UTMS 2008–5 February 12, 2008

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Carleman estimates for parabolic equations with nonhomogeneous boundary conditions

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November 20. 2007

Abstract

We prove a new Carleman estimate for general linear second order parabolic equation with nonhomogeneous boundary conditions. On basis of this estimate we obtain an improved Carleman estimate for the Stokes system and a system of parabolic equations with a parameter which can be viewed as an approximation of the Stokes system.

1 Introduction.

Local Carleman estimates for elliptic and parabolic equations are known since [1] and [7] and among other examples of applications they turn out to be essential to prove unique continuation properties. Global Carleman estimates for parabolic equations with homogeneous boundary conditions have been obtained by several authors in the recent years (see for example [9] for $L^2(0,T;L^2(\Omega))$ right-hand sides and [13] for $L^2(0,T;H^{-1}(\Omega))$ right-hand sides). They have been extensively used for obtaining observability inequalities in controllability theory and stability results for some inverse problems.

For the case of elliptic equations with nonhomogeneous boundary conditions and $H^{-1}(\Omega)$ right-hand sides, sharp Carleman estimates have been obtained in [12] and this result turned out to be essential for obtaining estimates on the pressure in the context of controllability for the Navier-Stokes equations (see [5]).

The main object of the present article is to obtain a similar result of global Carleman estimates for general parabolic equations with nonhomogeneous boundary conditions and right-hand sides in $L^2(0,T;H^{-1}(\Omega))$. To this aim, after localization and a change of coordinates, we use a factorization of the operator and successive estimates for first order pseudodifferential operators in order to obtain the Carleman estimate. The article is organized as follows : the main result is precisely given in Section 2 together with its complete proof. In Sections 3 this result is applied (using also the estimate for elliptic equations) to the Stokes operator, and in Section 4 it is applied to a compressible Stokes operator where the incompressibility condition is approximated by penalization. In the Appendix, we give some useful technical results on calculus for pseudodifferential operators depending on a parameter.

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First of all we need some notations which are introduced below.
Notations.
$B(x_0,\delta) = \{x|\ x \in \mathbb{R}^m,\ |x-x_0| \leq \delta\}$, ν is the outward unit normal vector to $\partial\Omega = \Gamma$, $\frac{\partial}{\partial \nu_A} = \sum_{i,j=1}^n a_{ij}\nu_i \frac{\partial}{\partial x_j}$, $Q = (0,T)\times\Omega$, $\Sigma = (0,T)\times\partial\Omega$, $Q_\omega = (0,T)\times\omega$, where ω is a subdomain of Ω, $\mathcal{L}(X,Y)$ is the space of linear continuous operators acting from a Banach space X into a Banach space Y, $[L,A] = LA - AL$ is the commutator of operators L and A.
We use the following functional spaces

$$H^{1,2}(Q) = \{y|\ \frac{\partial y}{\partial t}, \frac{\partial^2 y}{\partial x_i \partial x_j}, \frac{\partial y}{\partial x_i}, y \in L^2(Q) \quad \forall i,j \in \{1,\dots,n\}\},$$

$$W(Q) = \{y|\ y, \frac{\partial y}{\partial x_i} \in L^2(Q),\ \forall i \in \{1,\dots,n\},\ \frac{\partial y}{\partial t} \in L^2(0,T;H^{-1}(\Omega))\},$$

$$H^{\frac14,\frac12}(\mathbb{R}^n) = \{y(x_1,\dots,x_n)|\ \int_{\mathbb{R}^n}(1+|\xi_1|^{\frac12}+\sum_{i=2}^n|\xi_2|)|\hat{y}|^2 d\xi < +\infty\}.$$

The space $H^{\frac14,\frac12,s}(\mathbb{R}^n)$ is the space $H^{\frac14,\frac12}(\mathbb{R}^n)$ equipped with the norm

$$\|y\|_{H^{\frac14,\frac12,s}(\mathbb{R}^n)} = (\|y\|^2_{H^{\frac14,\frac12}(\mathbb{R}^n)} + |s|\|y\|^2_{L^2(\mathbb{R}^n)})^{\frac12}.$$

2 Precise statement of the result.

Let Ω be a bounded open set of $\mathbb{R}^n$ of class C^2 and let Γ be the boundary of Ω. We consider a solution $y \in W(Q)$ of the following linear second order parabolic equation

$$\begin{aligned} L(x,D)y &= \frac{\partial y}{\partial x_0} - \sum_{i,j=1}^n \frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial y}{\partial x_j}) + \sum_{j=1}^n b_j(x)\frac{\partial y}{\partial x_j} \\ &\quad + \sum_{i=1}^n \frac{\partial}{\partial x_i}(c_i(x)y) + d(x)y = f + \sum_{j=1}^n \frac{\partial f_j}{\partial x_j},\ \text{in } (0,T)\times\Omega, \end{aligned} \tag{2.1}$$

$$y = g \text{ on } (0,T)\times\Gamma, \tag{2.2}$$

where

$$a_{ij} \in C^2(\overline{Q}), \quad b_j, c_i, d \in L^\infty(Q) \ \text{ for } i,j \in \{1,...,n\}, \tag{2.3}$$

$$a_{ij} = a_{ji} \ \text{ for } i,j \in \{1,...,n\}, \tag{2.4}$$

and the coefficients a_{ij} satisfy the standard ellipticity condition

$$\exists\beta > 0,\ \forall\eta \in \mathbb{R}^n,\ \forall x \in \bar{Q},\ \sum_{i,j=1}^n a_{ij}(x)\eta_i\eta_j \geq \beta|\eta|^2. \tag{2.5}$$

On the other hand we assume that

$$f \in L^2(Q),\ f_j \in L^2(Q),\ \forall j = 1,...,n, \tag{2.6}$$

and g is the boundary value of a function in $W(Q)$. For simplicity we will assume

$$g \in H^{\frac{1}{4},\frac{1}{2}}((0,T)\times\Gamma) = H^{\frac{1}{4},\frac{1}{2}}(\Sigma). \tag{2.7}$$

Our goal is to obtain a sharp global Carleman inequality for solutions of (2.1), (2.2). In order to formulate our main result, we first have to introduce a suitable weight function.

Lemma 2.1. *Let ω be an arbitrary non empty open set such that $\omega \subset \Omega$. Then there exists a function $\psi \in C^2(\overline{\Omega})$ such that*

$$\psi = 0 \ \ on\ \Gamma, \tag{2.8}$$

$$\psi(\tilde{x}) \ > \ 0 \ \forall \tilde{x} \in \Omega, \tag{2.9}$$

$$|\nabla\psi(\tilde{x})| \ > \ 0 \ \forall \tilde{x} \in \overline{\Omega\setminus\omega}. \tag{2.10}$$

Proof. Let us consider a function $\theta(x) \in C^2(\mathbb{R}^n)$ such that

$$\Omega = \{x \big| \ \theta(x) < 0\}, \quad |\nabla\theta(x)| \neq 0 \quad \forall \ x \in \partial\Omega. \tag{2.11}$$

By virtue of the Theorem on density of Morse functions (see [2]) there exist a sequence of Morse functions $\{\theta_k(x)\}_{k=1}^{\infty}$ such that

$$\theta_k \to \theta \quad \text{in} \quad C^2(\overline{\Omega}) \quad \text{as} \quad k \to +\infty. \tag{2.12}$$

Let us construct a Morse function $\mu \in C^2(\overline{\Omega})$ such that

$$\mu(x)|_{\partial\Omega} = 0, \quad |\nabla\mu(x)| > 0 \quad \forall \ x \in \partial\Omega. \tag{2.13}$$

We denote by $\mathcal{B} = \{x \in \mathbb{R}^n | \nabla\theta(x) = 0\}$ the set of critical points of functions θ. Since $|\nabla\theta|\big|_{\partial\Omega} > 0$ there exists an open set $\Theta \subset \mathbb{R}^n$ such that

$$\overline{\Theta} \cap \mathcal{B} = \{\emptyset\}, \quad \partial\Omega \subset \Theta. \tag{2.14}$$

Let $e(x) \in C_0^{\infty}(\Theta)$, $e|_{\partial\Omega} \equiv 1$. Set $\mu_k(x) = \theta_k + e(\theta - \theta_k)$. It is obvious that

$$\mu_k|_{\partial\Omega} = 0. \tag{2.15}$$

By definition of the function $e(x)$ we have

$$\nabla\mu_k(x) = \nabla\theta_k(x) \quad \forall x \in \overline{\Omega\setminus\Theta}. \tag{2.16}$$

For all x from the set $\Theta \cap \Omega$

$$\nabla\mu_k(x) = \nabla\theta_k + e(\nabla\theta - \nabla\theta_k) + \nabla e(\theta - \theta_k). \tag{2.17}$$

By virtue of (2.12) and (2.17) we have: $\forall\epsilon > 0 \ \exists \ k_0(\epsilon)$ such that

$$|\nabla\mu_k| \geq |\nabla\theta_k| - ||e||_{C^1(\overline{\Omega})}|\nabla\theta - \nabla\theta_k| \tag{2.18}$$

$$-||e||_{C^1(\overline{\Omega})}|\theta - \theta_k| \geq |\nabla\theta_k| - \epsilon \quad \forall x \in \Theta \cap \Omega, \tag{2.19}$$

where $k > k_0$.

It follows from (2.12), (2.14), (2.16) and these last inequalities that there exists such $\epsilon > 0$ and $\hat{k}$ that

$$|\nabla \mu_{\hat{k}}| > 0 \quad \text{in} \quad \Theta \cap \Omega. \tag{2.20}$$

Set $\mu(x) = \mu_{\hat{k}}(x)$. By (2.15), (2.16) and (2.20) the Morse function $\mu_{\hat{k}}(x)$ satisfies (2.13).
We denote by $\mathfrak{M}$ the set of critical points of function $\mu(x)$:

$$\mathfrak{M} = \{\hat{x}_i \in \mathbb{R}^n \;\; i = 1, \dots r\}.$$

Let us consider the sequence of functions $\{l_i\}_{i=1}^r \subset C^\infty([0,1]; \mathbb{R}^n)$ such that

$$l_i(t) \in \Omega \;\; \forall \;\; t \in [0,1], \;\; l_i(t_1) \neq l_i(t_2) \;\; \forall t_1, t_2 \in [0,1] \;\; \& \;\; t_1 \neq t_2 \;\; i = 1, \cdots, r; \tag{2.21}$$

$$l_i(1) = \hat{x}_i, \;\; l_i(0) \in \omega_0 \;\; i = 1, \cdots, r; \tag{2.22}$$

$$l_i(t_1) \neq l_j(t_2) \quad \forall \; i \neq j \quad \forall \; t_1, t_2 \in [0,1]. \tag{2.23}$$

By (2.21) - (2.23) there exists a sequence of functions $\{\mathrm{w}^{(i)}\}_{i=1}^r \subset C^2(\mathbb{R}^n, \mathbb{R}^n)$ and $\{e_i\}_{i=1}^r \subset C_0^\infty(\Omega)$ such that

$$\frac{dl_i(t)}{dt} = \mathrm{w}^{(i)}(l_i(t)) \quad \forall t \in [0,1], \quad i = 1, \cdots, r; \tag{2.24}$$

$$\operatorname{supp} e_i \subset \Omega \quad i = 1, \cdots, r; \tag{2.25}$$

$$\operatorname{supp} e_i \cap \operatorname{supp} e_j = \{\emptyset\} \quad \forall \, i \neq j; \tag{2.26}$$

$$e_i(l_i(t)) = 1 \quad \forall \; t \in [0,1], \quad i = 1, \cdots, r. \tag{2.27}$$

We set

$$V^{(i)}(x) = e_i(x)\mathrm{w}^{(i)}(x).$$

Let us consider the system of the ordinary differential equations

$$\frac{dx}{dt} = V^{(i)}(x), \quad x(0) = x_0. \tag{2.28}$$

We denote by $S_t^{(i)} : \mathbb{R}^n \to \mathbb{R}^n$ the operator such that $S_t^{(i)}(x_0) = x(t)$, where $x(t)$ is the solution of problem (2.28).
By (2.22), (2.24) and (2.27) we have

$$S_1^{(i)}(l_i(0)) = \hat{x}_i \quad i = 1, \cdots, r.$$

We set

$$\psi(x) = \mu(g_r(x)), \quad g_r(x) = S_1^{(1)} \circ S_1^{(2)} \circ \cdots \circ S_1^{(r)}(x). \tag{2.29}$$

By (2.25) there exists a domain $\mathfrak{S} \subset \mathbb{R}^n$ such that $\partial\Omega \subset \mathfrak{S}$ and

$$S_1^{(i)}(x) = x \quad \forall \; x \in \mathfrak{S}, \quad i = 1, \cdots, r. \tag{2.30}$$

By (2.30) the mappings $S_1^{(i)}(x)$ are diffeomorphisms on the domain Ω. So $g_r(x)$ is a diffeomorphism on the domain Ω. By (2.30) $\psi(x)=\mu(x) \quad \forall x\in S$. Hence

$$\psi(x)|_{\partial\Omega}=0. \tag{2.31}$$

We denote by Ψ the set of critical points of function ψ. Since the mapping $g_r:\Omega\rightarrow\Omega$ is a diffeomorphism we have

$$\Psi=\{x\in\Omega|\ g_r(x)\in\mathfrak{M}\}. \tag{2.32}$$

By (2.26) and (2.30)

$$g_r(l_i(0))=\hat x_i \quad i=1,\cdots,r. \tag{2.33}$$

It follows from (2.32) and (2.33) that

$$\Psi\subset\omega_0.\blacksquare$$

We say that the function.

$$L_2(x,\xi)=i\xi_0+\sum_{j,k=1}^n a_{jk}(x)\xi_j\xi_k$$

is the principal symbol of the operator $L(x,D)$.
For functions $f(x,\xi),g(x,\xi)$ we introduce the Poisson bracket

$$\{f,g\}=\sum_{j=1}^n\left(\frac{\partial f}{\partial\xi_j}\frac{\partial g}{\partial x_j}-\frac{\partial g}{\partial\xi_j}\frac{\partial f}{\partial x_j}\right).$$

Definition. *We say that function $\alpha(x)$ is pseudoconvex with respect to the symbol $L_2(x,\xi)$ if there exists a constant $\hat C>0$such that*

$$\frac{\mathrm{Im}\{\bar L_2(x,\xi_0,\bar\zeta'),L_2(x,\xi_0,\zeta')\}}{|s|}>0 \quad \forall(x,\xi,s)\in\overline{Q\setminus Q_\omega}\times\mathcal{S},$$

where $\mathcal{S}=\{(x,\xi,s)|x\in\overline{Q\setminus Q_\omega},\ M(\xi,s)=1,L_2(x,\xi_0,\zeta')=0\}$, $\zeta'=(\xi_1+i|s|\alpha_{x_1},\dots,\xi_n+i|s|\alpha_{x_n})$ and $M(\xi,s)=(\xi_0^2+\sum_{i=1}^n\xi_i^4+s^4)^{\frac14}$.
Now, using this function ψ, we construct two weight functions

$$\varphi(x)=\frac{e^{\lambda\psi(x_1,\dots,x_n)}}{\ell^\kappa(x_0)},\quad \alpha(x)=\frac{e^{\lambda\psi(x_1,\dots,x_n)}-e^{2\lambda\|\psi\|_{C^0(\overline\Omega)}}}{\ell^\kappa(x_0)}, \tag{2.34}$$

where $\kappa\ge 2$, $\lambda\in\mathbb{R}$, $\lambda\ge1$ will be chosen later on large enough,

$$\ell\in C^\infty[0,T],\quad \ell(x_0)>0\ \forall x_0\in(0,T),$$

$$\ell(x_0)=x_0\quad\forall x_0\in[0,\frac T4],\quad \ell(x_0)=T-x_0\quad x_0\in[\frac{3T}{4},T].$$

We have

Proposition 2.2. *Let the function α be given by (2.34). Then there exist $\hat\lambda,\hat C$ such that*

$$Im\{\overline{L}_2(x,\xi_0,\xi'-i|s|\nabla'\alpha),L_2(x,\xi_0,\xi'+i|s|\nabla'\alpha)\}\ge\hat C|s|\lambda^4\frac{e^{\lambda\psi(x)}}{\ell^\kappa}M^2(\xi,s\frac{e^{\lambda\psi(x)}}{\ell^\kappa}) \tag{2.35}$$

for all $(x,\xi,s)\in\mathcal{S}$ and $\lambda\ge\hat\lambda$. Here $\hat\lambda$ is independent of s and $\hat C$ is independent of s and λ.

Proof. We introduce the following notations: $p^{(j)}(x,\xi) = \partial_{\xi_j} p(x,\xi)$, $p^{(j,i)}(x,\xi) = \partial^2_{\xi_j\xi_i} p(x,\xi)$, $p_{(j)}(x,\xi) = \partial_{x_j} p(x,\xi)$, $\tilde\nabla\alpha = (0,\alpha_{x_1},\dots,\alpha_{x_n})$, $\tilde\xi = (\xi_1,\dots,\xi_n)$. After short computations we obtain

$$\frac{\partial}{\partial\xi_0} L_2(x,\xi_0,\zeta') = i, \quad \frac{\partial}{\partial\xi_0} \bar L_2(x,\xi_0,\zeta') = -i.$$

$$\frac{\partial}{\partial x_m} L_2(x,\xi_0,\zeta') = L_{2,(m)}(x,\xi_0,\zeta') + i|s| \sum_{k=1}^n L_2^{(k)}(x,\xi_0,\zeta') \frac{\partial^2\alpha}{\partial x_k \partial x_m}.$$

$$\frac{\partial}{\partial x_m} \bar L_2(x,\xi_0,\bar\zeta') = L_{2,(m)}(x,\xi_0,\bar\zeta') - i|s| \sum_{k=1}^n L_2^{(k)}(x,\xi_0,\bar\zeta') \frac{\partial^2\alpha}{\partial x_k \partial x_m}.$$

Then

$$\mathrm{Im}\{\overline{L}_2(x,\xi_0,\bar\zeta'), L_2(x,\xi_0,\zeta')\} =$$
$$\mathrm{Im}\left(\sum_{k=0}^n \bar L_2^{(k)}(x,\xi_0,\bar\zeta') L_{2,(k)}(x,\xi_0,\zeta') - \bar L_{2,(k)}(x,\xi_0,\bar\zeta') L_2^{(k)}(x,\xi_0,\zeta')\right).$$

Simple computations provide the following formulae

$$\mathrm{Im}\left(\frac{\partial}{\partial\xi_0}\bar L_2(x,\xi_0,\bar\zeta')\frac{\partial}{\partial x_0}L_2(x,\xi_0,\zeta') - \frac{\partial}{\partial x_0}\bar L_2(x,\xi_0,\bar\zeta')\frac{\partial}{\partial\xi_0}L_2(x,\xi_0,\zeta')\right)$$
$$= \mathrm{Im}\Bigg((-i)(L_{2,(0)}(x,\xi_0,\zeta') + i|s|\sum_{m=1}^n L_2^{(m)}(x,\xi_0,\zeta')\frac{\partial^2\alpha}{\partial x_m\partial x_0})$$
$$-i(L_{2,(0)}(x,\xi_0,\bar\zeta') - i|s|\sum_{m=1}^n L_2^{(m)}(x,\xi_0,\bar\zeta')\frac{\partial\alpha}{\partial x_m\partial x_0})\Bigg\} =$$
$$= -2L_{2,(0)}(x,\xi) + 2s^2\sum_{m=1}^n L_2^{(m)}(x,\tilde\nabla\alpha)\frac{\partial^2\alpha}{\partial x_m\partial x_0} + 2s^2 a_{x_0}(x,\tilde\nabla\alpha,\tilde\nabla\alpha),$$

where $a_{x_0}(x,\eta,\eta) = \sum_{i,j=1}^n \frac{\partial a_{ij}}{\partial x_0}\eta_i\eta_j$ and

$$\mathrm{Im}\left(\frac{\partial}{\partial\xi_k}\bar L_2(x,\xi_0,\bar\zeta')\frac{\partial}{\partial x_k}L_2(x,\xi_0,\zeta') - \frac{\partial}{\partial x_k}\bar L_2(x,\xi_0,\bar\zeta')\frac{\partial}{\partial\xi_k}L_2(x,\xi_0,\zeta')\right)$$
$$= \mathrm{Im}\Bigg(\bar L_2^{(k)}(x,\xi_0,\bar\zeta')(L_{2,(k)}(x,\xi_0,\zeta') + i|s|\sum_{m=1}^n L_2^{(m)}(x,\xi_0,\zeta')\frac{\partial^2\alpha}{\partial x_k\partial x_m})$$
$$- L_2^{(k)}(x,\xi_0,\zeta')(L_{2,(k)}(x,\xi_0,\bar\zeta') - i|s|\sum_{m=1}^n L_2^{(k)}(x,\xi_0,\bar\zeta')\frac{\partial^2\alpha}{\partial x_k\partial x_m})\Bigg) =$$
$$-L_2^{(k)}(x,|s|\tilde\nabla\alpha)(L_{2,(k)}(x,\xi) - L_{2,(k)}(x,|s|\tilde\nabla\alpha)) + L_2^{(k)}(x,\xi)\mathrm{Im}L_{2,(k)}(x,\xi_0,\zeta')$$
$$+|s|L_2^{(k)}(x,\xi)\sum_{m=1}^n L_2^{(m)}(x,\xi)\frac{\partial^2\alpha}{\partial x_k\partial x_m} + |s|L_2^{(k)}(x,|s|\tilde\nabla\alpha)\sum_{m=1}^n L_2^{(m)}(x,|s|\tilde\nabla\alpha)\frac{\partial^2\alpha}{\partial x_k\partial x_m}$$
$$-L_2^{(k)}(x,|s|\tilde\nabla\alpha)(L_{2,(k)}(x,\xi) - L_{2,(k)}(x,|s|\tilde\nabla\alpha)) + L_2^{(k)}(x,\xi)\mathrm{Im}L_{2,(k)}(x,\xi_0,\zeta')$$
$$+|s|L_2^{(k)}(x,\xi)\sum_{m=1}^n L_2^{(m)}(x,\xi)\frac{\partial^2\alpha}{\partial x_k\partial x_m} + |s|L_2^{(k)}(x,|s|\tilde\nabla\alpha)\sum_{m=1}^n L_2^{(m)}(x,|s|\tilde\nabla\alpha)\frac{\partial^2\alpha}{\partial x_k\partial x_m}.$$

Therefore

$$\frac{1}{2}\mathrm{Im}\{\overline{L}_2(x,\xi_0,\bar{\zeta}'),L_2(x,\xi_0,\zeta')\} =$$

$$\frac{1}{2}\mathrm{Im}\left(\sum_{k=0}^{n}\frac{\partial}{\partial\xi_k}\bar{L}_2(x,\xi_0,\bar{\zeta}')\frac{\partial}{\partial x_k}L_{2,(k)}(x,\xi_0,\zeta')-\frac{\partial}{\partial x_k}\bar{L}_2(x,\xi_0,\bar{\zeta}')\frac{\partial}{\partial\xi_k}L_2(x,\xi_0,\zeta')\right)$$

$$= -L_{2,(0)}(x,\xi)+s^2\sum_{k=1}^{n}L_2^{(k)}(x,\tilde{\nabla}\alpha)\frac{\partial^2\alpha}{\partial x_k\partial x_0}+2s^2a_{x_0}(x,\tilde{\nabla}\alpha,\tilde{\nabla}\alpha)$$

$$-L_2^{(k)}(x,|s|\tilde{\nabla}\alpha)(L_{2,(k)}(x,\xi)-L_2^{(k)}(x,|s|\tilde{\nabla}\alpha))+L_2^{(k)}(x,|s|\tilde{\nabla}\alpha)\mathrm{Im}L_{2,(k)}(x,\xi_0,\zeta')$$

$$+\sum_{m,k=1}^{n}(|s|L_2^{(k)}(x,\xi)L_2^{(m)}(x,\xi)+|s|L_2^{(k)}(x,|s|\tilde{\nabla}\alpha)L_2^{(m)}(x,|s|\tilde{\nabla}\alpha))\frac{\partial^2\alpha}{\partial x_k\partial x_m}.$$

Observing that $\frac{\partial^2\alpha}{\partial x_k\partial x_m}=(\lambda^2\psi_{x_k}\psi_{x_m}+\lambda\psi_{x_ix_m})\frac{e^{\lambda\psi(x)}}{\ell^\kappa}$, we have

$$I=\sum_{m,k=1}^{n}(|s|L_2^{(k)}(x,\xi)L_2^{(m)}(x,\xi)+|s|L_2^{(k)}(x,|s|\tilde{\nabla}\alpha)L_2^{(m)}(x,|s|\tilde{\nabla}\alpha))\frac{\partial^2\alpha}{\partial x_k\partial x_m}$$

$$=\lambda^2|s|(a(x,\xi,\nabla\psi)^2+s^2\frac{e^{2\lambda\psi(x)}}{\ell^{2\kappa}}a(x,\nabla\psi,\nabla\psi)^2)\frac{e^{\lambda\psi}}{\ell^\kappa}+$$

$$\sum_{m,k=1}^{n}(|s|L_2^{(k)}(x,\xi)L_2^{(m)}(x,\xi)+|s|L_2^{(k)}(x,|s|\tilde{\nabla}\alpha)L_2^{(m)}(x,|s|\tilde{\nabla}\alpha))\lambda\psi_{x_kx_m}\frac{e^{\lambda\psi}}{\ell^\kappa}.$$

Since $(x,\xi,s)\in\mathcal{S}$ the following inequality holds true

$$a(x,\tilde{\xi},\tilde{\nabla}\alpha)^2=s^2a(x,\tilde{\nabla}\alpha,\tilde{\nabla}\alpha)^2\geq\hat{C}|(\tilde{\xi},s\frac{e^{\lambda\psi(x)}}{\ell^\kappa})|^2.$$

Taking $\hat{\lambda}$ sufficiently large, for all $\lambda\geq\hat{\lambda}$ we have

$$I\geq\frac{\lambda^4}{2}\hat{C}|s|\frac{e^{\lambda\psi(x)}}{\ell^\kappa}|(\tilde{\xi},s\frac{e^{\lambda\psi(x)}}{\ell^\kappa})|^2\quad\forall(x,\xi,s)\in\mathcal{S}\tag{2.36}$$

where $\hat{C}$ is independent of λ,ξ,s.
Finally observing that

$$|\xi_0|\leq|a(x,\tilde{\xi},|s|\tilde{\nabla}\alpha)|\quad\forall(x,\xi,s)\in\mathcal{S}$$

we obtain from (2.36)

$$I\geq\frac{\lambda^4}{2}\hat{C}\frac{e^{\lambda\psi(x)}}{\ell^\kappa}|s|M^2(\xi,s\frac{e^{\lambda\psi(x)}}{\ell^\kappa})\quad\forall(x,\xi,s)\in\mathcal{S}\tag{2.37}$$

where $\hat{C}$ is independent of λ,ξ,s. On the other hand

$$|-L_{2,(0)}(x,\xi)+2s^2a_{x_0}(x,\tilde{\nabla}\alpha,\tilde{\nabla}\alpha)+s^2\sum_{k=1}^{n}L_2^{(k)}(x,\tilde{\nabla}\alpha)\frac{\partial^2\alpha}{\partial x_k\partial x_0}$$

$$-L_2^{(m)}(x,|s|\tilde{\nabla}\alpha)(L_{2,(m)}(x,\xi)-L_{2,(m)}(x,|s|\tilde{\nabla}\alpha))+L_2^{(m)}(x,|s|\tilde{\nabla}\alpha)\mathrm{Im}L_{2,(m)}(x,\xi_0,\zeta')|$$

$$\leq C|s|\lambda^2\frac{e^{\lambda\psi(x)}}{\ell^\kappa}M^2(\xi,s\frac{e^{\lambda\psi(x)}}{\ell^\kappa}).\tag{2.38}$$

Inequalities (2.38), (2.37) imply (2.35). ■
Now we formulate our main observability estimate for the parabolic equation.

Theorem 2.3. *Let us assume that (2.3)-(2.6) holds true, and let $y \in W(Q)$ be a solution of (2.1), (2.2). Then there exists a constant $\hat{\lambda}$ such that for any $\lambda > \hat{\lambda}$ there exists $C > 0$ independent of s and $s_0(\lambda)$ that*

$$\frac{1}{s}\int_Q \frac{1}{\varphi}\sum_{j=1}^{n}|\frac{\partial y}{\partial x_j}|^2 e^{2s\alpha}dx + s\int_Q \varphi|y|^2 e^{2s\alpha}dx \leq C(s^{-\frac{1}{2}}||\varphi^{-\frac{1}{4}}ge^{s\alpha}||^2_{H^{\frac{1}{4},\frac{1}{2}}(\Sigma)}$$
$$+s^{-\frac{1}{2}}||\varphi^{-\frac{1}{4}+\frac{1}{\kappa}}ge^{s\alpha}||^2_{L^2(\Sigma)} + \frac{1}{s^2}\int_Q \frac{|f|^2}{\varphi^2}e^{2s\alpha}dx + \sum_{j=1}^{n}\int_Q |f_j|^2 e^{2s\alpha}dx$$
$$+\int_{Q_\omega} s\varphi|y|^2 e^{2s\alpha}dx) \quad \forall s \geq s_0 > 0. \tag{2.39}$$

Remark 2.4.

1) By a density argument, it suffices to prove the result when the solution y is supposed to be more regular, namely $y \in H^{1,2}(Q)$ and the right hand side has compact support. Actually, there exists a sequence of $\{f^k, f_1^k, \dots f_n^k, g^k\} \in (C_0^\infty(Q))^{n+1} \times C_0^\infty(\Sigma)$ converging to $\{(f, f_1, \dots f_n, g\}$ in $L^2(Q)^{n+1} \times H^{\frac{1}{4},\frac{1}{2}}(\Sigma)$ and the corresponding solution y to problem (2.1), (2.2) with right hand side $f^k + \sum_{j=1}^n \frac{\partial f_j^k}{\partial x_j}$ and boundary condition g^k is such that $y_k \in H^{1,2}(Q)$ and

$$y^k \to y \quad in\ L^2(0,T;H^1(\Omega)).$$

So it suffices to prove estimate (2.39) for solutions $y \in H^{1,2}(Q)$ and right hand sides with compact support in Q and Σ.

2) Without loss of generality, it is sufficient to consider the case where $b_j = 0$, $c_i = 0$ and $d = 0$ as the first and zero order terms in (2.1) can be added to the right hand side and the corresponding terms in (2.39) can be absorbed by the terms in the left hand side by choosing $\hat{s}$ and $\hat{\lambda}$ large enough.

The proof of Theorem 2.3 requires several steps and will be the content of the next subsections.

2.1 Localization in space and time.

For every $\delta > 0$ we can consider a covering of $Q = [0,T] \times \overline{\Omega}$ as follows

$$\overline{Q} \subset \bar{Q}_0 \cup (\bigcup_{k=1}^{I} B(\hat{x}_k, \delta)), \tag{2.40}$$

where $Q_0 = (0,T) \times \Omega_0, \overline{\Omega}_0 \subset \Omega$, $\hat{x}_k \in (0,T) \times \partial\Omega$.
Let $(e_k)_{k=0,\dots I}$ be a corresponding partition of unity, i.e.

$$e_0 \in C_0^\infty(Q_0),\ e_k \in C_0^\infty(B(\hat{x_k}, \delta)),\ k = 1, ..., I$$

$$e_k(x) \geq 0,\ k = 0, ..., I,\ \text{and}\ \sum_{k=0}^{I} e_k(x) = 1,\ \forall x \in \overline{Q}.$$

We now define

$$y_k(x) = y(x)e_k(x),\ k = 0, .., I.$$

Then if $L(x,D)y = \frac{\partial y}{\partial x_0} - \sum_{i,j=1}^n \frac{\partial}{\partial x_i}(a_{ij}\frac{\partial y}{\partial xj})$ (recall that from the previous remark this corresponds to the general case), we have for each $k = 0, ..., I$

$$L(x,D)y_k = [L(x,D), e_k]y + e_k f + \sum_{j=1}^n \frac{\partial(e_k f_j)}{\partial x_j} - \sum_{j=1}^n f_j \frac{\partial e_k}{\partial x_j}, \text{ in } Q, \tag{2.41}$$

$$y_k = g e_k \text{ on } (0,T)\times\Gamma, \tag{2.42}$$

and

$$supp\, y_0 \subset Q_0,\ supp\, y_k \subset B(\hat{x_k},\delta),\ k = 1, ..., I. \tag{2.43}$$

Notice that the commutator $[L, e_k]$ is a first order operator and that $e_k f_j$ and $f_j \frac{\partial e_k}{\partial x_j}$ have compact support in $B(\hat{x_k},\delta)$ for all $k = 1, ..., I$.
Let us suppose that Theorem 2.3 is true with the additional assumption that

$$supp\, y \subset Q_0,\ \text{or } supp\, y \subset B(\hat{x},\delta),\ \hat{x} \in (0,T)\times\partial\Omega.$$

Then, as $y = \sum_{k=0}^I y_k$, we have (the letter C will denote various constants independent of s)

$$\begin{aligned}
&\int_Q (\frac{1}{s\varphi}\sum_{j=1}^n |\frac{\partial y}{\partial y_j}|^2 + s\varphi|y|^2)e^{2s\alpha}dx \le C\sum_{k=0}^I \int_Q (\frac{1}{s\varphi}\sum_{j=1}^n |\frac{\partial y_k}{\partial x_j}|^2 + s\varphi|y_k|^2)e^{2s\alpha}dx \\
&\le C\sum_{k=0}^I \Bigg(s^{-\frac12}||\varphi^{-\frac14} g e_k e^{s\alpha}||^2_{H^{\frac14,\frac12}(\Sigma)} + s^{-\frac12}||\varphi^{-\frac14+\frac1\kappa} g e_k e^{s\alpha}||^2_{L^2(\Sigma)} + \int_Q \frac{|f e_k|^2}{s^2\varphi^2} e^{2s\alpha}dx \\
&+\sum_{j=0}^n \int_Q |f_j e_k|^2 e^{2s\alpha}dx + \frac{1}{s^2}\sum_{j=1}^n \int_Q \frac{|f_j|^2}{\varphi^2}e^{2s\alpha}dx + \int_Q |y|^2 e^{2s\alpha}dx \\
&+\sum_{k=1}^I \int_{Q_\omega} s\varphi|y_k|^2 e^{2s\alpha}dx \Bigg) \\
&\le C(s^{-\frac12}||\varphi^{-\frac14} g e^{s\alpha}||^2_{H^{\frac14,\frac12}(\Sigma)} + s^{-\frac12}||\varphi^{-\frac14+\frac1\kappa} g e^{s\alpha}||^2_{L^2(\Sigma)} + \frac{1}{s^2}\int_Q \frac{|f|^2}{\varphi^2}e^{2s\alpha}dx \\
&+2\sum_{j=1}^n \int_Q |f_j|^2 e^{2s\alpha}dx + \int_Q |y|^2 e^{2s\alpha}dx + \int_{Q_\omega} s\varphi|y|^2 e^{2s\alpha}dx).
\end{aligned} \tag{2.44}$$

Taking now $\hat{s}$ sufficiently large, we obtain (2.39) for $s \ge \hat{s}$. Therefore it is enough to prove Theorem 2.3 in the two cases :

- (i) $supp\, y \subset Q_0$,
- (ii) $supp\, y \subset B(\hat{x},\delta),\ \hat{x} \in (0,T)\times\partial\Omega$.

Case (i) immediately follows from [13] so, below, we concentrate on Case (ii).

2.2 Change of coordinates

Let us take $\hat{x} \in (0,T)\times\partial\Omega$, $\delta > 0$ and a solution y of (2.1), (2.2) such that $supp\, y \subset B(\hat{x},\delta)$. By (A.210) there exists $\delta > 0$ sufficiently small such that for some index $i_0 \in \{1,\dots,n\}$

$$\frac{\partial\psi}{\partial x_{i_0}}(x) \ne 0,\ \forall x \in B(\hat{x},\delta).$$

After renumbering we can assume that $i_0 = n$ and without loss of generality we can assume that

$$\frac{\partial \psi}{\partial x_n}(x) \neq 0, \ \forall x \in B(\hat{x}, \delta). \tag{2.45}$$

We now take the new coordinate system

$$\hat{x}_n = \psi(x_1, \dots, x_n), \ \hat{x}_i = x_i, \ i = 0, \dots, n-1. \tag{2.46}$$

As $\psi = 0$ on Γ, in the new coordinate system $((0,T) \times \partial\Omega) \cap B(\hat{x}, \delta)$ corresponds to $\hat{x}_n = 0$. Writing

$$\hat{y}(\hat{x}_0, \hat{x}_1, ..., \hat{x}_n) = y(x_0, x_1, ..., x_n)$$

we obtain from (2.1), (2.2)

$$\hat{L}(\hat{x}, D)\hat{y} = a_0 \frac{\partial \hat{y}}{\partial x_0} - \frac{\partial^2 \hat{y}}{\partial \hat{x}_n^2} - \sum_{j=1}^{n} \hat{a}_{nj} \frac{\partial^2 \hat{y}}{\partial \hat{x}_n \partial \hat{x}_j} - \hat{A}\hat{y} + \hat{B}\hat{y} = \hat{f} + \sum_{j=1}^{n} \frac{\partial \hat{f}_j}{\partial \hat{x}_j}, \ \text{in } [0,T] \times \mathbb{R}^{n-1} \times (0, \gamma), \tag{2.47}$$

$$\hat{y}(\hat{x'}, 0) = \hat{g}(\hat{x'}), \ \text{on } [0,T] \times \mathbb{R}^{n-1}, \quad \hat{x}' = (\hat{x}_0, \dots, \hat{x}_{n-1}) \tag{2.48}$$

and $\hat{y}$ vanishes in the neighborhood of the set $(\partial B'(0,\delta) \times [0,\gamma]) \cup (B'(0,\delta) \times \{\gamma\})$ with $B'(0,\delta) = \{\hat{x}' \in \mathbb{R}^n, \ |\hat{x}'| \leq \delta\}$ and $\hat{x}' = (\hat{x}_0, \dots, \hat{x}_{n-1})$.
We now want to show an inequality analogous to (2.39) corresponding to the weight function

$$\hat{\varphi}(\hat{x}) = e^{\lambda \hat{\psi}(\hat{x}_1, \dots, \hat{x}_n)}. \tag{2.49}$$

More precisely we want to show that there exists a constant $\hat{\lambda}$ such that for any $\lambda > \hat{\lambda}$ there exist s_0 and $\hat{C} > 0$ independent of s that for all $s \geq s_0$

$$\begin{aligned} \int_{\hat{Q}} \Big(\frac{1}{s\hat{\varphi}} \sum_{j=1}^{n} \Big|\frac{\partial \hat{y}}{\partial x_j}\Big|^2 + s\hat{\varphi}|\hat{y}|^2\Big) e^{2s\hat{\alpha}} d\hat{x} &\leq \hat{C}\Big(s^{-\frac{1}{2}} ||\hat{\varphi}^{-\frac{1}{4}} \hat{g} e^{s\hat{\alpha}}||^2_{H^{\frac{1}{4},\frac{1}{2}}(\Sigma)} \\ + \quad s^{-\frac{1}{2}} ||\hat{\varphi}^{-\frac{1}{4}+\frac{1}{\kappa}} \hat{g} e^{s\hat{\alpha}}||^2_{L^2(\Sigma)} &+ \int_{\hat{Q}} \frac{|\hat{f}|^2}{s^2\hat{\varphi}^2} e^{2s\hat{\alpha}} d\hat{x} + \sum_{j=1}^{n} \int_{\hat{Q}} |\hat{f}_j|^2 e^{2s\hat{\alpha}} d\hat{x}\Big). \end{aligned} \tag{2.50}$$

The operator A has the form

$$A\hat{y} = \sum_{i,j=1}^{n-1} \hat{a}_{ij}(\hat{x}) \frac{\partial^2 \hat{y}}{\partial \hat{x}_i \partial \hat{x}_j} \tag{2.51}$$

and operator $\hat{B}$ is a first order differential operator with L^∞ coefficients. We have already seen that we can ignore first order terms. We also omit from now on the notation $\hat{}$. We then obtain

$$L(x, D)y = a_0 \frac{\partial y}{\partial x_0} - \frac{\partial^2 y}{\partial x_n^2} - \sum_{j=1}^{n} a_{nj} \frac{\partial^2 y}{\partial x_n \partial x_j} - Ay = f + \sum_{j=1}^{n} \frac{\partial f_j}{\partial x_j}, \ \text{in } [0,T] \times \mathbb{R}^{n-1} \times (0, \gamma), \tag{2.52}$$

$$y(x', 0) = g(x'), \ \text{on } [0,T] \times \mathbb{R}^{n-1} \tag{2.53}$$

and

(2.54) y vanishes in the neigborhood of the set $(\partial B'(0,\delta)\times[0,\gamma])\cup(B'(0,\delta)\times\{\gamma\})$.

Here $a_0(x)\in C^1(\bar{Q})$ is a strictly positive function. Notice that f, f_j also have compact support in $B'(0,\delta)\times[0,\gamma)$ and that g has compact support in $B'(0,\delta)$. Moreover if we write for $x\in \mathbb{R}^{n+1}$ and $\xi^1,\xi^2\in\mathbb{R}^{n-1}$

$$\tilde{a}(x,\xi^1,\xi^2)=\sum_{i,j=1}^{n-1}a_{ij}(x)\xi_i^1\xi_j^2 \tag{2.55}$$

we have an ellipticity condition corresponding to (2.5) namely there exists $\beta>0$, such that for all $\xi\in\mathbb{R}^n$,

$$\xi_n^2+\sum_{j=1}^{n-1}a_{nj}(x)\xi_n\xi_j+\tilde{a}(x,\tilde{\xi},\tilde{\xi})\geq\beta|\xi|^2,\ \forall x\in\Pi_{\delta,\gamma}=B'(0,\delta)\times[0,\gamma], \tag{2.56}$$

where $\tilde{\xi}=(\xi_1,\ldots,\xi_{n-1})\in\mathbb{R}^{n-1}$. This shows that

$$\exists\hat{\gamma}>0,\ \forall\tilde{\xi}\in\mathbb{R}^{n-1},\ |\tilde{\xi}|=1,\ \tilde{a}(x,\tilde{\xi},\tilde{\xi})-(\sum_{j=1}^{n-1}a_{nj}(x)\xi_j)^2\geq\hat{\gamma}. \tag{2.57}$$

2.3 Localization in time.

From now on it is convenient for us to work with the function $w(x)=e^{|s|\alpha}y(x)$ instead of y. Function w verifies the equation

$$L(x,D_0,D'+i|s|\nabla'\alpha)w=F,\quad \text{in } [0,T]\times\mathbb{R}^{n-1}\times(0,\gamma), \tag{2.58}$$

$$w(x',0)=ge^{|s|\alpha}, \tag{2.59}$$

$$\operatorname{supp} w\subset\Pi_{\delta,\gamma}=B'(0,\delta)\times[0,\gamma), \tag{2.60}$$

where $F(x)=F_0+\sum_{i=1}^n\frac{\partial F_i}{\partial x_i}, F_0=e^{|s|\alpha}f-\sum_{j=1}^n|s|\alpha_{x_j}f_je^{|s|\alpha}-|s|\alpha_{x_0}w, F_i=f_je^{|s|\alpha}$. Since the function α has singularities at x_0 equal 0 and T it is the same for some coefficients of equation (2.58). We overcome this difficulty using a localization in time.
Let $\tilde{\psi}(x_0)\in C_0^\infty(\frac{1}{2},2)$ be a nonnegative function such that $\sum_{j=-\infty}^\infty\tilde{\psi}(2^{-j}x_0)=1$ (We may take $\tilde{\psi}(x_0)=\tilde{\beta}(x_0)-\tilde{\beta}(2x_0)$ where $\tilde{\beta}\in C_0^\infty(-2,2)$ and $\tilde{\beta}$ equals 1 for $|x_0|\leq 1$.) Denote $\tilde{\psi}_j(x_0)=\tilde{\psi}(2^{-j}x_0)$, $\mu_j(x_0)=\tilde{\psi}(2^{-j}/\ell^\kappa(x_0)), \tilde{\mu}_j(x_0)=\tilde{\psi}'(2^{-j}/\ell^\kappa(x_0)), \Psi_j(x_0)=\mu_{j+1}(x_0)+\mu_j(x_0)+\mu_{j-1}(x_0)$ and $w_j(x)=\mu_j(x_0)w(x), F_{i,j}(x)=\mu_j(x_0)F_i(x), g_j(x)=\mu_j(x_0)g(x)e^{|s|\alpha}$.
Function w_j satisfies the equation:

$$L(x,D_0,D'+i|s|\nabla'\alpha)w_j=\partial_{x_0}\mu_jw+F_{0,j}+\sum_{i=1}^n\frac{\partial F_{i,j}}{\partial x_i}\quad \text{in } G=\mathbb{R}^n\times(0,\gamma), \tag{2.61}$$

$$w_j(x',0) = g_j, \tag{2.62}$$

$$\operatorname{supp} w_j \subset \Pi_{\delta,\gamma}. \tag{2.63}$$

Suppose that for a function w_j the following Carleman estimate is already established:

$$\sum_{i=1}^{n} \|s^{-\frac12}\varphi^{-\frac12}\frac{\partial w_j}{\partial x_i}\|_{L^2(G)} + \|s^{\frac12}\varphi^{\frac12}w_j\|_{L^2(G)} \le C(\|s^{-\frac14}\varphi^{-\frac14}g_j\|_{H^{\frac14,\frac12}(\mathbb{R}^n)} +$$

$$\|s^{-\frac14}\varphi^{-\frac14+\frac1\kappa}g_j\|_{L^2(\mathbb{R}^n)} + \|(\partial_{x_0}\mu_j w + F_{0,j})/(s\varphi)\|_{L^2(G)} + \sum_{i=1}^{n}\|F_{i,j}\|_{L^2(G)}) \quad \forall s > s_0 > 1, \tag{2.64}$$

where C is independent of s and j and s_0 is independent of j.
Observe that

$$\operatorname{supp}\mu_j \cap \operatorname{supp}\mu_k \ne \{\emptyset\},\ \operatorname{supp}\tilde\mu_j \cap \operatorname{supp}\tilde\mu_k \ne \{\emptyset\} \quad \text{only if } |k-j| \le 1. \tag{2.65}$$

Let $\tilde f, \tilde h \in L^2(G)$ be an arbitrary function and $\tilde f_j = \mu_j \tilde f, \tilde h_j = \tilde\mu_j \tilde h$. By (2.65) we have

$$\sum_{j=-\infty}^{\infty} |\tilde f_j(x)|^2 \le C|\tilde f(x)|^2, \ \sum_{j=-\infty}^{\infty} |\tilde h_j(x)|^2 \le C|\tilde h(x)|^2 \quad \forall x \in G, \tag{2.66}$$

Therefore

$$\sum_{j=-\infty}^{\infty} (\|F_{0,j}/(s\varphi)\|_{L^2(G)} + \sum_{i=1}^{n}\|F_{i,j}\|_{L^2(G)}) \le 3(\|F_0/(s\varphi)\|_{L^2(G)} + \sum_{i=1}^{n}\|F_i\|_{L^2(G)}). \tag{2.67}$$

Observing that $\frac{\partial\mu_j}{\partial x_0} = -\kappa\tilde\mu_j\frac{2^{-j}}{\ell^{\kappa+1}}\ell'$ we have

$$\sum_{j=-\infty}^{\infty} \|\partial_{x_0}\mu_j\frac{w}{s\varphi}\|^2_{L^2(G)} = \sum_{j=-\infty}^{\infty}\|\kappa\tilde\mu_j\frac{2^{-j}}{\ell^{\kappa+1}}\ell'\frac{w}{s\varphi}\|^2_{L^2(G)} \le C\sum_{j=-\infty}^{\infty}\|\tilde\mu_j\frac{1}{\ell}\frac{w}{s\varphi}\|^2_{L^2(G)}.$$

Using (2.66) one more time we have

$$\sum_{j=-\infty}^{\infty} \|\partial_{x_0}\mu_j\frac{w}{s\varphi}\|^2_{L^2(G)} \le C\|\frac{w}{s\varphi^{1-\frac1\kappa}}\|^2_{L^2(G)}. \tag{2.68}$$

Since the restriction of a function φ on the hyperplane $\{x|\ x_n = 0\}$ is independent of $(x_1,\dots,x_{n-1})$ we have

$$\sum_{j=-\infty}^{\infty}\int_{-\infty}^{+\infty}\varphi^{-\frac12}\|g_j(x_0,\cdot)\|^2_{H^{\frac12}(\mathbb{R}^{n-1})}dx_0 = \sum_{j=-\infty}^{\infty}\int_{-\infty}^{+\infty}\varphi^{-\frac12}\mu_j^2\|g(x_0,\cdot)\|^2_{H^{\frac12}(\mathbb{R}^{n-1})}dx_0$$

$$\le C\int_{-\infty}^{+\infty}\varphi^{-\frac12}\|g(x_0,\cdot)\|^2_{H^{\frac12}(\mathbb{R}^{n-1})}dx_0 \le C\|\varphi^{-\frac14}g\|^2_{H^{0,\frac12}(\mathbb{R}^n)}. \tag{2.69}$$

Next we observe that

$$\sum_{j=-\infty}^{\infty}\|\varphi^{-\frac14}g_j\|^2_{H^{\frac14,0}(\mathbb{R}^n)} = \sum_{j=-\infty}^{\infty}\int_{\mathbb{R}^{n-1}}\|\varphi^{-\frac14}g_j(\cdot,x_1,\dots,x_{n-1})\|^2_{H^{\frac14}(\mathbb{R})}dx_1\dots dx_{n-1}.$$

Denote $\mathcal{G}_j = \operatorname{supp}\mu_j$, $h_i = \varphi^{-\frac{1}{4}} g_j$, $h = \varphi^{-\frac{1}{4}} g$. According to the definition of the norm in the space $H^{\frac{1}{4}}$ we have

$$\|h_j(\cdot, x_1, \dots, x_{n-1})\|^2_{H^{\frac{1}{4}}(\mathbb{R})} = C\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|(\mu_j h)(y_0,\cdot) - (\mu_j h)(x_0,\cdot)|^2}{|y_0 - x_0|^{\frac{3}{2}}} dx_0 dy_0$$

$$\le C\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|\mu_j(y_0,\cdot) - \mu_j(x_0,\cdot)|^2 |h(y_0,\cdot)|^2}{|y_0 - x_0|^{\frac{3}{2}}} dx_0 dy_0 + C\int_{\mathbb{R}}\int_{\mathcal{G}_j} \frac{|h(y_0,\cdot) - h(x_0,\cdot)|^2 |\mu_j(x_0)|^2}{|y_0 - x_0|^{\frac{3}{2}}} dx_0 dy_0$$

$$= C(I_1(j) + I_2(j)). \tag{2.70}$$

We estimate the terms I_1 and I_2 separately. By Young's inequality we have

$$I_1 = \int_{\mathcal{G}_j}\int_{[-\delta,\delta]} |\mu_j'(\zeta)|^2 |h(y_0, x_1, \dots, x_{n-1})|^2 |y_0 - x_0|^{\frac{1}{2}} dx_0 dy_0$$

$$\le \int_{\mathcal{G}_j}\int_{[-\delta,\delta]} |h(y_0, x_1, \dots, x_{n-1})|^2 |y_0 - x_0|^{\frac{1}{2}} dx_0 dy_0 \|\mu_j\|^2_{C^1(\mathbb{R})} \le C\|\mu_j\|^2_{C^1(\mathbb{R})} \|h(\cdot, x_1, \dots, x_{n-1})\|^2_{L^2(\mathcal{G}_j)}.$$

Obviously

$$\|\mu_j\|_{C^1(\mathbb{R})} \le C(1 + \|\varphi^{\frac{1}{\kappa}}(\cdot, x_1, \dots, x_{n-1}, 0)\|_{C^0(\mathcal{G}_j)}).$$

Next we notice that

$$2^{-j}/\ell^\kappa(x_0) \in [\frac{1}{2}, 2] \quad \forall x_0 \in \mathcal{G}_j.$$

Since $\varphi(\cdot, x_1, \dots, x_{n-1}, 0) = 1/\ell^\kappa(x_0)$ this implies

$$\varphi(\cdot, x_1, \dots, x_{n-1}, 0) \in [2^{j-1}, 2^j] \quad \forall x_0 \in \mathcal{G}_j$$

and

$$I_1 \le C\|\varphi^{-\frac{1}{4}+\frac{1}{\kappa}}(\cdot, x_1, \dots, x_{n-1}, 0) g(\cdot, x_1, \dots, x_{n-1})\|^2_{L^2(\mathcal{G}_j)} \tag{2.71}$$

and

$$\begin{aligned}
I_2 \quad &\le \|\mu_j\|^2_{C^0(\mathbb{R})} \int_{\mathbb{R}}\int_{\mathcal{G}_j} \frac{|h(y_0, x_1, \dots, x_{n-1}) - h(x_0, x_1, \dots, x_{n-1})|^2}{|y_0 - x_0|^{\frac{3}{2}}} dx_0 dy_0 \\
&\le \|\mu_j\|^2_{C^0(\mathbb{R})} (\|h(\cdot, x_1, \dots, x_{n-1})\|^2_{H^{\frac{1}{4}}(\cup_{l=j-2}^{j+2}\mathcal{G}_l)} \\
+ \quad &\int_{\mathbb{R}\setminus\cup_{l=j-1}^{j+1}\mathcal{G}_l}\int_{\mathcal{G}_j} \frac{|h(x_0, x_1, \dots, x_{n-1})|^2}{|y_0 - x_0|^{\frac{3}{2}}} dx_0 dy_0) \\
&\le C(\|h(\cdot, x_1, \dots, x_{n-1})\|^2_{H^{\frac{1}{4}}(\cup_{l=j-2}^{j+2}\mathcal{G}_l)} + \|\varphi^{-\frac{1}{4}+\frac{1}{\kappa}} g(\cdot, x_1, \dots, x_{n-1})\|^2_{L^2(\mathcal{G}_j)}).
\end{aligned} \tag{2.72}$$

Here we used the fact that

$$\int_{\mathbb{R}\setminus\cup_{l=j-1}^{j+1}\mathcal{G}_l} \frac{1}{|y_0 - x_0|^{\frac{3}{2}}} dy_0 \le C\int_{\{|y_0|^\kappa \ge 2^{-j-2}\}} \frac{1}{|y_0|^{\frac{3}{2}}} dy_0 \le C\|\varphi^{-\frac{1}{4}}(\cdot, x_1, \dots, x_{n-1}, 0)\|^2_{C^0(\mathcal{G}_j)} \quad \forall x_0 \in \mathcal{G}_j$$

By (2.65)

$$\begin{aligned}
\sum_{j=-\infty}^{\infty} \|h(\cdot, x_1, \dots, x_{n-1})\|^2_{H^{\frac{1}{4}}(\mathcal{G}_j)} &\le \sum_{j=-\infty}^{\infty} \|h(\cdot, x_1, \dots, x_{n-1})\|^2_{H^{\frac{1}{4}}(\mathcal{G}_{2j})} \\
+ \sum_{j=-\infty}^{\infty} \|h(\cdot, x_1, \dots, x_{n-1})\|^2_{H^{\frac{1}{4}}(\mathcal{G}_{2j+1})} &\le 2\|h(\cdot, x_1, \dots, x_{n-1})\|^2_{H^{\frac{1}{4}}(\mathbb{R})}.
\end{aligned} \tag{2.73}$$

By (2.71)-(2.73)

$$\sum_{j=-\infty}^{\infty}\|s^{-\frac14}\varphi^{-\frac14}g_j\|^2_{H^{\frac14,\frac12}(\mathbb{R}^n)}=\sum_{j=-\infty}^{\infty}\|s^{-\frac14}\varphi^{-\frac14}g_j\|^2_{H^{0,\frac12}(\mathbb{R}^n)}+\sum_{j=-\infty}^{\infty}\|s^{-\frac14}\varphi^{-\frac14}g_j\|^2_{H^{\frac14,0}(\mathbb{R}^n)}$$
$$\le C\left(\|s^{-\frac14}\varphi^{-\frac14}g\|^2_{H^{0,\frac12}(\mathbb{R}^n)}+s^{-\frac12}\sum_{j=-\infty}^{\infty}(I_1(j)+I_2(j))\right)$$
$$\le C(\|s^{-\frac14}\varphi^{-\frac14}g\|^2_{H^{\frac14,\frac12}(\mathbb{R}^n)}+\|s^{-\frac14}\varphi^{-\frac14+\frac1\kappa}g\|^2_{L^2(\mathbb{R}^n)}). \tag{2.74}$$

By (2.64) for any $s\ge s_0$

$$\sum_{k=1}^n\|s^{-\frac12}\varphi^{-\frac12}\frac{\partial w}{\partial x_k}\|_{L^2(G)}+\|s^{\frac12}\varphi^{\frac12}w\|_{L^2(G)}$$
$$\le\sum_{j=-\infty}^{\infty}\left(\sum_{k=1}^n\|s^{-\frac12}\varphi^{-\frac12}\frac{\partial w_j}{\partial x_k}\|_{L^2(G)}+\|s^{\frac12}\varphi^{\frac12}w_j\|_{L^2(G)}\right) \tag{2.75}$$
$$\le C\sum_{j=-\infty}^{\infty}(\|s^{-\frac14}\varphi^{-\frac14}g_j\|_{H^{\frac14,\frac12}(\mathbb{R}^n)}+\|s^{-\frac14}\varphi^{-\frac14+\frac1\kappa}g_j\|_{L^2(\mathbb{R}^n)}$$
$$+\|(\partial_{x_0}\mu_jw+F_{0,j})/(s\varphi)\|_{L^2(G)}+\sum_{i=1}^n\|F_{i,j}\|_{L^2(G)}).$$

Finally we estimate terms in the right hand side of (2.75) using (2.67), (2.68), (2.74)

$$\sum_{j=1}^n\|s^{-\frac12}\varphi^{-\frac12}\frac{\partial w}{\partial x_j}\|_{L^2(G)}+\|s^{\frac12}\varphi^{\frac12}w\|_{L^2(G)} \tag{2.76}$$
$$\le C(\|s^{-\frac14}\varphi^{-\frac14}g\|_{H^{\frac14,\frac12}(\mathbb{R}^n)}+\|s^{-\frac14}\varphi^{-\frac14+\frac1\kappa}g\|_{L^2(\mathbb{R}^n)}$$
$$+\|\frac{w}{s\varphi^{1-\frac1\kappa}}\|_{L^2(G)}+\|F_0/(s\varphi)\|_{L^2(G)}+\sum_{i=1}^n\|F_i\|_{L^2(G)})\quad\forall s>s_0.$$

Using the definition of the functions F_i and increasing the parameter s_0 if necessary, from (2.76) we obtain (2.50). Thus in order to prove (2.50) it suffices to prove (2.64). We concentrate on proving this estimate below.

2.4 Auxiliary problem.

In the previous subsection we showed that in order to prove the Carleman estimate (2.39) it suffices to establish a countable number of Carleman estimates for slightly simpler problems. We put all these problems in the following general framework : Consider the following partial differential equation

$$L(x,D_0,D'+i|\tau|\nabla'\beta)w=\tilde f\quad\text{in } G, \tag{2.77}$$
$$w(x',0)=\tilde g, \tag{2.78}$$
$$\operatorname{supp} w\subset\Pi_{\delta,\gamma}. \tag{2.79}$$

Here $\tilde f \in H^{-\frac12,-1,\tau}(G)$, $\text{supp}\,\tilde f \subset\subset \Pi_{\delta,\gamma}$, where

$$H^{-\frac12,-1,\tau}(G) = (H_0^{\frac12,1,\tau}(G))^*, \quad \|\tilde f\|_{H^{-\frac12,-1,\tau}(G)} = \sup_{w\in H_0^{\frac12,1,\tau}(G)} \frac{|<\tilde f, w>|}{\|w\|_{H_0^{-\frac12,-1,\tau}(G)}}.$$

We take a function β such that

$$\beta \in \mathcal{U}, \tag{2.80}$$

where the set $\mathcal{U}$ is constructed in the following way : First we extend a function $\psi(x_1,\dots,x_n)$ on the set $\mathbb{R}^{n-1}\times[0,\gamma]$ up to a C^2 function in such a way that ψ is a constant outside of a ball of a sufficiently large radius and $\psi(x) < 2\|\psi\|_{C^0(\bar\Omega)}$ on $\mathbb{R}^{n-1}\times[0,\gamma]$. (Here $\|\psi\|_{C^0(\bar\Omega)}$ is a norm of the function Ψ in original coordinates.) We fix a sequence $\{x_{0,j}\}$ is such that $x_{0,j}\in \text{supp}\,\mu_j$. The a set $\mathcal{U}$ consists of the functions of the form

$$(e^{\lambda\psi} - e^{2\lambda\|\psi\|_{C^0(\Omega)}})\ell^\kappa(x_{0,j})/\tilde\ell_j^\kappa(x_0).$$

The sequence of functions $\{\tilde\ell_j\}$ is constructed in the following way. We fix sufficiently large $\hat j$ such that for all $j\ge\hat j$ $\text{supp}\,\mu_j \subset [0,\frac T8]\cup[\frac{7T}{8},T]$. There exist $0<T_0(j)<T_1(j)<T$ such that $\text{supp}\,\mu_j\subset[T_0(j),T_1(j)]$. So for $j\le\hat j$ we define $\tilde\ell_j$ to be a smooth, strictly positive function on $[0,T]$ which coincides with ℓ on the segment $[T_0(j),T_1(j)]$ and equal to some constants on $[T,+\infty)$ and $(-\infty,0]$. If $j\ge\hat j$ then $\text{supp}\,\mu_j\subset[2^{-\frac{j+1}{\kappa}},2^{-\frac{j}{\kappa}}]\cup[T-2^{-\frac{j+1}{\kappa}},T-2^{-\frac{j}{\kappa}}]$. We set $\tilde\ell = 2^{-\frac{j+2}{\kappa}}$ on the segment $[0,2^{-\frac{j+2}{\kappa}}]$ and on the segment $[2^{-\frac{j+2}{\kappa}},2^{-\frac{j+1}{\kappa}}]$ we extend $\tilde\ell$ as a linear function in such a way that the resulting function is continuous. Similarly on the segment $[T-2^{-\frac{j-1}{\kappa}},T]$ we set $\tilde\ell(x_0) = T-2^{-\frac{j-1}{\kappa}}$ and on the segment $[T-2^{-\frac{j}{\kappa}},T-2^{-\frac{j-1}{\kappa}}]$ we let $\tilde\ell$ be a linear function, such that the resulting function is continuous. Finally on the segment $[2^{-\frac{j}{\kappa}},T-2^{-\frac{j+1}{\kappa}}]$ the function $\tilde\ell$ coincides with ℓ. It is not difficult to establish the following properties for functions of the set $\mathcal{U}$.
There exists a positive constant C such that for all $\beta\in\mathcal{U}$

$$\frac{\partial\beta}{\partial x_n}(x)\ge C>0 \quad \forall x\in\Pi_{\delta,\gamma}, \quad \frac{\partial\beta}{\partial x_i}=0 \quad \forall i\in\{1,\dots,n-1\},\ \forall x\in\{x_n=0\}. \tag{2.81}$$

There exists a positive constant $\hat C$ such that

$$\text{Im}\{\overline{L}_2(x,\xi_0,\xi'-i|\tau|\nabla'\beta), L_2(x,\xi_0,\xi'+i|\tau|\nabla'\beta)\} \ge \hat C|\tau|M^2(\xi,\tau) \tag{2.82}$$

for all $(x,\xi,\tau)\in\{(x,\xi,\tau)|x\in\Pi_{\delta_0,\gamma}\ L_2(x,\xi_0,\xi'+i|\tau|\nabla'\beta)=0\}$ where $\delta_0>\delta$ is some constant independent of β.
Let us show that problem (2.61)-(2.63) can be reduced to the problem (2.77)-(2.79). We set $w=w_j, \tilde f = \partial_{x_0}\mu_j w + F_{0,j} + \sum_{i=1}^n \frac{\partial F_{i,j}}{\partial x_i}, \tilde g = g_j$,

$$\tau = s/\ell^\kappa(x_{0,j}), \quad \beta(x) = \alpha(x)\ell^\kappa(x_{0,j})\ \forall x_0\in\text{supp}\,\mu_j,\ \text{where}\ x_{0,j}\in\text{supp}\,\mu_j. \tag{2.83}$$

Since

$$\beta(x) = \alpha(x)\ell^\kappa(x_{0,j}) \quad \forall x_0\in\text{supp}\,\mu_j$$

we have

$$\begin{aligned}&L(x,D_0,D'+i|\tau|\nabla'\beta)w = L(x,D_0,D'+i|\tau|\nabla'\beta)w_j =\\ &L(x,D_0,D'+i|\tau|(\nabla'\alpha)\ell^\kappa(x_{0,j}))w_j =\\ &L(x,D_0,D'+i|s|\nabla'\alpha)w_j = \partial_{x_0}\mu_j w + F_{0,j} + \sum_{i=1}^n\frac{\partial F_{i,j}}{\partial x_i}.\end{aligned}$$

Next we claim that for solutions of system (2.77)-(2.79) the following Carleman estimate holds true: There exist a constant C and τ_0, both independent of $\beta\in\mathcal{U}$, such that

$$\sum_{j=1}^{n}\|\tau^{-\frac12}\frac{\partial w}{\partial x_j}\|_{L^2(G)}+\|\tau^{\frac12}w\|_{L^2(G)}\le C(\tau^{-\frac14}\|\tilde g\|_{H^{\frac14,\frac12}(\mathbb{R}^n)}$$
$$+\tau^{-\frac14+\frac1\kappa}\|\tilde g\|_{L^2(\mathbb{R}^n)}+\|\tilde f\|_{H^{-\frac12,-1,\tau}(G)})\quad\forall\tau\ge\tau_0. \tag{2.84}$$

Suppose for the moment that this estimate is proved already. Let us show that it implies (2.64). We know already that functions $(w_j,g_j,\partial_{x_0}\mu_jw+F_{0,j}+\sum_{i=1}^n\frac{\partial F_{i,j}}{\partial x_i})$ satisfy (2.77)-(2.79) with τ and β defined in (2.83). Making the change of unknown in (2.84) we arrive to the inequality

$$\sum_{k=1}^{n}\|(\frac{s}{\ell^\kappa(x_{0,j})})^{-\frac12}\frac{\partial w_j}{\partial x_k}\|_{L^2(G)}+\|(\frac{|s|}{\ell^\kappa(x_{0,j})})^{\frac12}w_j\|_{L^2(G)}\le C((\frac{|s|}{\ell^\kappa(x_{0,j})})^{-\frac14}\|g_j\|_{H^{\frac14,\frac12}(\mathbb{R}^n)}+$$
$$(\frac{|s|}{\ell^\kappa(x_{0,j})})^{-\frac14+\frac1\kappa}\|g_j\|_{L^2(\mathbb{R}^n)}+\|\partial_{x_0}\mu_jw+F_{0,j}+\sum_{i=1}^n\frac{\partial F_{i,j}}{\partial x_i}\|_{H^{-\frac12,-1,(\frac{s}{\ell^k(x_{0,j})})}(G)}) \tag{2.85}$$

which holds for all $\tau\ge\tau_0$.
Note that there exist two constants $C_1>0,C_2>0$ independent of s,j such that

$$C_1\frac{1}{\varphi(x_0,\dots,x_{n-1},0)}\le\frac{1}{\ell^k(x_{0,j})}\le C_2\frac{1}{\varphi(x_0,\dots,x_{n-1},0)}\qquad\forall x_0\in\operatorname{supp}\mu_j, \tag{2.86}$$

Then previous inequality can be written in the form

$$\sum_{k=1}^{n}\||s|^{-\frac12}\varphi^{-\frac12}\frac{\partial w_j}{\partial x_k}\|_{L^2(G)}+\||s|^{\frac12}\varphi^{\frac12}w_j\|_{L^2(G)}\le C(\left(\frac{|s|}{\ell^\kappa(x_{0,j})}\right)^{-\frac14}\|g_j\|_{H^{\frac14,\frac12}(\mathbb{R}^n)}+$$
$$\|(|s|\varphi)^{-\frac14+\frac1\kappa}g_j\|_{L^2(\mathbb{R}^n)}+\|(\partial_{x_0}\mu_jw+F_{0,j})/(s\varphi)\|_{L^2(G)}+\sum_{i=1}^n\|F_{i,j}\|_{L^2(G)})\quad\forall s\ge s_0. \tag{2.87}$$

The last thing we need to show is the existence of a constant $C>0$ such that

$$\left(\frac{|s|}{\ell^k(x_{0,j})}\right)^{-\frac14}\|g_j\|_{H^{\frac14,\frac12}(\mathbb{R}^n)}\le C(\|(|s|\varphi)^{-\frac14}g_j\|_{H^{\frac14,\frac12}(\mathbb{R}^n)}+\|(|s|\varphi)^{-\frac14+\frac1\kappa}g_j\|_{L^2(\mathbb{R}^n)}).$$

Since (2.86) immediately implies the inequality

$$\left(\frac{|s|}{\ell^k(x_{0,j})}\right)^{-\frac14}\|g_j\|_{H^{0,\frac12}(\mathbb{R}^n)}\le C\|(|s|\varphi)^{-\frac14}g_j\|_{H^{0,\frac12}(\mathbb{R}^n)}$$

it suffices to prove

$$\left(\frac{|s|}{\ell^k(x_{0,j})}\right)^{-\frac14}\|g_j\|_{H^{\frac14,0}(\mathbb{R}^n)}\le C(\|(|s|\varphi)^{-\frac14}g_j\|_{H^{\frac14,0}(\mathbb{R}^n)}+\|(|s|\varphi)^{-\frac14+\frac1\kappa}g_j\|_{L^2(\mathbb{R}^n)}). \tag{2.88}$$

Then, elementary computations provide that for any $x_0\in\mathcal{G}_j$

$$\begin{aligned}&|h(x_0)\varphi^{-\frac14}(x_0,\dots,x_{n-1},0)-h(y_0)\varphi^{-\frac14}(y_0,\dots,y_{n-1},0)|^2=\\&|\varphi^{-\frac14}(x_0,\dots,x_{n-1},0)(h(x_0)-h(y_0)+h(y_0)(\varphi^{-\frac14}(x_0\dots,x_{n-1},0)-\varphi^{-\frac14}(y_0\dots,y_{n-1},0))|^2\\&\ge|\varphi^{-\frac14}(x_0,\dots,x_{n-1},0)|^2|h(x_0)-h(y_0)|^2\\&-4|h(y_0)|^2|\varphi^{-\frac14}(x_0,\dots,x_{n-1},0)-\varphi^{-\frac14}(y_0,\dots,y_{n-1},0)|^2\ge\end{aligned}$$
$$C(\frac{1}{\ell^k(x_{0,j})})^{-\frac14}|h(x_0)-h(y_0)|^2-4|h(y_0)|^2|\varphi^{-\frac14}(x_0,\dots,x_{n-1},0)-\varphi^{-\frac14}(y_0,\dots,y_{n-1},0)|^2. \tag{2.89}$$

Using (2.89) and definition of $H^{\frac14}$ norm we have

$$\left(\frac{|s|}{\ell^k(x_{0,j})}\right)^{-\frac14}\|g_j\|_{H^{\frac14,0}(\mathbb{R}^n)} \le C\Bigg(\|(|s|\varphi)^{-\frac14}g_j\|_{H^{\frac14,0}(\mathbb{R}^n)}$$

$$+\sqrt{\int_{\mathbb{R}^{n-1}}\int_{\mathcal{G}_j}\int_{\mathcal{G}_j}|s|^{-\frac14}\frac{|h(y_0)|^2|\varphi^{-\frac14}(x_0,\dots,x_{n-1},0)-\varphi^{-\frac14}(y_0,\dots,y_{n-1},0)|^2}{|x_0-y_0|^{\frac32}}dx_0dy_0dx_1\dots dx_{n-1}}\Bigg)$$

$$\le C\Bigg(\|(|s|\varphi)^{-\frac14}g_j\|_{H^{\frac14,0}(\mathbb{R}^n)}$$

$$+\sqrt{\int_{\mathbb{R}^{n-1}}\int_{\mathcal{G}_j}\int_{\mathcal{G}_j}|s|^{-\frac14}|h(y_0)|^2|(\ell^{\frac{\kappa}{4}})'(\zeta)|^2|x_0-y_0|^{\frac12}dx_0dy_0dx_1\dots dx_{n-1}}\Bigg)$$

$$\le \quad C(\|(|s|\varphi)^{-\frac14}g_j\|_{H^{\frac14,0}(\mathbb{R}^n)}+\||s|^{-\frac14}\varphi^{-\frac14+\frac1\kappa}g_j\|_{L^2(\mathbb{R}^n)}).$$

Thus the estimate (2.88) is established.
In (2.77) without a loss of generality one can assume that $a_{nn}\equiv 1$. The principal symbol of operator $L(x,D,\tau)$ can be written in the form

$$L_2(x,\xi,\tau)=ia_0(x)\xi_0+\sum_{i,j=1}^n a_{ij}(x)\zeta_i\zeta_j,$$

where $\zeta=\xi+i|\tau|\nabla\beta$. Consider the equation $L_2(x,\xi,\tau)=0$. The two roots in ξ_n of this equation are

$$-i|\tau|\beta_{x_n}(x)+\lambda^{\pm}(x,\xi',\tau),$$

where

$$\lambda^{\pm}(x,\xi',\tau)=-\sum_{j=1}^{n-1}a_{nj}(x)\zeta_j\pm\sqrt{-(a(x,\zeta',\zeta')+ia_0(x)\xi_0)+(\sum_{j=1}^{n-1}a_{nj}(x)\zeta_j)^2},$$

and $\zeta'=\xi'+i|\tau|\nabla'\beta$.
Let $\mathcal{M}=\{(\xi',\tau)|\xi_0^2+\tau^4+\sum_{i=1}^{n-1}\xi_i^4=1\}$.
If $(x,\xi',\tau)\in\Phi=\{(x,\xi',\tau)\in\Pi_{\delta,\gamma}\times\mathcal{M}|-(a(x,\zeta',\zeta')+ia_0(x)\xi_0)+(\sum_{j=1}^{n-1}a_{nj}(x)\zeta_j)^2\in\mathbb{C}\setminus\mathbb{R}^1_+\}$ we assume that $\mathrm{Im}\sqrt{-(a(x,\zeta',\zeta')+ia_0(x)\xi_0)+(\sum_{j=1}^{n-1}a_{nj}(x)\zeta_j)^2}$ is positive. Therefore outside of the set Φ functions $\lambda^{\pm}$ are smooth. In order to regularize this expression for $\lambda^{\pm}(x,\xi',\tau)$ near $(\xi',\tau)=0$, we consider $\upsilon\in C^\infty(\mathbb{R}^+)$ such that

$$\upsilon(t)=0,\ \text{for } t\in[0,\frac12],$$
$$\upsilon(t)=1 \text{ for } t>1,$$
$$0\le\upsilon(t)\le 1\ \forall t\in\mathbb{R}^+$$

and determine $\tilde\lambda^{\pm}(x,\xi',\tau)$ as

$$\tilde\lambda^{\pm}(x,\xi',\tau)=-\sum_{j=1}^{n-1}a_{nj}(x)\zeta_j\pm\upsilon(M(\xi',\tau))\sqrt{-(a(x,\zeta',\zeta')+ia_0(x)\xi_0)+(\sum_{j=1}^{n-1}a_{nj}(x)\zeta_j)^2},$$

$$M(\xi',\tau) = (\xi_0^2 + \sum_{j=1}^{n-1} \xi_j^4 + \tau^4)^{\frac{1}{4}}.$$

We set

$$r_\pm(x,\xi',\tau) = |\tau|\frac{\partial \beta}{\partial x_n}(x) + i\tilde{\lambda}^\pm(x,\xi',\tau). \tag{2.90}$$

We introduce the following sets

$\Upsilon = \{(\xi',\tau)|\exists\, x \in \Pi_{\delta,\gamma}$ such that $(x,\xi',\tau) \in \Phi\}$, $\Upsilon_\epsilon = \{(\xi',\tau)|\, dist((\xi',\tau),\Upsilon) \le \epsilon\}$ and $\Upsilon^1_\epsilon = \mathcal{M}\backslash\Upsilon_{\epsilon/2}$.

We claim that one can take a parameter $\gamma > 0$ small enough such that there exists a positive $\epsilon(\gamma)$, which can be taken arbitrarily small and a pair of functions $\{\chi_0, \chi_1\}$ independent of $\beta \in \mathcal{U}$ such that

$$\chi_0, \chi_1 \in C^\infty(\mathcal{M}) \tag{2.91}$$

$$\text{supp}\ \chi_0 \subset \Upsilon_\epsilon, \quad \text{supp}\chi_1 \subset \Upsilon^1_\epsilon. \tag{2.92}$$

$$\chi_0 \ge 0,\ \chi_1 \ge 0 \text{ and } \chi_0 + \chi_1 \ge 1 \quad \text{on } \mathcal{M}. \tag{2.93}$$

$$dist(\Upsilon, (0,\dots,0,1)) \to +0 \text{ as } \gamma \to +0. \tag{2.94}$$

Really we observe that $\beta_{x_i}(x',0) = 0$ for $i \in \{1,\dots,n-1\}$ and

$$\min_{x\in\{x|x_n=0,|x'|<\delta\}} \min\{\text{Re}Z, -|\text{Im Z}|\} < 0,$$

where $Z = -4(a(x,\zeta',\zeta') + ia_0(x)\xi_0) + (\sum_{j=1}^{n-1} a_{nj}(x)\zeta_j)^2$. So if a point (ξ',τ) belongs to Υ and $M(\xi',0) > 0$ the quantity $\sum_{j=1}^{n-1}|\tau\beta_{x_j}|$ can not be closed to zero. Really if $\tau = 0$ then ζ' is a real vector. Since a_0 is a strictly positive function then $\xi_0 = 0$ but in this case inequality (2.57) implies that Z is a real, negative number. This contradicts the fact that $(\xi,\tau) \in \Upsilon$. On the other hand

$$\max_{x\in\Pi_{\delta,\gamma}} \sum_{j=1}^{n-1} |\beta_{x_j}| \to +0 \quad \text{as } \gamma \to +0.$$

Therefore (2.94) holds true and for sufficiently small positive ϵ the choice of the functions χ_i is possible.
Next we extend χ_μ to the set $\{(\xi',\tau)|M(\xi',\tau) > 1\}$ by the formula

$$\chi_\mu(\xi',\tau) = \chi_\mu(\xi_0/M^2(\xi',\tau), \xi_1/M(\xi',\tau),\dots,\xi_n/M(\xi',\tau), \tau/M(\xi',\tau))$$

and we extend functions χ_μ up to C^∞ function on $\{(\xi',\tau)|M(\xi',\tau) < 1\}$. Let $\chi_\mu(D',\tau)$ be the pseudodifferential operator with symbol $\chi_\mu(\xi',\tau)$.

Applying the operator $\chi_\mu(D',\tau)$ to the both sides of the equation (2.77) we have

$$L(x,D_0,D'+i|\tau|\nabla'\beta)w_\mu = \chi_\mu\tilde f - [\chi_\mu, L(x,D_0,D'+i|\tau|\nabla'\beta)]w = \tilde f_\mu \quad \text{in } G, \tag{2.95}$$

$$w_\mu(x',0) = (\chi_\mu\tilde g)(x') \quad x'\in \mathbb{R}^n, \tag{2.96}$$

where $w_\mu = \chi_\mu(D',\tau)w$.
Observe that by Lemma A.5

$$\|[\chi_\mu, L(x,D_0,D'+i|\tau|\nabla'\beta)]w\|_{H^{-\frac12,-1,\tau}(G)} \le C\|w\|_{H^{\frac12,1,\tau}(G)}/(1+|\tau|)^{1-\frac1\kappa} \quad \mu\in\{0,1\}. \tag{2.97}$$

2.5 Proof of the main estimate.

First we obtain an a priori estimate for the function $w_0 = \chi_0(D',\tau)w$. We claim that there exists a constant $C>0$ such that

$$\|w_0\|_{H^{\frac12,1,\tau}(G)} \le C\left(\|\tilde f\|_{H^{-\frac12,-1,\tau}(G)} + \frac{\|w\|_{H^{\frac12,1,\tau}(G)}}{(1+|\tau|)^{1-\frac1\kappa}}\right). \tag{2.98}$$

Really, using the notations $W=(W_1,W_2)$, $F=(0,i\tilde f_0)$ where $W_1 = \tilde\Lambda(D',\tau)w_0, W_2 = \frac{\partial w_0}{\partial x_n} - |\tau|\beta_{x_n}w_0$ and $\tilde\Lambda(D',\tau)w_0 = \int_{\mathbb{R}^n}(1+M(\xi',\tau))\hat w_0 e^{i<\xi',x'>}d\xi'$ we rewrite system (2.95), (2.96) in the form

$$\frac{\partial W}{\partial x_n} = K(x,D',\tau)W + F \quad \text{in } G, \tag{2.99}$$

$$W(x',\gamma) = \frac{\partial}{\partial x_n}W(x',\gamma) = 0. \tag{2.100}$$

Here we set

$$K(x,D',\tau) = |\tau|\frac{\partial\beta}{\partial x_n}I + \begin{pmatrix} 0 & \tilde\Lambda(D',\tau) + [\tilde\Lambda, |\tau|\beta_{x_n}] \\ K_{12}(x,D',\tau) & K_{22}(x,D',\tau)\end{pmatrix},$$

where

$$K_{12}(x,D',\tau) = \sum_{j,k=1}^{n-1} a_{kj}(x)(D_j + i|\tau|\beta_{x_j})(D_k+i|\tau|\beta_{x_k})\tilde\Lambda^{-1}(D',\tau) + ia_0D_0\tilde\Lambda^{-1}(D',\tau)$$

and

$$K_{22}(x,D',\tau) = -i\sum_{j=1}^{n-1} a_{jn}(x)(D_j+i|\tau|\beta_{x_j}).$$

The eigenvalues of the matrix $K(x,\xi',\tau)$ are $ir_\pm(x,\xi',\tau)$. Hence by (2.94) there exists a positive constant C, independent of $\beta\in\mathcal U$ such that

$$\mathrm{Re}(K(x,\xi',\tau)\vec v,\vec v) \ge C|\vec v|^2 \ \forall (x,\xi',\tau)\in \Pi_{\delta,\gamma}\times\Upsilon_\epsilon \text{ and } \vec v\in\mathbb{R}^2. \tag{2.101}$$

We extend the symbol $K(x,\xi',\tau)$ from $\Pi_{\delta,\gamma}\times\Upsilon_\epsilon$ on $G\times\mathbb{R}^{n+1}$ in such a way that the new symbol $\tilde K(x,\xi',\tau)\in C^1_{cl}S^{\frac12,1,\tau}(G)$ and inequality (2.101) holds true on $G\times\mathbb{R}^{n+1}$ with the constant $C/2$. Moreover the symbol $\tilde K$ is independent of x' if $|x'|$ is sufficiently large. The function W verifies

$$\frac{\partial W}{\partial x_n}=\tilde K(x,D',\tau)W+\tilde F\quad\text{in } G, \tag{2.102}$$

$$W(x',\gamma)=\frac{\partial}{\partial x_n}W(x',\gamma)=0. \tag{2.103}$$

Applying Lemma A.8 and using (2.97) we obtain (2.98).
Next we obtain an estimate for the function $w_1=\chi_1(D',\tau)w$. The symbols $r_\pm(x,\xi',\tau)$ are smooth on $\overline{\Pi_{\delta,\gamma}\times\Upsilon^1_\epsilon}$. We would like to extend symbols $r_\pm$ on the set $G\times\mathbb{R}^{n+1}$. First we observe that for some positive constant C

$$\mathrm{Re}\, r_-(x,\xi',\tau)\ge CM(\xi',\tau)\quad\forall(x,\xi',\tau)\in\Pi_{\delta,\gamma}\times\Upsilon^1_\epsilon.$$

Therefore we extend the symbol r_- on the set $G\times\mathbb{R}^{n+1}$ in such a way that $r_-\in C^1_{cl}S^{\frac12,1}(G)$, the previous inequality holds true with a constant $C/2$ on $G\times\mathbb{R}^{n+1}$ and r_- is independent of x for all $|x'|\ge\hat K$:

$$\mathrm{Re}\, r_-(x,\xi',\tau)>\frac{C}{2}M(\xi',\tau)\quad\forall(x,\xi',\tau)\in G\times\mathbb{R}^{n+1}. \tag{2.104}$$

Next we extend the symbol r_+.
Observe that

$$-\mathrm{Re}\, r_+(x,\xi',\tau)\ge CM(\xi',\tau)\quad\forall(x,\xi',\tau)\in\{(x,\xi',\tau)|x\in\Pi_{\delta,\gamma},\ (\xi',\tau)\in\partial\Upsilon^1_\epsilon\}.$$

Therefore we extend r_+ on $\Pi_{\delta',\gamma}\times(\mathbb{R}^{n+1}\setminus\Upsilon^1_\epsilon)$ in such a way that

$$-\mathrm{Re}\, r_+(x,\xi',\tau)\ge CM(\xi',\tau)\quad\forall(x,\xi',\tau)\in\{(x,\xi',\tau)|x\in\Pi_{\delta',\gamma},\ (\xi',\tau)\in\mathbb{R}^{n+1}\setminus\Upsilon^1_\epsilon\}. \tag{2.105}$$

This is possible if the difference $\delta'-\delta>0$ is small. Then in the definition of the symbol of operator λ^+ we substitute the function β by by $\beta\chi_{-1}$ where $\chi_{-1}\in C_0^\infty(B(0,\delta'))$ and $\chi_{-1}|_{B(0,\delta)}=1$. Finally we extend r_+ from $\Pi_{\delta',\gamma}\times\mathbb{R}^{n+1}$ on $G\times\mathbb{R}^{n+1}$ up to a symbol of a class $C^1_{cl}S^{\frac12,1,\tau}(G)$ in such a way that the symbol r_+ is independent of x for all x' such that $|x'|>\hat C$ and

$$-\mathrm{Re}\, r_+(x,\xi',\tau)\ge CM(\xi',\tau)\quad\forall(x,\xi',\tau)\in\{(x,\xi',\tau)||x'|>\hat C,\ (\xi',\tau)\in\mathbb{R}^{n+1}\}.$$

We denote by $R_\pm(x,D',\tau)$ the pseudodifferential operator with symbol $r_\pm(x,\xi',\tau)$, namely

$$R_\pm(x,D',\tau)u(x)=\int_{\mathbb{R}^n} r_\pm(x,\xi',\tau)\hat u(\xi',x_n)e^{i<x',\xi'>}d\xi'. \tag{2.106}$$

Since $r_\pm(x,\xi',\tau)\in C^1_{cl}S^{\frac12,1,s}(G)$, by Lemma A.3, $R_\pm\in\mathcal{L}(H^{\frac12,1,\tau}(G);L^2(G))$. Using these pseudodifferential operators $R_\pm(x,D',\tau)$, we construct two operators

$$L_-(x,D,\tau)=\frac{\partial}{\partial x_n}-R_-(x,D',\tau),\quad L_+(x,D,\tau)=\frac{\partial}{\partial x_n}-R_+(x,D',\tau).$$

Symbols of the operators $L_-(x,D,\tau)$ and $L_+(x,D,\tau)$ are

$$L_-(x,\xi,\tau)=i\xi_n-r_-(x,\xi',\tau),\quad L_+(x,\xi,\tau)=i\xi_n-r_+(x,\xi',\tau). \tag{2.107}$$

If $\operatorname{supp} \hat{w}(\xi', x_n, \tau) \subset \Upsilon^1_\epsilon$ for any $x_n \in [0,\gamma]$, the operator $L(x,D,\tau)$ can be represented in the following form

$$(2.108)\qquad L(x,D,\tau)w = (\frac{\partial}{\partial x_n} - R_-(x,D',\tau))(\frac{\partial}{\partial x_n} - R_+(x,D',\tau))w + K(x_n)w \quad x_n \in [0,\gamma],$$

where,

$$(2.109)\quad Kw \in L^\infty(0,\gamma;\mathcal{L}(H^{\frac12,1,\tau}(\mathbb{R}^n), L^2(\mathbb{R}^n))), \quad \|K\|_{\mathcal{L}(H^{\frac12,1,\tau},L^2)} \le C(1+|\tau|)^{\frac{1}{\kappa}} \quad \forall x_n \in [0,\gamma].$$

Really, observe that

$$\begin{aligned}(\frac{\partial}{\partial x_n} - R_-(x,D',\tau))(\frac{\partial}{\partial x_n} - R_+(x,D',\tau)) &= \frac{\partial^2}{\partial x_n^2} + R_-(x,D',\tau)R_+(x,D',\tau)\\ &-(R_+(x,D',\tau)+R_-(x,D',\tau))\frac{\partial}{\partial x_n} - R_{+,(x_n)}(x,D',\tau).\end{aligned}\qquad(2.110)$$

According to Lemma A.4, the operator $R_-R_+ = R + K_1$ there R is the pseudodifferential operator of the form (A.203) with symbol $r_+(x,\xi',\tau)r_-(x,\xi',\tau)$ and the operator $K_1(x_n) \in \mathcal{L}(H^{\frac12,1,\tau}(G), L^2(G))$ is such that $\sup_{x_n\in[0,\gamma]}\|K_1(x_n)\| \le C(\pi_{C^1(r_+)}, \pi_{C^1(r_-)}) \le C(1+|\tau|)^{\frac{1}{\kappa}}$. The operator $R_{+,(x_n)}$ is the pseudodifferential operator with symbol $\frac{\partial}{\partial x_n} r_+(x,\xi',\tau) \in C^0_{cl}S^{\frac12,1,\tau}(\mathbb{R}^n)$ for any x_n in $[0,\gamma]$. By Lemma A.1 this operator belongs to $L^\infty(0,\gamma;\mathcal{L}(H^{\frac12,1,\tau}(\mathbb{R}^n), L^2(\mathbb{R}^n)))$. Next observe that $(R_+(x,D',\tau)+R_-(x,D',\tau))\frac{\partial}{\partial x_n} = \sum_{j=1}^{n-1} a_{nj}(D_j + i|\tau|\beta_{x_j})\frac{\partial}{\partial x_n}$. By Lemma A.4

$$R_-(x,D',\tau)R_+(x,D',\tau) = a(x, D'+i|\tau|\nabla'\beta, D'+i|\tau|\nabla'\beta) - iD_0 + K_2(x_n)$$

where the operator $K_2(x_n) \in \mathcal{L}(H^{\frac12,1,\tau}(G), L^2(G))$ is such that $\sup_{x_n\in[0,\gamma]}\|K_2(x_n)\| \le C(\pi_{C^1(r_+)}, \pi_{C^1(r_-)}) \le C(1+|\tau|)^{\frac{1}{\kappa}}$. This proves (2.108), (2.109).
Denote the function $L_+(x,D,\tau)w_1$ as z :

$$(2.111)\qquad L_+(x,D,\tau)w_1 = z \text{ in } G.$$

Consider the initial value problem

$$(2.112)\qquad L_-(x,D,\tau)z = \tilde{f}_1 - Kw_1 \text{ in } G, \quad z(\cdot,\gamma) = 0.$$

By (2.104) and Lemma A.8 there exists a constant C independent of β such that

$$(2.113)\qquad \|z\|_{L^2(G)} \le C(\|\tilde{f}_1\|_{H^{-\frac12,-1,\tau}(G)} + \|Kw_1\|_{H^{-\frac12,-1,\tau}(G)}).$$

Now we concentrate on obtaining an a priori estimate for equation (2.111). We introduce the operators

$$(2.114)\quad Q(x,D,\tau) = \frac12(L_+(x,D,\tau) + L_+(x,D,\tau)^*),$$

$$(2.115)\quad P(x,D,\tau) = \frac12(L_+(x,D,\tau) - L_+(x,D,\tau)^*) = \frac{\partial}{\partial x_n} - \frac12(R_+(x,D',\tau) - R_+(x,D',\tau)^*).$$

Therefore

$$Q = Q^*, \quad P^* = -P.$$

Then equation (2.111) can be written in the form

$$Q(x, D', \tau)w_1 + P(x, D, \tau)w_1 = z \quad \text{ in } G.$$

Taking the L^2-norm of the left and right hand side of this equation we have:

$$\|Qw_1\|^2_{L^2(G)} + \|Pw_1\|^2_{L^2(G)} + \mathrm{Re}(Qw_1, Pw_1)_{L^2(G)} = \|z\|^2_{L^2(G)}.$$

Observe that

$$\begin{aligned} \mathrm{Re}(Qw_1, Pw_1)_{L^2(G)} = (Qw_1, Pw_1)_{L^2(G)} + (Pw_1, Qw_1)_{L^2(G)} = \\ ([Q, P]w_1, w_1)_{L^2(G)} - (Q(x', 0, D', \tau)w_1(\cdot, 0), w_1(\cdot, 0))_{L^2(\mathbb{R}^n)}. \end{aligned} \tag{2.116}$$

Therefore

$$\begin{aligned} \|Qw_1\|^2_{L^2(G)} + \|Pw_1\|^2_{L^2(G)} + ([Q, P]w_1, w_1)_{L^2(G)} \\ -(Q(x', 0, D', \tau)w_1(\cdot, 0), w_1(\cdot, 0))_{L^2(\mathbb{R}^n)} = \|z\|^2_{L^2(G)}. \end{aligned} \tag{2.117}$$

By (2.82) there exists a positive constant $\hat{C}$ such that

$$\mathrm{Re}\{Q, P\}(x, \xi', \tau) > \hat{C}M(\xi', \tau) \quad \forall (x, \xi, \tau) \in \{(x, \xi, \tau)| x \in \Pi_{\delta', \gamma},\ Q(x, \xi', \tau) = 0, |\tau| > 1\}, \tag{2.118}$$

where $\delta' > \delta$.

Proposition 2.5. *Suppose that (2.118) holds true. Let $w \in H^{\frac{1}{2},1}(G)$ and $\mathrm{supp}\, w \subset \Pi_{\beta', \gamma}$. Then there exist positive constants C_0 and C_1 such that*

$$\|Qw\|^2_{L^2(G)} + \|Pw\|^2_{L^2(G)} + Re([Q, P]w, w)_{L^2(G)} \ge C_0\|w\|^2_{H^{\frac{1}{4}, \frac{1}{2}, \tau}(G)} - C_1\|w\|^2_{L^2(G)}.$$

Proof. The pseudodifferential operators P and Q have a symbols with C^1-smoothness in variable x. We approximate these operators by pseudodifferential operators with smooth symbols. The approximations are constructed in the following way:

$$Q_\tau = \frac{1}{2}(L_{+,\tau}(x, D, \tau) + L_{+,\tau}(x, D, \tau)^*),\ P_\tau = \frac{1}{2}(L_{+,\tau}(x, D, \tau) - L_{+,\tau}(x, D, \tau)^*).$$

$$L_{+,\tau} = \frac{\partial}{\partial x_n} - R_{+,\tau}(x, D', \tau).$$

The symbol of the operator $R_{+,\tau}$ given by

$$r_{+,\tau}(x, \xi', \tau) = |\tau|\frac{\partial \beta_\tau}{\partial x_n} + i\lambda^+_\tau(x, \xi', \tau),$$

$$\lambda^+_\tau(x, \xi', \tau) = \sum_{j=1}^{n-1} a_{\tau,0j}(x)\tilde{\zeta}_j \pm v(M(\xi', \tau))\sqrt{-(a_\tau(x, \tilde{\zeta}', \tilde{\zeta}') + ia_{\tau,0}(x)\xi_0) + (\sum_{j=1}^{n-1} a_{\tau,nj}(x)\tilde{\zeta}_j)^2}.$$

Here

$$a_{\tau,0} = a_0 * \eta_{\tau^{\frac{1}{2}+\epsilon}},\ a_{\tau,ij} = a_{ij} * \eta_{\tau^{\frac{1}{2}+\epsilon}},\ \beta_\tau = \beta * \eta_{\tau^{\frac{1}{2}+\epsilon}},\ \tilde{\zeta}_j = \xi_j + i|\tau|\frac{\partial \beta_\tau}{\partial x_j} \tag{2.119}$$

where the function $\eta_{\bar{\varepsilon}}$ is the standard mollifier (see e.g. [3] p. 620), and ϵ is a positive parameter. Using the properties of mollifiers we obtain

$$\|Q-Q_\tau\|_{\mathcal{L}(H^{\frac{1}{2},1,\tau}(\Omega);L^2(\Omega))}+\|P-P_\tau\|_{\mathcal{L}(H^{\frac{1}{2},1,\tau}(\Omega);L^2(\Omega))}\le C(\pi_{C^0(Q-O_\tau)}+\pi_{C^0(P-P_\tau)})$$
$$\le \frac{C_\epsilon}{(1+|\tau|)^{\frac{1}{2}+\epsilon}} \tag{2.120}$$

and

$$(1+|\tau|)\|\tilde{\Lambda}^{-\frac{1}{2}}Q_\tau w\|^2_{L^2(G)}\le \frac{\hat{C}}{2}\|Q_\tau w\|^2_{L^2(G)}, \tag{2.121}$$

where

$$\tilde{\Lambda}^{-\frac{1}{2}}(D)w=\int_{\mathbb{R}^n}\frac{1}{(1+\xi_0^2+\tau^4+\sum_{i=1}^{n-1}\xi_i^4)^{\frac{1}{4}}}\hat{w}e^{i<x',\xi'>}d\xi'.$$

By (2.118), (2.119) there exists $\tau_0>0$ such that

$$\tau_0|Q_\tau(x,\xi',\tau)|^2/M(\xi',\tau)+Re\{Q_\tau,P_\tau\}(x,\xi,\tau)>CM(\xi',\tau). \tag{2.122}$$

Some short computations provide

$$\tau_0\|Q_\tau\tilde{\Lambda}^{-\frac{1}{2}}w\|^2_{L^2(G)}+([Q_\tau,P_\tau]w,w)_{L^2(G)}=(\tau_0(Q_\tau\tilde{\Lambda}^{-\frac{1}{2}})^*Q_\tau\tilde{\Lambda}^{-\frac{1}{2}}w,w)_{L^2(G)}+([Q_\tau,P_\tau]w,w)_{L^2(G)}=$$
$$\mathrm{Re}((\tau_0(Q_\tau\tilde{\Lambda}^{-\frac{1}{2}})^*Q_\tau\tilde{\Lambda}^{-\frac{1}{2}}+[Q_\tau,P_\tau])w,w)_{L^2(G)}.$$

By (2.122) applying Gårdings inequality

$$\tau_0\|Q_\tau\tilde{\Lambda}^{-\frac{1}{2}}w\|^2_{L^2(G)}+\mathrm{Re}([Q_\tau,P_\tau]w,w)_{L^2(G)}$$
$$\ge C\int_0^\gamma\|w(\cdot,x_n)\|^2_{H^{\frac{1}{4},\frac{1}{2},\tau}(\mathbb{R}^n)}dx_n-C_1\|w\|^2_{L^2(G)}. \tag{2.123}$$

Next we observe that

$$\|Q_\tau\tilde{\Lambda}^{-\frac{1}{2}}w\|_{L^2(G)}=\|\tilde{\Lambda}^{-\frac{1}{2}}Q_\tau w+[Q_\tau,\tilde{\Lambda}^{-\frac{1}{2}}]w\|_{L^2(G)}$$
$$\le C_0(\|\tilde{\Lambda}^{-\frac{1}{2}}Q_\tau w\|_{L^2(G)}+\|w\|_{L^2(G)})$$
$$\le C\left(\frac{\|Q_\tau w\|_{L^2(G)}}{1+|\tau|}+\|w\|_{L^2(G)}\right), \tag{2.124}$$

where in the last inequality we used the estimate (2.121). Combining this inequality and (2.123) we obtain

$$\|Q_\tau w\|^2_{L^2(G)}+\mathrm{Re}([Q_\tau,P_\tau]w,w)_{L^2(G)}\ge C\int_0^\gamma\|w(\cdot,x_n)\|^2_{H^{\frac{1}{4},\frac{1}{2},\tau}(\mathbb{R}^n)}dx_n-C_1\|w\|^2_{L^2(G)}. \tag{2.125}$$

Since there exists a constant $\tilde{C}$ independent of τ such that

$$\mathrm{Re}\ Q_\tau^2(x,\xi',\tau)+\tilde{C}|\tau|^2\ge M(\xi',\tau)$$

by Gårdings inequality

$$\|Q_\tau w\|^2_{L^2(G)} + \tau^2 \|w\|^2_{L^2(G)} \ge C\|w\|^2_{H^{\frac12,1,\tau}(G)}.$$

This inequality and (2.125) imply
(2.126)

$$\|Q_\tau w\|^2_{L^2(G)} + \mathrm{Re}([Q_\tau, P_\tau]w, w)_{L^2(G)} \ge C\int_0^\gamma \|w(\cdot, x_n)\|^2_{H^{\frac12,1,\tau}(\mathbb{R}^n)}/(1+|\tau|)dx_n - C_1\|w\|^2_{L^2(G)}.$$

On the other hand

$$\|\frac{\partial w}{\partial x_n}\|_{L^2(G)} \le \|P_\tau w\|_{L^2(G)} + C\|w\|_{H^{\frac12,1,\tau}(G)}.$$

So (2.126) can be transformed into the following estimate

$$\|P_\tau w\|^2_{L^2(G)} + \|Q_\tau w\|^2_{L^2(G)} + \mathrm{Re}([Q_\tau, P_\tau]w, w)_{L^2(G)} \ge C\|w\|^2_{H^{\frac12,1,\tau}(G)} - C_1\|w\|^2_{L^2(G)}. \tag{2.127}$$

By (2.120) we can put in the left hand side of (2.127) the L^2-norms of functions Pw and Qw instead of $P_\tau w$ and $Q_\tau w$ respectively. The proof of proposition is finished. ■
By (2.117) and Proposition 2.5 there exist two positive constants C, C_1 independent of β, τ such that

$$\begin{aligned}\|z\|^2_{L^2(G)} \ge \frac12\|Qw_1\|^2_{L^2(G)} + \frac12\|Pw_1\|^2_{L^2(G)} + C\|w_1\|^2_{H^{\frac14,\frac12,\tau}(G)} - C_1\|w_1\|^2_{L^2(G)} \\ -\mathrm{Re}(Q(x', 0, D', \tau)w_1(\cdot, 0), w_1(\cdot, 0))_{L^2(\mathbb{R}^n)}.\end{aligned} \tag{2.128}$$

Depending on the sign of the fifth term in the right hand side of (2.128) we consider two cases.

Case 1. Assume that $\mathrm{Re}(Q(x', 0, D', \tau)w_1(\cdot, 0), w_1(\cdot, 0))_{L^2(\mathbb{R}^n)} \le 0$. Then from (2.128) we obtain

$$\|w_1\|_{H^{\frac14,\frac12,\tau}(G)} \le C(\|\tilde f_1\|_{H^{-\frac12,-1,\tau}(G)} + \|w_1\|_{L^2(G)}). \tag{2.129}$$

Case 2. Assume that $\mathrm{Re}(Q(x', 0, D', \tau)w_1(\cdot, 0), w_1(\cdot, 0))_{L^2(\mathbb{R}^n)} \ge 0$. Note that by (2.81), (2.90), (2.107) there exists a constant $C > 0$ independent of τ, β such that

$$\sqrt{1+|\tau|}\|w_1(\cdot, 0)\|_{L^2(\mathbb{R}^n)} \le C\|w_1(\cdot, 0)\|_{H^{\frac14,\frac12}(\mathbb{R}^n)}. \tag{2.130}$$

Let us consider the following (adjoint) problem

$$L_+(x, D, \tau)^* p = (-\frac{\partial}{\partial x_n} - R_+(x, D', \tau)^*)p = (1+|\tau|)w_1 + v \text{ in } G. \tag{2.131}$$

Denote

$$m_\tau(x) = 1 \quad x \in \Pi_{\delta_0,\gamma}, \quad m_1(x) = 1/(1+|\tau|)\ x \in \mathbb{R}^{n+1}\setminus \Pi_{\delta_0,\gamma},$$

where $\delta_0 \in (\delta, \delta')$. We have

Lemma 2.6. *There exists a constant $C > 0$ independent of τ and there exists a pair (p, v) satisfying (2.131) such that*

$$\sqrt{1+|\tau|}\int_G (|p|^2 + m_\tau^2|v|^2)dx + \int_{\mathbb{R}^n} m_\tau^2 |p(x', 0)|^2 dx' \le C(1+|\tau|)^{\frac32}\int_G |w_1|^2 dx. \tag{2.132}$$

Proof. For $\epsilon > 0$, let us consider the functional

$$(2.133)\quad J_\epsilon(p,v) = \frac{1}{2}\|p\|^2_{L^2(G)} + \frac{1}{2}\|m_\tau v\|^2_{L^2(G)} + \frac{1}{2\epsilon}\|\frac{\partial p}{\partial x_n} + R_+(x,D',\tau)^* p + (1+|\tau|)w_1 + v\|^2_{L^2(G)}.$$

Notice that there exists a pair (p,v) such that $J_\epsilon(p,v)$ is finite, for example $(p,v) = 0$. We consider the minimization problem

$$\min_{(p,v)\in U} J_\epsilon(p,v),$$

where

$$U = \{(p,v) \in L^2(G) \times L^2(G),\ |\ \frac{\partial p}{\partial x_n} + R_+(x,D',\tau)^* p + (1+|\tau|)w_1 + v \in L^2(G)\}.$$

There exists a minimizing sequence $\{(p_k,v_k)\}_{k=1}^\infty$ such that $(p_k,v_k) \in U$ and

$$J_\epsilon(p_k,v_k) \to \inf_{(p,v)\in U} J_\epsilon(p,v).$$

Then $\|(p_k,v_k)\|_{L^2(G)}$ is bounded and $\|\frac{\partial p_k}{\partial x_n} + R_+(x,D',\tau)^* p_k + (1+|\tau|)w_1 + v_k\|_{L^2(G)}$ is bounded. Therefore $R_+(x,D',\tau)^* p_k$ is bounded in $L^2(0,\gamma;H^{-\frac{1}{2},-1,\tau}(\mathbb{R}^n))$ and $\frac{\partial p_k}{\partial x_n}$ is bounded in $L^2(0,\gamma;H^{-\frac{1}{2},-1,\tau}(\mathbb{R}^n))$.
We can then extract a subsequence, still denoted by $\{(p_k,v_k)\}_{k=1}^\infty$ such that

$$\begin{aligned}
(p_k,v_k) &\rightharpoonup (p_\epsilon,v_\epsilon) \text{ in } L^2(G)\times L^2(G) \text{ weakly},\\
\frac{\partial p_k}{\partial x_n} &\rightharpoonup \frac{\partial p_\epsilon}{\partial x_n} \text{ in } L^2(0,\gamma;H^{-1}(\mathbb{R}^n)) \text{ weakly},\\
\frac{\partial p_k}{\partial x_n} + R_+(x,D',\tau)^* p_k + (1+|\tau|)w_1 + v_k &\rightharpoonup \frac{\partial p_\epsilon}{\partial x_n} + R_+(x,D',\tau)^* p_\epsilon + (1+|\tau|)w_1 + v_\epsilon\\
&\quad \text{in } L^2(0,\gamma;H^{-\frac{1}{2},-1}(\mathbb{R}^n)) \text{ weakly}.
\end{aligned}$$

But as $\|\frac{\partial p_k}{\partial x_n} + R_+(x,D',\tau)^* p_k + (1+|\tau|)w_1 + v_k\|^2_{L^2(G)}$ stays bounded, we have

$$\begin{aligned}
\frac{\partial p_k}{\partial x_n} + R_+(x,D',\tau)^* p_k + (1+|\tau|)w_1 + v_k &\rightharpoonup \frac{\partial p_\epsilon}{\partial x_n} + R_+(x,D',\tau)^* p_\epsilon + (1+|\tau|)w_1 + v_\epsilon\\
&\quad \text{in } L^2(G) \text{ weakly}.
\end{aligned}$$

Then (p_ϵ,v_ϵ) is a minimizer of J_ϵ that is to say $(p_\epsilon,v_\epsilon) \in U$ and

$$(2.134)\qquad J_\epsilon(p_\epsilon,v_\epsilon) = \min_{(p,v)\in U} J_\epsilon(p,v).$$

Writing the first order optimality conditions we have for every $r \in H^{\frac{1}{2},1}(G)$ and for every $\tilde{r} \in L^2(G)$

$$(2.135)\qquad <\partial_p J_\epsilon(p_\epsilon,v_\epsilon), r>= 0 \quad <\partial_v J_\epsilon(p_\epsilon,v_\epsilon), \tilde{r}>= 0.$$

Let us define q_ϵ by

$$(2.136)\qquad q_\epsilon = \frac{1}{\epsilon}(\frac{\partial p_\epsilon}{\partial x_n} + R_+(x,D',\tau)^* p_\epsilon + (1+|\tau|)w_1 + v_\epsilon).$$

We obtain from (2.135), for every $r \in H^{\frac{1}{2},1}(G)$

$$\int_G p_\epsilon \overline{r} dx + \int_G q_\epsilon \overline{(\frac{\partial r}{\partial x_n} + R_+(x, D', \tau)^* r)} dx = 0 \tag{2.137}$$

and for every $\tilde{r} \in L^2(G)$

$$\int_G v_\epsilon \overline{\tilde{r}} dx + \int_G q_\epsilon \overline{\tilde{r}} dx = 0. \tag{2.138}$$

Then q_ϵ satisfies the following problem

$$L_+(x, D, \tau) q_\epsilon = \frac{\partial q_\epsilon}{\partial x_n} - R_+(x, D', \tau) q_\epsilon = p_\epsilon \text{ in } G, \tag{2.139}$$

$$q_\epsilon = -m_\tau^2 v_\epsilon \quad \text{in } G, \tag{2.140}$$

$$q_\epsilon(x', 0) = 0, \; q_\epsilon(x', \gamma) = 0, \; x' \in \mathbb{R}^n. \tag{2.141}$$

We can also write (2.139)-(2.141) as follows

$$L_+(x, D, \tau) q_\epsilon = (P + Q) q_\epsilon = p_\epsilon \text{ in } G, \tag{2.142}$$

$$q_\epsilon = -m_\tau^2 v_\epsilon \quad \text{in } G, \tag{2.143}$$

$$q_\epsilon(x', 0) = 0, \; q_\epsilon(x', \gamma) = 0, \; x' \in \mathbb{R}^n. \tag{2.144}$$

Using Proposition 2.5 and (2.143) we obtain that there exists a constant $C > 0$ such that

$$\|Q q_\epsilon\|_{L^2(G)} + \|P q_\epsilon\|_{L^2(G)} + \|q_\epsilon\|_{H^{\frac{1}{4}, \frac{1}{2}, \tau}(G)} \le C \|(p_\epsilon, m_\tau v_\epsilon)\|_{L^2(G)}. \tag{2.145}$$

Notice that $Q q_\epsilon \in L^2(G)$ implies $q_\epsilon \in L^2(0, \gamma; H^{\frac{1}{2},1}(\mathbb{R}^n))$, which from (2.139) implies $\frac{\partial q_\epsilon}{\partial x_n} \in L^2(G)$. Now from the definition (2.136) of q_ϵ, p_ϵ satisfies

$$\frac{\partial p_\epsilon}{\partial x_n} + R_+(x, D', \tau)^* p_\epsilon = \epsilon q_\epsilon - (1 + |\tau|) w_1 - v_\epsilon$$

which can be written as

$$(P - Q) p_\epsilon = \frac{\partial p_\epsilon}{\partial x_n} + R_+(x, D', \tau)^* p_\epsilon = \epsilon q_\epsilon - (1 + |\tau|) w_1 - v_\epsilon. \tag{2.146}$$

Multiplying (2.146) by q_ϵ in $L^2(G)$, we obtain (using the boundary conditions on q_ϵ)

$$-\int_G p_\epsilon \overline{L_+(x, D, \tau) q_\epsilon} dx = \epsilon \int_G |q_\epsilon|^2 dx - \int_G (1 + |\tau|) w_1 \overline{q_\epsilon} dx + \int_G m_\tau^2 v_\epsilon^2 dx$$

so

$$\int_G |p_\epsilon|^2 dx + \int_G m_\tau^2 |v_\epsilon|^2 dx + \epsilon \int_G |q_\epsilon|^2 dx = \int_G (1 + |\tau|) w_1 \overline{q_\epsilon} dx$$

and by (2.145)

$$\begin{aligned} \|(p_\epsilon, m_\tau v_\epsilon)\|_{L^2(G)}^2 &\le (\int_G (1 + |\tau|) |w_1|^2 dx)^{\frac{1}{2}} (\int_G (1 + |\tau|) |q_\epsilon|^2 dx)^{\frac{1}{2}} \\ &\le C (\int_G (1 + |\tau|) |w_1|^2 dx)^{\frac{1}{2}} \|(p_\epsilon, m_\tau v_\epsilon)\|_{L^2(G)}. \end{aligned}$$

Therefore we obtain the first estimate on (p_ϵ, v_ϵ)

$$\|(p_\epsilon, m_\tau v_\epsilon)\|_{L^2(G)} \le C\sqrt{1+|\tau|}\|w_1\|_{L^2(G)}. \tag{2.147}$$

By (2.147) there exists a subsequence $\{(p_{\epsilon_m}, v_{\epsilon_m})\}_{m=1}^\infty$ such that

$$(p_{\epsilon_m}, v_{\epsilon_m}) \to (p, v) \quad \text{in } L^2(G)\times L^2(G), \tag{2.148}$$

$$x_n^2 p_{\epsilon_m} \to x_n^2 p \quad \text{in } H^{\frac{1}{2},1,\tau}(G),$$

$$q_\epsilon \to q \quad \text{in } H^{\frac{1}{2},1}(G).$$

Using the above relations we pass to the limit in (2.142)-(2.144). The pair $(p, v, q) \in L^2(G)\times L^2(G)\times L^2(0,\gamma; H^{\frac{1}{2},1,\tau}(\mathbb{R}^n))$ satisfies the optimality system

$$L_+(x, D, \tau)^* p = (1+|\tau|)w_1 + v, \quad \text{in } G, \tag{2.149}$$

$$L_+(x, D, \tau) q = p \quad \text{in } G, \tag{2.150}$$

$$q = -m_\tau^2 v \quad \text{in } G, \tag{2.151}$$

$$q(\cdot, 0) = q(\cdot, \gamma) = 0. \tag{2.152}$$

Using Proposition 2.5 and (2.151) we have

$$\|Qq\|_{L^2(G)} + \|Pq\|_{L^2(G)} + \|q\|_{H^{\frac{1}{4},\frac{1}{2},\tau}(G)} \le C\|(p, m_\tau v)\|_{L^2(G)}. \tag{2.153}$$

The inequality (2.147) and (2.148) imply

$$\|(p, m_\tau v)\|_{L^2(G)} \le C\sqrt{1+|\tau|}\|w_1\|_{L^2(G)}. \tag{2.154}$$

Notice that

$$p(\cdot, 0) = \frac{\partial}{\partial x_n} q(\cdot, 0).$$

By (2.149), (2.150)

$$\mathfrak{G}(x, D, \tau) q = L_+(x, D, \tau)^* L_+(x, D, \tau) q = (Q^2 - P^2 + [Q, P])q = (1+|\tau|)w_1 + v \quad \text{in } G.$$

Let $\theta(x_n) \in C^\infty[0,\gamma]$, $\theta(0) = 1$, $\theta \equiv 0$ in a neighborhood of $x_n = \gamma$ and $\chi(x_0, \dots, x_{n-1}) \in C_0^\infty(B(0, \delta'))$ such that $\chi|_{B(0,\delta_0)} \equiv 1$. Denote $\tilde\chi = \chi\theta$. Then function $\tilde\chi q$ verifies

$$\mathfrak{G}(x, D, \tau)(\tilde\chi q) = (1+|\tau|)\tilde\chi w_1 + \tilde\chi v - [\tilde\chi, \mathfrak{G}]q \text{ in } G.$$

Multiplying this equation by $P(x, D, \tau)(\tilde\chi q)$ we have

$$\begin{aligned} \operatorname{Re}(\mathfrak{G}(\tilde\chi q), P(\tilde\chi q))_{L^2(G)} &= (Q^2(\tilde\chi q), P(\tilde\chi q))_{L^2(G)} + (P(\tilde\chi q), Q^2(\tilde\chi q))_{L^2(G)} \\ -(P^2(\tilde\chi q), P(\tilde\chi q))_{L^2(G)} &- (P(\tilde\chi q), P^2(\tilde\chi q))_{L^2(G)} + \operatorname{Re}([Q, P](\tilde\chi q), P(\tilde\chi q))_{L^2(G)} \\ &= (1+|\tau|)(w_1, P(\tilde\chi q))_{L^2(G)} + (\tilde\chi v - [\tilde\chi, \mathfrak{G}]q, P(\tilde\chi q))_{L^2(G)}. \end{aligned} \tag{2.155}$$

By (2.153), (2.151)

$$\|Q(\tilde\chi q)\|_{L^2(G)}+\|P(\tilde\chi q)\|_{L^2(G)}+\|q\|_{H^{\frac12,1,\tau}(G)}/\sqrt{1+|\tau|}\le C\|(p,m_\tau v)\|_{L^2(G)}. \tag{2.156}$$

Using (2.156) and Lemma A.5 we have

$$\begin{aligned}|([Q,P](\tilde\chi q),P(\tilde\chi q))_{L^2(G)}|&\le\|[Q,P](\tilde\chi q)\|_{L^2(G)}\|P(\tilde\chi q)\|_{L^2(G)}\le C\|q\|_{H^{\frac12,1,\tau}(G)}\|P(\tilde\chi q)\|_{L^2(G)}\\ &\le C\|q\|_{H^{\frac12,1,\tau}(G)}\|(p,m_\tau v)\|_{L^2(G)}\le C\sqrt{1+|\tau|}\|(p,m_\tau v)\|^2_{L^2(G)}.\end{aligned} \tag{2.157}$$

Next

$$\begin{aligned}(Q^2(\tilde\chi q),P(\tilde\chi q))_{L^2(G)}+(P(\tilde\chi q),Q^2(\tilde\chi q))_{L^2(G)}&=-(PQ^2(\tilde\chi q),\tilde\chi q)_{L^2(G)}+(Q^2P(\tilde\chi q),\tilde\chi q)_{L^2(G)}=\\ -(QPQ(\tilde\chi q),\tilde\chi q)_{L^2(G)}&+(Q^2P(\tilde\chi q),\tilde\chi q)_{L^2(G)}+([Q,P]Q(\tilde\chi q),\tilde\chi q)_{L^2(G)}\\ &=([Q,P](\tilde\chi q),Q(\tilde\chi q))_{L^2(G)}+(Q(\tilde\chi q),[Q,P]^*(\tilde\chi q))_{L^2(G)}.\end{aligned} \tag{2.158}$$

Hence, by (2.156) and Lemma A.5 we have

$$|Re(Q^2(\tilde\chi q),P(\tilde\chi q))_{L^2(G)}|\le C\|q\|_{H^{\frac12,1,\tau}(G)}\|(p,m_\tau v)\|_{L^2(G)}\le C\sqrt{1+|\tau|}\|(p,m_\tau v)\|^2_{L^2(G)}. \tag{2.159}$$

Finally

$$\begin{aligned}\mathrm{Re}(P^2(\tilde\chi q),P(\tilde\chi q))_{L^2(G)}=(P^2(\tilde\chi q),P(\tilde\chi q))_{L^2(G)}+(P(\tilde\chi q),P^2(\tilde\chi q))_{L^2(G)}&=-(P^3(\tilde\chi q),\tilde\chi q)_{L^2(G)}\\ +(P^3(\tilde\chi q),\tilde\chi q)_{L^2(G)}-\|\chi p(\cdot,0)\|^2_{L^2(\mathbb{R}^n)}&=-\|\chi p(\cdot,0)\|^2_{L^2(\mathbb{R}^n)}.\end{aligned} \tag{2.160}$$

We observe that

$$\begin{aligned}|(\tilde\chi v-[\tilde\chi,\mathfrak{G}]q,P(\tilde\chi q))_{L^2(G)}|&\le C(\|v\|_{L^2(G)}+\|Pq\|_{L^2(G)}+\|Qq\|_{L^2(G)})\|P(\tilde\chi q)\|_{L^2(G)}\\ &\le C\|(p,m_\tau v)\|^2_{L^2(G)}.\end{aligned} \tag{2.161}$$

From (2.155)-(2.161) we have

$$\sqrt{1+|\tau|}\|\chi p(\cdot,0)\|^2_{L^2(\mathbb{R}^n)}\le C(1+|\tau|)\|(p,m_\tau v)\|^2_{L^2(G)}.$$

By (2.154) the right hand side of this inequality can be estimated as

$$\|\chi p(\cdot,0)\|^2_{L^2(\mathbb{R}^n)}\le C\sqrt{1+|\tau|}\|(p,m_\tau v)\|^2_{L^2(G)}\le C(1+|\tau|)^{\frac32}\|w_1\|^2_{L^2(G)}. \tag{2.162}$$

Proof of the Lemma 2.6 is complete. ■
We remind that

$$\operatorname{supp}w(\cdot,x_n)\subset B(0,\delta)\ \forall x_n\in[0,\gamma],\ \text{and}\quad \operatorname{supp}g(\cdot,x_n)\subset B(0,\delta). \tag{2.163}$$

Then by Lemma A.6 for any $\delta_2>\delta$

$$\|\chi_1 w(\cdot,x_n)\|_{L^2(\mathbb{R}^n\setminus B(0,\delta_2))}\le\frac{C(\delta_2)}{1+|\tau|}\|w(\cdot,x_n)\|_{L^2(\mathbb{R}^n)}, \tag{2.164}$$

$$\|\chi_1 g(\cdot,x_n)\|_{L^2(\mathbb{R}^n\setminus B(0,\delta_2))}\le\frac{C(\delta_2)}{1+|\tau|}\|g(\cdot,x_n)\|_{L^2(\mathbb{R}^n)}. \tag{2.165}$$

Taking the scalar product in $L^2(G)$ of (2.131) by w_1, integrating by parts and using (2.164) we have

$$
\begin{aligned}
&(1+|\tau|)\|w_1\|^2_{L^2(G)} = (w_1, L_+(x,D,\tau)^*p - v)_{L^2(G)} = (L_+(x,D,\tau)w_1, p)_{L^2(G)} - (w_1,v)_{L^2(G)} \\
&+(\chi_1\tilde{g}, p(\cdot,0))_{L^2(\mathbb{R}^n)} = -(w_1,v)_{L^2(G)} + (\chi_1\tilde{g},\, p(\cdot,0))_{L^2(\mathbb{R}^n)} + (\chi_1 z, p)_{L^2(G)} \\
&+([L_+,\chi_1]w,p)_{L^2(G)} \le C(\|z\|_{L^2(G)}\|p\|_{L^2(G)} + \|\tilde{g}\|_{L^2(\mathbb{R}^n)}\|m_\tau p(\cdot,0)\|_{L^2(\mathbb{R}^n)} + \|w\|_{L^2(G)}\|m_\tau v\|_{L^2(G)} \\
&+\|w\|_{L^2(G)}\|p\|_{L^2(G)}).
\end{aligned}
$$

By (2.132) we obtain from this inequality

$$
\begin{aligned}
&(1+|\tau|)\|w_1\|^2_{L^2(G)} \le C(\|z\|_{L^2(G)}\|p\|_{L^2(G)} + \|\tilde{g}\|_{L^2(\mathbb{R}^n)}(1+|\tau|)^{\frac{3}{4}}\|w_1\|_{L^2(G)} + \|w\|_{L^2(G)}\|m_\tau v\|_{L^2(G)}) \\
&\le C(\|z\|_{L^2(G)}\sqrt{1+|\tau|}\|w_1\|_{L^2(G)} + \|\tilde{g}\|_{L^2(\mathbb{R}^n)}(1+|\tau|)^{\frac{3}{4}}\|w_1\|_{L^2(G)} + \sqrt{1+|\tau|}\|w_1\|^2_{L^2(G)} \\
&+\sqrt{1+|\tau|}\|w\|^2_{L^2(G)}).
\end{aligned}
$$

This inequality and (2.113) imply

$$
\begin{aligned}
\|w_1\|_{L^2(G)} &\le C\left(\frac{\|\chi_1\tilde{g}\|_{L^2(\mathbb{R}^n)}}{(1+|\tau|)^{\frac{1}{4}}} + \frac{\|z\|_{L^2(G)}}{\sqrt{1+|\tau|}} + \frac{\|w\|_{L^2(G)}}{\sqrt{1+|\tau|}}\right) \\
&\le C\left(\frac{\|\chi_1\tilde{g}\|_{L^2(\mathbb{R}^n)}}{(1+|\tau|)^{\frac{1}{4}}} + \frac{\|\tilde{f}_1\|_{H^{-\frac{1}{2},-1,\tau}(G)}}{\sqrt{1+|\tau|}} + \frac{\|w\|_{L^2(G)}}{\sqrt{1+|\tau|}}\right). \qquad (2.166)
\end{aligned}
$$

By (2.130) we have

$$
\sqrt{1+|\tau|}\|w_1\|_{L^2(G)} \le C\left(\frac{\|\chi_1\tilde{g}\|_{H^{\frac{1}{4},\frac{1}{2}}(\mathbb{R}^n)}}{(1+|\tau|)^{\frac{1}{4}}} + \|\tilde{f}_1\|_{H^{-\frac{1}{2},-1,\tau}(G)} + \|w\|_{L^2(G)}\right). \qquad (2.167)
$$

Taking into account (2.167), (2.129) and (2.98) we obtain (2.84). Really

$$
\begin{aligned}
\sqrt{1+|\tau|}\|w\|_{L^2(G)} &= \sqrt{1+|\tau|}\sum_{\mu=0}^{1}\|w_\mu\|_{L^2(G)} \\
&\le C\sum_{\mu=0}^{1}(\|\chi_\mu\tilde{f} + [\chi_\mu, L(x,D,\tau)]w\|_{H^{-\frac{1}{2},-1,\tau}(G)} + (1+|\tau|)^{-\frac{1}{4}}\|\chi_\mu\tilde{g}\|_{H^{\frac{1}{4},\frac{1}{2}}(\mathbb{R}^n)}) \\
&\le C\left(\|\tilde{f}\|_{H^{-\frac{1}{2},-1,\tau}(G)} + (1+|\tau|)^{-\frac{1}{4}}\|\tilde{g}\|_{H^{\frac{1}{4},\frac{1}{2}}(\mathbb{R}^n)} + \frac{\|w\|_{H^{\frac{1}{2},1,\tau}(G)}}{(1+|\tau|)^{1-\frac{1}{\kappa}}}\right). \qquad (2.168)
\end{aligned}
$$

Then, from energy estimate for solutions of problem (2.77)-(2.79), we have

$$
\|w\|_{H^{\frac{1}{2},1}(G)} \le C(\|g\|_{H^{\frac{1}{4},\frac{1}{2}}(\mathbb{R}^n)} + (1+|\tau|)\|w\|_{L^2(G)} + \|\tilde{f}\|_{H^{-\frac{1}{2},-1,\tau}(G)}). \qquad (2.169)
$$

From (2.168), (2.169) we obtain (2.84). The proof of Theorem 2.3 is complete.■

3 Carleman estimate for the Stokes system.

Consider the Stokes system

$$P(D)y = \frac{\partial y}{\partial x_0} - \Delta y = \nabla p + f \quad \text{in } Q, \tag{3.170}$$
$$\operatorname{div} y = 0, \quad y = 0 \text{ on } (0,T)\times\Gamma,$$
$$y(0,x) = y_0. \tag{3.171}$$

We introduce the following spaces

$$H = \{u = (u_1,\dots,u_n) \in (L^2(\Omega))^n | \operatorname{div} u = 0, (u,\nu)|_{\partial\Omega} = 0\},\ n = \{2,3\}.$$

$$V = \{u = (u_1,\dots,u_n) \in (H^1_0(\Omega))^n | \ u \in H\}.$$

The proof of the following proposition can be found in the classical book [18].

Proposition 3.1. *A)Let $y_0 \in H$ and $f \in L^2(0,T;V')$. Then there exists a unique solution y to problem (3.170)-(3.171) with $y \in L^2(0,T;V)\cap C(0,T;H)$, $\frac{\partial y}{\partial x_0} \in L^2(0,T;V')$. B)Let $y_0 \in V$ and $f \in L^2(0,T;H)$. Then there exists a solution (y,p) to problem (3.170)-(3.171) with $(y,p) \in C(0,T;V)\cap H^{1,2}(Q) \times L^2(0,T;H^1(\Omega))$ and the following a priori estimate holds*

$$\|(y,p)\|_{H^{1,2}(Q)\times L^2(Q)} \le C(\|y_0\|_V + \|f\|_{L^2(0,T;H)}). \tag{3.172}$$

The goal of this section is to prove the following Carleman estimate for solutions of problem (3.170)-(3.171).

Theorem 3.2. *Let $\kappa = 6$, $f \in L^2(0,T,H)$ and $y_0 \in V$ and $y \in L^2(0,T,V)\cap H^{1,2}(Q)$ be a solution of (3.170), (3.171). Then there exists a constant $\hat\lambda$ such that for any $\lambda > \hat\lambda$ there exist constant $C > 0$ and $\hat s$ independent of s such that*

$$\|s^{\frac12}\varphi^{\frac12}(rot\, y)e^{s\alpha}\|_{L^2(Q)} + \|s\varphi y e^{s\alpha}\|_{L^2(Q)} \le C(\|fe^{s\alpha}\|_{L^2(Q)}$$
$$+\|s^{\frac12}\varphi^{\frac12}(rot\, y)e^{s\alpha}\|_{L^2(Q_\omega)} + \|s\varphi y e^{s\alpha}\|_{L^2(Q_\omega)})\ \forall s \ge \hat s. \tag{3.173}$$

First we prove the following simple proposition

Proposition 3.3. *Let $u \in H^{1,2}(Q)\cap L^2(0,T;H^1_0(\Omega))$. Then $\frac{\partial u}{\partial\nu} \in H^{\frac14,\frac12}(\Sigma)$.*

Proof. Let $\vec n(x_1,\dots,x_n) \in C^1(\bar\Omega)$ be a smooth vector field such that $\vec n = \nu$ on $\partial\Omega$. Since the function $\sum_{i=1}^n n_i\partial_{x_i}u \in L^2(0,T;H^1(\Omega))$ then $\frac{\partial y}{\partial\nu} \in L^2(0,T;H^{\frac12}(\partial\Omega))$. In order to show that $\frac{\partial u}{\partial\nu} \in H^{\frac14}(0,T;L^2(\partial\Omega))$ we observe that using a partition of unity and a local change of variables it suffices to consider a situation when $\Omega = \{(x_1,\dots,x_n)|x_n > 0\}$ and the function u has a support in $B(0,\delta)\cap\{(x_1,\dots,x_n)|x_n \ge 0\}$. Denote by $\hat u$ the Fourier transform respect to variables $x_0,\dots,x_{n-1}$. Then

$$\|u\|^2_{H^{\frac14}(0,T;L^2(\partial\Omega))} = -\int_{\mathbb{R}^{n+1}_+}\frac{\partial}{\partial x_n}|\frac{\partial\hat u}{\partial x_n}|^2\sqrt{1+|\xi_0|}d\xi_0\dots d\xi_{n-1}dx_n$$
$$= -\int_{\mathbb{R}^{n+1}_+}(\partial^2_{x_nx_n}\hat u\overline{\partial_{x_n}\hat u} + \partial_{x_n}\hat u\partial^2_{x_nx_n}\overline{\hat u})\sqrt{1+|\xi_0|}d\xi_0\dots d\xi_{n-1}dx_n$$
$$\le \int_{\mathbb{R}^{n+1}_+}(|\partial^2_{x_nx_n}\hat u|^2 + (1+|\xi_0|)|\partial_{x_n}\hat u|^2)d\xi_0\dots d\xi_{n-1}dx_n.$$

Integrating by parts and taking into account the Dirichlet boundary conditions we have

$$\int_{\mathbb{R}^{n+1}_+}(1+|\xi_0|)|\partial_{x_n}\hat u|^2 d\xi_0\dots d\xi_{n-1}dx_n=\int_{\mathbb{R}^{n+1}_+}(1+|\xi_0|)\hat u\overline{\partial^2_{x_nx_n}\hat u}d\xi_0\dots d\xi_{n-1}dx_n.$$

Applying the Cauchy-Bynakovskii inequality we obtain

$$\int_{\mathbb{R}^{n+1}_+}(1+|\xi_0|)|\partial_{x_n}\hat u|^2 d\xi_0\dots d\xi_{n-1}dx_n\le\int_{\mathbb{R}^{n+1}_+}(|\partial^2_{x_nx_n}\hat u|^2+(1+|\xi_0|)^2|\hat u|^2)d\xi_0\dots d\xi_{n-1}dx_n. \tag{3.174}$$

Combining (3.174) and (3.174) we have

$$\|u\|_{H^{\frac14}(0,T;L^2(\partial\Omega))}\le C\|u\|_{H^{1,2}(Q)}.$$

The proof of the Proposition is finished. ■

Proof. Applying to equation (3.170) the operator rot we have

$$\frac{\partial \mathrm{rot} y}{\partial x_0}-\Delta \mathrm{rot} y=\mathrm{rot} f\quad \text{in } Q. \tag{3.175}$$

Next we apply to (3.175) the Carleman estimate (2.39). There exists $s_0>0$ such that

$$\begin{aligned} s\int_Q\varphi|\mathrm{rot}y|^2e^{2s\alpha}dx &\le C(s^{-\frac12}||\varphi^{-\frac14}\mathrm{rot}ye^{s\hat\alpha}||^2_{H^{\frac14,\frac12}(\Sigma)}\\ &+s^{-\frac12}||\varphi^{-\frac14+\frac{1}{2\kappa}}\mathrm{rot}ye^{s\hat\alpha}||^2_{L^2(\Sigma)}\\ &+\int_Q|f|^2e^{2s\alpha}dx+\int_{Q_\omega}s\varphi|\mathrm{rot}\,y|^2e^{2s\alpha}dx)\quad\forall s\ge s_0, \end{aligned} \tag{3.176}$$

where $\hat\alpha(x_0)=\alpha(x)|_{\partial\Omega}$. Since $\mathrm{div}\,y=0$ we have $\Delta y=\mathrm{rot}\mathrm{rot}y$. Setting $u=ye^{s\alpha}$ we obtain

$$e^{s\alpha}\Delta e^{-s\alpha}u=e^{s\alpha}\mathrm{rot}\mathrm{rot}y=\mathrm{rot}(e^{s\alpha}\mathrm{rot}y)+[e^{s\alpha},\mathrm{rot}]\mathrm{rot}y.$$

Notice that

$$[e^{s\alpha},\mathrm{rot}]\mathrm{rot}y(t,x)=s\frac{c(x)}{\ell(x_0)^\kappa}(e^{s\alpha}\mathrm{rot}y), \tag{3.177}$$

where $c(x)\in(C^1(\bar\Omega))^3$ is some function.

Applying the Carleman estimate for elliptic equations obtained in [12] and using (4.194) we have:

$$\sqrt{\frac{s}{\ell^\kappa(x_0)}}\|ye^{s\alpha}\|_{L^2(\Omega)}\le C(\|\mathrm{rot}ye^{s\alpha}\|_{L^2(\Omega)}+\sqrt{\frac{s}{\ell^\kappa(x_0)}}\|ye^{s\alpha}\|_{L^2(\omega)})\quad\forall s\ge s_1,\forall x_0\in[0,T], \tag{3.178}$$

where constant C and s_0 are independent of s,x_0. Therefore combining (3.176) and (3.178) we have

$$\begin{aligned} \int_Q s\varphi|\mathrm{rot}y|^2e^{2s\alpha}dx+s^2\|\varphi ye^{s\alpha}\|^2_{L^2(Q)} &\le C(s^{-\frac12}||\varphi^{-\frac14}\frac{\partial y}{\partial\nu}e^{s\hat\alpha}||^2_{H^{\frac14,\frac12}(\Sigma)}\\ &+s^{-\frac12}||\varphi^{-\frac14+\frac{1}{2\kappa}}\frac{\partial y}{\partial\nu}e^{s\hat\alpha}||^2_{L^2(\Sigma)}+\int_Q|f|^2e^{2s\alpha}dx\\ &+\int_{Q_\omega}s\varphi|\mathrm{rot}\,y|^2e^{2s\alpha}dx+s^2\|\varphi ye^{s\alpha}\|^2_{L^2(Q_\omega)}),\ \forall s\ge\max\{s_0,s_1\}. \end{aligned} \tag{3.179}$$

We need to estimate the first term in the right hand side of (3.179). Denote $(w,q) = \ell^{\frac{\kappa-2}{4}}(ye^{s\hat\alpha(x_0)}, pe^{s\hat\alpha(x_0)})$. The pair (w,q) solves the following initial/boundary value problem :

$$(3.180)\quad P(D)w = \frac{\partial w}{\partial x_0} - \Delta w = \nabla q + f\ell^{\frac{\kappa-2}{4}}e^{s\hat\alpha} + s\hat\alpha'\ell^{\frac{\kappa-2}{4}}w + ye^{s\hat\alpha}\ell'\ell^{\frac{\kappa-6}{4}}\frac{\kappa-2}{4} \quad \text{in } Q,$$
$$\operatorname{div} w = 0, \quad w = 0 \text{ on } (0,T)\times\Gamma,$$

$$(3.181)\quad w(0,\cdot) = 0$$

By Proposition 3.1 and the fact that

$$\hat\alpha(x_0) \le \alpha(x) \quad \forall x \in Q$$

we have

$$\|(w,q)\|_{H^{1,2}(Q)\times L^2(0,T;H^1(\Omega))} \le C(\|fe^{s\hat\alpha}\|_{L^2(Q)} + \|s\varphi^{\frac34+\frac{3}{2\kappa}}ye^{s\hat\alpha}\|_{L^2(Q)})$$
$$(3.182)\quad \le C(\|fe^{s\alpha}\|_{L^2(Q)} + \|s\varphi^{\frac34+\frac{3}{2\kappa}}ye^{s\alpha}\|_{L^2(Q)}).$$

Using Proposition 3.3 observe that there exists a constant $C>0$ such that

$$(3.183)\quad \|\frac{\partial w}{\partial\nu}\|_{H^{\frac14,\frac12}(\Sigma)} \le C\|w\|_{H^{1,2}(Q)}.$$

We fix $\kappa = 6$ then

$$\|s\varphi^{\frac34+\frac{3}{2\kappa}}ye^{s\alpha}\|_{L^2(Q)} = \|s\varphi ye^{s\alpha}\|_{L^2(Q)}.$$

Combining (3.183) with (3.182) we have

$$(3.184)\quad \|\frac{\partial w}{\partial\nu}\|_{H^{\frac14,\frac12}(\Sigma)} \le C(\|fe^{s\alpha}\|_{L^2(Q)} + \|s\varphi ye^{s\alpha}\|_{L^2(Q)}).$$

Hence

$$s^{-\frac12}||\varphi^{-\frac14}\frac{\partial y}{\partial\nu}e^{s\hat\alpha}||^2_{H^{\frac14,\frac12}(\Sigma)} + s^{-\frac12}||\varphi^{-\frac14+\frac{1}{2\kappa}}\frac{\partial y}{\partial\nu}e^{s\hat\alpha}||^2_{L^2(\Sigma)}$$
$$\le Cs^{-\frac12}||\varphi^{-\frac14+\frac{1}{2\kappa}}\frac{\partial y}{\partial\nu}e^{s\hat\alpha}||^2_{H^{\frac14,\frac12}(\Sigma)} \le Cs^{-\frac12}\|w\|^2_{H^{1,2}(Q)}$$
$$(3.185)\quad \le Cs^{-\frac12}(\|fe^{s\alpha}\|^2_{L^2(Q)} + \|s\varphi ye^{s\alpha}\|^2_{L^2(Q)}).$$

The first two terms in the right hand side of (3.179) can be estimated by the right hand side of (3.185). Observe that the term $Cs^{-\frac12}\|s\varphi ye^{s\alpha}\|^2_{L^2(Q)}$ can be absorbed by the left hand side of (3.179) for all sufficiently large s. This proves the statement of the theorem.■

4 Observability estimate for parabolic system with parameter.

Consider the system of parabolic equations

$$(4.186)\quad P(D)y = \frac{\partial y}{\partial x_0} - \Delta y - \frac{1}{\varepsilon}\nabla \mathrm{div} y = f \quad \text{in } Q,$$

$$(4.187)\quad y = 0 \text{ on } (0,T)\times\Gamma,$$

$$(4.188)\quad y(0,x) = y_0.$$

Here ε is the positive parameter. The goal of this section is to obtain an observability estimate for system (4.186)-(4.188) which is uniform with respect to the small parameter ϵ.
We have

Theorem 4.1. *Let $\kappa = 6$, $f \in L^2(Q)$ and $y_0 \in H_0^1(\Omega)$ and $y \in L^2(0,T;H_0^1(\Omega)) \cap H^{1,2}(Q)$ be a solution of (4.186), (4.188). Then there exists a constant $\hat\lambda$ such that for any $\lambda > \hat\lambda$ there exist constant $C > 0$ and $\hat s$ independent of s such that*

$$\|s^{\frac12}\varphi^{\frac12}(div\, y)e^{s\alpha}\|_{L^2(Q)} + \|s^{\frac12}\varphi^{\frac12}(rot\, y)e^{s\alpha}\|_{L^2(Q)} + \|s\varphi y e^{s\alpha}\|_{L^2(Q)} \le C(\|fe^{s\alpha}\|_{L^2(Q)}$$
$$+\|s^{\frac12}\varphi^{\frac12}(div\, y)e^{s\alpha}\|_{L^2(Q_\omega)} + \|s^{\frac12}\varphi^{\frac12}(rot\, y)e^{s\alpha}\|_{L^2(Q_\omega)} + \|s\varphi y e^{s\alpha}\|_{L^2(Q_\omega)})\ \forall s \ge \hat s. \tag{4.189}$$

Proof. Applying to equation (4.186) the operators *rot* and *div* we have

$$\frac{\partial \mathrm{rot} y}{\partial x_0} - \Delta \mathrm{rot} y = \mathrm{rot} f \quad \text{in } Q. \tag{4.190}$$

$$\frac{\partial \mathrm{div} y}{\partial x_0} - (1 + \frac{1}{\epsilon})\Delta \mathrm{div} y = \mathrm{div} f \quad \text{in } Q. \tag{4.191}$$

Next we apply to (4.190) and to (4.191) the Carleman estimate (2.39). There exists $s_0 > 0$ such that

$$\begin{aligned} s\int_Q \varphi|\mathrm{rot} y|^2 e^{2s\alpha}dx \le C(s^{-\frac12}||\varphi^{-\frac14}\mathrm{rot} y e^{s\hat\alpha}||^2_{H^{\frac14,\frac12}(\Sigma)} \\ +s^{-\frac12}||\varphi^{-\frac14+\frac{1}{2\kappa}}\mathrm{rot} y e^{s\hat\alpha}||^2_{L^2(\Sigma)} \\ + \int_Q |f|^2 e^{2s\alpha}dx + \int_{Q_\omega} s\varphi|\mathrm{rot}\, y|^2 e^{2s\alpha}dx) \quad \forall s \ge s_0, \end{aligned} \tag{4.192}$$

$$\begin{aligned} s\int_Q \varphi|\mathrm{div} y|^2 e^{2s\alpha}dx \le C(s^{-\frac12}||\varphi^{-\frac14}\mathrm{div} y e^{s\hat\alpha}||^2_{H^{\frac14,\frac12}(\Sigma)} \\ +s^{-\frac12}||\varphi^{-\frac14+\frac{1}{2\kappa}}\mathrm{rot} y e^{s\hat\alpha}||^2_{L^2(\Sigma)} \\ + \int_Q |f|^2 e^{2s\alpha}dx + \int_{Q_\omega} s\varphi|\mathrm{div}\, y|^2 e^{2s\alpha}dx) \quad \forall s \ge s_0, \end{aligned} \tag{4.193}$$

where $\hat\alpha(x_0) = \alpha(x)|_{\partial\Omega}$. Using the formula $\Delta y = \mathrm{rot}\mathrm{rot} y + \nabla \mathrm{div}$ and setting $u = ye^{s\alpha}$ we obtain

$$e^{s\alpha}\Delta e^{-s\alpha}u = e^{s\alpha}(\mathrm{rot}\mathrm{rot} y + \nabla \mathrm{div} y) = \mathrm{rot}(e^{s\alpha}\mathrm{rot} y) + \nabla(e^{s\alpha}\mathrm{div} y) + [e^{s\alpha}, \mathrm{rot}]\mathrm{rot} y + [e^{s\alpha}, \nabla]\mathrm{div} y.$$

Notice that

$$[e^{s\alpha}, \mathrm{rot}]\mathrm{rot} y(t,x) = s\frac{c(x)}{\ell(x_0)^\kappa}(e^{s\alpha}\mathrm{rot} y), \tag{4.194}$$

where $c(x) \in (C^1(\bar\Omega))^3$ is some function.
Applying the Carleman estimate for elliptic equations obtained in [12] and using (4.194) we have $\forall s \ge s_1$ and $\forall x_0 \in [0,T]$:

$$\sqrt{\frac{s}{\ell^\kappa(x_0)}}\|ye^{s\alpha}\|_{L^2(\Omega)} \le C(\|\mathrm{rot} y e^{s\alpha}\|_{L^2(\Omega)} + \|\mathrm{div} y e^{s\alpha}\|_{L^2(\Omega)} + \sqrt{\frac{s}{\ell^\kappa(x_0)}}\|ye^{s\alpha}\|_{L^2(\omega)}) \tag{4.195}$$

where constant C and s_0 are independent of s, x_0. Therefore combining (4.192), (4.193) and (4.195) we have

$$\int_Q s\varphi(|\mathrm{rot} y|^2 + |\mathrm{div} y|^2)e^{2s\alpha}dx + s^2\|\varphi y e^{s\alpha}\|^2_{L^2(Q)} \le C(s^{-\frac{1}{2}}||\varphi^{-\frac{1}{4}}\frac{\partial y}{\partial \nu}e^{s\hat\alpha}||^2_{H^{\frac{1}{4},\frac{1}{2}}(\Sigma)}$$
$$+s^{-\frac{1}{2}}||\varphi^{-\frac{1}{4}+\frac{1}{2\kappa}}\frac{\partial y}{\partial \nu}e^{s\hat\alpha}||^2_{L^2(\Sigma)} + \int_Q |f|^2e^{2s\alpha}dx$$
$$+\int_{Q_\omega} s\varphi|\mathrm{rot}\, y|^2e^{2s\alpha}dx + \int_{Q_\omega} s\varphi|\mathrm{div}\, y|^2e^{2s\alpha}dx + s^2\|\varphi y e^{s\alpha}\|^2_{L^2(Q_\omega)})\ \forall s \ge \max\{s_0, s_1\}. \tag{4.196}$$

We need to estimate the first term in the right hand side of (4.196). Denote $w = \ell^{\frac{\kappa-2}{4}} y e^{s\hat\alpha(x_0)}$. The function w solves the following initial/boundary value problem:

$$P(D)w = \frac{\partial w}{\partial x_0} - \Delta w - \frac{1}{\varepsilon}\nabla \mathrm{div} w = f\ell^{\frac{\kappa-2}{4}}e^{s\hat\alpha} + s\hat\alpha'\ell^{\frac{\kappa-2}{4}}w + ye^{s\hat\alpha}\ell'\ell^{\frac{\kappa-6}{4}}\frac{\kappa-2}{4} \quad \text{in } Q, \tag{4.197}$$
$$\mathrm{div}\, w = 0, \quad w = 0 \text{ on } (0,T)\times\Gamma,$$
$$w(0,\cdot) = 0. \tag{4.198}$$

Using the standard a priori estimates for a parabolic equations and the fact that

$$\hat\alpha(x_0) \le \alpha(x) \quad \forall x \in Q$$

we have

$$\|w\|_{H^{1,2}(Q)} \le C(\|fe^{s\hat\alpha}\|_{L^2(Q)} + \|s\varphi^{\frac{3}{4}+\frac{3}{2\kappa}}ye^{s\hat\alpha}\|_{L^2(Q)})$$
$$\le C(\|fe^{s\alpha}\|_{L^2(Q)} + \|s\varphi^{\frac{3}{4}+\frac{3}{2\kappa}}ye^{s\alpha}\|_{L^2(Q)}). \tag{4.199}$$

Using Proposition 3.3 observe that there exists a constant $C > 0$ such that

$$\|\frac{\partial w}{\partial \nu}\|_{H^{\frac{1}{4},\frac{1}{2}}(\Sigma)} \le C\|w\|_{H^{1,2}(Q)}. \tag{4.200}$$

We fix $\kappa = 6$ then

$$\|s\varphi^{\frac{3}{4}+\frac{3}{2\kappa}}ye^{s\alpha}\|_{L^2(Q)} = \|s\varphi ye^{s\alpha}\|_{L^2(Q)}.$$

Combining (4.200) with (4.199) we have

$$\|\frac{\partial w}{\partial \nu}\|_{H^{\frac{1}{4},\frac{1}{2}}(\Sigma)} \le C(\|fe^{s\alpha}\|_{L^2(Q)} + \|s\varphi ye^{s\alpha}\|_{L^2(Q)}). \tag{4.201}$$

Hence

$$s^{-\frac{1}{2}}||\varphi^{-\frac{1}{4}}\frac{\partial y}{\partial \nu}e^{s\hat\alpha}||^2_{H^{\frac{1}{4},\frac{1}{2}}(\Sigma)} + s^{-\frac{1}{2}}||\varphi^{-\frac{1}{4}+\frac{1}{2\kappa}}\frac{\partial y}{\partial \nu}e^{s\hat\alpha}||^2_{L^2(\Sigma)}$$
$$\le Cs^{-\frac{1}{2}}||\varphi^{-\frac{1}{4}+\frac{1}{2\kappa}}\frac{\partial y}{\partial \nu}e^{s\hat\alpha}||^2_{H^{\frac{1}{4},\frac{1}{2}}(\Sigma)} \le Cs^{-\frac{1}{2}}\|w\|^2_{H^{1,2}(Q)}$$
$$\le Cs^{-\frac{1}{2}}(\|fe^{s\alpha}\|^2_{L^2(Q)} + \|s\varphi ye^{s\alpha}\|^2_{L^2(Q)}). \tag{4.202}$$

The first two terms in the right hand side of (4.196) can be estimated by the right hand side of (4.202). Observe that the term $Cs^{-\frac{1}{2}}\|s\varphi ye^{s\alpha}\|^2_{L^2(Q)}$ can be absorbed be the left hand side of (4.196) for all sufficiently large s. This proves the statement of the theorem.■

A Calculus for pseudodifferential operators with a parameter.

Let $\mathcal{O}$ be a domain in $\mathbb{R}^n$.
Definition. *We say that the symbol* $a(x',\xi',s)\in C^0(\bar{\mathcal{O}}\times\mathbb{R}^{n+1})$ *belongs to the class* $C^k_{cl}S^{\kappa/2,\kappa,s}(\mathcal{O})$ *if*

A) *There exists a compact set* $K\subset\subset\mathcal{O}$ *such that* $a(x',\xi',s)|_{\mathcal{O}\setminus K}=0$;
B) *For any* $\beta=(\beta_0,\dots,\beta_n)$ *there exists a constant* C_β

$$\Vert\frac{\partial^{\beta_0}}{\partial\xi_0^{\beta_0}}\cdots\frac{\partial^{\beta_{n-1}}}{\partial\xi_{n-1}^{\beta_{n-1}}}\frac{\partial^{\beta_n}}{\partial s^{\beta_n}}a(\cdot,\xi',s)\Vert_{C^k(\bar{\mathcal{O}})}\le C_\beta(|\xi_0|+s^2+\sum_{i=1}^{n-1}\xi_i^2)^{\frac{\kappa-|\beta|}{2}}\quad ,$$

where $|\beta|=2\beta_0+\sum_{j=1}^n\beta_j$ *and* $M(\xi',s)\ge 1$;
C) *For any* $N\in\mathbb{N}_+$ *the symbol* a *can be represented as*

$$a(x',\xi',s)=\sum_{j=1}^N a_j(x',\xi',s)+R_N(x',\xi',s)$$

where functions a_j *have the following properties*

$$a_j(x',\tau^2\xi_0,\tau\xi_1,\dots,\tau\xi_{n-1},\tau s)=\tau^{\kappa-j}a_j(x',\xi',s)\quad\forall\tau>1,\ \forall(x',\xi',s)\in\{(x',\xi',s)|x'\in K,\ M(\xi',s)>1\}$$

$$\Vert\frac{\partial_0^\beta}{\partial\xi_0^{\beta_0}}\cdots\frac{\partial_{n-1}^\beta}{\partial\xi_{n-1}^{\beta_{n-1}}}\frac{\partial_n^\beta}{\partial s^{\beta_n}}a_j(\cdot,\xi',s)\Vert_{C^k(\bar{\mathcal{O}})}\le C_\beta(|\xi_0|+s^2+\sum_{i=1}^{n-1}\xi_i^2)^{\frac{\kappa-j-|\beta|}{2}},\ \forall\beta\ \textit{and}\ \forall(\xi',s)\ \textit{such that}\ M(\xi',s)\ge 1$$

and the term R_N *satisfies the estimate*

$$\Vert R_N(\cdot,\xi',s)\Vert_{C^k(\bar{\mathcal{O}})}\le C_N(|\xi_0|+s^2+\sum_{i=1}^{n-1}\xi_i^2)^{\frac{\kappa-N}{2}}\quad\forall(\xi,s)\ \textit{such that}\ M(\xi',s)\ge 1.$$

For the symbol a we introduce the following seminorm

$$\pi_{C^k}(a)=\sum_{j=1}^{\hat N}\sup_{|\beta|\le\hat N}\sup_{M(\xi',s)\ge 1}\Vert\frac{\partial^{\beta_0}}{\partial\xi_0^{\beta_0}}\cdots\frac{\partial^{\beta_{n-1}}}{\partial\xi_{n-1}^{\beta_{n-1}}}\frac{\partial^{\beta_n}}{\partial s^{\beta_n}}a_j(\cdot,\xi',s)\Vert_{C^k(\bar{\mathcal{O}})}/(1+M(\xi',s))^{\kappa-j-|\beta|}$$
$$+\sup_{M(\xi',s)\le 1}\Vert a(\cdot,\xi',s)\Vert_{C^\kappa(\bar{\mathcal{O}})}$$

Let $\{\omega_j\}_{j=1}^\infty$ be a sequence of eigenfunctions of the operator Δ on $\mathcal{M}=\{(\xi_0,\dots,\xi_n)|\xi_0^2+\sum_{j=1}^n\xi_i^4=1\}$ and $\{\lambda_j\}_{j=1}^\infty$ be a sequence of corresponding eigenvalues. Assume that

$$(\omega_i,\omega_j)_{L^2(\mathcal{M})}=\delta_{i,j}.$$

The following asymptotic formula is established in [8]

$$\lambda_j=cj^{\frac{n}{2}}+O(j^{(n-1)/2})\quad\text{as } j\to+\infty.$$

For each k thanks to the standard elliptic estimate for the Laplace operator we have

$$\Vert\omega_j\Vert_{H^{2k}(\mathcal{M})}\le C_k\lambda_j^k. \tag{A.203}$$

Therefore by the Sobolev embedding theorem

$$\|\omega_j\|_{C^0(\mathcal{M})} \le C\lambda_j^n \quad \forall j \in \{1, \dots, \infty\}. \tag{A.204}$$

We extend the function ω_j on the set $\{\xi | \xi_0^2 + \sum_{i=1}^n \xi_i^4 \le 1\}$ as a smooth function and we set

$$\omega_j(\xi) = \omega_j(\xi_0/M^2(\xi), \xi_1/M(\xi), \dots, \xi_n/M(\xi)) \quad .$$

We introduce the pseudodifferential operator

$$\tilde{\omega}_j(D)w = \int_{\mathbb{R}^{n+1}} \omega_j(\xi)\hat{w}(\xi)e^{i<x,\xi>}d\xi, \quad \hat{w}(\xi) = \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n} w(x)e^{-i<\xi,x>}dx.$$

Below in order to distinguish the Fourier transforms respect to different variables we will use the following notations

$$F_{x'\to\xi'}u = \frac{1}{(2\pi)^{\frac{n-1}{2}}}\int_{\mathbb{R}^{n-1}} e^{-i\sum_{j=1}^{n-1}x_j\xi_j}u(x_1, \dots, x_{n-1})dx',$$

$$F_{x_n\to\xi_n}u = \frac{1}{(2\pi)^{\frac{1}{2}}}\int_{\mathbb{R}} e^{-ix_n\xi_n}u(x_n)dx_n.$$

First we define the operator $A(x'D', s)$ for functions in $C_0^\infty(\mathcal{O})$:

$$A(x', D', s)u = \int_{\mathbb{R}^{n-1}} a(x', \xi', s)F_{x'\to\xi'}ue^{i\sum_{j=1}^{n-1}x_j\xi_j}dx'.$$

The following lemma allows us to extend the definition of the operator A on Sobolev spaces.

Lemma A.1. *Let $a(x', \xi_0, ..., \xi_{n-1}, s) \in C^0_{cl}S^{\frac{1}{2},1,s}(\mathcal{O})$. Then $A \in \mathcal{L}(H_0^{\frac{1}{2},1,s}(\mathcal{O}); L^2(\mathcal{O}))$ and $\|A\|_{\mathcal{L}(H_0^{\frac{1}{2},1,s}(\mathcal{O});L^2(\mathcal{O}))} \le C(\pi_{C^0}(a))$.*

Proof. Thanks to the assumption **C** it suffices to consider the case when

$$a(x', \tau^2\xi_0, \tau\xi_1, \dots, \tau\xi_{n-1}, \tau s) = \tau a(x', \xi_0, \dots, \xi_{n-1}, s) \quad \forall \tau > 1. \tag{A.205}$$

The operator

$$\tilde{A}(x', D)v = \int_{\{\xi_0^2+\sum_{i=1}^n \xi_i^4 \le 1\}} a(x', \xi_0, \dots, \xi_n)\hat{v}(\xi)e^{i<x,\xi>}d\xi$$

is a continuous operator from $L^2(\mathcal{O}\times\mathbb{R})$ into $L^2(\mathcal{O}\times\mathbb{R})$ with a norm estimated as

$$\|\tilde{A}(x', D)\| \le C(\pi_{C^0}(a)).$$

Consider the symbol $b(x', \xi_0, \dots, \xi_n) = a(x', \xi_0, \dots, \xi_n)/(\xi_0^2 + \sum_{j=1}^n \xi_i^4)^{\frac{1}{4}}$. Then by (A.205)

$$b(x', \tau^2\xi_0, \tau\xi_1, \dots, \tau\xi_n) = b(x', \xi_0, \xi_1, \dots, \xi_n) \quad \forall \tau \ge 1.$$

We can represent the symbol b as

$$b(x', \xi) = \sum_{j=1}^\infty b_j(x')\omega_j(\xi_0/M^2(\xi), \xi_1/M(\xi), \dots, \xi_n/M(\xi)), \quad b_j(x') = (b(x', \xi), \omega_j(\xi))_{L^2(\mathcal{M})}.$$

Observe that $b_j(x') = (\Delta_\xi^k b(x',\xi), \omega_j(\xi))_{L^2(\mathcal{M})}/\lambda_j^k$. So

$$\|b_j\|_{C^0(\bar{\mathcal{O}})} \le C_m \lambda_j^{-m} \quad \forall m \in \{1,\dots,\infty\}. \tag{A.206}$$

By (A.204) and (A.206)

$$\|B(x',D)v\|_{L^2(\mathcal{O}\times\mathbb{R})} \le \sum_{j=1}^{\infty} \|b_j\|_{C^0(\bar{\mathcal{O}})} \|\tilde{\omega}_j(D)\| \|v\|_{L^2(\mathcal{O}\times\mathbb{R})} \le \sum_{j=1}^{\infty} C_m \lambda_j^{-m}\lambda_j^n \|v\|_{L^2(\mathcal{O}\times\mathbb{R})}.$$

Taking $m = 3n$ we have

$$\|B(x',D)v\|_{L^2(\mathcal{O}\times\mathbb{R})} \le \sum_{j=1}^{\infty} C_m \lambda_j^{-2n} \|v\|_{L^2(\mathcal{O}\times\mathbb{R})}.$$

Therefore the operator

$$A^b(x',D)v = \int_{\{\xi_0^2 + \sum_{i=1}^n \xi_i^4 \ge 1\}} a(x',\xi)\hat{v}(\xi) e^{i<x,\xi>} d\xi$$

is a continuous operator from $H_0^{\frac{1}{2},1}(\mathcal{O}\times\mathbb{R})$ into $L^2(\mathcal{O}\times\mathbb{R})$ with the norm satisfying the estimate

$$\|A^b\| \le \sum_{j=1}^{\infty} C_m \lambda_j^{-2n}.$$

Next we observe that for the function $v(x) = u(x_0,\dots,x_{n-1})w(x_n)$

$$\begin{aligned}\|\tilde{A}(x',D)v\|_{L^2(\mathcal{O}\times\mathbb{R})} &= \sqrt{2\pi}\|A(x',D_0,\dots,D_{n-1},\xi_n)uF_{x_n\to\xi_n}w\|_{L^2(\mathcal{O}\times\mathbb{R})} \\ &\le C(\pi_{C^0}(a))\left(\int_{-\infty}^{\infty} \|u\|^2_{H^{\frac{1}{2},1,\xi_n}(\mathcal{O})} |F_{x_n\to\xi_n}w|^2 d\xi_n\right)^{\frac{1}{2}}.\end{aligned} \tag{A.207}$$

We take a sequence $\{w_j(x_n)\}_1^\infty$ such that $F_{x_n\to\xi_n}w_j(\xi_n)$ has a compact support and $|F_{x_n\to\xi_n}w_j|^2 \to \delta(\xi_n - s)$ where $s\in\mathbb{R}$ be an arbitrary point. Since the function $\xi_n \to \|A(x',D',\xi_n)u\|_{L^2(\mathcal{O})}$ is continuous we have

$$\|A(x',D_0,\dots,D_{n-1},\xi_n)u\hat{w}\|^2_{L^2(\mathcal{O}\times\mathbb{R})} =$$
$$\int_{\mathbb{R}} \|A(x',D_0,\dots,D_{n-1},\xi_n)u\|^2_{L^2(\mathcal{O})} |F_{x_n\to\xi_n}w_j|^2 d\xi_n \to \|A(x',D_0,\dots,D_{n-1},s)u\|^2_{L^2(\mathcal{O})}.$$

This fact and (A.207) implies

$$\|A(x',D',s)u\|_{L^2(\mathcal{O})} \le C(\pi_{C^0}(a))\|u\|_{H^{\frac{1}{2},1,s}(\mathcal{O})}$$

for almost all s. Since the norm of the operator A is a continuous function of s we have this inequality for all s. ■

The following theorem provides an estimate for a commutator of a Lipschitz function and the pseudodifferential operator $\tilde{\omega}_j$.

Theorem A.2. *Let $f \in W^1_\infty(\mathcal{O})$ be a function with compact support then*

$$\|[f,\tilde{\omega}_j]\|_{\mathcal{L}(L^2(\mathcal{O}),H^{\frac{1}{2},1,s}(\mathcal{O}))} \le C\|f\|_{W^1_\infty(\mathcal{O})}\lambda_j^{4n},$$

where the constant C is independent of j.

The proof of this theorem is similar to the proof of Corollary in [15], page 309.

Lemma A.3. *Let $a(x',\xi_0,...,\xi_{n-1},s)\in C^1_{cl}S^{1/2,1,s}(\mathcal{O})$. Then $A(x',D',s)^*=A^*(x',D',s)+R$, where A^* is the pseudodifferential operator with symbol $\overline{a(x',\xi_0,\dots,\xi_{n-1},s)}$ and $R\in\mathcal{L}(L^2(\mathcal{O}),L^2(\mathcal{O}))$ is such that*

$$\|R\|_{\mathcal{L}(L^2(\mathcal{O}),L^2(\mathcal{O}))}\le C\pi_{C^1}(a).$$

Proof. Thanks to the assumption **C** it suffices to consider the case when

$$a(x',\tau^2\xi_0,\tau\xi_1,\dots,\tau\xi_{n-1},\tau s)=\tau a(x',\xi_0,\dots,\xi_{n-1},s)\quad\forall\tau>1.$$

The symbol $a(x',\xi)$ can be represented as

$$a(x',\xi)=\sum_{j=1}^\infty a_j(x')M(\xi)\tilde\omega_j(\xi).$$

Consider the operator

$$\tilde A(x',D)=\sum_{j=1}^\infty a_j(x')M(D)\tilde\omega_j(D),\quad M(D)w=\int_{\mathbb{R}^{n+1}}M(\xi)\hat w e^{i<x,\xi>}d\xi.$$

Then

$$\tilde A(x',D)^*=\sum_{j=1}^\infty (a_j(x')M(D)\tilde\omega_j(D))^*=\sum_{j=1}^\infty M(D)\tilde\omega_j(D)\overline{a_j(x')}$$
$$=\sum_{j=1}^\infty \overline{a_j(x')}M(D)\tilde\omega_j(D)+\sum_{j=1}^\infty[M(D)\tilde\omega_j(D),\overline{a_j(x')}].$$

Observe that $\sum_{j=1}^\infty\overline{a_j(x')}M(D)\tilde\omega_j(D)$ is the operator with symbol $\overline{a(x',\xi_0,\dots,\xi_n)}\in C^1_{cl}S^{\frac12,1,s}(\mathcal{O})$. Let us estimate the norm of the operator $\sum_{j=1}^\infty[M(D)\tilde\omega_j(D),\overline{a_j(x')}]$. By Theorem A.2

$$\|\sum_{j=1}^\infty[\overline{a_j(x')},M(D)\tilde\omega_j(D)]\|\le C_m\sum_{j=1}^\infty\|a_j\|_{C^1(\bar{\mathcal{O}})}\lambda_j^{\tilde\kappa(n)}\le C_m\sum_{j=1}^\infty\lambda_j^{\tilde\kappa(n)}\pi_{C^1}(a)\lambda_j^{-m}\le C\pi_{C^1}(a).$$

Denote $v=u(x_0,\dots,x_{n-1})w(x_n),\tilde v=\tilde u(x_0,\dots,x_{n-1})\tilde w(x_n)$. We have

$$(\tilde A(x',D)v,\tilde v)_{L^2(\mathcal{O}\times\mathbb{R})}=(v,\tilde A(x',D)^*\tilde v)_{L^2(\mathcal{O}\times\mathbb{R})}=(v,\tilde A^*(x',D)\tilde v)_{L^2(\mathcal{O}\times\mathbb{R})}+(v,R\tilde v)_{L^2(\mathcal{O}\times\mathbb{R})}.$$

On the other hand

$$(\tilde A(x',D)v,\tilde v)_{L^2(\mathcal{O}\times\mathbb{R})}=2\pi\int_{\mathbb{R}}(A(x',D',\xi_n)u,\tilde u)_{L^2(\mathcal{O})}w\bar{\tilde w}d\xi_n=2\pi\int_{\mathbb{R}}(u,A(x',D',\xi_n)^*\tilde u)_{L^2(\mathcal{O})}w\bar{\tilde w}d\xi_n.$$

Taking into account that $(v,A^*(x',D)\tilde v)_{L^2(\mathcal{O}\times\mathbb{R})}=\int_{\mathbb{R}}(u,A^*(x',D',\xi_n)\tilde u)_{L^2(\mathcal{O})}w\bar{\tilde w}d\xi_n$ we have

$$|\int_{\mathbb{R}}(u,(A(x',D',\xi_n)^*-A^*(x',D',\xi_n))\tilde u)_{L^2(\mathcal{O})}w\bar{\tilde w}d\xi_n|=|(v,R\tilde v)_{L^2(\mathcal{O}\times\mathbb{R})}|\le C\|v\|_{L^2(\mathcal{O}\times\mathbb{R})}\|\tilde v\|_{L^2(\mathcal{O}\times\mathbb{R})}.$$

We take a sequence $\{w_j\}_{j=1}^{\infty}$ such that $F_{x_n\to\xi_n}w_j$ has a compact support and $|F_{x_n\to\xi_n}w_j|^2\to\delta(\xi_n-s)$ where $s\in\mathbb{R}$ is an arbitrary point. Since the function $\xi_n\to\|A(x',D',\xi_n)u\|_{L^2(\mathcal{O})}$ is continuous we have

$$|\int_{\mathbb{R}}(u,(A(x',D',\xi_n)^*-A^*(x',D',\xi_n))\tilde u)_{L^2(\mathcal{O})}|w_j|^2d\xi_n|\to|(u,(A(x',D',s)^*-A^*(x',D',s))\tilde u)_{L^2(\mathcal{O})}|$$

Since

$$|(u,(A(x',D',s)^*-A^*(x',D',s))\tilde u)_{L^2(\mathcal{O})}|\le C\|u\|_{L^2(\mathcal{O})}\|\tilde u\|_{L^2(\mathcal{O})}.$$

the statement of the Lemma is proved. ■

Lemma A.4. *Let $a(x',\xi_0,...,\xi_{n-1},s)\in C^1_{cl}S^{\hat j/2,\hat j,s}(\mathcal{O})$ where $\hat j\in\{0,1\}$ and $b(x',\xi_0,...,\xi_{n-1},s)\in C^1_{cl}S^{\mu/2,\mu,s}(\mathcal{O})$. Then $A(x',D',s)B(x',D',s)=C(x',D',s)+R_0$ where $C(x',D',s)$ is the operator with symbol $a(x',\xi_0,...,\xi_{n-1},s)b(x',\xi_0,...,\xi_{n-1},s)$ and $R_0\in\mathcal{L}(H_0^{\frac{\mu+\tau}{2},\mu+\tau,s}(\mathcal{O}),H^{\frac{\tau+1}{2},\tau+1,s}(\mathcal{O}))$ for any $\tau\in[-1,0]$ if $\hat j=0$ and $R_0\in\mathcal{L}(H_0^{\frac{\mu}{2},\mu,s}(\mathcal{O}),L^2(\mathcal{O}))$ if $\hat j=1$. Moreover we have*

$$\|R_0\|_{\mathcal{L}(H_0^{\frac{\mu}{2},\mu,s}(\mathcal{O}),L^2(\mathcal{O}))}\le C\pi_{C^1}(\pi_{C^1}(a)\pi_{C^1}(b))\quad \textit{for } \hat j=1,$$

$$\|R_0\|_{\mathcal{L}(H_0^{\frac{\mu+\tau}{2},\mu+\tau,s}(\mathcal{O}),H^{\frac{\tau+1}{2},\tau+1,s}(\mathcal{O}))}\le C(\pi_{C^1}(a)\pi_{C^1}(b))\quad \textit{for } \hat j=0.$$

Proof. We set

$$A(x',D)=\sum_{j=1}^{\infty}a_j(x')M^{\hat j}(D)\tilde\omega_j(D),\quad B(x',D)=\sum_{j=1}^{\infty}b_j(x')M^{\mu}(D)\tilde\omega_j(D).$$

Observe that

$$A(x',D)B(x',D)=\sum_{m,k=1}^{\infty}a_m(x')b_k(x')M^{\hat j+\mu}(D)\tilde\omega_m(D)\tilde\omega_k(D)+\sum_{m,k=1}^{\infty}a_m(x')[M^{\hat j}\tilde\omega_m,b_k]M^{\mu}(D)\tilde\omega_k(D).$$

Since $C(x',D)=\sum_{m,k=1}^{\infty}a_m(x')b_k(x')M^{\hat j+\mu}(D)\tilde\omega_m(D)\tilde\omega_k(D)$, and for $\hat j=1$,

$$\|R_0\|_{\mathcal{L}(H_0^{\frac{\mu}{2},\mu,s}(\mathcal{O}),L^2(\mathcal{O}))}=\|\sum_{m,k=1}^{\infty}a_m(x')[M\tilde\omega_m,b_k]M^{\mu}(D)\tilde\omega_k(D)\|_{\mathcal{L}(H_0^{\frac{\mu}{2},\mu,s}(\mathcal{O}),L^2(\mathcal{O}))}$$

$$\le\sum_{m,k=1}^{\infty}\|a_m\|_{C^0(\bar{\mathcal{O}})}\|[M\tilde\omega_m,b_k]\|\|_{\mathcal{L}(L^2,L^2)}\tilde\omega_k(D)\|_{\mathcal{L}(L^2,L^2)}\le C_l\sum_{m,k=1}^{\infty}\lambda_m^{-l}\|[M\tilde\omega_m,b_k]\|_{\mathcal{L}(L^2,L^2)}\lambda_k^{\tilde\kappa(n)}.$$

Applying Theorem A.2 we obtain

$$\|R_0\|_{\mathcal{L}(H_0^{\frac{\mu}{2},\mu,s}(\mathcal{O}),L^2(\mathcal{O}))}\le\sum_{m,k=1}^{\infty}\|a_m\|_{C^0(\bar{\mathcal{O}})}\|[M\tilde\omega_m,b_k]\|\|\tilde\omega_k(D)\|$$

$$\le C_l\sum_{m,k=1}^{\infty}\lambda_m^{-l}\|b_k\|_{C^1(\bar{\mathcal{O}})}\lambda_m^{\tilde\kappa_1(n)}\lambda_k^{\tilde\kappa(n)}$$

$$\le C_{l,l_1}\sum_{m,k=1}^{\infty}\lambda_m^{-l}\lambda_k^{-l_1}\lambda_m^{\tilde\kappa_1(n)}\lambda_k^{\tilde\kappa(n)}\le C_{l,l_1}\sum_{k=1}^{\infty}\lambda_m^{-l}\lambda_m^{\tilde\kappa_1(n)}\sum_{m=1}^{\infty}\lambda_k^{-l_1}\lambda_k^{\tilde\kappa(n)}<\infty. \tag{A.208}$$

Let $v = v_j = u(x_0, \dots, x_{n-1})w_j(x_n)$. We take a sequence $\{w_j\}_{j=1}^\infty$ such that $F_{x_n\to\xi_n} w_j$ has a compact support and $|F_{x\to\xi_n} w_j|^2 \to \delta(\xi_n - s)$ where $s \in \mathbb{R}$ be an arbitrary point. Then for any $u \in H_0^{\frac12+\frac{\mu}{2},1+\mu}(\mathcal{O})$

(A.209) $$\|A(x',D)B(x',D)v_j - C(x',D)v_j\|^2_{L^2(\mathcal{O}\times\mathbb{R})} =$$

$$2\pi \int_{\mathbb{R}} \|(A(x',D',\xi_n)B(x',D',\xi_n) - C(x',D',\xi_n))u\|^2_{L^2(\mathcal{O})}|F_{x_n\to\xi_n}w_j|^2 d\xi_n \le C\|v_j\|^2_{H_0^{\frac{\mu}{2},\mu}(\mathbb{R}^{n+1})}.$$

Passing to the limit in (A.209) as $j \to +\infty$ we obtain

$$\|(A(x',D',s)B(x',D',s) - C(x',D',s))u\|^2_{L^2(\mathcal{O})} \le C\|u\|^2_{H_0^{\frac{\mu}{2},\mu,s}(\mathcal{O})}.$$

Let $\hat{j} = 0$.
(A.210)

$$\|R_0\|_{\mathcal{L}(H_0^{\frac{\mu+\tau}{2},\mu+\tau,s}(\mathcal{O}),H_0^{\frac{\tau+1}{2},\tau+1,s}(\mathcal{O}))} \le \sum_{m,k=1}^\infty \|a_m[\tilde{\omega}_m, b_k]M^{-\tau}\|_{\mathcal{L}(L^2(\mathcal{O}),H_0^{\frac{\tau+1}{2},\tau+1,s}(\mathcal{O}))}\|\tilde{\omega}_k(D)\|_{\mathcal{L}(L^2(\mathcal{O}),L^2(\mathcal{O}))}$$

In order to estimate the norm $\|a_m[\tilde{\omega}_m, b_k]M^{-\tau}\|_{\mathcal{L}(L^2(\mathcal{O}),H_0^{\frac{\tau+1}{2},\tau+1,s}(\mathcal{O}))}$ we observe that $M^{\tau+1}a_m[\tilde{\omega}_m, b_k]M^{-\tau} = a_m M^{\tau+1}[\tilde{\omega}_m, b_k]M^{-\tau} + [M^{\tau+1}, a_m][\tilde{\omega}_m, b_k]M^{-\tau}$. For the second term in this equality we have

(A.211) $$\|[\tilde{\omega}_m, b_k]M^{-\tau}\| \le \|b_k\|_{C^1(\bar{\mathcal{O}})}\|\tilde{\omega}_m\|_{\mathcal{L}(L^2(\mathcal{O}),L^2(\mathcal{O}))}, \quad \|[M^{\tau+1}, a_m]\| \le C\|a_m\|_{C^1(\bar{\mathcal{O}})}.$$

In order to estimate the first term we observe that $[\tilde{\omega}_m, b_k]^* = -[\tilde{\omega}_m, b_k]$. Then

$$[\tilde{\omega}_m, b_k] \in \mathcal{L}(L^2(\mathbb{R}^n), H^{\frac12,1}(\mathbb{R}^n)), \quad [\tilde{\omega}_m, b_k] \in \mathcal{L}(H^{-\frac12,-1}(\mathbb{R}^n), L^2(\mathbb{R}^n))$$

Using an interpolation argument

(A.212) $$[\tilde{\omega}_m, b_k] \in \mathcal{L}(H^{-\frac{\gamma}{2},-\gamma}(\mathbb{R}^n), H^{\frac{1-\gamma}{2},1-\gamma}(\mathbb{R}^n)) \quad \forall \gamma \in [0,1].$$

(A.213) $$\|[\tilde{\omega}_m, b_k]\|_{\mathcal{L}(H^{-\frac{\gamma}{2},-\gamma}(\mathbb{R}^n),H^{\frac{1-\gamma}{2},1-\gamma}(\mathbb{R}^n))} \le \|[\tilde{\omega}_m, b_k]\|_{\mathcal{L}(L^2(\mathbb{R}^n),H^{\frac12,1}(\mathbb{R}^n))} \quad \forall \gamma \in [0,1].$$

Applying (A.211)-(A.213) to (A.210) we obtain

$$\|R_0\|_{\mathcal{L}(H_0^{\frac{\mu+\tau}{2},\mu+\tau,s}(\mathcal{O}),H_0^{\frac{\tau+1}{2},\tau+1,s}(\mathcal{O}))}$$
$$\le C_l \sum_{m,k=1}^\infty \lambda_m^{-l}\|b_k\|_{C^1(\bar{\mathcal{O}})}\lambda_m^{\tilde{\kappa}_1(n)}\lambda_k^{\tilde{\kappa}(n)}$$

(A.214) $$\le C_{l,l_1} \sum_{m,k=1}^\infty \lambda_m^{-l}\lambda_k^{-l_1}\lambda_m^{\tilde{\kappa}_1(n)}\lambda_k^{\tilde{\kappa}(n)} \le C_{l,l_1}\sum_{k=1}^\infty \lambda_m^{-l}\lambda_m^{\tilde{\kappa}_1(n)} \sum_{m=1}^\infty \lambda_k^{-l_1}\lambda_k^{\tilde{\kappa}(n)} < \infty.$$

We finish the proof of Lemma using similar arguments as in case $\hat{j} = 1$.■
The direct consequence of the Lemma A.4 is the following commutator estimate.

Lemma A.5. *Let* $a(x', \xi_0, ..., \xi_{n-1}, s) \in C^1_{cl}S^{1/2,1,s}(\mathcal{O})$ *and* $b(x', \xi_0, ..., \xi_{n-1}, s) \in C^1_{cl}S^{0,0,s}(\mathcal{O})$*. Then* $[A, B] \in \mathcal{L}(L^2(\mathcal{O}); L^2(\mathcal{O}))$ *and*

$$\|[A, B]\|_{\mathcal{L}(L^2(\mathcal{O});L^2(\mathcal{O}))} \le C(\pi_{C^0}(a)\pi_{C^0}(b) + \pi_{C^0}(a)\pi_{C^1}(b) + \pi_{C^1}(a)\pi_{C^0}(b)).$$

Proof. By Lemma A.4 we have

$$A(x', D', s)B(x', D', s) = C(x', D', s) + R_0, \quad B(x', D', s)A(x', D', s) = C(x', D', s) + \tilde{R}_0,$$

where $R_0, \tilde{R}_0 \in \mathcal{L}(L^2(\mathcal{O}), L^2(\mathcal{O}))$. Since $[A, B] = R_0 - \tilde{R}_0$ we immediately obtain the statement of the Lemma. ■

Lemma A.6. *Let* $a(x', \xi', s) \in C^1_{cl}S^{\frac{1}{2},1,s}(\mathcal{O})$ *be a symbol with compact support in* $\mathcal{O}$*. Let* $u \in H^{\frac{1}{2},1,s}(\mathcal{O})$ *and* $\operatorname{supp} u \subset B(0, \delta)$*. Let* $\delta' > \delta$ *then there exists a constant* $C(\delta', \delta, \pi_{C^1}(a))$ *such that*

$$\|A(x', D', s)u\|_{H^{\frac{1}{2},1,s}(\mathcal{O}\setminus B(0,\delta'))} \le C\|u\|_{H^{\frac{1}{2},1,s}(\mathcal{O})}.$$

Proof. Consider the operator

$$\tilde{A}(x', D) = \sum_{j=1}^{\infty} a_j(x')M(D)\tilde{\omega}_j(D).$$

From (A.203), for any multiindex β we have

$$|\partial^\beta \omega_j(\xi)| \le C_\beta \lambda_j^{\tilde{\kappa}_2|\beta|}.$$

So, by Lemma 2.2 (see Chapter II [17]), we have for every function w such that $\operatorname{supp} w \subset B(0, \delta) \times [-1/4, 1/4]$

$$\|M(D)\tilde{\omega}_j(D)v\|_{H^{\frac{1}{2},1}(\mathcal{O}\setminus B(0,\delta')\times[-1,1]\setminus[-\frac{1}{2},\frac{1}{2}])} \le C\lambda_j^{\hat{\kappa}}\|v\|_{H^{\frac{1}{2},1}(B(0,\delta)\times[-\frac{1}{4},\frac{1}{4}])}.$$

Therefore

$$\begin{aligned}
\|\tilde{A}(x', D)v\|_{H^1(\mathcal{O}\setminus B(0,\delta')\times[-1,1]\setminus[-\frac{1}{2},\frac{1}{2}])} &\le \sum_{j=1}^{\infty} \|a_j(x')M(D)\tilde{\omega}_j(D)v\|_{H^1(\mathcal{O}\setminus B(0,\delta')\times[-1,1]\setminus[-\frac{1}{2},\frac{1}{2}])} \\
&\le \sum_{j=1}^{\infty} \|a_j\|_{C^1(\bar{\mathcal{O}})}\|M(D)\tilde{\omega}_j(D)v\|_{H^1(\mathcal{O}\setminus B(0,\delta')\times[-1,1]\setminus[-\frac{1}{2},\frac{1}{2}])} \\
&\le C\sum_{j=1}^{\infty} \lambda_j^{-m}\lambda_j^{\hat{\kappa}}\|v\|_{H^{\frac{1}{2},1}(B(0,\delta)\times[-\frac{1}{4},\frac{1}{4}])}. \qquad \text{(A.215)}
\end{aligned}$$

Let $v = v_j = u(x_0, \dots, x_{n-1})w_j(x_n)$. We take a sequence $\{w_j\}_{j=1}^{\infty}$ such that $F_{x_n\to\xi_n}w_j$ has a compact support and $|F_{x_n\to\xi_n}w_j|^2 \to \delta(\xi_n - s_0)$ where $s_0 \in \mathbb{R}$ is an arbitrary point. Then

$$\begin{aligned}
&\|\tilde{A}(x', D)v\|^2_{H^1(\mathcal{O}\setminus B(0,\delta')\times[-1,1]\setminus[-\frac{1}{2},\frac{1}{2}])} = \\
&\int_{-\frac{1}{4}}^{\frac{1}{4}} \|A(x', D', s)u\|^2_{H^{1,s}(\mathcal{O}\setminus B(0,\delta'))}|F_{x_n\to\xi_n}w_j|^2 d\xi_n \to \|A(x, D', s_0)u\|^2_{H^{1,s_0}(\mathcal{O}\setminus B(0,\delta'))} \qquad \text{(A.216)}
\end{aligned}$$

and

$$\|v_j\|_{H^{\frac{1}{2},1}(B(0,\delta)\times[-\frac{1}{4},\frac{1}{4}])} \to \|u\|_{H^{\frac{1}{2},1,s_0}(B(0,\delta))} \text{ as } j \to +\infty.$$

These relations and (A.215) prove the statement of the Lemma. ■

We shall use the following variant of Gårding's inequality:

Lemma A.7. *Let $p(x',\xi',s) \in C^1_{cl}S^{\frac{1}{2},1,s}(\mathcal{O})$ be a symbol with compact support in $\mathcal{O}$. Let $u \in H^{\frac{1}{2},1,s}(\mathcal{O})$ and $\operatorname{supp} u \subset B(0,\delta)$. Let $\delta' > \delta$ be such that $\overline{B(0,\delta')} \subset \mathcal{O}$ and $Re\, p(x',\xi',s) > \hat{C}|s|M(\xi',s)$ for any $x \in B(0,\delta')$. Then*

$$Re(Pu,u) \geq C|s|\|u\|^2_{H^{\frac{1}{4},\frac{1}{2},s}(\mathcal{O})} - C_1\|u\|^2_{L^2(\mathcal{O})}.$$

Proof. Let $\chi \in C^\infty_0(B(0,\delta'))$ be a function such that $\chi|_{B(0,\delta)} = 1$. Consider the pseudodifferential operator $A(x',D',s)$ with symbol $A(x',\xi',s) = (\operatorname{Re} p(x',\xi',s) - \chi\frac{\hat{C}}{2}|s|M(\xi',s))^{\frac{1}{2}} \in C^1_{cl}S^{\frac{1}{4},\frac{1}{2},s}(\mathcal{O})$. Then, according to Lemma A.4

$$A(x',D',s)^*A(x',D',s) = \operatorname{Re} p(x,\xi',s) - \chi\frac{\hat{C}}{2}|s|M(\xi',s) + R,$$

where $R \in \mathcal{L}(L^2(\mathcal{O}); L^2(\mathcal{O}))$. Therefore

$$\operatorname{Re}(Pu,u)_{L^2(\mathcal{O})} = \|A(x',D,s)\|^2_{L^2(\mathcal{O})} - ((1-\chi)M(D)u,u)_{L^2(\mathbb{R}^n)} + \frac{\hat{C}}{2}\|u\|^2_{H^{\frac{1}{4},\frac{1}{2},s}(\mathcal{O})} + (Ru,u)_{L^2(\mathcal{O})}.$$

Observing that $|(Ru,u)_{L^2(\mathcal{O})}| \leq C(\pi_{C^1}(p))\|u\|^2_{L^2(\mathcal{O})}$, and since by Lemma 2.2 (see Chapter II [17]), we have $|((1-\chi)M(D)u,u)_{L^2(\mathbb{R}^n)}| \leq C\|u\|^2_{L^2(\mathcal{O})}$, we obtain the statement of the Lemma. ■
Consider the following system of equations

$$\frac{\partial W}{\partial x_n} + K(x,D',s)W = F \quad \text{in } G, \tag{A.217}$$

$$W(x',0) = g, \tag{A.218}$$

where $W = (w_1,\dots,w_m)$, $F = (f_1,\dots,f_m)$, $g = (g_1,\dots,g_m)$. Let $K(x,D',s)$ be the $m\times m$ matrix pseudodifferential operator such that

$$K_{ij}(x,\xi',s) \in C^1_{cl}S^{\frac{1}{2},1,s}(G) \tag{A.219}$$

and there exists a constant $C > 0$ such that

$$(\operatorname{Re}K(x,\xi',s)v,v) \geq C|v|^2 \quad \forall(x,\xi',s) \in G\times\mathbb{R}^n\ \forall v \in \mathbb{R}^n \tag{A.220}$$

and the matrix

$$K(x,\xi',s) \text{ is independent of } x \text{ outside of a ball } B(0,\delta). \tag{A.221}$$

We have

Lemma A.8. *Suppose that assumptions (A.219) - (A.221) hold true. Then*
A) *For each $g \in H^{\frac{1}{4},\frac{1}{2},s}(\mathbb{R}^n)$, $F \in L^2(G)$ there exists a unique solution to problem (A.217), (A.218) $W \in H^{\frac{1}{2},1,s}(G)$ and the following a priori estimate holds true*

$$\|W\|_{H^{\frac{1}{2},1,s}(G)} + \|W\|_{L^\infty(0,\gamma;H^{\frac{1}{4},\frac{1}{2},s}(\mathbb{R}^n))} \leq C(\|g\|_{H^{\frac{1}{4},\frac{1}{2},s}(\mathbb{R}^n)} + \|F\|_{L^2(G)}). \tag{A.222}$$

B) *For each $g \in L^2(\mathbb{R}^n)$, $F \in H^{-\frac{1}{2},-1,s}(G)$ with supp$F \subset\subset G$ there exists a unique solution to problem (A.217), (A.218) $W \in L^2(G)$ and the following a priori estimate holds true*

$$\|W\|_{L^2(G)} + \|W\|_{L^\infty(0,\gamma;H^{-\frac{1}{4},-\frac{1}{2},s}(\mathbb{R}^n))} \leq C(\|g\|_{H^{-\frac{1}{4},-\frac{1}{2},s}(\mathbb{R}^n)} + \|F\|_{H^{-\frac{1}{2},-1,s}(G)}). \tag{A.223}$$

Remark. *For the initial data $g \in L^2(\mathbb{R}^n)$, $F \in H^{-\frac{1}{2},-1,s}(G)$ with suppF $\subset\subset G$ we understand solution of problem (A.217), (A.218) in the following way*

$$(W,(-\frac{\partial}{\partial x_n}+K(x,D',s)^*)\Phi)_{L^2(G)}=(g,\Phi(\cdot,0))_{L^2(\mathbb{R}^n)}+(F,\Phi)_{L^2(G)}$$

for any function $\Phi \in H^{\frac{1}{2},1,s}(G)$, $\Phi(\cdot,\gamma)=0$.

Proof. We set

$$\tilde{\Lambda}^{\frac{1}{2}}(D',s)w=\int_{\mathbb{R}^n}(1+\xi_0^2+\sum_{j=1}^{n-1}\xi_j^4+s^4)^{\frac{1}{8}}\hat{w}e^{<x',\xi'>}d\xi'.$$

Let $K_1(x,\xi,s)$ be the principal symbol of the operator K. Consider the matrix

$$P(x,\xi',s)=\int_0^\infty e^{-tK_1^*}e^{-tK_1}dt. \tag{A.224}$$

By (A.220) the integral in the right hand side of (A.224) is convergent. Then for the principal symbol of the operator K we have

$$PK_1+K_1^*P=I.$$

We also observe that

$$(P\vec{v},\vec{v})\geq C_1\|\vec{v}\|^2 \quad \forall v\in\mathbb{R}^m. \tag{A.225}$$

In order to show the solvability of (A.217), (A.218), we will first consider regularized problems. For $\epsilon \in]0,1[$, let us consider a family of the Friedrichs mollifiers $(\mathcal{J}_\epsilon)_\epsilon$ with $\mathcal{J}_\epsilon \in \mathcal{S}^{-\infty}(\mathbb{R}^n)$ (cf [16]) ($\mathcal{S}^p(\mathbb{R}^n)$ is the class of pseudodifferential operators of order p on $\mathbb{R}^n$). It is known that

$$\mathcal{J}_\epsilon u\rightarrow u \text{ in } L^2(\mathbb{R}^n) \text{ as } \epsilon\rightarrow +0,$$

$$\sup_{\epsilon\in[0,1]}\|\mathcal{J}_\epsilon\|_{\mathcal{L}(L^2(\mathbb{R}^n),L^2(\mathbb{R}^n))}\leq C,$$

$$\{[A,\mathcal{J}_\epsilon]:0<\epsilon<1\} \text{ is a bounded set in } \mathcal{L}(L^2(\mathbb{R}^n),L^2(\mathbb{R}^n))$$

for any $A(x,D,s)\in C^0([0,1],C^1_{cl}S^{\frac{1}{2},1,s}(\mathbb{R}^n))$. Let $\epsilon\in(0,1)$. We consider the following Cauchy problem for ordinary differential equation in a Banach space.

$$\frac{\partial W_\epsilon}{\partial x_n}+\mathcal{J}_\epsilon^*K(x,D',s)\mathcal{J}_\epsilon W_\epsilon=\mathcal{J}_\epsilon^*F \quad \text{in } G, \tag{A.226}$$

$$W_\epsilon(x',0)=g. \tag{A.227}$$

For any $g\in H^{\frac{1}{4},\frac{1}{2}}(\mathbb{R}^n)$ and $F\in L^2(G)$ there exists a unique solution W_ϵ to problem (A.226),(A.227) with $W_\epsilon\in L^\infty(0,\gamma;H^{\frac{1}{4},\frac{1}{2},s}(\mathbb{R}^n))$, $\frac{\partial W_\epsilon}{\partial x_n}\in L^2(G)$.

Simple computations provide the following

$$\begin{aligned}
&\frac{d}{dx_n}(P\tilde\Lambda^{\frac12}W_\epsilon,\tilde\Lambda^{\frac12}W_\epsilon)_{L^2(\mathbb{R}^n)}=(P\tilde\Lambda^{\frac12}\frac{\partial W_\epsilon}{\partial x_n},\tilde\Lambda^{\frac12}W_\epsilon)_{L^2(\mathbb{R}^n)}\\
&+(P\tilde\Lambda^{\frac12}W_\epsilon,\tilde\Lambda^{\frac12}\frac{\partial W_\epsilon}{\partial x_n})_{L^2(\mathbb{R}^n)}+(P'\tilde\Lambda^{\frac12}W_\epsilon,\tilde\Lambda^{\frac12}W_\epsilon)_{L^2(\mathbb{R}^n)}=\\
&(P\tilde\Lambda^{\frac12}(-\mathcal{J}_\epsilon^*K\mathcal{J}_\epsilon W_\epsilon+\mathcal{J}_\epsilon^*F),\tilde\Lambda^{\frac12}W_\epsilon)_{L^2(\mathbb{R}^n)}\\
&+(P\tilde\Lambda^{\frac12}W_\epsilon,\tilde\Lambda^{\frac12}(-\mathcal{J}_\epsilon^*K\mathcal{J}_\epsilon W_\epsilon+\mathcal{J}_\epsilon^*F))_{L^2(\mathbb{R}^n)}+(P'\tilde\Lambda^{\frac12}W_\epsilon,\tilde\Lambda^{\frac12}W_\epsilon)_{L^2(\mathbb{R}^n)}=\\
&-(P\mathcal{J}_\epsilon^*K\mathcal{J}_\epsilon\tilde\Lambda^{\frac12}W_\epsilon,\tilde\Lambda^{\frac12}W_\epsilon)_{L^2(\mathbb{R}^n)}+(P\tilde\Lambda^{\frac12}\mathcal{J}_\epsilon^*F,\tilde\Lambda^{\frac12}W_\epsilon)_{L^2(\mathbb{R}^n)}\\
&-(P\tilde\Lambda^{\frac12}W_\epsilon,\mathcal{J}_\epsilon^*K\mathcal{J}_\epsilon\tilde\Lambda^{\frac12}W_\epsilon)_{L^2(\mathbb{R}^n)}+(P\tilde\Lambda^{\frac12}W_\epsilon,\tilde\Lambda^{\frac12}\mathcal{J}_\epsilon^*F)_{L^2(\mathbb{R}^n)}+(P'\tilde\Lambda^{\frac12}W_\epsilon,\tilde\Lambda^{\frac12}W_\epsilon)_{L^2(\mathbb{R}^n)}=\\
&-((P\mathcal{J}_\epsilon^*K\mathcal{J}_\epsilon+\mathcal{J}_\epsilon^*K^*\mathcal{J}_\epsilon P)\tilde\Lambda^{\frac12}W_\epsilon,\tilde\Lambda^{\frac12}W_\epsilon)_{L^2(\mathbb{R}^n)}+(P\tilde\Lambda^{\frac12}\mathcal{J}_\epsilon^*F,\tilde\Lambda^{\frac12}W_\epsilon)_{L^2(\mathbb{R}^n)}\\
&+(P\tilde\Lambda^{\frac12}W_\epsilon,\tilde\Lambda^{\frac12}\mathcal{J}_\epsilon^*F)_{L^2(\mathbb{R}^n)}+(P'\tilde\Lambda^{\frac12}W_\epsilon,\tilde\Lambda^{\frac12}W_\epsilon)_{L^2(\mathbb{R}^n)}=\\
&-((PK+K^*P)\mathcal{J}_\epsilon\tilde\Lambda^{\frac12}W_\epsilon,\mathcal{J}_\epsilon\tilde\Lambda^{\frac12}W_\epsilon)_{L^2(\mathbb{R}^n)}+(P\tilde\Lambda^{\frac12}\mathcal{J}_\epsilon^*F,\tilde\Lambda^{\frac12}W_\epsilon)_{L^2(\mathbb{R}^n)}\\
&-(\mathcal{J}_\epsilon K^*[\mathcal{J}_\epsilon,P]\tilde\Lambda^{\frac12}W_\epsilon,\tilde\Lambda^{\frac12}W_\epsilon)_{L^2(\mathbb{R}^n)}+(P\tilde\Lambda^{\frac12}W_\epsilon,\tilde\Lambda^{\frac12}\mathcal{J}_\epsilon^*F)_{L^2(\mathbb{R}^n)}+(P'\tilde\Lambda^{\frac12}W_\epsilon,\tilde\Lambda^{\frac12}W_\epsilon)_{L^2(\mathbb{R}^n)}.
\end{aligned}$$

Here P' is the pseudodifferential operator with symbol $\frac{\partial}{\partial x_n}p(x,\xi',s)$. Note that

$$|(\mathcal{J}_\epsilon K^*[\mathcal{J}_\epsilon,P]\tilde\Lambda^{\frac12}W_\epsilon,\tilde\Lambda^{\frac12}W_\epsilon)_{L^2(\mathbb{R}^n)}|\le C\|W_\epsilon\|_{H^{\frac14,\frac12,s}(\mathbb{R}^n)} \tag{A.228}$$

and

$$|(P'\tilde\Lambda^{\frac12}W_\epsilon,\tilde\Lambda^{\frac12}W_\epsilon)_{L^2(\mathbb{R}^n)}|\le C\|W_\epsilon\|_{H^{\frac14,\frac12,s}(\mathbb{R}^n)}. \tag{A.229}$$

(Here and below all constants C are independent of $\epsilon\in(0,1)$.) After simple computations we obtain

$$\begin{aligned}
&(P\tilde\Lambda^{\frac12}\mathcal{J}_\epsilon^*F,\tilde\Lambda^{\frac12}W_\epsilon)_{L^2(\mathbb{R}^n)}=([P\tilde\Lambda^{\frac12},\mathcal{J}_\epsilon^*]F,\tilde\Lambda^{\frac12}W_\epsilon)_{L^2(\mathbb{R}^n)}+(P\tilde\Lambda^{\frac12}F,\mathcal{J}_\epsilon\tilde\Lambda^{\frac12}W_\epsilon)_{L^2(\mathbb{R}^n)}\\
&=([P\tilde\Lambda^{\frac12},\mathcal{J}_\epsilon^*]F,\tilde\Lambda^{\frac12}W_\epsilon)_{L^2(\mathbb{R}^n)}+(F,\tilde\Lambda^{\frac12}P^*\mathcal{J}_\epsilon\tilde\Lambda^{\frac12}W_\epsilon)_{L^2(\mathbb{R}^n)}=\\
&=([P\tilde\Lambda^{\frac12},\mathcal{J}_\epsilon^*]F,\tilde\Lambda^{\frac12}W_\epsilon)_{L^2(\mathbb{R}^n)}+(F,\tilde\Lambda^{\frac12}P^*\tilde\Lambda^{\frac12}\mathcal{J}_\epsilon W_\epsilon)_{L^2(\mathbb{R}^n)}+(F,\tilde\Lambda^{\frac12}P^*[\mathcal{J}_\epsilon,\tilde\Lambda^{\frac12}]W_\epsilon)_{L^2(\mathbb{R}^n)}.
\end{aligned}$$

Therefore for any positive δ

$$|(P\tilde\Lambda^{\frac12}\mathcal{J}_\epsilon F,\tilde\Lambda^{\frac12}W_\epsilon)_{L^2(\mathbb{R}^n)}|\le C(\delta)\|F(\cdot,x_n)\|^2_{L^2(\mathbb{R}^n)}+C\|\tilde\Lambda^{\frac12}W_\epsilon\|^2_{L^2(\mathbb{R}^n)}+\delta\|\mathcal{J}_\epsilon W_\epsilon\|^2_{H^{\frac12,1,s}(\mathbb{R}^n)}. \tag{A.230}$$

In similar way we have

$$|(P\tilde\Lambda^{\frac12}W_\epsilon,\tilde\Lambda^{\frac12}\mathcal{J}_\epsilon^*F)_{L^2(\mathbb{R}^n)}|\le C(\delta)\|F(\cdot,x_n)\|^2_{L^2(\mathbb{R}^n)}+C\|\tilde\Lambda^{\frac12}W_\epsilon\|^2_{L^2(\mathbb{R}^n)}+\delta\|\mathcal{J}_\epsilon W_\epsilon\|^2_{H^{\frac12,1,s}(\mathbb{R}^n)}. \tag{A.231}$$

Using (A.228)-(A.231) we obtain

$$\frac{d}{dx_n}(P\tilde\Lambda^{\frac12}W_\epsilon,\tilde\Lambda^{\frac12}W_\epsilon)_{L^2(\mathbb{R}^n)}+C_1\|\mathcal{J}_\epsilon W_\epsilon\|^2_{H^{\frac12,1,s}(\mathbb{R}^n)}\le C(\|\tilde\Lambda^{\frac12}W_\epsilon\|^2_{L^2(\mathbb{R}^n)}+\|F(\cdot,x_n)\|^2_{L^2(\mathbb{R}^n)}). \tag{A.232}$$

Applying Gronwall's inequality we obtain

$$\|W_\epsilon\|_{L^\infty(0,\gamma;H^{\frac14,\frac12,s}(\mathbb{R}^n))}+\|\mathcal{J}_\epsilon W_\epsilon\|_{L^2(0,\gamma;H^{\frac14,\frac12,s}(\mathbb{R}^n))}\le C(\|g\|_{H^{\frac14,\frac12,s}(\mathbb{R}^n)}+\|F\|_{L^2(G)}). \tag{A.233}$$

Using (A.233) from (A.217) we obtain the estimate for $\frac{\partial W_\epsilon}{\partial x_n}$:

$$\|\frac{\partial W_\epsilon}{\partial x_n}\|_{L^2(G)} \le C(\|g\|_{H^{\frac14,\frac12,s}(\mathbb{R}^n)} + \|F\|_{L^2(G)}). \tag{A.234}$$

Inequalities (A.233) and (A.234) imply

$$\|W_\epsilon\|_{H^{\frac12,1,s}(G)} + \|W_\epsilon\|_{L^\infty(0,\gamma;H^{\frac14,\frac12,s}(\mathbb{R}^n))} \le C(\|g\|_{H^{\frac14,\frac12,s}(\mathbb{R}^n)} + \|F\|_{L^2(G)}). \tag{A.235}$$

We can now extract a subsequence, still denoted by ϵ such that

$$W_\epsilon \rightharpoonup W \text{ in } L^\infty(0,\gamma;L^2(\mathbb{R}^n)) \text{ weakly}*, \tag{A.236}$$

$$\frac{\partial W_\epsilon}{\partial x_n} \rightharpoonup \frac{\partial W}{\partial x_n} \text{ in } L^2(G) \text{ weakly}, \tag{A.237}$$

$$\mathcal{J}_\epsilon W_\epsilon \rightharpoonup W \text{ in } L^2(0,\gamma;H^{\frac12,1}(\mathbb{R}^n)) \text{ weakly}, \tag{A.238}$$

$$\mathcal{J}_\epsilon W_\epsilon \rightharpoonup W \text{ in } L^\infty(0,\gamma;H^{\frac14,\frac12}(\mathbb{R}^n)) \text{ weakly} * . \tag{A.239}$$

Of course $W \in H^{\frac12,1}(G)$ and W is a solution of (A.217), (A.218). From (A.235) we obtain (A.222). Now we prove the statement **B** of this Lemma. Since the space $L^2(G)$ is dense in the space $\{F \in H^{-\frac12,-1,s}(G) | \operatorname{supp} F \subset\subset G\}$ in order to prove the statement **B** it suffices to establish the a priori estimate (A.223). Let Φ be a solution to the following boundary value problem

$$(-\frac{\partial}{\partial x_n} + K^*(x,D,s))\Phi = W \quad \text{in } G, \quad \Phi(\cdot,\gamma) = 0. \tag{A.240}$$

From the definition of a weak solution we have

$$\|W\|^2_{L^2(G)} = (g,\Phi(\cdot,\gamma))_{L^2(\mathbb{R}^n)} + (F,\Phi)_{L^2(G)}. \tag{A.241}$$

By the statement **A** of this Lemma solution to (A.240) satisfies the estimate

$$\|\Phi\|_{H^{\frac12,1,s}(G)} + \|\Phi\|_{L^\infty(0,\gamma;H^{\frac14,\frac12,s}(\mathbb{R}^n))} \le C\|W\|_{L^2(G)}. \tag{A.242}$$

Using in equality (A.241) estimate (A.242) we obtain (A.223). ∎

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