UTMS 2009–14 July 1, 2009

Decomposable extensions of difference fields

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Abstract

We define the decomposable extensions of difference fields and study the irreducibility of q-Painlevé equation of type $A_7^{(1)'}$. Every strongly normal extension or Liouville-Franke extension, the latter of which is a difference analogue of the Liouvillian extension, satisfies that its appropriate algebraic closure is a decomposable extension.

1 Introduction

Notation. Throughout the paper we say a set is a field only when it is a field of characteristic zero, namely when it contains the set of rational numbers. Terms used here will be seen in [2, 7]. For a difference field extension $\mathcal{L}/\mathcal{K}$ and $B \subset L$, $\mathcal{K}\langle B\rangle_{\mathcal{L}}$ denotes the difference field, the intersection of all difference intermediate fields of $\mathcal{L}/\mathcal{K}$ containing B.

In [8] the author introduced the definition and some examples of the $\mathcal{U}$-decomposable extensions of difference fields. In this paper we define the decomposable extensions of difference fields, which do not require the fixed difference field $\mathcal{U}$, and study the irreducibility of q-Painlevé equation of type $A_7^{(1)'}$.

We show that some algebraic closure of any $\mathcal{U}$-decomposable extension is decomposable in Proposition 4. Therefore some algebraic closure of Bialynicki-Birula's strongly normal extension or Infante's is decomposable (see [1, 5, 6, 8, 9]). Moreover Corollary 8 implies that any algebraic closure of the Liouville-Franke extension, a difference analogue of the Liouvillian extension, is decomposable (see [3, 4]).

We define the decomposable extensions and the $\mathcal{U}$-decomposable extensions.

Definition 1 (decomposable extension)**.** Let $\mathcal{K}$ be a difference field, and $\mathcal{L}$ an algebraically closed difference overfield of $\mathcal{K}$ satisfying $\operatorname{tr.deg} \mathcal{L}/\mathcal{K} < \infty$. We define decomposable extensions by induction on $\operatorname{tr.deg} \mathcal{L}/\mathcal{K}$.

(i) If $\operatorname{tr.deg} \mathcal{L}/\mathcal{K} \leq 1$, then $\mathcal{L}/\mathcal{K}$ is decomposable.

(ii) When $\operatorname{tr.deg} \mathcal{L}/\mathcal{K} \geq 2$, $\mathcal{L}/\mathcal{K}$ is decomposable if there exist a difference overfield $\mathcal{U}$ of $\mathcal{L}$, a difference overfield $\mathcal{E}$ of $\mathcal{K}$ in $\mathcal{U}$ of finite transcendence degree which is free from $\mathcal{L}$ over $\mathcal{K}$, and a difference intermediate field $\mathcal{M}$ of $\mathcal{L}\mathcal{E}/\mathcal{E}$ satisfying $\operatorname{tr.deg} \mathcal{L}\mathcal{E}/\mathcal{M} \geq 1$ and $\operatorname{tr.deg} \mathcal{M}/\mathcal{E} \geq 1$, such that $\overline{\mathcal{L}\mathcal{E}}/\mathcal{M}$ and $\overline{\mathcal{M}}/\mathcal{E}$ are decomposable, where $\overline{\mathcal{L}\mathcal{E}}$ is an algebraic closure of $\mathcal{L}\mathcal{E}$ and $\overline{\mathcal{M}}$ the algebraic closure of $\mathcal{M}$ in $\overline{\mathcal{L}\mathcal{E}}$.

Definition 2 ($\mathcal{U}$-decomposable extension)**.** Let $\mathcal{U}$ be a difference field and $\mathcal{L}/\mathcal{K}$ a difference field extension in $\mathcal{U}$ of finite transcendence degree. We define $\mathcal{U}$-decomposable extensions by induction on $\operatorname{tr.deg} L/K$.

(i) If $\operatorname{tr.deg} L/K \leq 1$ then $\mathcal{L}/\mathcal{K}$ is $\mathcal{U}$-decomposable.

(ii) When $\operatorname{tr.deg} \mathcal{L}/\mathcal{K} \geq 2$, $\mathcal{L}/\mathcal{K}$ is $\mathcal{U}$-decomposable if there exist a difference overfield $\mathcal{E}$ of $\mathcal{K}$ in $\mathcal{U}$ of finite transcendence degree which is free from $\mathcal{L}$ over $\mathcal{K}$, and a difference intermediate field $\mathcal{M}$ of $\mathcal{L}\mathcal{E}/\mathcal{E}$ such that $\operatorname{tr.deg} LE/M \geq 1$, $\operatorname{tr.deg} M/E \geq 1$, $\mathcal{L}\mathcal{E}/\mathcal{M}$ is $\mathcal{U}$-decomposable, and $\mathcal{M}/\mathcal{E}$ is $\mathcal{U}$-decomposable.

This work was supported by KAKENHI ($20 \cdot 4941$).

2 Decomposable extension

Proposition 3. *Let $\mathcal{K}$ be a difference field, and $\mathcal{L}/\mathcal{K}$ and $\mathcal{N}/\mathcal{L}$ be decomposable extensions. Then $\mathcal{N}/\mathcal{K}$ is decomposable.*

Proof. (i) If $\operatorname{tr.deg} N/K \leq 1$, then we find that $\mathcal{N}/\mathcal{K}$ is decomposable by the definition.

(ii) Suppose $\operatorname{tr.deg} N/K \geq 2$.

(ii-1) If $\operatorname{tr.deg} N/L = 0$, then we obtain $\mathcal{N} = \mathcal{L}$ because $\mathcal{L}$ is algebraically closed. Therefore $\mathcal{N}/\mathcal{K}$ is decomposable.

(ii-2) Suppose $\operatorname{tr.deg} L/K = 0$. Since $\mathcal{N}/\mathcal{L}$ is a decomposable extension of $\operatorname{tr.deg} N/L \geq 2$, there exist a difference overfield $\mathcal{U}$ of $\mathcal{N}$, a difference overfield

$\mathcal{E}$ of $\mathcal{L}$ in $\mathcal{U}$ of finite transcendence degree which is free from $\mathcal{N}$ over $\mathcal{L}$, and a difference intermediate field $\mathcal{M}$ of $\mathcal{N}\mathcal{E}/\mathcal{E}$ satisfying $\operatorname{tr.deg}\mathcal{N}\mathcal{E}/\mathcal{M} \geq 1$ and $\operatorname{tr.deg}\mathcal{M}/\mathcal{E} \geq 1$, such that $\overline{\mathcal{N}\mathcal{E}}/\overline{\mathcal{M}}$ and $\overline{\mathcal{M}}/\mathcal{E}$ are decomposable, where $\overline{\mathcal{N}\mathcal{E}}$ is an algebraic closure of $\mathcal{N}\mathcal{E}$ and $\overline{\mathcal{M}}$ the algebraic closure of $\mathcal{M}$ in $\overline{\mathcal{N}\mathcal{E}}$.

Note $\operatorname{tr.deg} E/K = \operatorname{tr.deg} E/L < \infty$ and that N and E are free over K. Then we find that $\mathcal{N}/\mathcal{K}$ is decomposable.

(ii-3) Suppose $\operatorname{tr.deg} N/L \geq 1$ and $\operatorname{tr.deg} L/K \geq 1$. Putting $\mathcal{U} = \mathcal{N}$, $\mathcal{E} = \mathcal{K}$ and $\mathcal{M} = \mathcal{L}$, we find that $\mathcal{N}/\mathcal{K}$ is decomposable by the definition. □

Therefore chains of decomposable extensions are decomposable.

Proposition 4. *Let $\mathcal{L}/\mathcal{K}$ be a $\mathcal{U}$-decomposable extension, $\overline{\mathcal{U}}$ an algebraic closure of $\mathcal{U}$, and $\overline{\mathcal{L}}$ the algebraic closure of $\mathcal{L}$ in $\overline{\mathcal{U}}$. Then $\overline{\mathcal{L}}/\mathcal{K}$ is decomposable.*

Proof. We prove this by induction on $\operatorname{tr.deg} L/K$. If $\operatorname{tr.deg} L/K \leq 1$, then $\operatorname{tr.deg} \overline{L}/K \leq 1$, and so $\overline{\mathcal{L}}/\mathcal{K}$ is decomposable.

Suppose $\operatorname{tr.deg} L/K \geq 2$ and that the statement is true for ones of less transcendence degree. Since $\mathcal{L}/\mathcal{K}$ and $\overline{\mathcal{L}}/\mathcal{L}$ are $\overline{\mathcal{U}}$-decomposable, we find that $\overline{\mathcal{L}}/\mathcal{K}$ is $\overline{\mathcal{U}}$-decomposable (see [8]). Therefore there exist a difference overfield $\mathcal{E} \subset \overline{\mathcal{U}}$ of $\mathcal{K}$ satisfying $\operatorname{tr.deg} E/K < \infty$ and that E is free from $\overline{L}$ over K, and a difference intermediate field $\mathcal{M}$ of $\overline{\mathcal{L}}\mathcal{E}/\mathcal{E}$ satisfying $\operatorname{tr.deg} \overline{L}E/M \geq 1$ and $\operatorname{tr.deg} M/E \geq 1$, such that $\overline{\mathcal{L}}\mathcal{E}/\mathcal{M}$ and $\mathcal{M}/\mathcal{E}$ are $\overline{\mathcal{U}}$-decomposable.

Let $\overline{\overline{\mathcal{L}}\mathcal{E}}$ and $\overline{\mathcal{M}}$ be the algebraic closures of $\overline{\mathcal{L}}\mathcal{E}$ and $\mathcal{M}$ in $\overline{\mathcal{U}}$ respectively. By the induction hypothesis we find that $\overline{\overline{\mathcal{L}}\mathcal{E}}/\mathcal{M}$ and $\overline{\mathcal{M}}/\mathcal{E}$ are decomposable, which implies that $\overline{\mathcal{L}}/\mathcal{K}$ is decomposable. □

The remaining results in this section are on a linear difference equation. We include the following Lemma for readers convenience.

Lemma 5. *Let $\mathcal{K}$ be a difference field, $C = C_{\mathcal{K}}$, $n \in \mathbb{Z}_{\geq 1}$, and $b^{(1)}, \ldots, b^{(n)} \in K$. Then the following are equivalent.*

(i) *$b^{(1)}, \ldots, b^{(n)}$ are linearly dependent over C.*

(ii) $\operatorname{Cas}(b^{(1)}, \ldots, b^{(n)}) = 0$.

Proof. Let $\mathcal{K} = (K, \tau)$. If $b^{(1)}, \ldots, b^{(n)}$ are linearly dependent over C, there are $c_1, \ldots, c_n \in C$ such that $(c_1, \ldots, c_n) \neq 0$ and $\sum_{i=1}^n c_i b^{(i)} = 0$. Then we obtain $\sum_{i=1}^n c_i b_j^{(i)} = 0$ for all $0 \leq j \leq n-1$, which implies $\operatorname{Cas}(b^{(1)}, \ldots, b^{(n)}) = 0$.

Suppose $\mathrm{Cas}\,(b^{(1)}, \ldots, b^{(n)}) = 0$. We prove (i) by induction on n. The statement is true in the case $n = 1$. Suppose $n \geq 2$ and the statement is true for $n-1$. There are $c_1, \ldots, c_n \in K$ such that $(c_1, \ldots, c_n) \neq 0$ and

$$\begin{pmatrix} b^{(1)} & \cdots & b^{(n)} \\ \vdots & \ddots & \vdots \\ b^{(1)}_{n-1} & \cdots & b^{(n)}_{n-1} \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = 0.$$

We may suppose $c_1 = 1$. From $\sum_{i=1}^n c_i b_j^{(i)} = 0$ for any $0 \leq j \leq n-1$, we obtain $\sum_{i=1}^n \tau(c_i) b_j^{(i)} = 0$ for any $1 \leq j \leq n$. Therefore it follows that for any $1 \leq j \leq n-1$,

$$\sum_{i=2}^n (\tau(c_i) - c_i) b_j^{(i)} = \sum_{i=1}^n (\tau(c_i) - c_i) b_j^{(i)} = 0,$$

which implies

$$\begin{pmatrix} b^{(2)}_1 & \cdots & b^{(n)}_1 \\ \vdots & \ddots & \vdots \\ b^{(2)}_{n-1} & \cdots & b^{(n)}_{n-1} \end{pmatrix} \begin{pmatrix} \tau(c_2) - c_2 \\ \vdots \\ \tau(c_n) - c_n \end{pmatrix} = 0.$$

Case 1. The case $\mathrm{Cas}\,(b_1^{(2)}, \ldots, b_1^{(n)}) \neq 0$. In this case we find that $\tau(c_i) = c_i$ for all $2 \leq i \leq n$, which implies $c_i \in C$ for all $1 \leq i \leq n$. Since we have $\sum_{i=1}^n c_i b^{(i)} = 0$, we conclude that $b^{(1)}, \ldots, b^{(n)}$ are linearly dependent over C.

Case 2. The case $\mathrm{Cas}\,(b_1^{(2)}, \ldots, b_1^{(n)}) = 0$. We obtain $\mathrm{Cas}\,(b^{(2)}, \ldots, b^{(n)}) = 0$. By the induction hypothesis we find that $b^{(2)}, \ldots, b^{(n)}$ are linearly dependent over C, which implies $b^{(1)}, b^{(2)}, \ldots, b^{(n)}$ are linearly dependent over C. $\square$

Lemma 6. *Let $\mathcal{K}$ be a difference field,*

$$y_n + a_{n-1} y_{n-1} + \cdots + a_0 y = 0 \tag{1}$$

a linear homogeneous difference equation over $\mathcal{K}$, where $n \geq 1$, f a solution of (1)*, and $\mathcal{L}$ an algebraic difference overfield of $\mathcal{K}\langle f \rangle$. Then $\mathcal{L}/\mathcal{K}$ is $\mathcal{U}$-decomposable for some difference overfield $\mathcal{U}$ of $\mathcal{L}$.*

Proof. We may suppose $\mathrm{tr.deg}\,\mathcal{K}\langle f \rangle / \mathcal{K} \geq 2$. Let $\mathcal{L} = (L, \tau_L)$ and choose $b_j^{(i)}$, $1 \leq i \leq n$, $0 \leq j \leq n-1$ to be algebraically independent over L. Put

$B = \{b_j^{(i)} \mid 1 \le i \le n,\ 0 \le j \le n-1\}$, $b^{(i)} = b_0^{(i)}$ and $b_n^{(i)} = -a_{n-1}b_{n-1}^{(i)} - \cdots - a_0 b^{(i)}$. Define the isomorphism τ of $L(B)$ into $L(B)$ sending $b_j^{(i)}$ to $b_{j+1}^{(i)}$ for all $1 \le i \le n$ and $0 \le j \le n-1$, and $x \in L$ to $\tau_L x \in L$. Put $\mathcal{N} = (L(B), \tau)$. Then $\mathcal{N}$ is difference overfield of $\mathcal{L}$. Note the following,

$$n^2 = \operatorname{tr.deg} \mathcal{N}/\mathcal{L} = \operatorname{tr.deg} \mathcal{K}\langle f, B\rangle/\mathcal{K}\langle f\rangle = \operatorname{tr.deg} \mathcal{K}\langle B\rangle/\mathcal{K},$$

which implies $\mathcal{K}\langle f\rangle$ and $\mathcal{L}$ are free from $\mathcal{K}\langle B\rangle$ over $\mathcal{K}$.

Since we have

$$\begin{pmatrix} f & f_1 & \cdots & f_n \\ b^{(1)} & b_1^{(1)} & \cdots & b_n^{(1)} \\ \vdots & \vdots & \ddots & \vdots \\ b^{(n-1)} & b_1^{(n-1)} & \cdots & b_n^{(n-1)} \\ b^{(n)} & b_1^{(n)} & \cdots & b_n^{(n)} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix},$$

we obtain $\operatorname{Cas}(f, b^{(1)}, \ldots, b^{(n)}) = 0$. Put $C = C_{\mathcal{K}\langle f,B\rangle}$. By Lemma 5 we find that $f, b^{(1)}, \ldots, b^{(n)}$ are linearly dependent over C. On the other hand we have $\operatorname{Cas}(b^{(1)}, \ldots, b^{(n)}) \ne 0$, which implies that $b^{(1)}, \ldots, b^{(n)}$ are linearly independent over C. Therefore we find that there are $c_1, \ldots, c_n \in C$ such that $f = \sum_{i=1}^{n} c_i b^{(i)}$.

Put $\mathcal{M}_i = \mathcal{K}\langle B, c_1, \ldots, c_i\rangle \subset \mathcal{K}\langle f, B\rangle$ for all $1 \le i \le n$ and $\mathcal{M}_0 = \mathcal{K}\langle B\rangle$. Note $\mathcal{M}_n = \mathcal{K}\langle f, B\rangle$ and $M_i = M_0(c_1, \ldots, c_i)$. Then we obtain $\operatorname{tr.deg} M_i/M_{i-1} \le 1$ for any $1 \le i \le n$, and so

$$\operatorname{tr.deg} \mathcal{K}\langle f, B\rangle/\mathcal{K}\langle B\rangle = \operatorname{tr.deg} \mathcal{K}\langle f\rangle/\mathcal{K} \ge 2$$

implies that there is some $1 \le k \le n-1$ such that $\operatorname{tr.deg} M_k/M_0 = 1$. We also find that $\mathcal{M}_i/\mathcal{M}_{i-1}$ is $\mathcal{N}$-decomposable for any $1 \le i \le n$, and so $\mathcal{M}_k/\mathcal{M}_0$ and $\mathcal{M}_n/\mathcal{M}_k$ are $\mathcal{N}$-decomposable. Since $\mathcal{N}/\mathcal{K}\langle f, B\rangle$ is algebraic, $\mathcal{N}/\mathcal{M}_k$ is $\mathcal{N}$-decomposable of $\operatorname{tr.deg} \ge 1$.

Note $\mathcal{N} = \mathcal{L}\mathcal{M}_0$, and it follows that $\mathcal{L}/\mathcal{K}$ is $\mathcal{N}$-decomposable. □

Proposition 7. *Let $\mathcal{K}$ be a difference field,*

$$y_n + a_{n-1}y_{n-1} + \cdots + a_0 y = b \tag{2}$$

a linear difference equation over $\mathcal{K}$, where $n \ge 1$, f a solution of (2), and $\mathcal{L}$ an algebraic difference overfield of $\mathcal{K}\langle f\rangle$. Then $\mathcal{L}/\mathcal{K}$ is $\mathcal{U}$-decomposable for some difference overfield $\mathcal{U}$ of $\mathcal{L}$.

Proof. We may suppose $b \neq 0$. Let $\mathcal{L} = (L, \tau)$, and put $a_n = 1$. The solution f satisfies $\sum_{i=0}^{n} a_i f_i = b$ and $\sum_{i=1}^{n+1} \tau(a_{i-1}) f_i = b_1$. Then we have

$$\begin{aligned} 0 &= \sum_{i=1}^{n+1} \tau(a_{i-1}) f_i - \frac{b_i}{b} \sum_{i=0}^{n} a_i f_i \\ &= f_{n+1} + \sum_{i=1}^{n} (\tau(a_{i-1}) - \frac{b_i}{b} a_i) f_i - \frac{b_1}{b} a_0 f. \end{aligned}$$

Therefore by Proposition 6 there is a difference overfield $\mathcal{U}$ of $\mathcal{L}$ such that $\mathcal{L}/\mathcal{K}$ is $\mathcal{U}$-decomposable. □

Corollary 8. *Let $\mathcal{K}$ be a difference field,*

$$y_n + a_{n-1} y_{n-1} + \cdots + a_0 y = b \tag{3}$$

be a linear difference equation over $\mathcal{K}$, where $n \geq 1$, and f a solution of (3)*. Then $\overline{\mathcal{K}\langle f \rangle}/\mathcal{K}$ is decomposable for any algebraic closure $\overline{\mathcal{K}\langle f \rangle}$ of $\mathcal{K}\langle f \rangle$.*

Proof. Let $\mathcal{L} = \overline{\mathcal{K}\langle f \rangle}$ be an algebraic closure of $\mathcal{K}\langle f \rangle$. By Proposition 7 we find that $\mathcal{L}/\mathcal{K}$ is $\mathcal{U}$-decomposable for some difference overfield $\mathcal{U}$ of $\mathcal{L}$. Let $\overline{\mathcal{U}}$ be an algebraic closure of $\mathcal{U}$. The algebraic closure of $\mathcal{L}$ in $\overline{\mathcal{U}}$ equals $\mathcal{L}$ because $\mathcal{L}$ is algebraically closed. Therefore by Proposition 4 we conclude that $\mathcal{L}/\mathcal{K}$ is decomposable. □

3 Irreducibility of q-$P(A_7')$

Notation. Throughout this section let C be an algebraically closed field of characteristic zero, t transcendental over C and $q \in C^{\times}$.

The q-Painlevé equation of type $A_7^{(1)'}$, the object here, appears in Sakai's paper [11]. The system over $(C(t), t \mapsto qt)$ is the following,

$$\begin{aligned} y_1 y &= z_1^2, \\ z_1 z &= \frac{y(1 - ty)}{t(y - 1)}. \end{aligned}$$

We prove that if q is not a root of unity and (f, g) a solution in a decomposable extension of $(\mathbb{C}(t), t \mapsto qt)$, then f and g are algebraic functions of the form $c/\sqrt{t}$, $c \in \mathbb{C}$.

Lemma 9. *Let $\mathcal{K}$ be a difference field, $\mathcal{D}$ a decomposable extension of $\mathcal{K}$ and $f \in \mathcal{D}$. Suppose that if $\mathcal{L}$ is a difference overfield of $\mathcal{K}$ of finite transcendence degree and $\mathcal{U}$ a difference overfield of $\mathcal{L}$ such that $\mathcal{K}\langle f\rangle_{\mathcal{D}} \subset \mathcal{U}$, then the following holds,*

$$\operatorname{tr.deg} \mathcal{L}\langle f\rangle_{\mathcal{U}}/\mathcal{L} \leq 1 \Rightarrow f \text{ is algebraic over } L.$$

Then f is algebraic over K.

Proof. Assume that f is transcendental over K. Choose $(\mathcal{L}, \mathcal{N})$ be an element of

$$\{(\mathcal{L}, \mathcal{N}) \mid \mathcal{K} \subset \mathcal{L} \subset \mathcal{N},\ \operatorname{tr.deg} L/K < \infty,\ \mathcal{N}/\mathcal{L} \text{ is decomposable},\\ \mathcal{K}\langle f\rangle_{\mathcal{D}} \subset \mathcal{N}, \text{ and } f \text{ is transcendental over } L\}$$

which has the minimal transcendence degree $\operatorname{tr.deg} N/L$. The choice is guaranteed because $(\mathcal{K}, \mathcal{D})$ satisfies the conditions. If we assume $\operatorname{tr.deg} N/L \leq 1$, we obtain $\operatorname{tr.deg} \mathcal{L}\langle f\rangle_{\mathcal{N}}/\mathcal{L} \leq 1$, which implies that f is algebraic over L by the hypothesis, a contradiction. Therefore it follows that $\operatorname{tr.deg} N/L \geq 2$.

Since $\mathcal{N}/\mathcal{L}$ is decomposable, there exist a difference overfield $\mathcal{U}$ of $\mathcal{N}$, a difference overfield $\mathcal{E}$ of $\mathcal{L}$ in $\mathcal{U}$ of finite transcendence degree which is free from $\mathcal{N}$ over $\mathcal{L}$, and a difference intermediate field $\mathcal{M}$ of $\mathcal{N}\mathcal{E}/\mathcal{E}$ satisfying $\operatorname{tr.deg} \mathcal{N}\mathcal{E}/\mathcal{M} \geq 1$ and $\operatorname{tr.deg} \mathcal{M}/\mathcal{E} \geq 1$, such that $\overline{\mathcal{N}\mathcal{E}}/\mathcal{M}$ and $\overline{\mathcal{M}}/\mathcal{E}$ are decomposable, where $\overline{\mathcal{N}\mathcal{E}}$ is an algebraic closure of $\mathcal{N}\mathcal{E}$ and $\overline{\mathcal{M}}$ the algebraic closure of $\mathcal{M}$ in $\overline{\mathcal{N}\mathcal{E}}$.

From $\mathcal{K}\langle f\rangle_{\mathcal{D}} \subset \mathcal{N} \subset \overline{\mathcal{N}\mathcal{E}}$ and $\operatorname{tr.deg} \overline{\mathcal{N}\mathcal{E}}/\mathcal{M} < \operatorname{tr.deg} \mathcal{N}/\mathcal{L}$ we find that f is algebraic over M, namely $f \in \overline{M}$. Note that

$$\mathcal{K}\langle f\rangle_{\mathcal{D}} = \mathcal{K}\langle f\rangle_{\overline{\mathcal{N}\mathcal{E}}} = \mathcal{K}\langle f\rangle_{\overline{\mathcal{M}}} \subset \overline{\mathcal{M}}.$$

Then from $\operatorname{tr.deg} \overline{\mathcal{M}}/\mathcal{E} < \operatorname{tr.deg} \mathcal{N}/\mathcal{L}$ we find that f is algebraic over E.

Since N and E are free over L, we find that f is transcendental over E, a contradiction. Therefore f is algebraic over K. □

Lemma 10. *Let $q \in C^{\times}$ be not a root of unity, $\mathcal{K}$ an inversive difference overfield of $(C(t), t \mapsto qt)$, $\mathcal{U} = (U, \tau)$ a difference overfield of $\mathcal{K}$, $\mathcal{L} \subset \mathcal{U}$ a difference overfield of $\mathcal{K}$ satisfying $\operatorname{tr.deg} \mathcal{L}/\mathcal{K} < \infty$, and $f \in \mathcal{U}$ a solution of the equation over $\mathcal{K}$,*

$$q^2 t^2 (y_1 - 1)^2 y_2 y = (1 - qty_1)^2.$$

Then we obtain

$$\operatorname{tr.deg} \mathcal{L}\langle f\rangle/\mathcal{L} \leq 1 \Rightarrow f \text{ is algebraic over } L.$$

Proof. We may suppose that L is algebraically closed. Then $\mathcal{L}$ is inversive. Assume $\operatorname{tr.deg}\mathcal{L}\langle f\rangle/\mathcal{L} = 1$. We find that f and f_1 are transcendental over L. Choose an irreducible polynomial over L,

$$F = \sum_{i=0}^{n_0}\sum_{j=0}^{n_1} a_{ij}Y^iY_1^j \in L[Y, Y_1] \setminus \{0\}, \quad n_0 = \deg_Y F,\, n_1 = \deg_{Y_1} F,$$

such that $F(f, f_1) = 0$, and $a_{n_0n_1} = 0$ or 1. Define the following three polynomials,

$$F^* = \sum_{i=0}^{n_0}\sum_{j=0}^{n_1} \tau(a_{ij})Y^iY_1^j,$$

$$F_1 = (q^2t^2Y(Y_1-1)^2)^{n_1}F^*\left(Y_1, \frac{(1-qtY_1)^2}{q^2t^2Y(Y_1-1)^2}\right) \in L[Y, Y_1] \setminus \{0\},$$

$$F_0 = (q^2t^2Y_1(Y-1)^2)^{n_0}F\left(\frac{(1-qtY)^2}{q^2t^2Y_1(Y-1)^2}, Y\right) \in L[Y, Y_1] \setminus \{0\}.$$

Since the solution f satisfies

$$q^2t^2(f_1-1)^2f_2f = (1-qtf_1)^2,$$

we obtain $F_1(f, f_1) = F_0(f_1, f_2) = 0$, and so $F \mid F_1$ and $F^* \mid F_0$. These imply

$$n_1 = \deg_{Y_1} F^* \le \deg_{Y_1} F_0 \le n_0 = \deg_Y F \le \deg_Y F_1 \le n_1.$$

Therefore we obtain $n_0 = n_1$. Put $n = n_0 = n_1 \ge 1$. Let $P \in L[Y, Y_1] \setminus \{0\}$ be the polynomial such that $F_1 = PF$. We find $P \in L[Y_1]$ because $\deg_Y P = \deg_Y F_1 - \deg_Y F = 0$.

We have

$$\begin{aligned}
F_1 &= (q^2t^2Y(Y_1-1)^2)^n \sum_{i=0}^{n}\sum_{j=0}^{n} \tau(a_{ij})Y_1^i\left(\frac{(1-qtY_1)^2}{q^2t^2Y(Y_1-1)^2}\right)^j \\
&= \sum_{j=0}^{n}(q^2t^2Y(Y_1-1)^2)^{n-j}(1-qtY_1)^{2j}\sum_{i=0}^{n}\tau(a_{ij})Y_1^i \\
&= \sum_{j=0}^{n}(q^2t^2Y(Y_1-1)^2)^{j}(1-qtY_1)^{2(n-j)}\sum_{i=0}^{n}\tau(a_{i,n-j})Y_1^i \\
&= \sum_{j=0}^{n}\left\{(qt)^{2j}(Y_1-1)^{2j}(1-qtY_1)^{2(n-j)}\sum_{i=0}^{n}\tau(a_{i,n-j})Y_1^i\right\}Y^j
\end{aligned}$$

and

$$PF = P\sum_{i=0}^{n}\sum_{j=0}^{n} a_{ij}Y^iY_1^j = P\sum_{j=0}^{n}\sum_{i=0}^{n} a_{ji}Y^jY_1^i = \sum_{j=0}^{n}\left\{P\sum_{i=0}^{n} a_{ji}Y_1^i\right\}Y^j.$$

Therefore for all $k \in \{0, 1, \ldots, n\}$ we obtain

$$(*k) \qquad (qt)^{2k}(Y_1-1)^{2k}(1-qtY_1)^{2(n-k)}\sum_{i=0}^{n}\tau(a_{i,n-k})Y_1^i = P\sum_{i=0}^{n} a_{ki}Y_1^i.$$

The equation $(*n)$ and $(*0)$ are the following,

$$(*n) \qquad (qt)^{2n}(Y_1-1)^{2n}\sum_{i=0}^{n}\tau(a_{i0})Y_1^i = P\sum_{i=0}^{n} a_{ni}Y_1^i \quad (\neq 0),$$

$$(*0) \qquad (1-qtY_1)^{2n}\sum_{i=0}^{n}\tau(a_{in})Y_1^i = P\sum_{i=0}^{n} a_{0i}Y_1^i \quad (\neq 0).$$

Note that $\sum_{i=0}^{n} a_{ni}Y_1^i \neq 0$ and $\sum_{i=0}^{n}\tau(a_{in})Y_1^i \neq 0$.

By $(*n)$ we find $(Y_1-1)^n \mid P$, and so by $(*0)$, $(Y_1-1)^n \mid \sum_{i=0}^{n}\tau(a_{in})Y_1^i$. Therefore we obtain $\sum_{i=0}^{n}\tau(a_{in})Y_1^i = \tau(a_{nn})(Y_1-1)^n$, which implies $a_{nn} = 1$. Comparing the terms of degree 0 of the equation

$$(4) \qquad \sum_{i=0}^{n}\tau(a_{in})Y_1^i = (Y_1-1)^n,$$

we find

$$(5) \qquad a_{0n} = (-1)^n \neq 0.$$

By this the equation $(*0)$ yields $\deg P = 2n$, and so

$$P = p(Y_1-1)^n(1-qtY_1)^n, \quad p \in L^\times.$$

Then from $(*0)$ we obtain

$$(1-qtY_1)^n = p\sum_{i=0}^{n} a_{0i}Y_1^i,$$

which implies $1 = pa_{00}$ and $(-qt)^n = pa_{0n}$. By (5) we find $p = (qt)^n$ and $a_{00} = (qt)^{-n}$.

Since we have $(1 - qtY_1)^n \mid P$, we obtain $(1 - qtY_1)^n \mid \sum_{i=0}^n \tau(a_{i0})Y_1^i$ by the equation $(*n)$, and so

$$\sum_{i=0}^{n} \tau(a_{i0})Y_1^i = \tau(a_{00})(1 - qtY_1)^n = (q^2t)^{-n}(1 - qtY_1)^n.$$

Then from $(*n)$ we obtain

$$(Y_1 - 1)^n q^{-n} = \sum_{i=0}^{n} a_{ni} Y_1^i.$$

Comparing the terms of degree n, we find $q^{-n} = a_{nn} = 1$, a contradiction.

Therefore we conclude that $\operatorname{tr.deg} \mathcal{L}\langle f\rangle/\mathcal{L} \neq 1$, which implies

$$\operatorname{tr.deg} \mathcal{L}\langle f\rangle/\mathcal{L} \leq 1 \Rightarrow \operatorname{tr.deg} \mathcal{L}\langle f\rangle/\mathcal{L} = 0 \Rightarrow f \text{ is algebraic over } L,$$

the required. □

Theorem 11. *Let $q \in C^\times$ be not a root of unity, $\mathcal{K}$ an inversive difference overfield of $(C(t), t \mapsto qt)$, $\mathcal{D}$ a decomposable extension of $\mathcal{K}$, and $f, g \in \mathcal{D}$ satisfy two equations,*

$$f_1 f = g_1^2, \quad g_1 g = \frac{f(1 - tf)}{t(f - 1)}.$$

Then f and g are algebraic over K.

Proof. We may suppose $f \neq 0$ and $g \neq 0$. The two equations yield

$$f_2 f_1^2 f = (f_2 f_1)(f_1 f) = g_2^2 g_1^2 = (g_2 g_1)^2 = \frac{f_1^2(1 - qtf_1)^2}{q^2t^2(f_1 - 1)^2},$$
$$q^2t^2(f_1 - 1)^2 f_2 f = (1 - qtf_1)^2.$$

If we let $\mathcal{L}$ be a difference overfield of $\mathcal{K}$ satisfying $\operatorname{tr.deg} \mathcal{L}/\mathcal{K} < \infty$, and $\mathcal{U}$ a difference overfield of $\mathcal{L}$ satisfying $\mathcal{K}\langle f\rangle_{\mathcal{D}} \subset \mathcal{U}$, by Lemma 10 we obtain the following,

$$\operatorname{tr.deg} \mathcal{L}\langle f\rangle_{\mathcal{U}}/\mathcal{L} \leq 1 \Rightarrow f \text{ is algebraic over } L.$$

Therefore by Lemma 9 we find that f is algebraic over K, which implies g is also algebraic over K. □

It remains to find the algebraic solutions. We have

Lemma 12 (Lemma 9 in [10]). *Let $q \in C^\times$ be not a root of unity, t transcendental over C, $F/C(t)$ a finite algebraic extension of degree n, and τ an isomorphism of F into F over C sending t to qt. Then $F = C(x)$, $x^n = t$.*

Theorem 13. *Let $q \in C^\times$ be not a root of unity, put $\mathcal{K} = (C(t), t \mapsto qt)$, and let $\overline{\mathcal{K}} = (\overline{C(t)}, \tau)$ be an algebraic closure of $\mathcal{K}$. Suppose that $f, g \in \overline{\mathcal{K}}$ satisfy the following two equations,*

$$f_1 f = g_1^2, \tag{6}$$

$$g_1 g = \frac{f(1-tf)}{t(f-1)}. \tag{7}$$

Then one of the following holds.

(i) $(f, g) = (0, 0)$.

(ii) $(f, g) = (-1/x, -\alpha/x)$, $(-1/x, \alpha/x)$, $(1/x, -\alpha/x)$ *or* $(1/x, \alpha/x)$, *where* $\alpha \in C^\times$ *satisfies* $\alpha^4 = q$ *and* $x \in \overline{C(t)}$ *satisfies* $x^2 = t$ *and* $\tau x = \alpha^2 x$.

Proof. We may suppose $f \neq 0$ and $g \neq 0$. Put $\mathcal{L} = \mathcal{K}\langle f, g\rangle \subset \overline{\mathcal{K}}$. Then we have $L = C(t)(f, g)$. Put $n = [L : C(t)] < \infty$. By Lemma 12 we find $L = C(x)$, $x^n = t$. Since we have $(\tau x/x)^n = q \in C^\times$, we obtain $\tau x/x \in C^\times$. Put $r = \tau x/x \in C^\times$, which satisfies $r^n = q$ and $\tau x = rx$. Note that $f, g \in L = C(x)$ and $\mathcal{L}$ is inversive.

Express f and g as $f = P/Q$ and $g = R/S$, where $P, Q, R, S \in C[x] \setminus \{0\}$, P and Q are relatively prime, R and S are relatively prime, and Q and S are monic. From the equation (6) we obtain

$$P_1 P S_1^2 = Q_1 Q R_1^2 \quad (\neq 0), \tag{8}$$

and from the equation (7),

$$x^n (P - Q) Q R_1 R = P(Q - x^n P) S_1 S \quad (\neq 0). \tag{9}$$

By these equations we find $x \mid P(Q - x^n P)S_1 S$, and so $x \mid P$ or $x \mid Q$.

Let v_0 be the normalized discrete valuation of $C(x)/C$ with the prime element x. We prove $x \mid Q$ in $C[x]$. Assume $x \mid P$. Put $m = v_0(P) \in \mathbb{Z}_{>0}$, namely $x^m \mid P$ and $x^{m+1} \nmid P$. We obtain $x \mid R$ from (8), and so $x \nmid S$. Then it follows that

$$2m = v_0(P_1 P S_1^2) = v_0(Q_1 Q R_1^2) = v_0(R_1^2) = 2v_0(R_1),$$

which implies $v_0(R) = m$. Therefore by (9) we find $n + 2m = m$, a contradiction.

Put $m = v_0(Q) \in \mathbb{Z}_{>0}$. From the equation (8) we obtain $x \mid S$ and $x \nmid R$, and so $v_0(S) = m$. Then from the equation (9) we obtain $v_0(Q - x^n P) = n - m$. Since we have $0 \leq n - m < n$, we find $v_0(Q) = n - m$, which implies $n = 2m$.

Express f and g as $f = \sum_{i=-m}^{\infty} a_i x^i$, $a_{-m} \neq 0$ and $g = \sum_{i=-m}^{\infty} b_i x^i$, $b_{-m} \neq 0$. Seeing the first terms of the equation (6), we obtain $a_{-m}^2 = b_{-m}^2 r^{-m}$. On the other hand from the equation (7) we obtain $b_{-m}^2 r^{-m} = 1$. Then it follows that $a_{-m}^2 = 1$.

Combining the equations (6) and (7) as

$$f_2 f_1^2 f = (f_2 f_1)(f_1 f) = g_2^2 g_1^2 = (g_2 g_1)^2 = \frac{f_1^2 (1 - qtf_1)^2}{q^2 t^2 (f_1 - 1)^2},$$

we obtain

$$q^2 t^2 (f_1 - 1)^2 f_2 f = (1 - qtf_1)^2. \tag{10}$$

We prove that for any $i \geq -m$,

$$m \nmid i \Rightarrow a_i = 0,$$

which yields $f \in C(x^m)$. Assume that there is $i \geq -m$ such that $m \nmid i$ and $a_i \neq 0$. Let

$$km + l = \min\{i \geq -m \mid m \nmid i \text{ and } a_i \neq 0\}, \quad 0 < l < m.$$

The left side of the equation (10) is

$$\begin{aligned}
&q^2 x^{4m} (-1 + a_{-m} r^{-m} x^{-m} + \cdots + a_{km} r^{km} x^{km} + a_{km+l} r^{km+l} x^{km+l} + \cdots)^2 \\
&\times (a_{-m} r^{-2m} x^{-m} + \cdots + a_{km} r^{2km} x^{km} + a_{km+l} r^{2(km+l)} x^{km+l} + \cdots) \\
&\times (a_{-m} x^{-m} + \cdots + a_{km} x^{km} + a_{km+l} x^{km+l} + \cdots)
\end{aligned}$$

and the right side is

$$(-1 + qa_{-m} r^{-m} x^m + \cdots + qa_{km} r^{km} x^{(k+2)m} + qa_{km+l} r^{km+l} x^{(k+2)m+l} + \cdots)^2.$$

On the one hand the first term of the right side whose exponent is not divisible by m is $2(-1)qa_{km+l}r^{km+l}x^{(k+2)m+l}$. On the other hand the term of degree $(k+1)m+l$ of the left side is

$$\begin{aligned}
&q^2x^{4m}(2a_{km+l}r^{km+l}x^{km+l}\cdot a_{-m}r^{-m}x^{-m}\cdot a_{-m}r^{-2m}x^{-m}\cdot a_{-m}x^{-m}\\
&\qquad + a_{km+l}r^{2(km+l)}x^{km+l}(a_{-m}r^{-m}x^{-m})^2a_{-m}x^{-m}\\
&\qquad + a_{km+l}x^{km+l}(a_{-m}r^{-m}x^{-m})^2a_{-m}r^{-2m}x^{-m})\\
&= q^2x^{(k+1)m+l}a_{km+l}a_{-m}^3(2r^{(k-3)m+l}+r^{2((k-1)m+l)}+r^{-4m}).
\end{aligned}$$

Therefore it follows that

$$(r^{(k-1)m+l}+r^{-2m})^2 = r^{2((k-1)m+l)}+2r^{(k-3)m+l}+r^{-4m}=0,$$

which implies $q^{2((k+1)m+l)}=1$, a contradiction.

Put $z=x^m$. Then we have $f=\sum_{i=-1}^{\infty}a_{mi}z^i$. The left side of the equation (10) is

$$\begin{aligned}
&q^2z^4(a_{-m}r^{-m}z^{-1}+(a_0-1)+a_mr^mz+\cdots)^2\\
&\times(a_{-m}r^{-2m}z^{-1}+a_0+a_mr^{2m}z+\cdots)\\
&\times(a_{-m}z^{-1}+a_0+a_mz+\cdots)
\end{aligned}$$

and the right side is

$$(-1+qa_{-m}r^{-m}z+qa_0z^2+qa_mr^mz^3+\cdots)^2.$$

Comparing the terms of degree 1, we find $a_0(r^m+1)^2=0$. Since $r^m+1=0$ implies $q=1$, we obtain $a_0=0$.

We prove that $a_{mi}=0$ for all $i\geq 1$ by induction. Firstly we deal with the case $i=1$. Comparing the terms of degree 2 of the above two expansions, we find $a_m(r^{-2m}+1)^2=0$, which implies $a_m=0$. Secondly we suppose $i\geq 2$ and the statement is true for the numbers $<i$. Comparing the terms of degree $i+1$, we find $a_{mi}(r^{m(i+1)}+1)^2=0$, which implies $a_{mi}=0$.

Therefore we obtain $f=a_{-m}/z=a_{-m}/x^m\in C(x^m)$. The equation (6) yields $S^2=r^{-m}x^{2m}R^2$. Since S is monic, we find $S^2=x^{2m}$, and so $S=x^m$. Then we have $R^2=r^m\in C^\times$, which implies $R\in C^\times$. Therefore we obtain $g=R/S\in C(x^m)$.

By $L=C(t)(f,g)\subset C(x^m)\subset C(x)=L$ we find $L=C(x^m)$. Then we have

$$2\leq 2m=n=[L:C(t)]=[C(x^m):C(x^{2m})]\leq 2,$$

which implies $n = 2$ and $m = 1$. Let $\alpha \in C^\times$ be a root of the polynomial $X^2 - r \in C[X]$. We have $f = a_{-1}/x$, $a_{-1} = -1$ or 1, and $g = R/x$, $R = -\alpha$ or α. Note that $\alpha^4 = r^2 = q$. $\square$

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