,x)k_{(s)}^{(m)}(t,x,u_1))$$

for any $f \in C_b^\infty(\mathbf{R}^N)$, $t > 0$ and $x \in \mathbf{R}^N$. Moreover, $(\mathcal{R}_{(s)}F)(t,x;u,\varphi)\psi(\delta_{(s)}^{(m)}(t,x))$ is independent of a choice of ψ.

Let $\varphi, \psi \in C_0^\infty((-1/2,1/2))$ such that $\psi = 1$ in the neighborhood of the clousure of $\{z \in (-1/2,1/2);\ \varphi(z) > 0\}$. Then for any $n \geqq 1$ we can find $\varphi_k \in C_0^\infty((-1/2,1/2))$, $k = 0,1,\ldots,n$, such that $\varphi_0 = \varphi$, $\varphi_n = \psi$, and that $\varphi_k = 1$ in the neighborhood of the clousure of $\{z \in (-1/2,1/2);\ \varphi_{k-1}(z) > 0\}$, $k = 1,\ldots,n$. Then we see that for any $u_1,\ldots,u_n \in A_{\leqq m}^{**}$ and continuous map $F : (0,\infty) \times \mathbf{R}^N \to \hat{W}^{\infty,\infty-}$

$$E^P[F(t,x)\varphi(\delta_{(s)}^{(m)})(\Phi(r(u_1)\cdots r(u_n)U_{(s)}(t)f)(x)]$$

$$= E^P[(\mathcal{R}_{(s)}F)(t,x;u_1,\ldots,u_n,\varphi)\psi(\delta_{(s)}^{(m)}(t,x))(U_{(s)}(t)f)(x)],$$

where

$$(\mathcal{R}_{(s)}F)(t,x;u_1,\ldots,u_n,\varphi) = (\mathcal{R}_{(s)}(u_n,\varphi_{n-1})\cdots\mathcal{R}_{(s)}(u_1,\varphi_0)F)(t,x)$$

for any $f \in C_b^\infty(\mathbf{R}^N)$, $t > 0$ and $x \in \mathbf{R}^N$.

So conbining this with Proposition 31 we have the following.

Proposition 32 *Let $\varphi, \psi \in C_0^\infty((-1/2,1/2))$ such that $\psi = 1$ in the neighborhood of the clousure of $\{z \in (-1/2,1/2);\ \varphi(z) > 0\}$. For any $i,j \geqq 0$, $T_1 > T_0 > 0$ and $F \in \tilde{\mathcal{K}}_0$, there is an $F' \in \tilde{\mathcal{K}}_0$, such that*

$$E^P[F(t,x)\varphi(\delta_{(s)}^{(m)}(t,x))(\Phi(e_i)U_{(s)}(t)\Phi(e_j)f)(x))]$$

$$= E^P[(F'(t,x)\psi(\delta_{(s)}^{(m)}(t,x))(U(t)f)(x)],$$

for any $f \in C_b^\infty(\mathbf{R}^N)$, $t \in [T_0,T_1]$, and $x \in \mathbf{R}^N$.

Now let us prove our lemma.

Note that

$$(Q_{(s)}(t)f)(x) = E^P[(U_{(s)}(t)f)(x)].$$

Let us fix $\varphi \in C_0^\infty((-1/2, 1/2))$ such that $\varphi(z) = 1$ for $z \in (-1/4, 1/4)$, and let $Q_{(s),i}(t)$, $i = 0, 1$, $t > 0$, be linear operators in $C_b^\infty(\mathbf{R}^N)$ given by

$$(Q_{(s),0}(t)f)(x) = E^P[\varphi(\delta_{(s)}^{(m)}(t,x))(U_{(s)}(t)f)(x)],$$

and

$$(Q_{(s),1}(t)f)(x) = E^P[(1 - \varphi(\delta_{(s)}^{(m)}(t,x)))(U_{(s)}(t)f)(x)].$$

Since

$$P(|\delta_{(s)}^{(m)}(t,x))| > 1/4) \leqq 4^n E[|\delta_{(s)}^{(m)}(t,x))|^n],$$

we have by Equation (29)

$$\sup_{s\in(0,1]} s^{-n/3} \sup_{x\in\mathbf{R}^N} \sup_{t\in(0,T]} t^{n\gamma_m} P(|\delta_{(s)}(t,x))| > 1/4) < \infty$$

for any $n \geqq 1$. Then we see that for any $n \geqq 1$ and $T_1 > T_0 > 0$,

$$\sup_{s\in(0,1]} s^{-n} \sup_{x\in\mathbf{R}^N} \sup_{t\in[T_0,T_1]} || \frac{\partial^\alpha}{\partial x^\alpha}(1 - \varphi(\delta_{(s)}^{(m)}(t,x)))||_{\hat{W}^{r,p}} < \infty \tag{32}$$

Now our lemma is a consequence of Proposition 32 and Equations (29) and (32).

10 Commutation and Infinitesimal Difference

Let $\tilde{A}_j : A^* \times \mathbf{R}\langle A\rangle \to \mathbf{R}\langle A\rangle$, $j = 0, 1$, be a map inductively defined by

$$\tilde{A}_j(1)w = w,\ \tilde{A}_j(v_i)w = v_i w,\ i = 1, \dots, d,\ j = 0, 1,$$

$$\tilde{A}_0(v_0)w = [v_0, w],\ \tilde{A}_0(v_0)w = \frac{1}{2}\sum_{i=1}^d v_i^2 + [v_0, w],$$

and

$$\tilde{A}_j(uv_i)w = \tilde{A}_j(v_i)(\tilde{A}_j(u)w), \qquad i = 0, \dots, d,\ u\ A^*,\ w \in \mathbf{R}\langle A\rangle.$$

Then we have the following.

Proposition 33 *$\tilde{A}_j(u)w \in \mathbf{R}^{**}\langle A\rangle_{n+||u||}$ for any $j = 0, 1$, $w \in \mathbf{R}^{**}\langle A\rangle_n$, $n \geqq 0$, and $u \in A^*$.*

Proof. We have our assertion, noting that

$$[v_0, r(u_1)\cdots r(u_n)] = \sum_{k=1}^n r(u_1)\cdots r(u_{k-1})r(v_0 u_k)r(u_{k+1})\cdots r(u_n).$$

∎

It is easy to see that

$$U(t)\Phi(w)U_0(t)^{-1} = \Phi(\tilde{A}_0(1)w) + \sum_{i=0}^{d} \int_0^t U(r)\Phi(\tilde{A}_0(v_i)w)U_0(r)^{-1} \circ dB^i(r)$$

$$= \Phi(\tilde{A}_1(1)w) + \sum_{i=0}^{d} \int_0^t U(r)\Phi(\tilde{A}_1(v_i)w)U_0(r)^{-1} dB^i(r)$$

for any $w \in A^*$. Therefore we have for any $n \geqq 0$

$$U(t)U_0(t)^{-1} = \sum_{u \in A^*_{\leqq n}} I(1;u)(t)\Phi(\tilde{A}_1(u)1)$$

$$+ \sum_{u \in R^*_n} I(U(\cdot)\Phi(\tilde{A}_0(u)1)U_0(\cdot)^{-1}.$$

Here R^*_n is as in (23).

Remind that $\hat{X}(t)$ is a solution to the following SDE over $\mathbf{R}\langle\langle A\rangle\rangle$.

$$\hat{X}(t) = 1 + \sum_{i=0}^{d} \int_0^t \hat{X}(r)v_i \circ dB^i(r).$$

$$= 1 + \sum_{i=1}^{d} \int_0^t \hat{X}(r)v_i dB^i(r) + \int_0^t \hat{X}(r)(\frac{1}{2}\sum_{i=1}^{d} v_i^2 + v_0)dB^0(r).$$

Let $\hat{X}_0(t)$ $\hat{Y}_{(s)}(t)$ and $\hat{Y}_{(s),0}(t)$ are solutions to the following ordinary differential equations over $\mathbf{R}\langle\langle A\rangle\rangle$

$$\hat{X}_0(t) = 1 + \int_0^t \hat{X}_0(r)v_0 \circ dB^0(r),$$

$$\hat{Y}_{(s)}(t) = 1 + \sum_{i=0}^{d} \int_0^t \hat{Y}_{(s)}(r)v_i\eta^i_{(s)}(r)dr,$$

and

$$\hat{Y}_{(s),0}(t) = 1 + \int_0^t \hat{Y}_{(s)}(r)v_0\eta^0_{(s)}(r)dr.$$

Then we see that

$$\hat{X}(t)w\hat{X}_0(t)^{-1} = \tilde{A}_1(1)w + \sum_{i=0}^{d} \int_0^t \hat{X}(r)(\tilde{A}_1(v_i)w)\hat{X}_0(r)^{-1}dB^i(r)$$

for any $w \in A^*$. So we see that for any $n \geqq 0$

$$\hat{X}(t)\hat{X}_0(t)^{-1} = \sum_{u \in A^*_{\leqq n}} I(1;u)(t)(\tilde{A}(u)1) + \sum_{u \in R^*_n} I(\hat{X}(\cdot)(\tilde{A}_1(u)1)\hat{X}_0(\cdot)^{-1}.$$

Noting that

$$j_{\leqq n}(\hat{X}(t)\hat{X}_0(t)^{-1}) = j_{\leqq n}(j_{\leqq n}(\hat{X}(t))\hat{X}_0(t)^{-1}),$$

we have

$$U(t)U_0(t)^{-1}$$
$$= \Phi(j_{\leqq n}(j_{\leqq n}(\hat{X}(t))\hat{X}_0(t)^{-1}))) + \sum_{u\in R_n^*} I(U(\cdot)\Phi(\tilde{A}_1(u)1)U_0(\cdot)^{-1}. \tag{33}$$

Similarly we have

$$U_{(s)}(t)U_0(t)^{-1} = \Phi(j_{\leqq n}(j_{\leqq n}(\hat{Y}_{(s)}(t))\hat{Y}_{(s),0}0(t)^{-1})))$$

$$+ \sum_{v_{i_1}\cdots v_{i_m}\in R_n^*} \int_0^t \eta_{(s)}^{i_1}(r_1)dr_1 \cdots \int_0^{r_{m-1}} \eta_{(s)}^{i_1}(r_m)dr_m U_{(s)}(r_{m+1})\Phi(\tilde{A}_0(v_{i_m}\cdots v_{i_1}1)U_0(r_m)^{-1}. \tag{34}$$

Note that by the assumption (G-3) we have

$$U_{(s),0}(s) = U_0(s) \text{ and } \hat{Y}_{(s),0}(s) = \hat{X}_0(s) = \exp(sv_0). \tag{35}$$

Then by Propositions 21, 22, and Equation (33), we have the following.

Proposition 34 *For any* $n \geqq 0$, *and* $i, i' \geqq 0$, *there are* $M \geqq 1$ *and measurable functions* $d_{n,i,i',j,k} : R_n^* \to \hat{K}_0(\mathbf{R})$, $j = 0, 1, \ldots, M$, $k = 0, 1$, *such that*

$$\Phi(e_i)U(t)\Phi(e_{i'}) - \Phi(e_i j_{\leqq n}(j_{\leqq n}(\hat{X}(t))\exp(-tv_0))))U_0(t)\Phi(e_{i'})$$

$$= \sum_{u\in R_n^*}\sum_{j=0}^{M} \Phi(e_j)I(d_{n,i,i',j,0}(\cdot,*,u)U(\cdot)U_0(t-\cdot);u)(t).$$

$$= \sum_{u\in R_n^*}\sum_{j=0}^{M} I(U(\cdot)U_0(t-\cdot)d_{n,i,i',j,1}(\cdot,*,u);u)(t)\Phi(e_j).$$

By Equation (34), we see that for any $m \geqq 1$ and $w, w' \in \mathbf{R}^{**}\langle A\rangle$

$$\Phi(w)(U_{(s)}(s))\Phi(w')$$

$$= \Phi(wj_{\leqq n}(j_{\leqq n}(\hat{Y}_{(s)}(s))\exp(-sv_0))))U_0(s)\Phi(w')$$

$$+ \sum_{v_{i_1},\ldots v_{i_q}\in A: v_{i_1}\cdots v_{i_q}\in R_n^*} \int_0^s \eta_{(s)}^{i_1}(r_1)dr_1 \cdots \int_0^{r_{q-1}} \eta_{(s)}^{i_1}(r_q)dr_q$$

$$\Phi(w)U_{(s)}(r_{q+1})\Phi(\tilde{A}_0(v_{i_q}\cdots v_{i_1}1)Exp((\int_{r_q}^s \eta_{(s)}^{i_q}(\tilde{r})d\tilde{r})V_0)\Phi(w').$$

Then by Propositions 21, 22, and Equations (34) and (35) we have the following.

Proposition 35 *For any* $n \geqq 0$, *and* $i, i' \geqq 0$, *there are* $M \geqq 1$ *and measurable functions* $d_{n,(s),i,i',j,k} : (0,s] \times R_n^* \to \hat{K}_0(\mathbf{R})$, $j = 0, 1, \ldots, M$, $s \in (0,1]$, $k = 0, 1$, *such that*

$$\Phi(e_i)U_{(s)}(s)\Phi(e_{i'}) - \Phi(e_i j_n(j_{\leqq n}(\hat{Y}_{(s)}(s))\exp(-sv_0))))U_0(s)\Phi(e_{i'})$$

$$= \sum_{v_{i_1},\ldots v_{i_q}\in A: v_{i_1}\cdots v_{i_q}\in R_n^*} \sum_{j=0}^{M} \Phi(e_j) \int_0^s \eta_{(s)}^{i_1}(r_1)dr_1 \cdots \int_0^{r_{q-1}} \eta_{(s)}^{i_1}(r_q)dr_q$$

$$d_{n,(s),i,i',j,0}(\cdot,*,v_{i_1}\cdots v_{i_q})(r_q)U_{(s)}(r_q)Exp((\int_{r_q}^s \eta_{(s)}(\tilde{r})d\tilde{r})V_0)$$

$$= \sum_{v_{i_1},\ldots v_{i_q}\in A: v_{i_1}\cdots v_{i_q}\in R_m^*} \sum_{j=0}^{M} \int_0^s \eta_{(s)}^{i_1}(r_1)dr_1 \cdots \int_0^{r_{q-1}} \eta_{(s)}^{i_1}(r_q)dr_q$$

$$U_{(s)}(r_q)Exp((\int_{r_q}^s \eta_{(s)}(\tilde{r})d\tilde{r})V_0).d_{n,(s),i,i',1}^k(\cdot,*,v_{i_1}\cdots v_{i_q})\Phi(e_j).$$

Note that

$$E[(\int_0^s |\eta_{(s)}^{i_1}(r_1)|dr_1 \cdots \int_0^{r_{q-1}} dr_q|\eta_{(s)}^{i_q}(r_q)||g(r_q)|)^p]^{1/p}$$

$$\leqq E[\{(\int_0^s |\eta_{(s)}^{i_1}(r)|dr)\cdots(\int_0^s |\eta_{(s)}^{i_{q-1}}(r)|dr)(\int_0^s |\eta_{(s)}^{i_q}(r)||g(r)|dr)\}^p]^{1/p}$$

$$\leqq s^{||v_{i_1}\cdots v_{i_q}||/2} E[\{(s^{-1}\int_0^s |\eta_{(s)}^i(r)|^2dr + \sum_{i=1}^d \int_0^s |\eta_{(s)}^i(r)|^2dr)\}^{qp}]^{1/2p} E[(\int_0^s |g(r)|^2dr)^{2p}]^{1/2p}$$

for any $v_{i_1}\cdots v_{i_q} \in R_n^*$ and progressively measurable function g.

Therefore as a corollary to the above propositions, we have the following.

Corollary 36 *For any $n \geqq 0$, and $i,i' \geqq 0$, there are $M \geqq 1$ and linear operators $R_{s,k,j} = R_{n,s,k,i,i',j}$, $\tilde{R}_{(s),j,k} = \tilde{R}_{n,(s),i,i',j,k}$, $j = 0,1\ldots,M$, $k = 0,1$, $s \in (0,1]$ defined in $C_b^\infty(\mathbf{R}^N)$ satisfying the following.*
(1) *There is a $C > 0$ such that*

$$||R_{s,k,0}f||_\infty + ||R_{s,k,1}f||_\infty + ||\tilde{R}_{(s),j,0}f||_\infty + ||\tilde{R}_{(s),j.1}f||_\infty \leqq Cs^{(n+1)/2}||f||_\infty$$

for any $f \in C_b^\infty(\mathbf{R}^N)$, $s \in (0,1]$ and $j = 0,1\ldots,M$.
(2)

$$\Phi(e_i)P_s\Phi(e_{i'}) - \Phi(e_i j_{\leqq n}(E[j_{\leqq n}(\hat{X}(s))])\exp(-sv_0)))U_0(s)\Phi(e_{i'})$$

$$= \sum_{j=0}^M \Phi(e_j)R_{s,j,0} = \sum_{j=0}^M R_{s,j,1}\Phi(e_j)$$

(3)

$$\Phi(e_i)Q_{(s)}\Phi(e_{i'}) - \Phi(e_i j_{\leqq n}(E[j_{\leqq n}(\hat{Y}_{(s)}(s))])\exp(-sv_0)))U_0(s)\Phi(e_{i'})$$

$$= \sum_{j=0}^M \Phi(e_j)\tilde{R}_{(s),j,0} = \sum_{j=0}^M \tilde{R}_{(s),j,1}\Phi(e_j).$$

11 Proof of Theorems 3

Let us assume the assumption of Theorem 3. Note that

$$P_{ns} - Q^n_{(s)} = \sum_{k=1}^{n} P_{(k-1)s}(P_s - Q_{(s)})Q_{(s)}((n-k)s)$$
$$= R_{(s),n,0} + R_{(s),n,1},$$

where

$$R_{(s),n,0} = \sum_{k=1}^{[n/2]} P_{(k-1)s}(P_s - Q_{(s)})Q_{(s),0}((n-k)s) + \sum_{k=[n/2]+1}^{n} P_{(k-1)s}(P_s - Q_{(s)})Q_{(s)}((n-k)s),$$

and

$$R_{(s),n,1} = \sum_{k=1}^{[n/2]} P_{(k-1)s}(P_s - Q_{(s)})Q_{(s),1}((n-k)s).$$

Here $Q_{(s),0}(t)$ and $Q_{(s),1}(t)$ are as in Lemma 30.

Then we have the following.

Proposition 37 *Let $T_1 > T_0 > 0$. Then we have the following.*
(1) *For any $w \in \mathbf{R}^{**}\langle A\rangle$, there is a $C > 0$ such that*

$$||\Phi(w)R_{(s),n,0}f|| \leqq Cs^{(m-1)/2}||f||_\infty$$

for any $f \in C_b^\infty(\mathbf{R}^N)$, $s \in (0,1]$, $n \geqq 1$ with $T_0 \leqq ns \leqq T_1$.
(2) *For any $\gamma > 0$, there is a $C > 0$ such that*

$$||R_{(s),n,1}f|| \leqq Cs^\gamma||f||_\infty$$

for any $f \in C_b^\infty(\mathbf{R}^N)$, $s \in (0,1]$, $n \geqq 1$ with $T_0 \leqq ns \leqq T_1$.
(3) *There is a $C > 0$ such that*

$$||(P_{ns} - Q^n_{(s)})f|| \leqq Cs^{(m-1)/2}||f||_\infty$$

for any $f \in C_b^\infty(\mathbf{R}^N)$, $s \in (0,1]$, $n \geqq 1$ with $T_0 \leqq ns \leqq T_1$.

Proof. The assertion (2) is an easy consequence of Lemma 30. The assertion (3) follows from the assertions (1) and (2). So it is sufficient to prove the assertion (1).

Fix $w \in \mathbf{R}^{**}\langle A\rangle$. Applying Proposition 22, we see that there are $I \geqq 1$ and linear operators $\tilde{P}_{t,i}$ in $C_b^\infty(\mathbf{R}^N)$ such that

$$\Phi(w)P_t = \sum_{i=0}^{M} \tilde{P}_{t,i}\Phi(e_i)$$

and that there is a $C_0 > 0$ such that

$$||\tilde{P}_{t,i}f||_\infty \leqq C_0||f||_\infty$$

for any $f \in C_b^\infty(\mathbf{R}^N)$ and $i = 0, \dots, I$.

Applying Corollary 36 to $n = m$, we see that for there are $K \geqq 1$ and linear operators $\tilde{R}_{(s),i,k,j}$ in $C_b^\infty(\mathbf{R}^N)$, $s \in (0,1]$, $i = 0, \dots, I$ $k = 0, 1$, $j = 0, \dots, J$ such that

$$\Phi(e_i)(P_s - Q_{(s)}) = \sum_{j=0}^{J} \Phi(e_j)\tilde{R}_{(s),i,0,j} = \sum_{j=0}^{J} \tilde{R}_{(s),i,1,j}\Phi(e_j)$$

and that there is a $C_1 > 0$ such that

$$||\tilde{R}_{(s),i,k,j} f||_\infty \leqq C_1 s^{(m+1)/2} ||f||_\infty$$

for any $f \in C_b^\infty(\mathbf{R}^N)$, $s \in (0,1]$, $i = 0, \dots, I$ and $k = 0, 1$, and $j = 0, \dots, J$. Then we see that

$$||\Phi(w)R_{(s),n,0}f||_\infty$$

$$\leqq \sum_{i=0}^{I}\sum_{j=0}^{J}\sum_{k=1}^{[n/2]} ||P_{(k-1)s,i}\tilde{R}_{(s),i,1,j}\Phi(e_j)Q_{(s),0}((n-k)s)f||_\infty$$

$$+ \sum_{j=0}^{J}\sum_{k=[n/2]+1}^{n} ||\Phi(w)P_{(k-1)s}\Phi(e_j)\tilde{R}_{(s),0,j}Q_{(s)}((n-k)s)f||_\infty$$

$$\leqq \sum_{i=0}^{I}\sum_{j=0}^{J} s^{(m+1)/2} \sum_{k=1}^{[n/2]} C_0C_1||\Phi(e_j)Q_{(s),0}((n-k)s)f||_\infty$$

$$+ \sum_{j=0}^{J} s^{(m+1)/2} \sum_{k=[n/2]+1}^{n} C_1 \sup\{||\Phi(w)P_{(k-1)s}\Phi(e_j)\tilde{f}||_\infty;\ f \in C_b^\infty(\mathbf{R}^N), ||\tilde{f}||_\infty \leqq 1\}||f||_\infty.$$

Then we have the assertion (1) from Proposition 29 and Lemma 30 . ∎

Theorem 3 is an easy consequence of the above Proposition.

12 Proof of Theorem 4

We assume the assumption of Theorem 4. Note that

$$\langle \log(\hat{X}(s)\exp(-sv_0)), v_0\rangle = \langle \log(\hat{Y}_{(s)}(s)\exp(-sv_0)), v_0\rangle = 0$$

with probability 1. Therefore we see that

$$w_0 = s^{-(m+1)/2}E[j_{\leqq m+1}((\hat{X}(s) - \hat{Y}_{(s)}(s))\exp(-sv_0))] \in \mathbf{R}^{**}\langle A\rangle.$$

Also, by Corollary 36, there are $M \geqq 1$ and linear operators $\hat{R}_{s,k,j}$, $j = 0, 1 \dots, M$, $k = 0, 1$, $s \in (0,1]$ defined in $C_b^\infty(\mathbf{R}^N)$ satisfying the following.

(1) There is a $C > 0$ such that

$$||R_{s,k,0}f||_\infty + ||R_{s,k,1}f||_\infty \leqq Cs^{(m+2)/2}||f||_\infty$$

for any $f \in C_b^\infty(\mathbf{R}^N)$, $s \in (0,1]$ and $j = 0, 1 \ldots, M$.
(2)

$$P_s - Q_{(s)} + s^{(m+1)/2}\Phi(w_0)U_0(s)$$
$$= \sum_{j=0}^M \Phi(e_j)\hat{R}_{s,j,0} = \sum_{j=0}^M \hat{R}_{s,j,1}\Phi(e_j)$$

Now by applying Corollary 36 for $n = m + 2$, we see that

$$P_{ns} - Q_{(s)}^n$$
$$= \sum_{k=1}^n P_{(k-1)s}(P_s - Q_{(s)})P_{(n-k)s} - \sum_{k=1}^n P_{(k-1)s}(P_s - Q_{(s)})(P_{(n-k)s} - Q_{(s)}((n-k)s))$$
$$= \sum_{i=0} I_{(s),n,i},$$

where

$$I_{(s),n,0} = s^{(m+1)/2}\sum_{k=1}^n P_{(k-1)s}\Phi(w_0)U_0(s)P_{(n-k)s}$$

$$I_{(s),n,1} = \sum_{j=0}^M \sum_{k=1}^{[n/2]} P_{(k-1)s}\hat{R}_{q+2,s,j,1}\Phi(e_j)P_{(n-k)s} + \sum_{j=0}^M \sum_{k=[n/2]+1}^n P_{(k-1)s}\Phi(e_j)\hat{R}_{q+2,s,j,0}P_{(n-k)s}$$

$$I_{(s),n,2} = -\sum_{k=1}^n \sum_{\ell=1}^{n-k} P_{(k-1)s}(P_s - Q_{(s)})P_{\ell-1}(P_s - Q_{(s)})Q_{(s),n-k,0}$$

$$I_{(s),n,3} = -\sum_{k=1}^n \sum_{\ell=1}^{n-k} P_{(k-1)s}(P_s - Q_{(s)})P_{\ell-1}(P_s - Q_{(s)})Q_{(s),n-k,1}$$

Then by using a similar argument in the proof of Proposition, we see that for any $T_1 > T_0 > 0$, there is a $C > 0$ such that

$$||I_{(s),n,1}f||_\infty \leqq Cs^{(m+1)/2}||f||_\infty$$
$$||I_{(s),n,2}f||_\infty \leqq Cs^{m-1}||f||_\infty$$
$$||I_{(s),n,3}f||_\infty \leqq Cs^{2m}||f||_\infty$$

for any $f \in C_b^\infty(\mathbf{R}^N)$, $n \geqq 1$, $s \in (0,1]$, with $ns \in [T_0, T_1]$.

Also, note that

$$\int_0^{ns} P_r\Phi(w_0)P_{ns-r}dr - s^{-(m-1)/2}I_{(s),n,0}$$
$$= \sum_{k=1}^n P_{(k-1)s}(\int_0^s (P_r\Phi(w_0)P_{s-r} - \Phi(w_0)U_0(s))dr)P_{(n-k)s}.$$

Note that for $r \in (0,s)$,

$$P_r\Phi(w_0)P_{s-r}U_0(s)^{-1}$$
$$= P_rU_0(r)^{-1}(U_0(r)\Phi(w_0)U_0(r)^{-1})U_0(r)(P_{s-r}U_0(s-r)^{-1})U_0(r)^{-1}.$$

Therefore appliying Corollary 36 for $n = 1$, we have the following.

Proposition 38 *For any $s \in (0,1]$, there are $M \geqq 1$, and linear operators defined $R_{s,i,j}$, $= 0,1,\ldots,M$, $j = 0,1$, in $C_b^\infty(\mathbf{R}^N)$ satisfying the following.*
(1) *There is a $C > 0$ such that*

$$||R_{s,i,j}f||_\infty \leqq Cs||f||_\infty$$

for any $f \in C_b^\infty(\mathbf{R}^N)$, $s \in (0,1]$, $i = 0,1,\ldots,M$, $j = 0,1$.
(2)

$$\int_0^s (P_r\Phi(w_0)P_{s-r}U_0(s)^{-1})ds$$

$$= \Phi(w_0) + \sum_{i=0}^M \Phi(e_i)R_{s,j,0} = \Phi(w_0) + \sum_{i=0}^M R_{s,j,1}\Phi(e_i).$$

Then again similarly to the proof of Proposition 37 we see that for any $T_1 > T_0 > 0$, there is a $C > 0$ such that

$$||s^{(m-1)/2}\int_0^{ns} P_r\Phi(w_0)P_{ns-r}f dr - I_{(s),n,0}f||_\infty \leqq Cs^{(m+1)/2}||f||_\infty$$

for any $f \in C_b^\infty(\mathbf{R}^N)$, $n \geqq 1$, $s \in (0,1]$, with $ns \in [T_0, T_1]$.

So we have Theorem 4.

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ADDRESS:
Graduate School of Mathematical Sciences, The University of Tokyo
3–8–1 Komaba Meguro-ku, Tokyo 153-8914, JAPAN
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