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Asymptotic Behavior of distributions of the sum of i.i.d. random variables with fat tail II

by

Hirotaka Fushiya and Shigeo Kusuoka

UNIVERSITY OF TOKYO
GRADUATE SCHOOL OF MATHEMATICAL SCIENCES
KOMABA, TOKYO, JAPAN

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1 Introduction

It is a classical problem to find an efficient approximation formula for distributions of sums of independent identically distributed random variables. The well-known one is the central limit theorem. Let $(\Omega, \mathcal{F}, P)$ be a probability space and X_n, $n = 1, 2, \dots,$ be independent identically distributed random variables. If we assume that $E[X_1^2] = 1$ and $E[X_1] = 0$ we have

$$\sup_{s \in \mathbf{R}} |P(\sum_{k=1}^{n} X_k > sn^{1/2}) - \Phi_0(s)| \to 0, \qquad n \to \infty, \tag{1}$$

where

$$\Phi_0(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} \exp(-\frac{y^2}{2}) dy, \qquad x \in \mathbf{R}.$$

Recently people in finance are interested in computing the quantile of the distribution of $\sum_{k=1}^{n} X_k$ for the purpose of measuring market risk. However, it is said that the central limit theorem is not efficient for their purpose. For large $s > 0$, both $P(\sum_{k=1}^{n} X_k > sn^{1/2})$ and $\Phi_0(s)$ are small, and so Equation (1) does not give us a good information. Our aim in the present paper is to give a new approximation formula which gives more efficient information for $P(\sum_{k=1}^{n} X_k > sn^{1/2})$.

Now let us explain our result. Let $(\Omega, \mathcal{F}, P)$ be a probability space, and let X_n, $n = 1, 2, \dots,$ be independent random variables with the same probability law μ. Also, let $F : \mathbf{R} \to [0, 1]$ and $\bar{F} : \mathbf{R} \to [0, 1]$ be given by

$$F(x) = \mu((-\infty, x]) \text{ and } \bar{F}(x) = \mu((x, \infty)), \qquad x \in \mathbf{R}.$$

Throughout this paper we assume the following assumptions (A1), (A2), (A3) and (A4).

(A1) $\bar{F}(x)$ is a regularly varying function of index $-\alpha$ for some $\alpha > 2$, as $x \to \infty$, i.e., if we let

$$L(x) = x^{\alpha} \bar{F}(x), \qquad x \geqq 1,$$

*Graduate School of Mathematical Sciences, The University of Tokyo, Komaba 3-8-1, Meguro-ku, Tokyo 153-8914, Japan, research supported by the 21st century COE project, Graduate School of Mathematical Sciences, The University of Tokyo

then $L(x) > 0$ for any $x \geqq 1$, and for any $a > 0$

$$\frac{L(ax)}{L(x)} \to 1, \qquad x \to \infty.$$

Also we assume the following.

(A2) $|x|^{\alpha+2} F(x) \to 0, \quad x \to -\infty.$

(A3) The probability law μ is absolutely continuous and has a density function $\rho : \mathbf{R} \to [0, \infty)$ which is right continuous and has a finite total variation.

Since $\alpha > 2$, we see that $E[|X_1|^2] < \infty$. We assume furthermore the following.

(A4) $E[X_1] = 0$ and $E[X_1^2] = 1$.

Let K be an integer such that $K - 1 < \alpha \leqq K$. Then $K \geqq 3$. From the assumptions (A1) and (A2), we see that the probability law μ has $(K-1)$-th moment. So let η_k, $k = 1, \dots, K-1$, be given by

$$\eta_k = \int_{\mathbf{R}} x^k \mu(dx).$$

Then we see that $\eta_1 = 0$ and $\eta_2 = 1$. Also, let us define $\Phi_k : \mathbf{R} \to \mathbf{R}$, $k = 1, 2, \dots$, by

$$\Phi_1(x) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{x^2}{2}) = -\frac{d}{dx}\Phi_0(x),$$

and

$$\Phi_k(x) = (-1)^{k-1} \frac{d^{k-1}}{dx^{k-1}} \Phi_1(x), \qquad k = 2, 3, \dots.$$

Our main result is the following.

Theorem 1 *There are $\delta > 0$ and $C > 0$ such that*

$$\sup_{s \in [1, \log n]} |P(\sum_{k=1}^n X_k > s n^{1/2}) - G(n,s)| \leqq C n^{-(\alpha-2)/2-\delta}, \qquad n = 3, 4, \dots.$$

Here

$$G(n,s)$$

$$= \Phi_0(s) + n \int_{-\infty}^s \bar{F}((s-x)n^{1/2}) \Phi_1(x) dx - \sum_{k=1}^{K-1} \frac{n^{-(k-2)/2}}{k!} \Phi_k(s) \int_0^\infty x^k \mu(dx)$$

$$+ \frac{n^{-(K-2)/2}}{K!} \Phi_K(s) \int_{-\infty}^0 x^K \mu(dx) + \sum_{k=3}^{K-1} n^{-k/6} q_k(n^{-1/3}, \eta_2, \dots, \eta_k) \Phi_k(s)$$

$$+ \sum_{k=K}^{3(K-1)} n^{-k/6} q_k(n^{-1/3}, \eta_2, \dots, \eta_{K-1}, 0, \dots, 0) \Phi_k(s),$$

where q_k's are polynomials defined in the next section.

We also prove the following, as a consequence of the above theorem and Theorem 1 in [1]

Theorem 2 *Let $\gamma \in (0, \alpha/2-1)$. Then we have*

$$\sup_{s\in\mathbf{R}} \left|\frac{P(\sum_{k=1}^{n} X_k > s)}{\Phi_0(n^{-1/2}s) + ((n\bar{F}(s)) \wedge n^{-\gamma})} - 1\right| \to 0, \qquad n \to \infty.$$

2 Algebraic preparation

In this section, we think of formal power series in z. First, we think of the following formal power series in z.

$$\log(1 + \sum_{k=2}^{\infty} \frac{a_k}{k!} z^k) = \sum_{\ell=1}^{\infty} \frac{(-1)^{\ell-1}}{\ell} (\sum_{k=2}^{\infty} \frac{a_k}{k!} z^k)^\ell = \sum_{\ell=2}^{\infty} c_\ell(a_2, \dots, a_\ell) \frac{z^\ell}{\ell!} \tag{2}$$

Then we see that $c_\ell(a_2, \dots, a_\ell)$, $\ell \geqq 2$, are polynomials in $a_2, \dots, a_\ell$, and

$$c_\ell(t^2 a_2, \dots, t^\ell a_\ell) = t^\ell c_\ell(a_1, \dots, a_\ell)$$

for any $t, a_1, \dots, a_\ell \in \mathbf{R}$. Moreover, we see that

$$c_2(a_2) = a_2 \quad \text{and} \quad c_\ell(a_2, \dots, a_{\ell-1}, a_\ell) = c_\ell(a_2, \dots, a_{\ell-1}, 0) + a_\ell, \quad \ell \geqq 2.$$

We also think of the following formal power series in z.

$$\exp(y^{-3} \sum_{\ell=3}^{\infty} c_\ell(a_2, \dots, a_\ell) \frac{(yz)^\ell}{\ell!})$$

$$= 1 + \sum_{k=1}^{\infty} \frac{1}{k!} (\sum_{\ell=3}^{\infty} c_\ell(a_2, \dots, a_\ell) \frac{y^{\ell-3} z^\ell}{\ell!})^k = 1 + \sum_{k=3}^{\infty} q_k(y, a_2, \dots, a_k) z^k. \tag{3}$$

Then we see that $q_k(y, a_2, \dots, a_k)$, $k \geqq 3$, are polynomials in y, $a_2, \dots, a_\ell$. Note that

$$q_k(y, t^2 a_2 \dots, t^k a_k) = t^k q_k(y, a_2, \dots, a_k)$$

and that

$$q_k(y, a_2, \dots, a_k) = q_k(y, a_2, \dots, a_{k-1}, 0) + \frac{y^{k-3}}{k!} a_k, \qquad k \geqq 3.$$

Also we have

$$\exp(y^{-6} \sum_{\ell=3}^{\infty} c_\ell(a_2, \dots, a_\ell) \frac{(y^3 z)^\ell}{\ell!})$$

$$= \exp((y^2)^{-3} \sum_{\ell=3}^{\infty} c_\ell(y^2 a_2, \dots, y^\ell a_\ell) \frac{(y^2 z)^\ell}{\ell!})$$

$$= 1 + \sum_{k=3}^{\infty} q_k(y^2, y^2 a_2, \dots, y^k a_k) z^k = 1 + \sum_{k=3}^{\infty} y^k q_k(y^2, a_2, \dots, a_k) z^k \tag{4}$$

as a formal power series in z.

3 Preliminary facts

Proposition 3 *We have*

$$\sup_{1/2\leqq a\leqq 2}\frac{L(ax)}{L(x)}\to 1, \qquad x\to\infty,$$

and

$$\inf_{1/2\leqq a\leqq 2}\frac{L(ax)}{L(x)}\to 1, \qquad x\to\infty.$$

Proof. Since the proof is similar, we prove the first equation only. If not, there are $\varepsilon>0$, $\{a_n\}_{n=1}^{\infty}$ and $\{x_n\}_{n=1}^{\infty}$ such that $1/2\leqq a_n\leqq 2$, $x_n\geqq 1$, $n=1,2,\dots$, $x_n\to\infty$, $n\to 1$, and that

$$\frac{L(a_nx_n)}{L(x_n)}>1+\varepsilon, \qquad n=1,2,\dots.$$

Then taking a subsequence if necessary, we may assume that there is an $a\in[1/2,2]$ such that $a_n\to a$, $n\to\infty$. Then we see that for any $m\geqq 3$ there is a $n(m)\geqq 1$ such that

$$(a-\frac{1}{m})^{-\alpha}L((a-\frac{1}{m})x_n)=\bar{F}((a-\frac{1}{m})x_n)\geqq \bar{F}(a_nx_n)=a_n^{-\alpha}L(a_nx_n), \qquad n\geqq n(m).$$

So we have

$$(1-\frac{1}{ma})^{-\alpha}\geqq \lim_{n\to\infty}\frac{L(a_nx_n)}{L(ax_n)}\geqq 1+\varepsilon, \qquad m\geqq 3.$$

Since m is arbitrary, this implies a contradiction. ∎

Proposition 4 *For any $\varepsilon\in(0,1)$, there is an $M\geqq 1$ such that*

$$M^{-1}y^{-\varepsilon}\leqq\frac{L(yx)}{L(x)}\leqq My^{\varepsilon} \qquad x,y\geqq 1.$$

Proof. For any $\varepsilon\in(0,1)$ there is an $m\geqq 1$ such that

$$|\frac{L(ex)}{L(x)}-1|\leqq\varepsilon \qquad x\geqq e^m.$$

Let

$$C=\sup_{x\in[1,e^m]}(\frac{L(ex)}{L(x)}+\frac{L(x)}{L(ex)})<\infty.$$

Then we have

$$C^{-m}(1-\varepsilon)^n\leqq\frac{L(e^nx)}{L(x)}\leqq C^m(1+\varepsilon)^n, \qquad x\geqq 1,\ n\geqq 0.$$

For any $y\geqq 1$, there is an $n\geqq 1$ such that $e^{n-1}\leqq y\leqq e^n$. Then we have

$$\bar{F}(e^{n-1}x)\geqq\bar{F}(yx)\geqq\bar{F}(e^nx).$$

So we have for any $x,y\geqq 1$

$$(e^{-1}yx)^{-\alpha}L(e^{n-1}x)\geqq(e^{n-1}x)^{-\alpha}L(e^{n-1}x)\geqq(yx)^{-\alpha}L(yx)$$

$$\geqq (e^n x)^{-\alpha} L(e^n x) \geqq (eyx)^{-\alpha} L(e^n x),$$

which implies

$$C^{-m} e^{-\alpha} (1-\varepsilon)^n \leqq \frac{L(yx)}{L(x)} \leqq C^m e^{\alpha} (1+\varepsilon)^{n-1}.$$

Therefore we have

$$C^{-m} e^{-\alpha} (1-\varepsilon) y^{\log(1-\varepsilon)} \leqq \frac{L(yx)}{L(x)} \leqq C^m e^{\alpha} y^{\log(1+\varepsilon)}, \qquad x \geqq 1,\ y \geqq 1.$$

This implies our assertion. ∎

The following is known as Karamata's theorem, but we give a proof.

Proposition 5 (1) *For any* $\beta < -1$,

$$\frac{1}{t^{\beta+1} L(t)} \int_t^\infty x^\beta L(x) dx \to -\frac{1}{\beta+1}, \qquad t \to \infty.$$

(2) *For any* $\beta > -1$,

$$\frac{1}{t^{\beta+1} L(t)} \int_1^t x^\beta L(x) dx \to \frac{1}{\beta+1}, \qquad t \to \infty.$$

(3) *Let* $f : [1, \infty) \to (0, \infty)$ *be given by*

$$f(t) = \int_1^t x^{-1} L(x) dx \qquad t \geqq 1.$$

Then f *is slowly varying.*

Proof. Note that for $t > 1$

$$\frac{1}{t^{\beta+1} L(t)} \int_t^\infty x^\beta L(x) dx = \int_1^\infty x^\beta \frac{L(tx)}{L(t)} dx, \text{ if } \beta < -1$$

and

$$\frac{1}{t^{\beta+1} L(t)} \int_1^t x^\beta L(x) dx, = \int_{1/t}^1 x^\beta (\frac{L(t)}{L(tx)})^{-1} dx \text{ if } \beta > -1$$

Then the assertions (1) and (2) follow from this equation and Proposition 3.

Let us prove (3). If $\lim_{t\to\infty} f(t) < \infty$, the assertion is obvious. So we assume that $\lim_{t\to\infty} f(t) = \infty$. Then for any $a > 0$ and $t_0 > 1$

$$f(at) = \int_{1/a}^t x^{-1} L(ax) dx = \int_{1/a}^{t_0} x^{-1} L(ax) dx + \int_{t_0}^t x^{-1} L(x) \frac{L(ax)}{L(x)} dx.$$

So we have

$$\inf_{x \geqq t_0} \frac{L(ax)}{L(x)} \leqq \varliminf_{t\to\infty} \frac{f(at)}{f(t)} \leqq \varlimsup_{t\to\infty} \frac{f(at)}{f(t)} \leqq \sup_{x \geqq t_0} \frac{L(ax)}{L(x)}.$$

Therefore by Proposition 4 and Lebesgue's convergence theorem, we have our assertion. ∎

4 Estimate for moments and characteristic functions

Remind that K is an integer such that $K-1<\alpha \leqq K$ and

$$\eta_k = \int_{-\infty}^{\infty} x^k \mu(dx), \qquad k=1,2,\ldots,K-1.$$

Then by the assumption (A4) we have $\eta_1 = 0$ and $\eta_2 = 1$. Note that

$$1-\bar{F}(t) \geqq 1 - \int_2^{\infty} \frac{x^2}{4}\mu(dx) \geqq \frac{3}{4}$$

for any $t \geqq 2$. Let

$$\eta_k(t) = \int_{(-\infty,t]} x^k \mu(dx), \qquad t>0,\ k=1,2,\ldots,K+1,$$

and

$$\bar{\eta}_k(t) = \int_{(t,\infty)} x^k \mu(dx), \qquad t>0,\ k=1,2,\ldots,K-1.$$

Then we have

$$\eta_k(t) = \int_{(-\infty,0)} x^k \mu(dx) + k\int_0^t x^{k-1}\bar{F}(x)dx - t^k\bar{F}(t), \qquad t>0,\ k=1,2,\ldots,K+1,$$

and

$$\bar{\eta}_k(t) = k\int_t^{\infty} x^{k-1}\bar{F}(x)dx + t^k\bar{F}(t) \qquad t>0,\ k=1,2,\ldots,K-1.$$

Then by Propositions 4 and 5 we have the following.

Proposition 6 *For any $\varepsilon>0$, there is a $C(\varepsilon)>0$ such that*

$$L(t) \leqq C(\varepsilon)t^{\varepsilon},$$

$$|\eta_K(t)| \leqq C(\varepsilon)t^{-\alpha+K+\varepsilon},$$

$$|\bar{\eta}_k(t)| \leqq C(\varepsilon)t^{-\alpha+k+\varepsilon}, \qquad k=1,2,\ldots K-1,$$

and

$$\int_{(-\infty,t]} |x|^{K+1}\mu(dx) \leqq C(\varepsilon)t^{-\alpha+K+1+\varepsilon}$$

for any $t \geqq 1$.

The following is well known.

Proposition 7 (1) *For any $m \geqq 1$, let $r_{e,m}: \mathbf{R} \to \mathbf{C}$ be given by*

$$r_{e,m}(t) = \exp(it) - (1+\sum_{k=1}^{m}\frac{(it)^k}{k!}), \qquad t \in \mathbf{R}.$$

Then we have

$$|r_{e,m}(t)| \leqq \frac{|t|^{m+1}}{(m+1)!} \qquad t \in \mathbf{R}.$$

(2) *For any* $m \geqq 1$, *let* $r_{l,m} : \{z \in \mathbf{C};\ |z| \leqq 1/2\} \to \mathbf{C}$ *be given by*

$$r_{l,m}(z) = \log(1+z) - \sum_{k=1}^{m} \frac{(-1)^{k-1}}{k} z^k, \qquad z \in \mathbf{C},\ |z| \leqq 1/2.$$

Then we have

$$|r_{l,m}(z)| \leqq 2|z|^{m+1}, \qquad z \in \mathbf{C},\ |z| \leqq 1/2.$$

Let $\mu(t)$, $t > 0$, be a probability measure on $(\mathbf{R}, \mathcal{B}(\mathbf{R}))$ given by

$$\mu(t)(A) = (1 - \bar{F}(t))^{-1} \mu(A \cap (-\infty, t]),$$

for any $A \in \mathcal{B}(\mathbf{R})$.

Let $\varphi(\cdot; \mu(t))$, $t > 0$, be the characteristic function of the probability measure $\mu(t)$, i.e.,

$$\varphi(\xi; \mu(t)) = \int_{\mathbf{R}} \exp(ix\xi)\mu(t)(dx), \qquad \xi \in \mathbf{R}.$$

By the assumption (A3), we see that the density function $\rho(x) \to 0$ as $|x| \to \infty$. Also we see that the probability measure $\mu(t)$, $t \geqq 2$, is absolutely continuous and its density function is $(1 - \bar{F}(t))^{-1}\rho(x)1_{(-\infty,t]}(x)$, whose total variation is dominated by twice of that of ρ.

Therefore we have the following.

Proposition 8 (1) *For any* $t \geqq 2$ *and* $\xi \in \mathbf{R}$,

$$i\xi\varphi(\xi; \mu(t)) = (1 - \bar{F}(t))^{-1} \int_{\mathbf{R}} i\xi e^{i\xi x} \rho(x) 1_{(-\infty,t]}(x) dx$$

$$= -(1 - \bar{F}(t))^{-1} \int_{\mathbf{R}} e^{i\xi x} d(\rho(x) 1_{(-\infty,t]}(x)).$$

(2) *There is a* $C > 0$ *such that*

$$|\varphi(\xi, \mu(t))| \leqq C(1 + |\xi|)^{-1} \text{ for any } t \geqq 2 \text{ and } \xi \in \mathbf{R}.$$

Then we have the following.

Proposition 9 (1) *There is a* $c_0 > 0$ *such that*

$$|\varphi(\xi, \mu(t))| \leqq (1 + c_0|\xi|^2)^{-1/4} \text{ for any } t \geqq 2 \text{ and } \xi \in \mathbf{R}.$$

(2) *For any* $t \geqq 2$, $\xi \in \mathbf{R}$, *and integers* n, m *with* $n \geqq m$,

$$|\varphi(n^{-1/2}\xi, \mu(t))|^n \leqq (1 + \frac{c_0}{m}|\xi|^2)^{-m/4}.$$

Proof. Let $g(x) = \rho(x)1_{(-2,2)}(x)$, $x \in \mathbf{R}$. Then we have

$$p = \int_{\mathbf{R}} g(x)dx \geqq 1 - \int_{\mathbf{R}} \frac{x^2}{4}\rho(x)dx \geqq 3/4.$$

Note that

$$|\varphi(\xi, \mu(t))|^2 = (1 - \bar{F}(t))^{-2} \int_{\mathbf{R}} \int_{\mathbf{R}} \exp(i\xi(x-y))\rho(x)1_{(-\infty,t]}(x)\rho(y)1_{(-\infty,t]}(y)dxdy$$

$$\leqq (1 - p^2) + \int_{\mathbf{R}} \int_{\mathbf{R}} \exp(i\xi(x-y))g(x)g(y)dxdy = 1 - f(\xi),$$

where

$$f(\xi) = \int_{\mathbf{R}} \int_{\mathbf{R}} (1 - \cos(\xi(x-y)))g(x)g(y)dxdy.$$

So we see that

$$\lim_{\xi \to 0} |\xi|^{-2} f(\xi) = \frac{1}{2} \int_{\mathbf{R}} \int_{\mathbf{R}} (x-y)^2 g(x)g(y)dxdy > 0.$$

Also, it is easy to see that $f(\xi) > 0$, for all $\xi \in \mathbf{R} \setminus \{0\}$, and so we see that

$$a(r) = \inf_{|\xi| \leqq r} |\xi|^{-2} f(\xi) > 0 \qquad \text{for all} \quad r > 0.$$

Therefore we see that

$$|\varphi(\xi, \mu(t))| \leqq (1 - a(r)|\xi|^2)^{1/2} \leqq (1 + a(r)|\xi|^2)^{-1/4}, \qquad |\xi| \leqq r.$$

Also by Proposition 8(2), we see that there is an $r_0 > 0$ such that

$$|\varphi(\xi, \mu(t))| \leqq (1 + |\xi|^2)^{-1/4}, \qquad |\xi| \geqq r_0$$

So we have the assertion (1).

It is easy to chack that $(1 + x/\beta)^\beta \geqq 1 + x$ for any $\beta \geqq 1$ and $x \geqq 0$. Therefore if $n \geqq m$, we have

$$(1 + c_0|n^{-1/2}\xi|^2)^{n/m} \geqq 1 + \frac{c_0}{m}|\xi|^2.$$

This implies the assertion (2). ∎

5 Asymptotic expansion of characteristic functions

Let

$$\varphi_1(\xi, t) = -\sum_{k=1}^{K-1} \frac{(i\xi)^k}{k!}\bar{\eta}_k(t) + \frac{(i\xi)^K}{K!}\eta_K(t)$$

and

$$\psi_0(n, \xi) = \sum_{k=3}^{K-1} n^{-k/6} q_k(n^{-1/3}, \eta_2, \ldots, \eta_k)(i\xi)^k$$

$$+ \sum_{k=K}^{3(K-1)} n^{-k/6} q_k(n^{-1/3}, \eta_2, \ldots, \eta_{K-1}, 0, \ldots, 0)(i\xi)^k$$

for $t \geqq 2$, $n \geqq 1$ and $\xi \in \mathbf{R}$. Let $\delta = ((\alpha-2)\wedge 1)/(4(K+2))$, $\delta' = \delta/(4(K+2))$, and $t_n = n^{1/2-\delta}, n = 1, 2, 3, \ldots$. Then $t_n \geqq 2$ for any $n \geqq 8$.

In this section, we prove the following.

Lemma 10 *Let*

$$R_{n,0}(\xi) = \exp(\frac{1}{2}\xi^2)\varphi(n^{-1/2}\xi, \mu(t_n))^n - (1 + \psi_0(n,\xi) + n\varphi_1(n^{-1/2}\xi, t_n))$$

$$R_{n,1}(\xi) = \exp(\frac{1}{2}\xi^2)\varphi(n^{-1/2}\xi, \mu(t_n))^n - 1$$

$$R_{n,2}(\xi) = \exp(\frac{1}{2}\xi^2)\varphi(n^{-1/2}\xi, \mu(t_n))^{n-1} - 1$$

Then there is a $C > 0$ such that

$$|R_{n,0}(\xi)| \leqq Cn^{-(\alpha-2)/2-\delta/4}|\xi|$$

and

$$|R_{n,1}(\xi)| + |R_{n,2}(\xi)| \leqq Cn^{-2K\delta}|\xi|$$

for any $n \geqq 8$ and $\xi \in \mathbf{R}$ with $|\xi| \leqq n^{\delta'}$.

We make some preparations to prove this lemma. First we prove the following.

Proposition 11 *Let*

$$\varphi_0(\xi) = \sum_{k=2}^{K-1} \frac{(i\xi)^k}{k!}\eta_k,$$

and

$$R_0(\xi, t) = \varphi(\xi; \mu(t)) - (1 + \varphi_0(\xi) + \varphi_1(\xi, t)).$$

Then we have for any $n \geqq 8$, and $\xi \in \mathbf{R}$ with $|\xi| \leqq n^{\delta'}$,

$$|\varphi(n^{-1/2}\xi; \mu(t_n)) - 1| \leqq \frac{2\sqrt{3}}{3}n^{-1/2}|\xi|,$$

$$|\varphi_1(n^{-1/2}\xi, t_n)| \leqq KC(\delta)n^{-\alpha/2+(K+1)\delta}|\xi|$$

and

$$|R_0(n^{-1/2}\xi, t_n)| \leqq 3C(\delta)n^{-\alpha/2-\delta/4}|\xi|.$$

Here $C(\delta)$ is as in Proposition 6.

Proof. We can easily see that

$$\varphi(\xi; \mu(t)) = \int_{\mathbf{R}} \exp(ix\xi)\mu(t)(dx)$$

$$= 1 + \sum_{k=1}^{K} \frac{(i\xi)^k}{k!}\eta_k(t) + \int_{(\infty,t]} r_{e,K}(x\xi)\mu(dx) + \bar{F}(t)(1-\bar{F}(t))^{-1}\int_{(\infty,t]} r_{e,0}(x\xi)\mu(dx)$$

So we see that

$$R_0(\xi,t) = \bar{F}(t)(1-\bar{F}(t))^{-1}\int_{(\infty,t]} r_{e,0}(x\xi)\mu(dx) + \int_{(\infty,t]} r_{e,K}(x\xi)\mu(dx).$$

By Proposition 6 we have

$$|\varphi_1(\xi,t)| \leqq C(\delta)\sum_{k=1}^{K}\frac{|\xi|^k}{k!}|t|^{-\alpha+k+\delta}, \qquad \xi\in\mathbf{R},\ t\geqq 2,$$

and

$$|R_0(\xi,t)| \leqq 2C(\delta)|\xi|t^{-\alpha+\delta}(\int_{\mathbf{R}}|x|\mu(dx)) + C(\delta)|\xi|^{K+1}t^{-\alpha+K+1+\delta}, \qquad \xi\in\mathbf{R},\ t\geqq 2.$$

Also, we have

$$|\varphi(\xi;\mu(t))-1| \leqq |\xi|\int_{\mathbf{R}}|x|\mu(t)(dx) \leqq (1-\bar{F}(t))^{-1/2}|\xi| \leqq \frac{2\sqrt{3}}{3}|\xi|, \qquad \xi\in\mathbf{R},\ t\geqq 2.$$

Note that

$$(n^{-1/2+\delta'})^k(n^{1/2-\delta})^{-\alpha+k+\delta} = n^{-\alpha/2+(\alpha+1/2)\delta-k(\delta-\delta')-\delta^2}.$$

So we have our assertion. ∎

Proposition 12 *Let*

$$\psi_1(\xi) = \sum_{k=3}^{K-1}\frac{(i\xi)^k}{k!}c_k(\eta_2,\ldots,\eta_{k-1}) + \frac{(i\xi)^K}{K!}c_K(\eta_2,\ldots,\eta_{K-1},0), \qquad \xi\in\mathbf{R}.$$

Also, for any $n\geqq 8$, and $\xi\in\mathbf{R}$ with $|\xi|\leqq n^{\delta'}$, let

$$R_1(n,\xi) = \log(\varphi(n^{-1/2}\xi,\mu(t_n))) - \{-\frac{1}{2n}\xi^2 + \psi_1(n^{-1/2}\xi) + \varphi_1(n^{-1/2}\xi,t_n)\}.$$

Then there is a constant $C>0$ such that

$$|R_1(n,\xi)| \leqq Cn^{-\alpha/2-\delta/4}|\xi|$$

for any $n\geqq 8$, and $\xi\in\mathbf{R}$ with $|\xi|\leqq n^{\delta'}$.

Proof. Let

$$R_{1,1}(\xi) = \sum_{k=1}^{K}\frac{(-1)^{k-1}}{k}(\varphi_0(\xi))^k + \frac{1}{2}\xi^2 - \psi_1(\xi).$$

Note that

$$\log(1+\sum_{k=2}^{K-1}\eta_k\frac{z^k}{k!})$$

$$=\sum_{k=2}^{K-1}c_k(\eta_2,\ldots,\eta_k)\frac{z^k}{k!} + \sum_{k=K}^{\infty}c_k(\eta_2,\ldots,\eta_{K-1},0,\ldots,0)\frac{z^k}{k!}$$

as a formal power series of z. So we see that there is a constant $C > 0$ such that

$$|R_{1,1}(\xi)| \leqq C|\xi|^{K+1} \tag{5}$$

for any $\xi \in \mathbf{R}$ with $|\xi| \leqq 1$.

We can easily see that

$$\begin{aligned} & R_1(n,\xi) \\ = & \log(1+\varphi_0(n^{-1/2}\xi)+\varphi_1(n^{-1/2}\xi,t_n)+R_0(n^{-1/2}\xi,t_n)) \\ & \qquad -\{-\frac{1}{2n}\xi^2+\psi_1(n^{-1/2}\xi)+\varphi_1(n^{-1/2}\xi,t_n)\} \\ = & R_{1,1}(n^{-1/2}\xi)+r_{l,K}(\varphi(n^{-1/2}\xi,\mu(t_n))-1)+R_0(n^{-1/2}\xi,t_n) \\ & +\sum_{k=2}^{K}(-1)^{k-1}(\varphi_0(n^{-1/2}\xi))^{k-1}(\varphi_1(n^{-1/2}\xi,t_n)+R_0(n^{-1/2}\xi,t_n)) \\ & +\sum_{k=1}^{K}\frac{(-1)^{k-1}}{k}\sum_{j=2}^{k}\binom{k}{j}(\varphi_0(n^{-1/2}\xi))^{k-j}(\varphi_1(n^{-1/2}\xi,t_n)+R_0(n^{-1/2}\xi,t_n))^j. \end{aligned}$$

Then we have our assertion from Equation (5) and Proposition 11.

Proposition 13 *Let*

$$R_2(n,\xi) = \exp(n\psi_1(n^{-1/2}\xi)) - (1+\psi_0(n,\xi)).$$

Then there is a constant $C > 0$ such that

$$|R_2(n,\xi)| \leqq Cn^{-(\alpha-2)/2-1/4}|\xi|$$

for any $n \geqq 8$, and $\xi \in \mathbf{R}$ with $|\xi| \leqq n^{\delta'}$.

Proof. Note that

$$\begin{aligned} & \exp(y^{-6}(\sum_{k=3}^{K-1}\frac{(y^3z)^k}{k!}c_k(\eta_2,\ldots,\eta_k)+\sum_{k=K}^{\infty}\frac{(y^3z)^k}{k!}c_k(\eta_2,\ldots,\eta_{K-1},0,\ldots,0))) \\ & = 1+\sum_{k=3}^{K-1}y^kq_k(y^2,\eta_2,\ldots,\eta_k)z^k+\sum_{k=K}^{\infty}y^kq_k(\eta_2,\ldots,a_{K-1},0,\ldots,0)))z^k \end{aligned}$$

as a formal power series in z. This implies our assertion. ∎

Now let us prove Lemma 10.

Note that for any $n \geqq 8$, and $\xi \in \mathbf{R}$ with $|\xi| \leqq n^{\delta'}$,

$$\begin{aligned} & \exp(\frac{1}{2}\xi^2)\varphi(n^{-1/2}\xi;\mu(t_n))^n \\ & = \exp(n\varphi_1(n^{-1/2}\xi,t_n)+n\psi_1(n^{-1/2}\xi)+nR_1(n,\xi))) \\ = & (1+n\varphi_1(n^{-1/2}\xi,t_n)+r_{e,1}(n\varphi_1(n^{-1/2}\xi,t_n)))(1+\psi_0(n,\xi)+R_2(n,\xi))(1+r_{e,0}(nR_1(n,\xi))). \end{aligned}$$

So we see that

$$R_{n,0}(n,\xi)$$
$$= r_{e,0}(nR_1(n,\xi))\exp(n\varphi_1(n^{-1/2}\xi,t_n)+n\psi_1(n^{-1/2}\xi))+R_2(n,\xi)\exp(n\varphi_1(n^{-1/2}\xi,t_n))$$
$$+r_{e,1}(n\varphi_1(n^{-1/2}\xi,t_n)).$$

Thus we have the first equation from Propositions 11, 12, 13.

Also, we have

$$R_{n,1}(n,\xi)=\exp(n\varphi_1(n^{-1/2}\xi,t_n)+n\psi_1(n^{-1/2}\xi)+nR_1(n,\xi)))-1,$$

and

$$R_{n,2}(n,\xi)=\exp((n-1)\varphi_1(n^{-1/2}\xi,t_n)+(n-1)\psi_1(n^{-1/2}\xi)+(n-1)R_1(n,\xi))-\frac{\xi^2}{n})-1.$$

So, again from Propositions 11, 12, 13 we have the second equation.

6 Proof of Theorem 1

First, we prove the following.

Lemma 14 *Let ν be a probability measure on $(\mathbf{R},\mathcal{B}(\mathbf{R})$ such that $\int_{\mathbf{R}} x^2\nu(dx)<\infty$. Also, assume that there is a constant $C>0$ such that the characteristic function $\varphi(\cdot,\nu):\mathbf{R}\to\mathbf{C}$ satisfies*

$$|\varphi(\xi;\nu)|\leqq C(1+|\xi|)^{-2},\qquad \xi\in\mathbf{R}.$$

Then for any $x\in\mathbf{R}$

$$\nu((x,\infty))=\Phi_0(x)+\frac{1}{2\pi}\int_{\mathbf{R}}\frac{e^{-ix\xi}}{i\xi}(\varphi(\xi,\nu)-\exp(-\frac{|\xi|^2}{2}))d\xi.$$

Proof. From the assumption, ν has a continuous density function β and

$$\beta(x)=\frac{1}{2\pi}\int_{\mathbf{R}}e^{-ix\xi}\varphi(\xi,\nu)d\xi.$$

So we have

$$\nu((x,x+n]))=\Phi_0(x)-\Phi_0(x+n)+\frac{1}{2\pi}\int_{\mathbf{R}}(\int_x^{x+n}e^{-iz\xi}dz)(\varphi(\xi,\nu)-\exp(-\frac{|\xi|^2}{2}))d\xi.$$

$$=\Phi_0(x)-\Phi_0(x+n)+\frac{1}{2\pi}\int_{\mathbf{R}}\frac{e^{-ix\xi}-e^{-i(x+n)\xi}}{i\xi}(\varphi(\xi,\nu)-\exp(-\frac{|\xi|^2}{2}))d\xi.$$

Since

$$\int_{\mathbf{R}}\frac{1}{|\xi|}|\varphi(\xi,\nu)-\exp(-\frac{|\xi|^2}{2})|d\xi<\infty,$$

letting $n\to\infty$, we have the assertion. ∎

We remark that

$$\Phi_k(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}(i\xi)^{k-1}\exp(-i\xi x-\frac{\xi^2}{2})d\xi,\qquad k=1,2,\dots.$$

Note that

$$P(\sum_{k=1}^{n} X_k > sn^{1/2}) = \sum_{m=0}^{n} I_m(n,s),$$

where

$$I_m(n,s) = P(\sum_{k=1}^{n} X_k > sn^{1/2}, \sum_{k=1}^{n} 1_{\{X_k > t_n\}} = m), \qquad m = 0, 1, \dots, n.$$

Then we have

$$I_m(n,s) = \binom{n}{m} P(\sum_{k=1}^{n} X_k > sn^{1/2},\ X_i > t_n,\ i = 1, \dots, m,\ X_j \leqq t_n,\ j = m+1, \dots, n),$$

for $m = 0, 1, \dots, n$.

Proposition 15 *There is a $C > 0$ such that*

$$\sum_{m=2}^{n} I_m(n,s) \leqq Cn^{-(\alpha-2)/2-\delta}$$

for any $s \geqq 1$ and $n \geqq 8$.

Proof. We see that

$$\sum_{m=2}^{n} I_m(n,s) \leqq \sum_{m=2}^{n} \frac{n(n-1)}{m(m-1)} \binom{n-2}{m-2} \bar{F}(t_n)^m (1 - \bar{F}(t_n))^{n-m}$$

$$\leqq \frac{n(n-1)}{2} \bar{F}(t_n)^2 \leqq C(\delta)^2 n^{2-\alpha+2(K+1)\delta} \leqq C(\delta)^2 n^{-(\alpha-2)/2-\delta}.$$

This implies our assertion. ∎

Proposition 16 *There is a $C > 0$ such that*

$$\sup_{s \in [1, \log n]} |I_0(n,s) - \{(1 - n\bar{F}(t_n))\Phi_0(s) - \sum_{k=1}^{K-1} \frac{n^{-(k-2)/2}}{k!} \bar{\eta}_k(t_n) \Phi_k(s)$$

$$+ \frac{(n^{1/2})^{K-2}}{K!} \eta_K(t_n) \Phi_K(s) + g(n,s)\}| \leqq Cn^{-(\alpha-2)/2-\delta/4}$$

for any $n \geqq 8$. Here

$$g(n,s) = \sum_{k=3}^{K-1} n^{-k/6} q_k(n^{-1/3}, \eta_2, \dots, \eta_k) \Phi_k(s)$$

$$+ \sum_{k=K}^{3(K-1)} n^{-k/6} q_k(n^{-1/3}, \eta_2, \dots, \eta_{K-1}, 0, \dots, 0) \Phi_k(s).$$

Proof. Note that

$$I_0(n,s) = (1-\bar{F}(t_n))^n \mu(t_n)^{*n}((sn^{1/2},\infty))$$
$$= I_{0,0}(n,s) + I_{0,1}(n,s) + I_{0,2}(n,s),$$

where

$$I_{0,0}(n,s) = \mu(t_n)^{*n}((sn^{1/2},\infty)),$$
$$I_{0,1}(n,s) = -n\bar{F}(t_n)\mu(t_n)^{*n}((sn^{1/2},\infty)),$$
$$I_{0,2}(n,s), = ((1-\bar{F}(t_n))^n - 1 + n\bar{F}(t_n))\mu(t_n)^{*n}((sn^{1/2},\infty)).$$

By Proposition 9 and Lemma 14, we have

$$I_{0,0}(n,s)$$
$$= \Phi_0(s) + (\frac{1}{2\pi})\int_{\mathbf{R}} \frac{e^{-is\xi}}{i\xi}(\varphi(n^{-1/2}\xi,\mu(t_n))^n - \exp(-\frac{\xi^2}{2}))d\xi.$$

Let

$$\tilde{R}_{0,,0}(n,s) = I_{0,0}(n,s) - \{\Phi_0(s) + (\frac{1}{2\pi})\int_{\mathbf{R}} \frac{e^{-is\xi}}{i\xi}(\psi_0(n,\xi) + n\varphi_1(n^{-1/2}\xi,t_n))e^{-\xi^2/2}d\xi\}$$

Then by Lemma 10 we have

$$|\tilde{R}_{0,0}(n,s)|$$
$$\leqq \int_{|\xi|\leqq n^{\delta'}} \frac{|R_{n,0}(\xi)|}{|\xi|}\exp(-\frac{\xi^2}{2})d\xi + \int_{|\xi|>n^{\delta'}} \frac{1}{|\xi|}(|\varphi(n^{-1/2}\xi,\mu(t_n))|^n + \exp(-\frac{\xi^2}{2}))d\xi$$
$$+ \int_{|\xi|>n^{\delta'}} (|\psi_0(n,\xi)| + n|\varphi_1(n^{-1/2}\xi,t_n)|)e^{-\xi^2/2}d\xi$$

So by Proposition 9 and Lemma 10, we see that there is a $C_0 > 0$ such that

$$|\tilde{R}_{0,0}(n,s)| \leqq C_0 n^{-(\alpha-2)/2-\delta/4}, \qquad n \geqq 8,\ s \geqq 1. \tag{6}$$

Also, we see that

$$(\frac{1}{2\pi})\int_{\mathbf{R}} \frac{e^{-is\xi}}{i\xi} n\varphi_1(n^{-1/2}\xi,t_n)\exp(-\frac{1}{2}|\xi|^2)d\xi$$
$$= -\sum_{k=1}^{K-1} \frac{(n^{-1/2})^{k-2}}{k!}\bar{\eta}_k(t_n)\Phi_k(s) + \frac{(n^{1/2})^{K-2}}{k!}\eta_K(t_n)\Phi_K(s),$$

and

$$(\frac{1}{2\pi})\int_{\mathbf{R}} \frac{e^{-is\xi}}{i\xi}\psi_0(n,\xi)\exp(-\frac{1}{2}|\xi|^2)d\xi = g(n,s)$$

Similarly by Lemma 10, we see that there is a $C_1 > 0$ such that

$$\sup_{s\in[1,\log n]} |I_{0,1}(n,s) - n\bar{F}(sn^{1/2})\Phi_0(s)| \leqq C_1 n^{-(\alpha-2)/2-\delta}, \qquad n \geqq 8. \tag{7}$$

Note that $|(1-x)^n - (1-nx)| \leqq n^2x^2$ for any $x \in [0,1]$, $n \geqq 1$. So we have

$$|I_{0,2}(n,s)| \leqq n^2\bar{F}(t_n)^2 \leqq C(\delta)^2 n^{(\alpha-2)/2-\delta}.$$

This and Equations 6, 7 imply our assertion. ∎

Proposition 17 *There is a $C > 0$ such that*

$$\sup_{s\in[1,\log n]} |I_1(n,s) - \{n \int_{-\infty}^{s} \bar{F}((s-x)n^{1/2})\Phi_1(x)dx + n\bar{F}(t_n)\Phi_0(s)$$

$$- \sum_{k=1}^{K} \frac{n^{-(k-2)/2}}{k!}\Phi_k(s) \int_0^{t_n} x^k \mu(dx)\}| \leqq Cn^{-(\alpha-2)/2-\delta/4}.$$

Proof. We see that

$$\begin{aligned} I_1(n,s) &= n(1-\bar{F}(t_n))^{n-1} \int_{\mathbf{R}} P(X_1 + x > sn^{1/2},\ X_1 > t_n)\ \mu(t_n)^{*(n-1)}(dx) \\ &= n(1-\bar{F}(t_n))^{n-1} \int_{-\infty}^{\infty} \bar{F}((sn^{1/2} - x) \vee t_n)\mu(t_n)^{*(n-1)}(dx) \\ &= nJ_0(n,s) + nJ_1(n,s) + nJ_2(n,s), \end{aligned}$$

where

$$J_0(n,s) = \int_{-\infty}^{\infty} \bar{F}((s-x)n^{1/2} \vee t_n)\Phi_1(x)dx, \tag{8}$$

$$J_1(n,s) = \int_{-\infty}^{\infty} \bar{F}((sn^{1/2} - x) \vee t_n)(\mu(t_n)^{*(n-1)}(dx) - n^{-1/2}\Phi_1(xn^{1/2})dx), \tag{9}$$

and

$$J_2(n,s) = -(1 - (1-\bar{F}(t_n))^{n-1})I_1(n,s). \tag{10}$$

Note that

$$J_0(n,s) = J_{0,0}(n,s) + J_{0,1}(n,s) + J_{0,2}(n,s),$$

where

$$J_{0,0}(n,s) = \int_{-\infty}^{s} \bar{F}((s-x)n^{1/2})\Phi_1(x)dx,$$

$$J_{0,1}(n,s) = -\int_{s-n^{-\delta}}^{s} \bar{F}((s-x)n^{1/2})\Phi_1(x)dx = -\int_0^{n^{-\delta}} \bar{F}(xn^{1/2})\Phi_1(s-x)dx,$$

and

$$J_{0,2}(n,s) = \bar{F}(t_n)\int_{s-n^{-\delta}}^{\infty} \Phi_1(x)dx = \bar{F}(t_n)\Phi_0(s-n^{-\delta}).$$

We see that

$$J_{0,1}(n,s) = -\sum_{k=1}^{K} \frac{1}{(k-1)!}\Phi_k(s) \int_0^{n^{-\delta}} \bar{F}(xn^{1/2})x^{k-1}dx + R_{J,1}(n,s),$$

where

$$R_{J,1}(n,s) = -\int_0^{n^{-\delta}} \bar{F}(xn^{1/2})(\Phi_1(s-x) - \sum_{k=1}^{K} \frac{x^{k-1}}{(k-1)!}\Phi_k(s))dx.$$

Then

$$|R_{J,1}(n,s)|$$

$$\leqq \sup_{x\in[0,n^{-\delta}]} |\Phi_{K+1}(s-x)|(\int_{n^{-1/2}}^{n^{-\delta}} x^K (xn^{1/2})^{-\alpha} L(xn^{1/2}) dx + \int_0^{n^{-1/2}} x^K dx).$$

$$\leqq \sup_{x\in\mathbf{R}} |\Phi_{K+1}(x)|(C(\delta) n^{-\alpha/2+\delta/2} \int_0^{n^{-\delta}} x^{\delta+(K-\alpha)} dx + n^{-(K+1)/2})$$

$$\leqq \sup_{x\in\mathbf{R}} |\Phi_{K+1}(x)|(C(\delta)+1) n^{-\alpha/2-\delta/2}. \tag{11}$$

Also, we see that

$$J_{0,2}(n,s)$$

$$= \bar{F}(t_n)\Phi_0(s) + \sum_{k=1}^{K} \bar{F}(t_n) \frac{(n^\delta)^k}{k!} \Phi_k(s) + R_{J,2}(n,s),$$

where

$$R_{J,2}(n,s) = \bar{F}(t_n)(\Phi_0(s-n^{-\delta}) - \sum_{k=0}^{K} \frac{(-n^\delta)^k}{k!} \frac{d^k\Phi_0}{dx^k}(s)).$$

We see that

$$|R_{J,2}(n,s)| \leqq \bar{F}(t_n) n^{-(K+1)\delta} \sup_{x\in\mathbf{R}} |\Phi_{K+1}(x)| \leqq C(\delta) \sup_{x\in\mathbf{R}} |\Phi_{K+1}(x)| n^{-\alpha/2-\delta/4}. \tag{12}$$

It is easy to see that

$$\int_0^{n^{-\delta}} \bar{F}(xn^{1/2}) x^{k-1} dx = n^{-k/2} \int_0^{t_n} \bar{F}(x) x^{k-1} dx$$

$$= n^{-k/2}(-\frac{1}{k} \int_0^{t_n} x^k \mu(dx) + \frac{n^{\delta k}}{k} \bar{F}(t_n)), \qquad k = 1, \ldots, K.$$

So we have

$$J_{0,1}(n,s) + J_{0,2}(n,s)$$

$$= \bar{F}(t_n)\Phi_0(s) - \sum_{k=1}^{K} \frac{n^{-k/2}}{k!} \Phi_k(s) \int_0^{t_n} x^k \mu(dx) + R_{J,1}(n,s) + R_{J,2}(n,s) \tag{13}$$

Also, we have

$$J_1(n,s) = J_{1,1}(n,s) + J_{1,2}(n,s)$$

where

$$J_{1,1}(n,s) = \bar{F}(t_n)(\mu(t_n)^{*(n-1)}((s-n^{-\delta})n^{1/2}, \infty) - \Phi_0(s-n^{-\delta}))$$

and

$$J_{1,2}(n,s) = \int_{-\infty}^{s-n^{-\delta}} dx \bar{F}((s-x)n^{1/2}) \frac{1}{2\pi} \int_{\mathbf{R}} e^{-ix\xi} (\varphi(n^{-1/2}\xi; \mu(t_n))^{n-1} - \exp(-\frac{\xi^2}{2})) d\xi$$

By Proposition 9 and Lemma 14, we see that there is a $C_1 > 0$ such that

$$|\mu(t_n)^{*(n-1)}((xn^{1/2}, \infty)) - \Phi_1(x)| \leqq |\int_{-\infty}^{\infty} \frac{e^{-ix\xi}}{\xi} (\varphi(n^{-1/2}\xi; \mu(t_n))^{n-1} - \exp(-\frac{\xi^2}{2})) d\xi|$$

$$\leqq \int_{|\xi|>n^{\delta'}} \frac{1}{|\xi|}(|\varphi(\xi;\mu(t_n))|^{n-1} + \exp(-\frac{\xi^2}{2}))d\xi + \int_{|\xi|<n^{\delta'}} \frac{1}{|\xi|}|R_{n,2}(\xi)|\exp(-\frac{\xi^2}{2}))d\xi$$
$$\leqq C_1 n^{-2K\delta}, \text{ for any } x \in \mathbf{R} \text{ and } n \geqq 8.$$

Therefore we have

$$|J_{1,1}(n,s)| \leqq C_1 \bar{F}(t_n) n^{-2K\delta} \leqq C(\delta) C_1 n^{-\alpha/2-\delta}.$$

Similarly by Lemma 14, we see that there is a $C_2 > 0$ such that

$$|\int_{\mathbf{R}} e^{-ix\xi}(\varphi(n^{-1/2}\xi;\mu(t_n))^{n-1} - \exp(-\frac{\xi^2}{2}))d\xi|$$
$$\leqq C_2 n^{-2K\delta}, \text{ for any } x \in \mathbf{R} \text{ and } n \geqq 8.$$

Then we have

$$|J_{1,2}(n,s)| \leqq C_2 n^{-2K\delta} C(\delta) \int_{n^{-\delta}}^{\infty} (xn^{1/2})^{-\alpha+\delta} dx \leqq C_2 C(\delta) n^{-\alpha/2-\delta}$$

So we see that there is a $C > 0$ such that

$$\sup_{s\in[1,\log n]} |J_1(n,s)| \leqq C n^{-(\alpha-2)/2-\delta} \tag{14}$$

Note that

$$|J_2(n,s)| \leqq n^2 \bar{F}(t_n)^2 \tag{15}$$

So Equations (8) - (15) imply our assertion. ∎

Now Theorem 1 is an easy consequence of Propositions 15, 16, 17, since

$$\bar{\eta}_k(t_n) + \int_0^{t_n} x^k \mu(dx) = \int_0^{\infty} x^k \mu(dx), \qquad k = 1, 2, \ldots, K-1,$$

and

$$\eta_K(t_n) - \int_0^{t_n} x^K \mu(dx) = \int_{-\infty}^0 x^K \mu(dx).$$

This completes the proof of Theorem 1.

7 Proof of Theorem 2

It is well known (e.g. Williams [2]) that there is a $C_0 > 0$ such that

$$|\Phi_k(x)| \leqq C_0(1+x)^{k-1}\Phi_1(x), \qquad x \geqq 0, \ k = 1, \ldots, 3K,$$

and

$$C_0^{-1}\Phi_1(x) \leqq x\Phi_0(x) \leqq C_0\Phi_1(x), \qquad x \geqq 1.$$

Let

$$H(n,s) = \Phi_0(s) + n\bar{F}(n^{1/2}s),$$

and

$$A(n,s) = n\int_{-\infty}^{s} \bar{F}((s-x)n^{1/2})\Phi_1(x)dx - \sum_{k=1}^{2} \frac{n^{-(k-2)/2}}{k!}\Phi_k(s)\int_0^{\infty} x^k \mu(dx).$$

First we prove the following.

Proposition 18

$$\sup_{s\in[1,\log n]} \frac{|A(n,s) - n\bar{F}(n^{1/2}s)|}{H(n,s)} \to 0, \qquad n \to \infty.$$

Proof. Let us take a $\gamma \in (0, (\alpha-2)/(4\alpha))$ and fix it. Let $s \geqq 0$ and $n \geqq 3$. Note that

$$\int_{-\infty}^{s} \bar{F}((s-x)n^{1/2})\Phi_1(x)dx = \sum_{k=1}^{4} I_k(n,s),$$

where

$$I_1(n,s) = \int_{s-n^{-\gamma}}^{s} \bar{F}((s-x)n^{1/2})\Phi_1(x)dx,$$

$$I_2(n,s) = \int_{-s}^{7s/8} \bar{F}((s-x)n^{1/2})\Phi_1(x)dx,$$

$$I_3(n,s) = \int_{7s/8}^{s-n^{-\gamma}} \bar{F}((s-x)n^{1/2})\Phi_1(x)dx,$$

and

$$I_4(n,s) = \int_{-\infty}^{-s} \bar{F}((s-x)n^{1/2})\Phi_1(x)dx.$$

Note that

$$I_1(n,s) = n^{-1/2}\int_0^{n^{(1/2-\gamma)}} \bar{F}(y)\Phi_1(s - n^{-1/2}y)dy.$$

Let

$$R(n,s,y) = \Phi_1(s - n^{-1/2}y) - (\Phi_1(s) + n^{-1/2}y\Phi_2(s))$$

Then for $y \in [0, sn^{1/2-\gamma}]$

$$|R(n,s,y)| \leqq n^{-1}y^2 \sup_{z\in[s-n^{-\gamma},s]} |\Phi_3(z)|$$

$$\leqq C_0 n^{-1}y^2(1+s)^2\Phi_1(s-n^{-\gamma}) = C_0 n^{-1}y^2(1+s)^2\Phi_1(s)\exp(sn^{-\gamma} - n^{-2\gamma}/2)$$
$$\leqq C_0 n^{-1}y^2(1+s)^3\exp(n^{-\gamma}s)\Phi_0(s).$$

So we see that

$$n|I_1(n,s) - \sum_{k=1}^{2} \frac{n^{-k/2}}{k!}\Phi_k(s)\int_0^\infty x^k\mu(dx)|$$

$$\leqq C_0(1+s)^3 n^{-1/2}\exp(n^{-\gamma}s)(\int_0^{n^{1/2-\gamma}} y^2\bar{F}(y)dy)\Phi_0(s)$$

$$+C_0(1+s)n^{1/2}(\int_{n^{1/2-\gamma}}^{\infty} \bar{F}(y)dy)\Phi_0(s) + C_0(1+s)^2(\int_{n^{1/2-\gamma}}^{\infty} y\bar{F}(y)dy)\Phi_0(s)$$

This implies that

$$\sup_{s\in[1,\log n]} \Phi_0(s)^{-1}|nI_1(n,s) - \sum_{k=1}^{2} \frac{n^{-(k-2)/2}}{k!}\Phi_k(s)\int_0^\infty x^k\mu(dx)| \to 0, \qquad n\to\infty. \quad (16)$$

Note that

$$I_2(n,s) = \bar{F}(sn^{1/2}) \int_{-s}^{7s/8} (1 - \frac{x}{s})^{-\alpha} \frac{L((s-x)n^{1/2})}{L(sn^{1/2})} \Phi_1(x) dx$$

It is easy to see that

$$\sup_{s \in [(\log n)^{1/4}, \log n]} | \int_{-s}^{7s/8} (1 - \frac{x}{s})^{-\alpha} \frac{L((s-x)n^{1/2})}{L(sn^{1/2})} \Phi_1(x) dx - 1| \to 0, \qquad n \to \infty$$

Also we see that

$$n|I_2(n,s)| \leqq n\bar{F}(sn^{1/2}) 8^{\alpha} \int_{-s}^{7s/8} \frac{L((s-x)n^{1/2})}{L(sn^{1/2})} \Phi_1(x) dx$$

Therefore we have

$$\sup_{s \in [1, (\log n)^{1/4}]} \Phi_0(s)^{-1} (n|I_2(n,s)| + n\bar{F}(sn^{1/2})) \to 0, \qquad n \to \infty.$$

Thus we have

$$\sup_{s \in [1, \log n]} H(n,s)^{-1} |nI_2(n,s) - n\bar{F}(sn^{1/2})| \to 0, \qquad n \to \infty. \tag{17}$$

Note that $\sqrt{3}/2 \leqq 7/8$. Then we have

$$\Phi_1(7s/8) \leqq (\Phi_1(s))^{3/4},$$

and so we have

$$nI_3(n,s) \leqq ns\bar{F}(n^{1/2-\gamma}) \Phi_1(7s/8) \leqq (n\bar{F}(n^{1/2} \log n))^{1/2} (s\Phi_1(s))^{3/4} \frac{ns^{1/4} \bar{F}(n^{1/2-\gamma})}{(n\bar{F}(n^{1/2} \log n))^{1/2}}.$$

Since

$$\sup_{n \geqq 3} \sup_{s \in [1, \log n]} \frac{ns^{1/4} \bar{F}(n^{(1/2-\gamma)})}{(n\bar{F}(n^{1/2} \log n))^{1/2}} < \infty,$$

we see that there is a constant $C > 0$ such that

$$nI_3(n,s) \leqq C(n\bar{F}(sn^{1/2}))^{1/2} \Phi_0(s)^{3/4} \leqq C(n\bar{F}(sn^{1/2}))^{1/4} H(n,s), \qquad n \geqq 3, \; s \in [1, \log n].$$

So we have

$$\sup_{s \in [1, \log n]} H(n,s)^{-1} |nI_3(n,s)| \to 0, \qquad n \to \infty. \tag{18}$$

Also we have

$$n|I_4(n,s))| \leqq n\bar{F}(2sn^{1/2}) \Phi_0(s).$$

So this equation, Equations (16) (17) and (18) imply our assertion. ∎

Proposition 19

$$\sup_{s \in [1, \infty)} |\frac{P(\sum_{k=1}^{n} X_k > sn^{1/2})}{\Phi_0(s) + n\bar{F}(n^{1/2}s)} - 1| \to 0, \qquad n \to \infty.$$

Proof. It is easy to see that there is a $C > 0$ such that

$$|G(n,s) - (\Phi_0(s) + A(n,s))| \leqq Cn^{-1/2} \max\{|\Phi_k(s)|; k = 3, \ldots, 3K\}$$

So we see that

$$\sup_{s \in [1, \log n]} H(n,s)^{-1} |G(n,s) - (\Phi_0(s) + A(n,s))| \to 0, \qquad n \to \infty.$$

Therefore by Proposition 18, we see that

$$\sup_{s \in [1, \log n]} |H(n,s)^{-1} G(n,s) - 1| \to 0, \qquad n \to \infty.$$

So by Theorem 1, we see that

$$\sup_{s \in [1, \log n]} |\frac{P(\sum_{k=1}^n X_k > sn^{1/2})}{H(n,s)} - 1| \to 0 \qquad n \to \infty.$$

Now it is obvious that

$$\sup_{s \in [\log n, \infty)} |\frac{n\bar{F}(sn^{1/2})}{H(n,s)} - 1| \to 0 \qquad n \to \infty.$$

Then by Theorem 1 in [1], we have

$$\sup_{s \in [\log n, \infty)} |\frac{P(\sum_{k=1}^n X_k > sn^{1/2})}{H(n,s)} - 1| \to 0 \qquad n \to \infty.$$

So we have our assertion.

Now let us prove Theorem 2. It is well known that

$$P(\sum_{k=1}^n X_k > sn^{1/2}) \to \Phi_0(s), \qquad n \to \infty$$

for any $s \in \mathbf{R}$. Since both of $P(\sum_{k=1}^n X_k > sn^{1/2})$ and $\Phi_0(s)$ are nondecreasing in s, and $\Phi_0(s)$ is continuous in s, we see that

$$\sup_{s \in \mathbf{R}} |P(\sum_{k=1}^n X_k > sn^{1/2}) - \Phi_0(s)| \to 0, \qquad n \to \infty.$$

So we have

$$\sup_{s \in (-\infty, 1]} |\frac{P(\sum_{k=1}^n X_k > sn^{1/2})}{\Phi_0(s)} - 1| \to 0, \qquad n \to \infty.$$

Since

$$\sup_{s \in [1, \infty)} |\frac{n\bar{F}(sn^{1/2}) \wedge n^{-\gamma}}{n\bar{F}(sn^{1/2})} - 1| \to 0, \qquad n \to \infty,$$

we have Theorem 2 from Proposition 19.

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