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On the uniform simplicity of diffeomorphism groups

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ON THE UNIFORM SIMPLICITY OF DIFFEOMORPHISM GROUPS

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Abstract. We show the uniform simplicity of the identity component $\mathrm{Diff}^r(M^n)_0$ of the group of C^r diffeomorphisms $\mathrm{Diff}^r(M^n)$ ($1 \leq r \leq \infty$, $r \neq n+1$) of the compact connected n-dimensional manifold M^n with handle decomposition without handles of the middle index $n/2$. More precisely, for any elements f and g of such $\mathrm{Diff}^r(M^n)_0 \setminus \{\mathrm{id}\}$, f can be written as a product of at most $16n+28$ conjugates of g or g^{-1}, which we denote by $f \in (C_g)^{16n+28}$. We have better estimates for several manifolds. For the n-dimensional sphere S^n, for any elements f and g of $\mathrm{Diff}^r(S^n)_0 \setminus \{\mathrm{id}\}$ ($1 \leq r \leq \infty$, $r \neq n+1$), $f \in (C_g)^{12}$, and for a compact connected 3-manifold M^3, for any elements f and g of $\mathrm{Diff}^r(M^3)_0 \setminus \{\mathrm{id}\}$ ($1 \leq r \leq \infty$, $r \neq 4$), $f \in (C_g)^{44}$.

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1. Introduction

In 1947, Ulam and von Neumann ([22]) announced the following theorem.

Theorem 1.1 (Ulam-von Neumann [22]). *The group of orientation preserving homeomorphisms of the* 2*-dimensional sphere* S^2 *is a simple group. Moreover there is a positive integer* N *such that for any orientation preserving homeomorphisms* f *and* g *of* S^2, f *can be written as a product of* N *conjugates of* g *if* g *is not the identity.*

In 1958, Anderson ([1]) showed the following theorem.

Theorem 1.2 (Anderson [1]). *Let* $\mathrm{Homeo}(S^n)_0$ *denote the identity component of the group of homeomorphisms of the* n*-dimensional sphere* S^n*. For* $n = 1, 2, 3$ *and for elements* f *and* $g \in \mathrm{Homeo}(S^n)_0 \setminus \{\mathrm{id}\}$, f *can be written as a product of at most* 6 *conjugates of* g *or* g^{-1}.

In 1960, Fisher ([5]) showed that for a compact connected manifold M^n of dim $n \leq 3$, $\mathrm{Homeo}(M^n)_0$ is a simple group.

Here, a group G is said to be *simple* if G contains no nontrivial proper normal subgroups. Equivalently, G is simple if, for $f \in G$ and $g \in G \setminus \{e\}$, f can be written as a product of conjugates of g or g^{-1}.

In 1970, Epstein ([4], [2]) showed that for certain groups such as the group of C^r diffeomorphisms ($r \leq \infty$) where we can apply the fragmentation technique, the perfectness implies the simplicity.

Here a group G is said to be *perfect* if the abelianization of G is a trivial group. Equivalently, G is perfect if any element of G can be written as a product of commutators.

For a manifold M^n, let $\mathrm{Diff}^r(M^n)$ denote the group of C^r diffeomorphisms of M^n, and $\mathrm{Diff}^r_c(M^n)$, the group of C^r diffeomorphisms of M^n with compact support ($1 \leq r \leq \infty$). Here the *support* $\mathrm{supp}(f)$ of a diffeomorphism f of M^n is defined to be the *closure* of $\{x \in M^n \mid f(x) \neq x\}$. Let $\mathrm{Diff}^r(M^n)_0$ and $\mathrm{Diff}^r_c(M^n)_0$ denote the identity components of $\mathrm{Diff}^r(M^n)$ and $\mathrm{Diff}^r_c(M^n)$ with respect to the C^r topology, respectively ([2]).

Herman-Mather-Thurston ([7], [10], [11], [15], [2]) showed the perfectness of the identity component $\mathrm{Diff}^r_c(M^n)_0$ of the group of C^r diffeomorphisms ($1 \leq r \leq \infty$, $r \neq n+1$) of an n-dimensional manifold M^n with compact support, which implies the simplicity of the group when M^n is connected.

For $g \in G$, let C_g denote the union of the conjugate classes of g and of g^{-1}. Then G is simple if $G = \bigcup_{k=1}^{\infty}(C_g)^k$ for any element $g \in G \setminus \{e\}$. For a simple group G, we can define an interesting distance function on the set $\{C_g \mid g \in G \setminus \{e\}\}$ by

$$d(C_f, C_g) = \log \min\{k \mid C_f \subset (C_g)^k \text{ and } C_g \subset (C_f)^k\}.$$

Definition 1.3. We say that G is uniformly simple if there is a positive integer N such that, for $f \in G$ and $g \in G \setminus \{e\}$, f can be written as a product of at most N conjugates of g or g^{-1}: $G = \bigcup_{k=1}^{N}(C_g)^k$.

In other words, G is uniformly simple if the distance function d on $\{C_g \mid g \in G \setminus \{e\}\}$ is bounded.

There are simple groups which are not uniformly simple. For example, the direct limit A_∞ of the alternate groups A_n, the identity component of the group of volume preserving diffeomorphisms with compact support of $\boldsymbol{R}^n$ ($n \geq 3$), etc.

If an infinite group is uniformly simple, then it is uniformly perfect. Here a group G is said to be *uniformly perfect* if there is a positive integer N such that any element $f \in G$ can be written as a product of at most N commutators. By using the results of Herman-Mather-Thurston ([7], [10], [11], [15], [2]), we showed in [21] the uniform perfectness of $\mathrm{Diff}^r(M^n)_0$ ($1 \leq r \leq \infty$, $r \neq n+1$) for the compact n-dimensional manifold M^n with handle decomposition without handles of the middle index $n/2$.

We show in this paper, the uniform simplicity of the identity component $\mathrm{Diff}^r(M^n)_0$ ($1 \leq r \leq \infty$, $r \neq n+1$) of the group of diffeomorphisms of the compact connected n-dimensional manifold M^n with handle decomposition without handles of the middle index $n/2$. This uniform simplicity (in particular, the estimates on the number of conjugates) follows from certain improvement of the proof in [21] of the uniform perfectness of $\mathrm{Diff}^r(M^n)_0$ (see also Remark 3.4).

Our results in this paper are as follows.

Theorem 1.4. *For the n-dimensional sphere S^n ($n \geq 1$), for any elements f and g of $\mathrm{Diff}^r(S^n)_0 \setminus \{\mathrm{id}\}$ ($1 \leq r \leq \infty$, $r \neq n+1$), f can be written as a product of at most 12 conjugates of g or g^{-1}.*

For a handle decomposition, let c be the order of the set of indices which appears as the indices of handles in the handle decomposition. In the following theorems, for a manifold M^n, $c(M^n)$ denotes the minimum of such numbers c among the handle decompositions of M^n without the middle index $n/2$ (if n is even). Of course, $c(M^n) \leq n+1$.

Theorem 1.5. *Let M^{2m} be a compact connected $(2m)$-dimensional manifold with handle decomposition without handles of index m, then for any elements f and g of $\mathrm{Diff}^r(M^{2m})_0 \setminus \{\mathrm{id}\}$ ($1 \leq r \leq \infty$, $r \neq 2m+1$), f can be written as a product of at most $16c(M^{2m}) + 8$ conjugates of g or g^{-1}.*

Theorem 1.6. *Let M^{2m+1} be a compact connected $(2m+1)$-dimensional manifold, then for any elements f and g of $\mathrm{Diff}^r(M^{2m+1})_0 \setminus \{\mathrm{id}\}$ ($1 \leq r \leq \infty$, $r \neq 2m+2$), f can be written as a product of at most $16c(M^{2m+1}) + 12$ conjugates of g or g^{-1}.*

Since $c(M^n) \leq n+1$, we have the following corollary.

Corollary 1.7. *Let M^n be a compact connected n-dimensional manifold with handle decomposition without handles of index $n/2$. For any elements f and g of $\mathrm{Diff}^r(M^n)_0 \setminus \{\mathrm{id}\}$ ($1 \leq r \leq \infty$, $r \neq n+1$), f can be written as a product of at most $16n + 28$ conjugates of g or g^{-1}.*

In many cases, we have a better estimate on the number of conjugates. In particular, for a compact connected 3-dimensional manifolds M^3, we have the following.

Corollary 1.8. *Let M^3 be a compact connected 3-dimensional manifold. For any elements f and g of $\mathrm{Diff}^r(M^3)_0 \setminus \{\mathrm{id}\}$ ($1 \leq r \leq \infty$, $r \neq 4$), f can be written as a product of at most 44 conjugates of g or g^{-1}.*

In Section 2, we review the results of our previous paper [21] and give the necessary improvement. In Section 3, we give the proofs of theorems. There we also remark that for the n-dimensional sphere S^n, any element $f \in \mathrm{Diff}(S^n)_0$ can be written as a product of 3 commutators, and for a compact $(2m+1)$-dimensional manifold M^{2m+1}, any element $f \in \mathrm{Diff}(M^{2m+1})_0$ can be written as a product of 5 commutators.

2. Uniform perfectness of diffeomorphism groups

In [21, Theorem 4.1], we showed the following theorem.

Theorem 2.1 ([21])**.** *Let M^n be the interior of a compact n-dimensional manifold with handle decomposition with handles of indices not greater than $(n-1)/2$, then any element of $\mathrm{Diff}^r_c(M^n)_0$ $(1 \leq r \leq \infty,\ r \neq n+1)$ can be written as a product of two commutators.*

To discuss the uniform simplicity, we use an improvement of this theorem. In the proof of this theorem, we used a nice Morse function on M^n to find a k-dimensional complex K^k differentiably embedded in M^n $(k \leq (n-1)/2)$ which is a deformation retract of M^n, and an isotopy $\{H_t\}_{t\in[0,1]}$ $(H_0 = \mathrm{id})$ with a neighborhood V of K^k such that $(H_1)^j(V)$ $(j \in \boldsymbol{Z})$ are disjoint. We will use the Morse function on M^n and the associated handle decomposition to show the following theorem.

Theorem 2.2. *Let M^n be the interior of a compact n-dimensional manifold with handle decomposition with handles of indices not greater than $(n-1)/2$. Let c be the order of the set of indices appearing in the handle decomposition. Then any element of $\mathrm{Diff}^r_c(M^n)_0$ $(1 \leq r \leq \infty,\ r \neq n+1)$ can be written as a product of two commutators. Moreover, if M^n is connected, any element of $\mathrm{Diff}^r_c(M^n)_0$ can be written as a product of $4c+1$ commutators with support in balls.*

To prove Theorem 2.2, we review the Morse functions and handle decompositions. Before the beginning of the proof of Theorem 2.2, let f denote a Morse function and we fix notations as in [21].

Let $f : M^n \longrightarrow \boldsymbol{R}$ be a Morse function on a compact connected n-dimensional manifold M^n such that $f(M^n) = [0, n]$, the set of critical points of index k is contained in $f^{-1}(k)$ $(k = 0, \ldots, n)$ and $f^{-1}(0)$ and $f^{-1}(n)$ are one point sets.

Put $W_k = f^{-1}([0, k + 1/2])$, and then this W_k is a compact manifold with boundary $\partial W_k = f^{-1}(k + 1/2)$. Let c_k be the number of critical points of index k. Then the manifold W_k is diffeomorphic to the manifold obtained from W_{k-1} by attaching c_k handles of index k $(k = 0, \ldots, n)$. This means the following.

Let $D^k \times D^{n-k}$ be the product of the k-dimensional disk D^k and the $(n-k)$-dimensional disk D^{n-k}. Let $\varphi_i : (\partial D^k) \times D^{n-k} \longrightarrow \partial W_{k-1}$ $(i = 1, \ldots, c_k)$ be diffeomorphisms with disjoint images. Let

$$W'_k = W_{k-1} \cup_{\bigsqcup_{i=1}^{c_k} \varphi_i} \bigsqcup_{i=1}^{c_k} (D^k \times D^{n-k})_i$$

be the space obtained from the disjoint union $W_{k-1} \sqcup \bigsqcup_{i=1}^{c_k} (D^k \times D^{n-k})_i$ by identifying

$$x \in ((\partial D^k) \times D^{n-k})_i \subset (D^k \times D^{n-k})_i$$

with $\varphi_i(x) \in \partial W_{k-1} \subset W_{k-1}$.

In this paper, we consider that W'_k is a submanifold with corner of W_k and $W_k \setminus W'_k$ is diffeomorphic to $\partial W_k \times (-\infty, k+1/2]$ (which is shown by using the flowlines of the gradient flow Ψ_t). The handles $(D^k \times D^{n-k})_i$ ($i = 1, \ldots, c_k$) of index k are contained in the interior of W_k. Then we have the sequence

$$D^n \cong W_0 \subset W'_1 \subset W_1 \subset \cdots \subset W'_k \subset W_k \subset \cdots \subset W'_n = W_n = M^n.$$

By choosing a Riemannian metric on the manifold M^n, the Morse function f defines the gradient vector field and the gradient flow Ψ_t. The fixed points of the gradient flow Ψ_t are precisely the critical points of f. The core disk and the co-core disk of a handle of a handle decomposition of M^n correspond to the local stable manifold and the local unstable manifold of the corresponding fixed point p of the gradient flow Ψ_t, respectively ([13], [14]). Let e_i^k and ${e'}_i^{n-k}$ denote the global stable manifold and the global unstable manifold, respectively, for the fixed point p of Ψ_t which is a critical point of index k of f. Then e_i^k and ${e'}_i^{n-k}$ are diffeomorphic to $\boldsymbol{R}^k$ and $\boldsymbol{R}^{n-k}$, respectively. Then we know that the global stable manifolds and the global unstable manifolds of fixed points of Ψ_t form the cell decomposition $\bigcup_{k=0}^n \bigcup_{i=1}^{c_k} e_i^k$ and the dual cell decomposition $\bigcup_{k=0}^n \bigcup_{i=1}^{c_k} {e'}_i^{n-k}$ of M^n, respectively ([[13]]). The dual cell decomposition is the cell decomposition for the Morse function $n - f$. Consider the k-skeleton $X^{(k)}$ of the cell decomposition and the $(n-k-1)$-skeleton $X'^{(n-k-1)}$ of the dual cell decomposition:

$$X^{(k)} = \bigcup_{j \le k} \bigcup_{i=1}^{c_j} e_i^j \quad \text{and} \quad X'^{(n-k-1)} = \bigcup_{j \ge k+1} \bigcup_{i=1}^{c_j} {e'}_i^{n-j}.$$

$X^{(k)}$ and $X'^{(n-k-1)}$ are compact sets. The boundary ∂W_k of W_k is transverse to the gradient flow Ψ_t, and hence $M \setminus (X^{(k)} \cup X'^{(n-k-1)})$ is diffeomorphic to $\partial W_k \times \boldsymbol{R}$ by the map

$$\partial W_k \times \boldsymbol{R} \ni (x,t) \longmapsto \Psi_t(x) \in M \setminus (X^{(k)} \cup X'^{(n-k-1)}).$$

Moreover $\Psi_t(\partial W_k)$ converges to $X^{(k)}$ as $t \longrightarrow -\infty$ and to $X'^{(n-k-1)}$ as $t \longrightarrow \infty$. Hence, $M \setminus X'^{(n-k-1)}$ is diffeomorphic to the interior $\mathrm{int}(W_k)$ of W_k and $X^{(k)}$ is a deformation retract of both W_k and $M \setminus X'^{(n-k-1)}$:

$$X^{(k)} \subset \mathrm{int}(W_k) \subset W_k \subset M \setminus X'^{(n-k-1)}.$$

Hence we call $X^{(k)}$ the core complex of W_k.

The core disks $(D^k \times \{0\})_i$ is in the stable manifold for the gradient flow Ψ_t of the critical point $(\{0\} \times \{0\})_i$ of index k. We may consider the flow Ψ_t on the handle $(D^k \times D^{n-k})_i$ of index k is in the form of a direct product of linear flows. Then the stable manifold e_i^k is written as

$$e_i^k = \bigcup_{t\in(-\infty,0]} \Psi_t((D^k \times \{0\})_i) \quad \text{or} \quad e_i^k = \bigcap_{\tau\in(-\infty,0]} \bigcup_{t\in(-\infty,\tau]} \Psi_t((D^k \times D^{n-k})_i).$$

Using the gradient flow Ψ_t, for any neighborhood V of $X^{(k)}$ and for any compact subset A in $\mathrm{int}(W_k)$, we can construct an isotopy $\{G_t : \mathrm{int}(W_k) \longrightarrow \mathrm{int}(W_k)\}_{t\in[0,1]}$ with compact support such that $G_0 = \mathrm{id}_{\mathrm{int}(W_k)}$, $G_t|X^{(k)} = \mathrm{id}_{X^{(k)}}$ $(t \in [0, 1])$ and $G_1(A) \subset V$. A similar statement is true for $X^{(k)} \subset M \setminus X'^{(n-k-1)}$.

We prove the following lemma which is the core complex version of [21, Lemma 4.3].

Lemma 2.3. *Let M^n be a compact n-dimensional manifold. Let $X^{(k)}$ be the k skeleton of the cell decomposition associated with a Morse function on M^n. Let L^ℓ be a compact set which is a union of finitely many images of $\boldsymbol{R}^s$ $(s < \ell)$ under differentiable maps. If $k+\ell+1 \leq n$ then there is an isotopy $\{F_t : M^n \longrightarrow M^n\}_{t\in[0,1]}$ $(F_0 = \mathrm{id})$ such that $F_1(X^{(k)}) \cap L^\ell = \emptyset$.*

Proof. We construct the isotopy F_t, skeleton by skeleton. Assume that for $u \leq k-1$, there is an isotopy $\{F_t^u\}_{t\in[0,1]}$ $(F_0^u = \mathrm{id})$ such that $F_1^u(X^{(u)}) \cap L^\ell = \emptyset$. Then there is a neighborhood U_u of $X^{(u)}$ such that $F_1^u(U_u) \cap L^\ell = \emptyset$.

Let $u + 1 \leq k$. Since the number of $(u + 1)$-dimensional cells of $X^{(k)}$ is c_{u+1}, there is a negative real number τ_{u+1} such that, for the $(u + 1)$-dimensional cells e_i^{u+1} $(i = 1, \ldots, c_k)$ of $X^{(k)}$, $\Psi_{\tau_{u+1}}((\partial D^{u+1} \times D^{n-u-1})_i) \subset U_u$. Since there are only finitely many handles of index $u + 1$, we can take τ_{u+1} uniformly on i.

We define F_t^{u+1} with support in $\bigcup_{i=1}^{c_{u+1}} \Psi_{\tau_{u+1}}((D^{u+1} \times D^{n-u-1})_i)$. Note that

$$\bigcup_{i=1}^{c_{u+1}} \Psi_{\tau_{u+1}}((D^{u+1} \times D^{n-u-1})_i) \quad (\subset W'_{u+1})$$

is a union of disjoint closed balls in M^n. Since $\Psi_{\tau_{u+1}}((\partial D^{u+1} \times D^{n-u-1})_i) \subset U_u$, there is a disk $(D'^{u+1} \times \{0\})_i \subset (\mathrm{int}(D^{u+1}) \times \{0\})_i$ such that

$$\Psi_{\tau_{u+1}}(((D^{u+1} \setminus \mathrm{int}(D'^{u+1})) \times D^{n-u-1})_i) \subset U_u.$$

Hence

$$X^{(u+1)} \cap L^\ell \subset \bigcup_{i=1}^{c_{u+1}} \Psi_{\tau_{u+1}}((\mathrm{int}(D'^{u+1}) \times \{0\})_i).$$

We have the projection

$$p = \mathrm{proj}_2 \circ \Psi_{-\tau_{u+1}} : \Psi_{\tau_{u+1}}((D^{u+1} \times D^{n-u-1})_i) \longrightarrow D^{n-u-1}.$$

Since $p(\Psi_{\tau_{u+1}}((D^{u+1} \times D^{n-u-1})_i) \cap L^\ell)$ is a finite union of images of $\boldsymbol{R}^s$ $(s \leq \ell \leq n - k - 1 \leq n - u - 2)$ under differentiable maps, it is a measure zero subset of

D^{n-u-1}, and since L^ℓ is compact, it is a nowhere dense subset of D^{n-u-1}. Take a point q close to 0 in the complement of

$$p(\Psi_{\tau_{u+1}}((D^{u+1} \times D^{n-u-1})_i) \cap L^\ell).$$

Let $\{F_t^{u+1}\}_{t\in[0,1]}$ ($F_0^{u+1} = \mathrm{id}$) be the isotopy with support in $\bigcup_{i=1}^{c_{u+1}} \Psi_{\tau_{u+1}}((D^{u+1} \times D^{n-u-1})_i)$ such that

$$F_t^{u+1}(\Psi_{\tau_{u+1}}(x,0)) = \Psi_{\tau_{u+1}}(x, t\mu(x))$$

for $\Psi_{\tau_{u+1}}(x,0) \in (D'^{u+1} \times D^{n-u-1})_i)$, where $\mu : \mathrm{int}(D'^{u+1}) \longrightarrow [0,1]$ is a C^∞ function with compact support such that $\mu(x) = 1$ for $x \in D''^{u+1} \subset \mathrm{int}(D'^{u+1})$ such that

$$\Psi_{\tau_{u+1}}(((D^{u+1} \setminus \mathrm{int}(D''^{u+1})) \times D^{n-u-1})_i) \subset U_u$$

and

$$X^{(u+1)} \cap L^\ell \cap \Psi_{\tau_{u+1}}((D^{u+1} \times D^{n-u-1})_i) \subset \Psi_{\tau_{u+1}}((D''^{u+1} \times \{0\})_i).$$

Thus we obtain an isotopy $\{F_t^{u+1}\}_{t\in[0,1]}$ such that $F_1^{u+1}(X^{(u+1)}) \cap L^\ell = \emptyset$.

Then we define F_t to be the composition of $F_t^k, \dots, F_t^0$. □

Remark 2.4. Note that the support of the isotopy $\{F_t^u\}_{t\in[0,1]}$ is contained in a disjoint union of balls, hence it is contained in a larger embedded ball V_u. Note also that we can choose F_1^u which is a commutator with support in the ball. It is because we can take a ball $V_u' \subset \overline{V_u'} \subset V_u$ which contains the support of the isotopy $\{F_t^u\}_{t\in[0,1]}$, and choose an element $\alpha \in \mathrm{Diff}_c^r(V_u)$ such that $\alpha(V_u') \cap V_u' = \emptyset$ and $\alpha(V_u') \cap X^{(k)} = \emptyset$, Then $F_1^u \alpha (F_1^u)^{-1} \alpha^{-1}$ coincides with F_1^u on $X^{(k)}$.

Proof of Theorem 2.2. By applying Lemma 2.3 to the core complex $X^{(k)}$ of M^n with respect to $X^{(k)}$ itself, there is an isotopy $\{F_t\}_{t\in[0,1]}$ ($F_0 = \mathrm{id}$) such that $F_1(X^{(k)}) \cap X^{(k)} = \emptyset$. Then there is a neighborhood W of $X^{(k)}$ such that $W \cap F_1(W) = \emptyset$. By using the gradient flow, we can construct an isotopy $\{G_t\}_{t\in[0,1]}$ ($G_0 = \mathrm{id}$) such that $G_1(F_1(\overline{W})) \subset W$. Then for $g = G_1 \circ F_1$ and $U = W \setminus G_1(F_1(\overline{W}))$, $g^j(U)$ ($j \in \boldsymbol{Z}$) are disjoint (see [21, Lemma 4.5]).

Note here that $F_1 = F_1^k \circ \cdots \circ F_1^0$ is a product of c commutators with support in balls by Remark 2.4, where $F_t^u = \mathrm{id}$ if there are no handles of index u.

On the other hand, G_1 is defined by using the gradient flow. However, G_1 can also be written as a product of isotopies with support in neighborhoods of

$$(D^u \times D^{n-u})_i \cup \bigcup_{t\in[0,\infty)} \Psi_t((D^u \times \partial D^{n-u})_i)$$

which shrink these sets to the core disks $(D^u \times \{0\})_i$, where $i = 1, \dots, c_u$; $u = 0, \dots, k$. These neighborhoods are balls and the product G_1^u of these isotopies for the handles of the same index u is with support in a disjoint union of balls. Hence it is also supported in a larger embedded ball. By an argument similar to that in Remark 2.4, G_1^u can be replaced by a commutator with support in the ball without changing $G_1^u|(F_1(\overline{W})$. Hence $G_1 = G_1^0 \circ \cdots \circ G_1^k$ is also a product of c commutators with support in balls.

Now any element $f \in \mathrm{Diff}_c^r(M^n)_0$ $(1 \leq r \leq \infty,\ r \neq n+1)$ is conjugate to an element with support in U by an isotopy constructed from the gradient flow Ψ_t. We may assume that the support of f is contained in U.

By the results of Herman-Mather-Thurston ([7], [10], [11], [15], [2]), f can be written as a product of commutators such that the support of each commutator is contained in an embedded ball.

Hence we can write $f = [a_1, b_1] \cdots [a_k, b_k]$, where the supports of a_i and b_i are contained in a ball V_i in U. We put

$$H = \prod_{i=1}^{k} g^{k-i}([a_1, b_1] \cdots [a_i, b_i]) g^{i-k},$$

where $g = G_1 \circ F_1$. Then H is an element of $\mathrm{Diff}_c^r(M^n)_0$ and

$$\begin{aligned} H^{-1} g H g^{-1} &= ([a_1, b_1] \cdots [a_k, b_k])^{-1} \prod_{i=0}^{k-1} g^{k-i} [a_{i+1}, b_{i+1}] g^{i-k} \\ &= f^{-1} \prod_{i=0}^{k-1} g^{k-i} [a_{i+1}, b_{i+1}] g^{i-k} \\ &= f^{-1} \Big[\prod_{i=0}^{k-1} g^{k-i} a_{i+1} g^{i-k}, \prod_{i=0}^{k-1} g^{k-i} b_{i+1} g^{i-k} \Big]. \end{aligned}$$

By putting $A = \displaystyle\prod_{i=0}^{k-1} g^{k-i} a_{i+1} g^{i-k}$ and $B = \displaystyle\prod_{i=0}^{k-1} g^{k-i} b_{i+1} g^{i-k}$, f can be written as a product of two commutators: $f = [A, B][g, H^{-1}]$.

Now, note that the supports of A and B are contained in a disjoint union $\bigcup_{i=1}^k g(V_i)$ of balls $g(V_i)$. Thus the supports of A and B are contained in a larger embedded ball.

Since F_1 and G_1 can be written as products of c commutators with support in balls, $g = G_1 \circ F_1$ can be written as a product of $2c$ commutators with support in balls and $[g, H^{-1}] = g(H^{-1} g^{-1} H)$ can be written as a product of $4c$ commutators with support in balls. Thus f can be written as a product of $4c + 1$ commutators with support in balls. □

Remark 2.5. In many cases, we can construct F_1 such that $(F_1)^j(W)$ $(j \in \boldsymbol{Z})$ are disjoint. In this case, we use F_1 and W in the place of g and U, and f is written as a product of $2c + 1$ commutators with support in balls. In particular, for a 3-dimensional handle body H^3, this is the case, where $c = 2$. Hence any element of $\mathrm{Diff}_c^r(H^3)_0$ $(1 \leq r \leq \infty,\ r \neq 4)$ can be written as a product of 5 commutators with support in balls.

3. Uniform simplicity of the diffeomorphism groups

First we review how the perfectness of $\mathrm{Diff}^r_c(\boldsymbol{R}^n)_0$ implies the simplicity of $\mathrm{Diff}^r_c(M^n)_0$ for a connected manifold M. That is, we have the following lemma which is now well known.

Lemma 3.1. *Let M^n be a connected n-dimensional manifold. Let g be a nontrivial element of $\mathrm{Diff}^r_c(M^n)_0$. Assume that $f \in \mathrm{Diff}^r_c(M^n)_0$ is written as a product of commutators $[a_i, b_i]$ $(i = 1, \ldots, k)$: $f = [a_1, b_1] \cdots [a_k, b_k]$, where a_i and b_i are with support in an embedded ball $U_i \subset \overline{U_i} \subset M^n$. Then f can be written as a product of $4k$ conjugates of g and g^{-1}.*

Proof. Since g is a nontrivial element of $\mathrm{Diff}^r_c(M^n)_0$, there is an open ball $U \subset \overline{U} \subset M^n$ such that $g(U) \cap U = \emptyset$. Then any commutator $[a, b]$ in $\mathrm{Diff}^r_c(U)_0$ can be written as a product of 4 conjugates of g or g^{-1}. For, if a, $b \in \mathrm{Diff}^r_c(U)_0$, then by putting $c = g^{-1}ag$, we have $cb = bc$ and

$$\begin{aligned} aba^{-1}b^{-1} &= gcg^{-1}bgc^{-1}g^{-1}b^{-1} \\ &= gcg^{-1}c^{-1}cbgc^{-1}b^{-1}bg^{-1}b^{-1} \\ &= g(cg^{-1}c^{-1})(bcgc^{-1}b^{-1})(bg^{-1}b^{-1}). \end{aligned}$$

Now for $f = [a_1, b_1] \cdots [a_k, b_k]$, there are balls U_i such that $\mathrm{supp}(a_i)$, $\mathrm{supp}(b_i) \subset U_i$. By the ball theorem, there is a diffeomorphism $h_i \in \mathrm{Diff}^r(M^n)_0$ such that $h_i(U_i) = U$. Since $h_i[a_i, b_i]{h_i}^{-1}$ is with support in U, it can be written as a product of 4 conjugates of g or g^{-1}: $h_i[a_i, b_i]{h_i}^{-1} \in (C_g)^4$. Hence $[a_i, b_i] \in (C_g)^4$ and $f = [a_1, b_1] \cdots [a_k, b_k] \in (C_g)^{4k}$. □

Before proving Theorem 1.4, we give a remark which makes a better estimate on the number of commutators than our previous one ([21, Theorem 5.2]).

Remark 3.2. In [21, Theorem 5.2], we showed that any element $f \in \mathrm{Diff}(S^n)_0$ can be written as a product of 4 commutators. However, we can in fact write $f \in \mathrm{Diff}(S^n)_0$ as a product of 3 commutators with support in embedded balls. The reason is as follows: By [21, Theorem 5.1], for $f \in \mathrm{Diff}^r(S^n)_0$, we have the decomposition $f = g \circ h$, where $g \in \mathrm{Diff}^r_c(S^n \setminus Q^0)_0$ and $h \in \mathrm{Diff}^r_c(S^n \setminus P^0)_0$ for some points P^0 and $Q^0 \in S^n$. We have a closed ball $\overline{V}$ containing the support of the isotopy of g and take a diffeomorphism $\alpha \in \mathrm{Diff}^r_c(S^n \setminus Q^0)_0$, such that $\alpha(V) \cap V = \emptyset$ and $P^0 \notin \alpha(V)$. Then

$$f = (g\alpha g^{-1}\alpha^{-1}) \circ (\alpha g^{-1}\alpha^{-1}h)$$

and

$$\mathrm{supp}(\alpha g^{-1}\alpha^{-1}h) \subset \alpha(V) \cup \mathrm{supp}(h) \not\ni P^0.$$

Thus $g\alpha g^{-1}\alpha^{-1} \in \mathrm{Diff}^r_c(S^n \setminus Q^0)_0$ and $\alpha g^{-1}\alpha^{-1}h \in \mathrm{Diff}^r_c(S^n \setminus P^0)_0$. Here $\alpha g^{-1}\alpha^{-1}h$ can be written as a product of 2 commutators by Theorem 2.1 ([21, Theorem 4.1]). Since $S^n \setminus Q^0$ and $S^n \setminus P^0$ are diffeomorphic to $\boldsymbol{R}^n$ and any commutator of $\mathrm{Diff}^r_c(\boldsymbol{R}^n)_0$ is with support in a ball, f can be written as a product of 3 commutators with support in embedded balls.

Proof of Theorem 1.4. By Remark 3.2, any element $f \in \mathrm{Diff}^r(S^n)_0$ can be written as a product of 3 commutators with support in embedded balls. By Lemma 3.1, f is written as a product of $4 \cdot 3 = 12$ conjugates of γ or γ^{-1} for any nontrivial element $\gamma \in \mathrm{Diff}^r(S^n)_0$. □

By using Theorem 2.2 and [21, Theorem 5.2], the proof of Theorem 1.5 is straightforward.

Proof of Theorem 1.5. Let M^{2m} be a compact connected $(2m)$-dimensional manifold with handle decomposition without handles of index m. For M^{2m}, from the handle decomposition, we obtain P^{m-1} and $Q^{m-1} \subset M^{2m}$ such that $P^{m-1} \subset M^{2m} \setminus Q^{m-1}$ and $Q^{m-1} \subset M^{2m} \setminus P^{m-1}$ are deformation retracts. By [21, Theorem 5.2], any element f of $\mathrm{Diff}^r(M^{2m})_0$ can be decomposed as $f = g \circ h$, where $g \in \mathrm{Diff}^r_c(M^{2m} \setminus k(Q^{m-1}))_0$ and $h \in \mathrm{Diff}^r_c(M^{2m} \setminus P^{m-1})_0$. Then by Theorem 2.2, g and h can be written as products of $4c(M^{2m} \setminus k(Q^{m-1}))+1$ and $4c(M^{2m} \setminus P^{m-1})+1$ commutators with support in balls if $1 \leq r \leq \infty$, $r \neq 2m+1$, respectively. Since

$$c(M^{2m} \setminus k(Q^{m-1}) + c(M^{2m} \setminus P^{m-1}) = c(M^{2m}),$$

f can be written as $4c(M^{2m}) + 2$ commutators with support in balls. By Lemma 3.1, for any nontrivial element $\gamma \in \mathrm{Diff}^r(M^{2m})_0$, f can be written as a product of $16c(M^{2m}) + 8$ conjugates of γ or γ^{-1}. □

Before proving Theorem 1.6, we give a better estimate on the number of commutators than our previous one ([21, Theorem 6.1]).

Remark 3.3. In [21, Theorem 6.1], we showed that for a compact $(2m+1)$-dimensional manifold M^{2m+1}, any element $f \in \mathrm{Diff}^r(M^{2m+1})_0$ ($1 \leq r \leq \infty$, $r \neq 2m+2$) can be written as a product of 6 commutators. We can in fact write $f \in \mathrm{Diff}^r(M^{2m+1})_0$ as a product of 5 commutators. The reason is just as follows: For a compact connected $(2m+1)$-dimensional manifold M^{2m+1}, we obtain P^m and $Q^m \subset M^{2m+1}$ from the handle decomposition such that $P^m \subset M^{2m+1} \setminus Q^m$ and $Q^m \subset M^{2m+1} \setminus P^m$ are deformation retracts. By [21, Theorem 6.2], any element f of $\mathrm{Diff}^r(M^{2m+1})_0$ can be decomposed as $f = a \circ g \circ h$, where a is with support in a disjoint union of balls, $g \in \mathrm{Diff}^r_c(M^{2m+1} \setminus k(Q^m))_0$ and $h \in \mathrm{Diff}^r_c(M^{2m+1} \setminus k'(P^m))_0$. By an argument similar to that in Remark 2.4 or 3.2, the diffeomorphism a can be replaced by a commutator with support in the ball by changing g. Since g and h can be written as products of two commutators by Theorem 2.1 ([21, Theorem 4.1]), f can be written as products of 5 commutators.

Proof of Theorem 1.6. By Remark 3.3, any element f of $\mathrm{Diff}^r(M^{2m+1})_0$ is decomposed as $f = a \circ g \circ h$, where a is a commutator with support in the ball, $g \in \mathrm{Diff}_c^r(M^{2m+1} \setminus k(Q^m))_0$ and $h \in \mathrm{Diff}_c^r(M^{2m+1} \setminus k'(P^m))_0$. Then by Theorem 2.2, g and h can be written as products of $4c(M^{2m+1} \setminus k(Q^m)) + 1$ and $4c(M^{2m+1} \setminus k'(P^m)) + 1$ commutators with support in balls if $1 \leq r \leq \infty$, $r \neq 2m+2$, respectively. Since

$$c(M^{2m} \setminus k(Q^m) + c(M^{2m+1} \setminus k'(P^m)) = c(M^{2m}),$$

f can be written as $4c(M^{2m+1}) + 3$ commutators with support in balls. By Lemma 3.1, for any nontrivial element $\gamma \in \mathrm{Diff}^r(M^{2m+1})_0$, f can be written as a product of $16c(M^{2m+1}) + 12$ conjugates of γ or γ^{-1}. □

Proof of Corollary 1.8. By Remark 2.5, for a 3-dimensional open handle body H^3, any element of $\mathrm{Diff}_c^r(H^3)$ ($1 \leq r \leq \infty$, $r \neq 4$) can be written as a product of 5 commutators with support in balls. Now any element $f \in \mathrm{Diff}^r(M^3)_0$, can be decomposed as $f = a \circ g \circ h$ as in the proof of Theorem 1.6. Since g and h can be written as products of 5 commutators with support in balls, f can be written as 11 commutators with support in balls. By Lemma 3.1, for any nontrivial element $\gamma \in \mathrm{Diff}^r(M^3)_0$, f can be written as a product of 44 conjugates of γ or γ^{-1}. □

Remark 3.4. The uniform simplicity of the groups we treated also follows from a proposition of Burago-Ivanov-Polterovich ([3, Proposition 1.15]), our previous remark ([21, Remark 6.6]) and Lemma 3.1. We note here that the fragmentation norm ([3]) of an element of $\mathrm{Diff}^r(S^n)_0$ is at most 2, that of an element of $\mathrm{Diff}^r(M^{2m})_0$ for M^{2m} with handle decomposition without handles of index m is at most $2c(M^{2m})+2$, that of an element of $\mathrm{Diff}^r(M^{2m+1})_0$ is at most $2c(M^{2m+1}) + 3$. The reason is that for $g = G_1 \circ F_1$ which we used in the proof of Theorem 2.2,

$$\begin{aligned} G_1 \circ F_1 &= G_1^0 \circ \cdots \circ G_1^k \circ F_1^k \circ \cdots \circ F_1^0 \\ &= (G_1^0 \circ F_1^0) \circ (F_1^0)^{-1} \circ (G_1^1 \circ F_1^1) \circ (F_1^0) \\ &\qquad \circ (F_1^1 \circ F_1^0)^{-1} \circ (G_1^2 \circ F_1^2) \circ (F_1^1 \circ F_1^0) \\ &\qquad \circ \cdots \circ (F_1^{k-1} \circ \cdots \circ F_1^0)^{-1} \circ (G_1^k \circ F_1^k) \circ (F_1^{k-1} \circ \cdots \circ F_1^0) \end{aligned}$$

and $G_1^u \circ F_1^u$ ($0 \leq u \leq k$) is with support in a union of disjoint balls, hence is with support in a larger ball. Hence $g = G_1 \circ F_1$ can be written as a product of c diffeomorphisms with support in embedded balls.

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