UTMS 2008–23 August 19, 2008

A Completeness Theorem for the Logical System MPCL Designed for Mathematical Psychology

by

Kensaku Gomi and Yasuaki Mizumura

UNIVERSITY OF TOKYO
GRADUATE SCHOOL OF MATHEMATICAL SCIENCES
KOMABA, TOKYO, JAPAN

A Completeness Theorem for the Logical System MPCL Designed for Mathematical Psychology

Kensaku GOMI * and Yasuaki MIZUMURA †

August 19, 2008

Abstract

Our end in mathematical psychology is to construct and analyze and utilize a mathematical model of the human system of thinking, the outer world which human cognizes, and the relationship between them, from mechanists' viewpoint. As the core of our mathematical model, we need some good logical system, and MPCL is our present tentative one. We will define it, give an account of its relationship with natural languages, and prove a completeness theorem for it, based on a newly formulated universal algebraic logic.

Key words: thinking machine, logic, language, algebra

2000 Mathematics Subject Classification: 03B80

Contents

*Graduate School of Mathematical Sciences, University of Tokyo, 3-8-1 Komaba, Meguro, Tokyo 153-8914 Japan. email:mathpsy@mb.infoweb.ne.jp.

†Information System Dept., Corporate Planning & Coordination, Asahi Kasei Chemicals Corporation, 1-105 Kanda-Jinbocho, Chiyoda, Tokyo 101-8101 Japan. email:mizumura.yc@om.asahi-kasei.co.jp

1 Introduction

The purpose of this paper is to prove a completeness theorem for the logical system MPCL which is designed for mathematical psychology. Our end is to construct and analyze and utilize a mathematical model of the human system of thinking, the outer world which human cognizes, and the relationship between them, from mechanists' viewpoint. Therefore we have nothing to do with old mathematical psychology which centers around statistical treatments of experimental data. Our mathematical psychology is rather a close relative of metamathematics, philosophy of logic, theoretical linguistics, and so on. In particular, Richard Montague's theory [9] of natural languages had a great influence on us at the outset, although mathematical psychology is not the theory of natural languages.

As the core of our mathematical model, we need some good logical system. At present we are in the process of trial and error in order to find out the still unknown ideal logical system. This process may well be compared to the one which past metamathematicians had experienced before they invented the logical system PL of predicate logic [6]. Nevertheless we believe that our present tentative logical system MPCL is of worth, which we will describe in §3.6.

"MPCL" is an abbreviation for **monophasic case logic**. The set K of "cases" is one of the parameters of the formal language of MPCL. Roughly speaking, cases are mathematical models of substances which are supposed to exist in human brain, and some of which are supposed to be expressed by some of Japanese postpositional particles called **teniwoha** such as "ga," "wa," "wo," "ni," some of English prepositions, and so on.

Human thinks about various phases of the entities in the outer world such as specific location, direction, time, the recipient of an action, and so on, while for instance, one Japanese postpositional particle "ni" may be used to indicate those various phases. Therefore one postpositional particle "ni" is considered an expression of various substances in various phases in human brain, and so K must be divided into various phases. However, a mathematical model with such various phases seemed difficult to study in the first attempt, and thus we have begun with the model MPCL with only one phase.

In fact, this paper is an abridged translation of an impermanent aspect of the personal electronic publication *Mathematical Psychology* [3] by the first author, where our work in progress has been shown for more than decade by frequent revisions, and in particular, the logical system CL of the coming generation has already been born with arbitrary number of phases.

The formal language of MPCL is quite different from that of PL as indicated by the existence of the parameter K. It is also notably different from the PL language in that it has plenty of quantifiers and that they are not accompanied by variables but by cases, just as quantifiers in orthodox Japanese are accompanied by teniwoha. However, those are rather superficial linguistic differences. As a logical system, MPCL is deeply different from PL in that consistent sets of "closed predicates" do not necessarily possess models [8].

In spite of those differences, MPCL is an extension of PL in the sense that PL is embedded in MPCL, just as we read and understand PL sentences translating them into natural languages. For this and some other reasons, our strategy for MPCL completeness is an extension of that for PL completeness, while tactics not obtained from PL theory were supplied by the second author [7].

2 Definitions for stating the main result

Here we show definitions other than that of MPCL to the extent necessary for stating the main result of this paper. We expect that the readers will have a variety of backgrounds such as mathematical logic, philosophy, linguistics, and so on. Although our mathematical psychology overlaps with them, it is a new branch different from any one of them, and as such, free to use new terminologies and formulations. Therefore, we will make this paper as self-contained as tolerated. However, we expect that our set-theoretical notation and terminology will be standard except that we denote the set of all mappings of a set Y into a set Z by $Y \to Z$. Thus we write $f \in Y \to Z$ instead of $f : Y \to Z$.

2.1 Sorted algebras

For each set A and each natural number n, an n-ary **operation** on A is a mapping α of a subset D of A^n into A. The set D is called the **domain** of α and denoted by $\mathrm{Dom}\,\alpha$, while the image αD is denoted by $\mathrm{Im}\,\alpha$. The number n is called an **arity** of α, and so if $D = \emptyset$, every natural number is an arity of α. We say that α is **global** if $D = A^n$. A subset B of A is said to be **closed** under the operation α if $\alpha(a_1, \ldots, a_n) \in B$ for each $(a_1, \ldots, a_n) \in B^n \cap D$. If B is closed under α, the **restriction** $\alpha|_{B^n \cap D}$ of α to B is an operation on B.

An **algebra** is a set A equipped with a family $(\alpha_\lambda)_{\lambda \in \Lambda}$ of operations on A, which we call the **operation system** or **OS** of the algebra A. We often identify the operation α_λ with its index λ. The algebra $(A, (\alpha_\lambda)_{\lambda \in \Lambda})$ is said to be **global** if α_λ is global for every $\lambda \in \Lambda$.

The algebra $(A, (\alpha_\lambda)_{\lambda \in \Lambda})$ has two kinds of **subalgebras**. The first is an algebra $(A, (\alpha_\mu)_{\mu \in M})$ obtained by reducing the OS of A from $(\alpha_\lambda)_{\lambda \in \Lambda}$ to $(\alpha_\mu)_{\mu \in M}$ for a subset M of Λ. Such an algebra will be called an **operational subalgebra**. Also, if a subset B of A is closed under α_λ for each $\lambda \in \Lambda$, then B becomes an algebra equipped with the operation system $(\beta_\lambda)_{\lambda \in \Lambda}$ consisting of restrictions β_λ of α_λ to B. Such an algebra $(B, (\beta_\lambda)_{\lambda \in \Lambda})$ is called a **support subalgebra**.

Let $(A, (\alpha_\lambda)_{\lambda \in \Lambda})$ be an algebra. Then the intersection of support subalgebras of A is also a support subalgebra of A, and A itself is a support subalgebra of A. Therefore, for each subset S of A, the intersection of all support subalgebras of A which contain S is the smallest of the support subalgebras of A which contain S. We denote it by $[S]$ and call it the **closure** of S or the support subalgebra **generated** by S. A support subalgebra B is said to be **finitely generated** if $B = [S]$ for some finite subset S of A. Define the subsets S_n $(n = 0, 1, \ldots)$ of A inductively as follows. First $S_0 = S$. Next for each $n \geq 1$, S_n is the set of all elements $\alpha_\lambda(a_1, \ldots, a_m)$ with $\lambda \in \Lambda$, $(a_1, \ldots, a_m) \in \mathrm{Dom}\,\alpha_\lambda$, and $a_i \in S_{l_i}$ $(i = 1, \ldots, m)$ for some non-negative integers $l_1, \ldots, l_m$ such that $n = 1 + \sum_{i=1}^m l_i$. Then it is easy to show $[S] = \bigcup_{n \geq 0} S_n$. We call S_n $(n = 0, 1, \ldots)$ the **descendants** of S.

Two algebras A and B are said to be **similar**, if they have operation systems $(\alpha_\lambda)_{\lambda \in \Lambda}$ and $(\beta_\lambda)_{\lambda \in \Lambda}$ indexed by the same set Λ, and α_λ and β_λ have a common arity for each $\lambda \in \Lambda$.

Let $(A, (\alpha_\lambda)_{\lambda \in \Lambda})$ and $(B, (\beta_\lambda)_{\lambda \in \Lambda})$ be similar algebras. Then a mapping f of A into B is called a **homomorphism** if it satisfies the following two conditions for all $\lambda \in \Lambda$, where n_λ denotes an arity common to α_λ and β_λ.

- If $(a_1, \ldots, a_{n_\lambda}) \in \mathrm{Dom}\,\alpha_\lambda$, then $(fa_1, \ldots, fa_{n_\lambda}) \in \mathrm{Dom}\,\beta_\lambda$ and $f(\alpha_\lambda(a_1, \ldots, a_{n_\lambda})) = \beta_\lambda(fa_1, \ldots, fa_{n_\lambda})$.
- If $(a_1, \ldots, a_{n_\lambda}) \in A^{n_\lambda}$ and $(fa_1, \ldots, fa_{n_\lambda}) \in \mathrm{Dom}\,\beta_\lambda$, then $(a_1, \ldots, a_{n_\lambda}) \in \mathrm{Dom}\,\alpha_\lambda$.

A bijective homomorphism is called an **isomorphism**. If both A and B are global algebras, a mapping f of A into B is a homomorphism iff it satisfies the following condition for all $\lambda \in \Lambda$ and all $(a_1, \dots, a_{n_\lambda}) \in A^{n_\lambda}$:

$$f(\alpha_\lambda(a_1, \dots, a_{n_\lambda})) = \beta_\lambda(fa_1, \dots, fa_{n_\lambda}).$$

A **sorted algebra** is an algebra A equipped with an algebra T similar to A and a homomorphism σ of A into T. We call T and σ the **type algebra** and the **sort mapping** of the sorted algebra A. For each subset S of A and each $t \in T$, we define the **t-part** S_t of S to be the inverse image $\{a \in S \mid \sigma a = t\}$ of t in S by σ.

Let (A, T, σ) and (B, T, τ) be sorted algebras with the same type algebra T. Then a mapping f of A into B is said to be **sort-consistent**, if it satisfies $\tau f = \sigma$, or equivalently $f(A_t) \subseteq B_t$ for all $t \in T$.

A sorted algebra (A, T, σ) is said to be **universal** or called a **USA** if A has a subset S which satisfies the following two conditions, the latter being called the **universality**.

- $A = [S]$.
- If (A', T, σ') is a sorted algebra and φ is a mapping of S into A' satisfying $\sigma'\varphi = \sigma|_S$, then there exists a sort-consistent homomorphism f of A into A' which extends φ.

We call S as above the set of the **primes** of A. It is known that every sorted algebra has at most one prime set and that f in the above condition is uniquely determined by φ.

The following theorem is known to hold.

Theorem 2.1 (Unique Existence of USA) Let S be a set, T be an algebra, and τ be a mapping of S into T. Then there exists a USA (A, T, σ, S) with $\sigma|_S = \tau$. If (A', T, σ', S) is also a USA with $\sigma'|_S = \tau$, then there exists a sort-consistent isomorphism of A onto A' extending id_S.

For a proof, we refer the reader to [3][4]. In the course of the proof, it is shown that if (A, T, σ, S) is a USA then A is the direct union of the descendants S_n $(n = 0, 1, \dots)$ of S. Therefore, for each element a of A, there exists a unique non-negative integer n such that $a \in S_n$. We call it the **rank** of a. It is also shown that if $a \in S$ then a has no expression $a = \alpha(a_1, \dots, a_k)$ by an operation α in the OS of A, while if $a \in A - S$ then a has a unique such expression and rank $a = 1 + \sum_{i=1}^{k}$ rank a_i.

Let (A, T, σ) be a sorted algebra and V be a non-empty set. Define $A^V = \bigcup_{t \in T}(V \to A_t)$. Then we can construct a sorted algebra (A^V, T, ρ) as follows. First define the sort mapping ρ of A^V into T by $\rho b = t$ for each $b \in V \to A_t$ and each $t \in T$. Then

$$\rho b = \sigma(b\nu)$$

for each $b \in A^V$ and each $v \in V$. Let $(\alpha_\lambda)_{\lambda\in\Lambda}$ and $(\tau_\lambda)_{\lambda\in\Lambda}$ be the OS's of A and T respectively, and let n_λ be an arity of α_λ and τ_λ. For each $\lambda \in \Lambda$, define the operation β_λ on A^V as follows. First define the domain of β_λ to be

$$D_\lambda = \big\{(b_1, \dots, b_{n_\lambda}) \in (A^V)^{n_\lambda} \mid (\rho b_1, \dots, \rho b_{n_\lambda}) \in \mathrm{Dom}\,\tau_\lambda\big\}.$$

If $(b_1, \dots, b_{n_\lambda}) \in D_\lambda$, then $\big(\sigma(b_1 v), \dots, \sigma(b_{n_\lambda} v)\big) = (\rho b_1, \dots, \rho b_{n_\lambda}) \in \mathrm{Dom}\,\tau_\lambda$ so $(b_1 v, \dots, b_{n_\lambda} v) \in \mathrm{Dom}\,\alpha_\lambda$ for each $v \in V$, and we can define the mapping $\beta_\lambda(b_1, \dots, b_{n_\lambda})$ of V into A by

$$\big(\beta_\lambda(b_1, \dots, b_{n_\lambda})\big)v = \alpha_\lambda(b_1 v, \dots, b_{n_\lambda} v)$$

for each $v \in V$. Furthermore

$$\sigma\big(\alpha_\lambda(b_1 v, \dots, b_{n_\lambda} v)\big) = \tau_\lambda\big(\sigma(b_1 v), \dots, \sigma(b_{n_\lambda} v)\big) = \tau_\lambda(\rho b_1, \dots, \rho b_{n_\lambda}),$$

and $t = \tau_\lambda(\rho b_1, \dots, \rho b_{n_\lambda})$ is not varied by $v \in V$, hence $\beta_\lambda(b_1, \dots, b_{n_\lambda}) \in V \to A_t \subseteq A^V$. Thus β_λ certainly is an operation on A^V for each $\lambda \in \Lambda$, and so $(A^V, (\beta_\lambda)_{\lambda\in\Lambda})$ becomes an algebra. Furthermore

$$\begin{aligned}\rho(\beta_\lambda(b_1, \dots, b_{n_\lambda})) &= \sigma\big(\big(\beta_\lambda(b_1, \dots, b_{n_\lambda})\big)v\big)\\ &= \sigma\big(\alpha_\lambda(b_1 v, \dots, b_{n_\lambda} v)\big) = \tau_\lambda(\rho b_1, \dots, \rho b_{n_\lambda})\end{aligned}$$

with any element $v \in V$, and so ρ is a homomorphism of A^V into T. Thus we have constructed the sorted algebra (A^V, T, ρ), which we call the **power algebra** of A with **exponent** V. Furthermore, it follows from the above definition that for each $v \in V$ the mapping $b \mapsto bv$ of A^V into A is a sort-consistent homomorphism, which we call the **projection** by v.

Let $(A, (\alpha_\lambda)_{\lambda\in\Lambda})$ be an algebra. If, for two elements a and b of A, there exists an element $\lambda \in \Lambda$ such that $a = \alpha_\lambda(\dots, b, \dots)$, then we write $b \prec a$. If $b \prec a$ or $b = a$, we write $b \preceq a$. If there exists a sequence $b_0, \dots, b_n$ of elements of A such that $b_0 = a$, $b_n = b$ and $b_i \preceq b_{i-1}$ for $i = 1, \dots, n$, then we say that b **occurs** in a and call the sequence an **occurrence** of b in a.

In the rest of this subsection, let (A, T, σ, S) be a USA, $(\alpha_\lambda)_{\lambda\in\Lambda}$ and $(\tau_\lambda)_{\lambda\in\Lambda}$ be the OS's of A and T respectively, and assume that Λ is contained in the set of all formal products of elements of $\Gamma \amalg S$ for some set Γ. More precisely, Λ is a subset of the free semigroup over $\Gamma \amalg S$. For each element λ of Λ, let S^λ denote the set of the elements of S which occur in λ.

Let $a \in A$ and $s \in S$. Then an occurrence $s_0, \dots, s_n$ of s in a is said to be **free**, if $\{s_0, \dots, s_n\} \cap \mathrm{Im}\,\alpha_\lambda = \emptyset$ for each $\lambda \in \Lambda$ such that $s \in S^\lambda$. If there exists a free occurrence of s in a, we write $s \ll a$ or say that s **occurs free** in a. For each subset X of S, we define $X^a_{\mathrm{free}} = \{x \in X \mid x \ll a\}$. Let $b \in A$. Then the occurrence $s_0, \dots, s_n$ of s in a is said to be **free from** b, if $\{s_0, \dots, s_n\} \cap \mathrm{Im}\,\alpha_\lambda = \emptyset$ for each $\lambda \in \Lambda$ such that $(S^\lambda)^b_{\mathrm{free}} \neq \emptyset$. We say that s is **free from** b **in** a, if every free occurrence of s in a is free from b.

Let $s \in S$ and $c \in A$ with $\sigma s = \sigma c$. Then, for each element a of A, we can define the element $a(s/c)$ of A with $\sigma\big(a(s/c)\big) = \sigma a$ by induction on the rank

r of a as follows. If $r = 0$, then $a \in S$, and so we define

$$a(s/c) = \begin{cases} c & \text{if } a = s, \\ a & \text{if } a \neq s, \end{cases} \tag{2.1}$$

hence $\sigma\big(a(s/c)\big) = \sigma a$ as desired. Suppose $r \geq 1$. Then a has a unique expression $a = \alpha_\lambda(a_1, \dots, a_k)$ and r is greater than the ranks of $a_1, \dots, a_k$, so $a_i(s/c)$ has already been defined and satisfies $\sigma\big(a_i(s/c)\big) = \sigma a_i$ for $i = 1, \dots, k$. Since $(\sigma a_1, \dots, \sigma a_k)$ belongs to $\operatorname{Dom} \tau_\lambda$, so does $\big(\sigma\big(a_1(s/c)\big), \dots, \sigma\big(a_k(s/c)\big)\big)$, hence $\big(a_1(s/c), \dots, a_k(s/c)\big) \in \operatorname{Dom} \alpha_\lambda$, and so we define

$$a(s/c) = \begin{cases} \alpha_\lambda\big(a_1(s/c), \dots, a_k(s/c)\big) & \text{if } s \notin S^\lambda, \\ a & \text{if } s \in S^\lambda. \end{cases} \tag{2.2}$$

Then even when $a(s/c) \neq a$, we have

$$\begin{aligned} \sigma\big(a(s/c)\big) &= \sigma\big(\alpha_\lambda\big(a_1(s/c), \dots, a_k(s/c)\big)\big) \\ &= \tau_\lambda\big(\sigma\big(a_1(s/c)\big), \dots, \sigma\big(a_k(s/c)\big)\big) = \tau_\lambda(\sigma a_1, \dots, \sigma a_k) = \sigma a \end{aligned}$$

as desired. The definition of $a(s/c)$ by induction is complete. We call the transformation $a \mapsto a(s/c)$ on A the **substitution of** c **for** s. Since $\sigma\big(a(s/c)\big) = \sigma a$, the substitution is sort-consistent.

2.2 Formal languages

By definition of moderate generality, a **formal language** is a universal sorted algebra (A, T, σ, S) equipped with subsets C and $X \neq \emptyset$ of S and a set Γ which satisfy the following three conditions.

- The prime set S is the direct sum $C \amalg X$ of C and X.
- Let $(\tau_\lambda)_{\lambda \in \Lambda}$ be the OS of the type algebra T. Then its index set Λ is contained in the direct sum $\Gamma \amalg \Gamma X$ of Γ and ΓX, where ΓX is the set of all formal products γx of $\gamma \in \Gamma$ and $x \in X$.
- The arity of each operation τ_λ with $\lambda \in \Lambda \cap \Gamma X$ is equal to 1.

We call C and X the sets of the **constants** and **variables** respectively. Henceforth, we identify each index $\lambda \in \Lambda \cap \Gamma X$ with the operation τ_λ, call it a **variable operation**, and denote its domain by T_λ.

Let $(A, T, \sigma, S, C, X, \Gamma)$ be a formal language. Define $\Lambda' = \Lambda \cap \Gamma$ and let T' be the operational subalgebra of T obtained by reducing the OS of T from $(\tau_\lambda)_{\lambda \in \Lambda}$ to $(\tau_\lambda)_{\lambda \in \Lambda'}$. Then, a sorted algebra W is called a **cognizable world** for A, if it satisfies the following two conditions.

- The type algebra of W is equal to T'.
- $W_t \neq \emptyset$ for each $t \in \sigma S$.

Furthermore, an arbitrarily chosen non-empty collection $\mathcal{W}$ of cognizable worlds for A is called the domain of the **actual worlds** for A.

For each actual world $W \in \mathcal{W}$ for A, a C-**denotation** into W is a mapping Φ of C into W which satisfies $\Phi C_t \subseteq W_t$ for each $t \in T$. There is at least one C-denotation. If $C = \emptyset$, then since $\emptyset \to W = \{\emptyset\}$ by the set-theoretical definition of $Y \to Z$, $\emptyset$ is the unique C-denotation. Similarly, an X-**denotation** into W is a mapping ν of X into W which satisfies $\nu X_t \subseteq W_t$ for each $t \in T$. We denote the set of all X-denotations into W by $V_{X,W}$, because denotations are alternatively called **valuations**. Then $V_{X,W} \neq \emptyset$, and so we can construct the power algebra $(W^{V_{X,W}}, T', \rho)$ of W with exponent $V_{X,W}$ as described in §2.1. Let $(\beta_\lambda)_{\lambda\in\Lambda'}$ be its OS.

Suppose that, for an actual world $W \in \mathcal{W}$ for A and for each variable operation $\lambda \in \Lambda \cap \Gamma X$ and the variable x such that $\lambda \in \Gamma x$, we are given a mapping

$$\lambda_W \in \left(\bigcup_{t\in T_\lambda} (W_{\sigma x} \to W_t) \right) \to W$$

which satisfies

$$\lambda_W(W_{\sigma x} \to W_t) \subseteq W_{\lambda t}$$

for each $t \in T_\lambda$. Then we can define the unary operation β_λ on $W^{V_{X,W}}$ for each $\lambda \in \Lambda \cap \Gamma X$ as follows, and extending the OS of $W^{V_{X,W}}$ from $(\beta_\lambda)_{\lambda\in\Lambda'}$ to $(\beta_\lambda)_{\lambda\in\Lambda}$, we can construct the sorted algebra $(W^{V_{X,W}}, T, \rho)$. First we define, for each pair x, w of $x \in X$ and $w \in W_{\sigma x}$, the transformation $\nu \mapsto (x/w)\nu$ on $V_{X,W}$ by

$$\big((x/w)\nu\big)y = \begin{cases} \nu y & \text{when } X \ni y \neq x, \\ w & \text{when } y = x. \end{cases} \tag{2.3}$$

We call the transformation (x/w) the **redenotation** for x by w. Next we define, for each quadruple t, φ, x, ν consisting of $t \in T$, $\varphi \in V_{X,W} \to W_t$, $x \in X$ and $\nu \in V_{X,W}$, the mapping $\varphi\big((x/\square)\nu\big)$ of $W_{\sigma x}$ into W_t by

$$\Big(\varphi\big((x/\square)\nu\big)\Big)\, w = \varphi\big((x/w)\nu\big)$$

for each $w \in W_{\sigma x}$. We finally define for each $\lambda \in \Lambda \cap \Gamma X$ the unary operation β_λ on $W^{V_{X,W}}$ as follows. Suppose $\lambda \in \Gamma x$ with $x \in X$. First we define

$$\operatorname{Dom} \beta_\lambda = \bigcup_{t\in T_\lambda} (V_{X,W} \to W_t).$$

Next for each $t \in T_\lambda$ and each $\varphi \in V_{X,W} \to W_t$ we define $\beta_\lambda\varphi$ to be the element of $V_{X,W} \to W_{\lambda t}$ such that

$$(\beta_\lambda\varphi)\nu = \lambda_W\Big(\varphi\big((x/\square)\nu\big)\Big)$$

for each $\nu \in V_{X,W}$. Since $\varphi\big((x/\square)\nu\big) \in W_{\sigma x} \to W_t$ and $\lambda_W(W_{\sigma x} \to W_t) \subseteq W_{\lambda t}$, certainly $(\beta_\lambda \varphi)\nu \in W_{\lambda t}$. Since $V_{X,W} \to W_t$ is the t-part of $W^{V_{X,W}}$ for each $t \in T$, we have thus constructed the sorted algebra $(W^{V_{X,W}}, T, \rho)$. We call the mapping λ_W used above for $\lambda \in \Lambda \cap \Gamma X$ an **interpretation** of λ on W.

Now let Φ be a C-denotation into W. Then we can construct the sort-consistent homomorhism Φ^* of A into $W^{V_{X,W}}$ as follows. First we define the mapping φ of S into $V_{X,W} \to W$ so that

$$(\varphi a)\nu = \begin{cases} \Phi a & \text{when } a \in C, \\ \nu a & \text{when } a \in X \end{cases}$$

for each $\nu \in V_{X,W}$. Then $\varphi S_t \subseteq V_{X,W} \to W_t$ for each $t \in T$ because $\Phi C_t \subseteq W_t$ and $\nu X_t \subseteq W_t$, and so φ maps S into $W^{V_{X,W}}$ and satisfies $\rho\varphi = \sigma|_S$. Therefore by the universality of A, there exists a unique sort-consistent homomorphism of A into $W^{V_{X,W}}$ which extends φ. We call it the **semantic mapping** determined by Φ and denote it by Φ^*. Since Φ^* is an extension of φ,

$$(\Phi^* a)\nu = \begin{cases} \Phi a & \text{when } a \in C, \\ \nu a & \text{when } a \in X \end{cases} \tag{2.4}$$

for each $\nu \in V_{X,W}$.

By definition, a **logical system** is a triple $A, \mathcal{W}, (\lambda_W)_{\lambda,W}$ of a formal language $(A, T, \sigma, S, C, X, \Gamma)$, a domain $\mathcal{W}$ of actual worlds for A, and a family $(\lambda_W)_{\lambda,W}$ of interpretations λ_W of variable operations $\lambda \in \Lambda \cap \Gamma X$ on $W \in \mathcal{W}$.

Suppose the logical system $A, \mathcal{W}, (\lambda_W)_{\lambda,W}$ satisfies the following condition.

- For an element $\phi \in T$, the ϕ-part A_ϕ of A is non-empty, and the ϕ-part W_ϕ of each $W \in \mathcal{W}$ is equal to $\mathbb{T} = \{0, 1\}$.

Then we call ϕ a **truth** and call the elements of A_ϕ the **sentences**.

Suppose $A, \mathcal{W}, (\lambda_W)_{\lambda,W}$ is a logical system with a truth ϕ. Then we can construct a non-empty subset $\mathcal{F}_\mathcal{W}$ of $A_\phi \to \mathbb{T}$ as follows. Let $W \in \mathcal{W}$ be an actual world and Φ be a C-denotation into W. Then since the semantic mapping Φ^* is sort-consistent and the ϕ-part $V_{X,W} \to W_\phi$ of $W^{V_{X,W}}$ is equal to $V_{X,W} \to \mathbb{T}$ because $W_\phi = \mathbb{T}$, we have $\Phi^* A_\phi \subseteq V_{X,W} \to \mathbb{T}$, and so for each $\nu \in V_{X,W}$, we obtain the mappping $a \mapsto (\Phi^* a)\nu$ of A_ϕ into $\mathbb{T}$. We define $\mathcal{F}_\mathcal{W}$ to be the set of all those mappings obtained from all possible triples W, Φ, ν of actual worlds $W \in \mathcal{W}$ and C-denotations Φ into W and $\nu \in V_{X,W}$.

Thus we have seen above that each logical system $A, \mathcal{W}, (\lambda_W)_{\lambda,W}$ with a truth ϕ yields the pair $(A_\phi, \mathcal{F}_\mathcal{W})$ of A_ϕ and the subset $\mathcal{F}_\mathcal{W} \neq \emptyset$ of $A_\phi \to \mathbb{T}$. We call $(A_\phi, \mathcal{F}_\mathcal{W})$ the **sentence logical space**.

2.3 Completeness for logical spaces

The theory of completeness for sentence logical spaces may be remarkably generalized [3][5]. Let A be a set. A **logic** on A is a relation between A^* and A, where A^* is the set of all sequences $a_1 \cdots a_n$ of elements $a_1, \dots, a_n$ of A of arbitrary finite length $n \geq 0$. Logics on A are regarded as subsets of $A^* \times A$, and so we can discuss their intersections, unions, and inclusion.

Let R be a logic on A. If a subset B of A satisfies the condition

$$b_1, \ldots, b_n \in B,\ a \in A,\ b_1 \cdots b_n \, R \, a \implies a \in B,$$

then we call B an **R-theory** or say that B is **closed** under R. Similarly to support subalgebras of algebras, the intersection of R-theories is an R-theory, and A is an R-theory. Therefore, for each subset S of A, there exists the smallest R-theory containing S, which we denote by $[S]_R$ and call the **R-closure** of S. It is easy to show that $[S]_R$ is the union of the **R-descendants** S_n $(n = 0, 1, \ldots)$ of S, where $S_0 = S$ and S_n $(n \geq 1)$ is the inductively defined set of elements $a \in A$ such that $a_1 \cdots a_m \, R \, a$ for some elements $a_i \in S_{l_i}$ $(i = 1, \ldots, m)$ with $n = 1 + \sum_{i=1}^{m} l_i$. Also, an element $a \in A$ belongs to $[S]_R$ iff there exist elements $a_1, \ldots, a_n \in A$ such that $a_n = a$ and, for each $i \in \{1, \ldots, n\}$, either $a_i \in S$ or there exist numbers $j_1, \ldots, j_k \in \{1, \ldots, i-1\}$ satisfying $a_{j_1} \cdots a_{j_k} \, R \, a_i$.

A **deduction pair** on A is a pair (R, D) of a logic R on A and a subset D of A. We call R and D the **rule** and **basis** of the deduction pair. For each deduction pair (R, D) on A, we define the logic R^D on A by

$$a_1 \cdots a_n \, R^D \, a \iff [\{a_1, \ldots, a_n\} \cup D]_R \ni a$$

for each $(a_1 \cdots a_n, a) \in A^* \times A$ with $a_1, \ldots, a_n \in A$.

A **logical space** in a wider sense is a pair $(A, \mathcal{F})$ of a non-empty set A and a subset $\mathcal{F}$ of $A \to \mathbb{T}$. Since the power set $\mathcal{P}A$ of A is identified with $A \to \mathbb{T}$, a pair $(A, \mathcal{B})$ of A and a subset $\mathcal{B}$ of $\mathcal{P}A$ is also called a logical space.

Let $(A, \mathcal{B})$ be a logical space with $\mathcal{B} \subseteq \mathcal{P}A$. A logic R on A is **$\mathcal{B}$-sound**, if every element of $\mathcal{B}$ is closed under R. The union of $\mathcal{B}$-sound logics on A is $\mathcal{B}$-sound, and so there exists the greatest $\mathcal{B}$-sound logic on A, which we denote by Q for the time being. Then a deduction pair (R, D) on A is said to be **$\mathcal{B}$-complete** if $R^D = Q$.

Let $(A, \mathcal{F})$ be a logical space with $\mathcal{F} \subseteq A \to \mathbb{T}$. Then $\mathcal{F}$ is identified with the subset $\mathcal{B} = \{\varphi^{-1}1 \mid \varphi \in \mathcal{F}\}$ of $\mathcal{P}A$, so we say that a deduction pair (R, D) on A is **$\mathcal{F}$-complete** if it is $\mathcal{B}$-complete. Furthermore we define

$$C = \bigcap_{B \in \mathcal{B}} B = \{a \in A \mid \varphi a = 1 \text{ for all } \varphi \in \mathcal{F}\}, \tag{2.5}$$

which we call the **$\mathcal{F}$-core**. Elements and subsets of A are said to be **$\mathcal{F}$-sound** if they are contained in the $\mathcal{F}$-core. An $\mathcal{F}$-sound element is also called an **$\mathcal{F}$-tautology**.

The above definition of $\mathcal{F}$-completeness is justified and amplified by the following theorem [3][5].

Theorem 2.2 Let $(A, \mathcal{F})$ be a logical space with $\mathcal{F} \subseteq A \to \mathbb{T}$ and (R, D) be a deduction pair on A. Then (R, D) is $\mathcal{F}$-complete iff the following two conditions are equivalent for each $(a_1 \cdots a_n, a) \in A^* \times A$.

- $\inf\{\varphi a_1, \ldots, \varphi a_n\} \leq \varphi a$ for all $\varphi \in \mathcal{F}$.
- $[\{a_1, \ldots, a_n\} \cup D]_R \ni a$.

When $n = 0$, the above two conditions are identical with $a \in C$ and $a \in [D]_R$ respectively. Therefore, $\mathcal{F}$-complete deduction pairs (R, D) have the property of satisfying $C = [D]_R$. We call it the **$\mathcal{F}$-core-completeness**.

Now let $A, \mathcal{W}, (\lambda_W)_{\lambda,W}$ be a logical system with a truth ϕ. Then it yields the sentence logical space $(A_\phi, \mathcal{F}_\mathcal{W})$ as described in §2.2, and so we may discuss the $\mathcal{F}_\mathcal{W}$-completeness of deduction pairs on A_ϕ. As for this paper, the purpose of a **completeness theorem** for $A, \mathcal{W}, (\lambda_W)_{\lambda,W}$ is to present an $\mathcal{F}_\mathcal{W}$-complete deduction pair on A_ϕ.

3 Definition of MPCL

Here we define the logical system MPCL for which we will prove a completeness theorem.

3.1 Quantities and measures

A **quantitative system** is a set $\mathbb{P}$ equipped with a global binary associative and commutative operation $(x, y) \mapsto x + y$ with the identity element 0 and a total order $\leq$ which satisfy the following two conditions.

- If elements $p, p', q, q' \in \mathbb{P}$ satisfy $p \leq p'$ and $q \leq q'$, then $p + q \leq p' + q'$.
- $0 \leq p$ for every element p of $\mathbb{P}$, that is to say, $0 = \min \mathbb{P}$.

Naturally, the sets $\mathbb{Z}_{\geq 0}$ and $\mathbb{R}_{\geq 0}$ of non-negative integers and of non-negative real numbers are typical quantitative systems.

Let S be a set and $(\mathbb{P}, +, 0, \leq)$ be a quantitative system. Then a **$\mathbb{P}$-measure** on S is a mapping $X \mapsto |X|$ of $\mathcal{P}S$ into $\mathbb{P}$ which satisfies the following three conditions for all $X, Y \in \mathcal{P}S$.

- $X \neq \emptyset \iff |X| > 0$.
- $X \subseteq Y \implies |X| \leq |Y|$.
- $|X \cup Y| \leq |X| + |Y|$.

If $S \neq \emptyset$ and $\#\mathbb{P} > 1$, there exists at least one $\mathbb{P}$-measure on S.

3.2 MPC language

Here we define the formal language of MPCL. First we take arbitrary three sets $\mathbb{S}, \mathbb{C}, \mathbb{X}$ satisfying the conditions

$$\mathbb{S} = \mathbb{C} \amalg \mathbb{X}, \qquad \mathbb{X} \neq \emptyset.$$

Next we take an arbitrary set K equipped with a specific element π. We call K the set of **cases** and in particular call π the **nominative case**. Next we take two arbitrary distinct symbols δ and ε not contained in K, and define

$$\mathsf{T} = \{\delta, \varepsilon\} \cup \mathcal{P}\mathsf{K}.$$

Next we take a mapping τ of $\mathbb{S}$ into T such that the inverse image

$$\mathbb{X}_\varepsilon = \{x \in \mathbb{X} \mid \tau x = \varepsilon\}$$

of ε in $\mathbb{X}$ is not empty. Next we take an arbitrary quantitative system $(\mathbb{P}, +, 0, \leq)$ with $\#\mathbb{P} > 1$, and define $\mathfrak{P}$ to be the set of the unions of a finite number of intervals of $\mathbb{P}$ on the following list:

$$\begin{aligned}
(p \rightarrow) &= \{x \in \mathbb{P} \mid p < x\}, \\
(p, q] &= \{x \in \mathbb{P} \mid p < x \leq q\}, \qquad\qquad \text{where } p, q \in \mathbb{P}. \\
(\leftarrow q] &= \{x \in \mathbb{P} \mid x \leq q\},
\end{aligned}$$

Next we take a copy

$$\neg\mathfrak{P} = \{\neg\mathfrak{p} \mid \mathfrak{p} \in \mathfrak{P}\}$$

of the set $\mathfrak{P}$ such that $\neg\mathfrak{P} \cap \mathfrak{P} = \emptyset$, and define

$$\mathfrak{Q} = \neg\mathfrak{P} \cup \mathfrak{P},$$

which we call the set of the **quantifiers**. Next we take an arbitrary symbol $\breve{o} \notin \mathfrak{Q}$. Finally we define the eight kinds of operations on T as follows.

1. The family of binary operations $\breve{o}k$ $(k \in \mathsf{K})$.

$$\operatorname{Dom} \breve{o}k = \{\varepsilon\} \times \{P \in \mathcal{P}\mathsf{K} \mid k \in P\}, \qquad\qquad \varepsilon\, \breve{o}k\, P = P - \{k\}.$$

2. The family of binary operations λk $\big((\lambda, k) \in \mathfrak{Q} \times \mathsf{K}\big)$.

$$\operatorname{Dom} \lambda k = \{\delta, \varepsilon\} \times \{P \in \mathcal{P}\mathsf{K} \mid k \in P\}, \qquad\qquad \delta\, \lambda k\, P = \varepsilon\, \lambda k\, P = P - \{k\}.$$

3. The three binary operations $\wedge, \vee, \Rightarrow$.

$$\operatorname{Dom} \wedge = \operatorname{Dom} \vee = \operatorname{Dom} \Rightarrow\, = (\mathcal{P}\mathsf{K})^2, \qquad P \wedge Q = P \vee Q = P \Rightarrow Q = P \cup Q.$$

4. The unary operation $\Diamond$.

$$\operatorname{Dom} \Diamond = \mathcal{P}\mathsf{K}, \qquad\qquad P^{\Diamond} = P.$$

5. The unary operation $\triangle$.

$$\operatorname{Dom} \triangle = \{\delta, \varepsilon\}, \qquad\qquad \delta\triangle = \varepsilon\triangle = \{\pi\}.$$

6. The two binary operations $\sqcap, \sqcup$.

$$\mathrm{Dom}\,\sqcap = \mathrm{Dom}\,\sqcup = \{\delta, \varepsilon\}^2, \qquad \xi \sqcap \eta = \xi \sqcup \eta = \delta \text{ for } (\xi, \eta) \in \{\delta, \varepsilon\}^2.$$

7. The unary operation $\square$.

$$\mathrm{Dom}\,\square = \{\delta, \varepsilon\}, \qquad \delta^{\square} = \varepsilon^{\square} = \delta.$$

8. The family of unary operations Ωx $(x \in \mathbb{X}_\varepsilon)$.

$$\mathrm{Dom}\,\Omega x = \{\emptyset\}, \qquad \emptyset\,\Omega x = \delta.$$

We let T be the algebra equipped with the above eight kinds of operations. Thus we have chosen a set $\mathbb{S}$, an algebra T, and a mapping τ of $\mathbb{S}$ into T. Therefore by Theorem 2.1, there exists the USA $(A, T, \sigma, \mathbb{S})$ with $\sigma|_{\mathbb{S}} = \tau$, which is unique up to sort-consistent isomorphism. The OS's of T and A are both indexed by the set

$$\Lambda = \{\lambda k, \wedge, \vee, \Rightarrow, \Diamond, \triangle, \sqcap, \sqcup, \square, \Omega x \mid \lambda \in \{\breve{o}\} \cup \mathfrak{Q},\ k \in K,\ x \in \mathbb{X}_\varepsilon\},$$

and so if we define

$$\Gamma = \{\lambda k, \wedge, \vee, \Rightarrow, \Diamond, \triangle, \sqcap, \sqcup, \square, \Omega \mid \lambda \in \{\breve{o}\} \cup \mathfrak{Q},\ k \in K\},$$

then $\Lambda \subseteq \Gamma \cup \Gamma\mathbb{X}$ with $\Lambda \cap \Gamma\mathbb{X} = \{\Omega x \mid x \in \mathbb{X}_\varepsilon\}$. Therefore $(A, T, \sigma, \mathbb{S}, \mathbb{C}, \mathbb{X}, \Gamma)$ is a formal language, which we call the **MPC language**. Its variable operations Ωx $(x \in \mathbb{X}_\varepsilon)$ are called the **nominalizers**.

Since (A, T, σ) is a sorted algebra, A is divided into its t-parts A_t $(t \in T)$, and since $T = \{\delta, \varepsilon\} \cup \mathcal{P}K$, we have

$$A = A_\delta \cup A_\varepsilon \cup \bigcup_{P \in \mathcal{P}K} A_P,$$

so we define

$$G = A_\delta \cup A_\varepsilon, \qquad H = \bigcup_{P \in \mathcal{P}K} A_P.$$

We call G the set of the **nominals** and call H the set of the **predicates**. For each $f \in H$, we denote by K_f the element $P \in \mathcal{P}K$ satisfying $f \in A_P$ and call it the **range** of f.

Since (A, T, σ) is a sorted algebra, the following also holds on the domains and images of the operations in the operation system Λ of A.

(1) $\mathrm{Dom}\,\breve{o}k = A_\varepsilon \times \bigcup_{k \in P \in \mathcal{P}K} A_P$.
If $a \in A_\varepsilon$ and $f \in A_P$ with $k \in P \in \mathcal{P}K$, then $a\,\breve{o}k\,f \in A_{P-\{k\}}$.

(2) $\mathrm{Dom}\,\lambda k = G \times \bigcup_{k \in P \in \mathcal{P}K} A_P$.
If $a \in G$ and $f \in A_P$ with $k \in P \in \mathcal{P}K$, then $a\,\lambda k\,f \in A_{P-\{k\}}$.

(3) $\mathrm{Dom} \wedge = \mathrm{Dom} \vee = \mathrm{Dom} \Rightarrow = \mathsf{H}^2$.
If $f \in A_P$ and $g \in A_Q$ with $P, Q \in \mathcal{P}K$, then $f \wedge g$, $f \vee g$, $f \Rightarrow g \in A_{P \cup Q}$.

(4) $\mathrm{Dom} \lozenge = \mathsf{H}$.
If $f \in A_P$ with $P \in \mathcal{P}K$, then $f^{\lozenge} \in A_P$.

(5) $\mathrm{Dom} \triangle = G$, $\mathrm{Im} \triangle \subseteq A_{\{\pi\}}$.

(6) $\mathrm{Dom} \sqcap = \mathrm{Dom} \sqcup = G^2$, $\mathrm{Im} \sqcap \subseteq A_\delta$, $\mathrm{Im} \sqcup \subseteq A_\delta$.

(7) $\mathrm{Dom} \square = G$, $\mathrm{Im} \square \subseteq A_\delta$.

(8) $\mathrm{Dom}\, \Omega x = A_\emptyset$, $\mathrm{Im}\, \Omega x \subseteq A_\delta$.

Consequently, the following also holds:

(9) Let $a_1, \ldots, a_n \in G$, $f \in \mathsf{H}$, $\lambda_1, \ldots, \lambda_n \in \{\breve{o}\} \cup \mathfrak{Q}$, and $k_1, \ldots, k_n$ be distinct cases in K_f. Assume $a_i \in A_\varepsilon$ for all $i \in \{1, \ldots, n\}$ with $\lambda_i = \breve{o}$. Then $a_1\, \lambda_1 k_1\, (a_2\, \lambda_2 k_2\, (\cdots\, (a_n\, \lambda_n k_n\, f) \cdots))$ belongs to $A_{P - \{k_1, \ldots, k_n\}}$.

(10) H and A_P ($P \in \mathcal{P}K$) are closed under the operations $\wedge, \vee, \Rightarrow, \lozenge$, whose restrictions to H and A_P are global.

(11) G and A_δ are closed under the operations $\sqcap, \sqcup, \square$, whose restrictions to G and A_δ are global.

(12) $A_\delta - \mathbb{S}$ is non-empty and consists of nominals in one of the shapes $b \sqcap c$, $b \sqcup c$, $b^{\square}$, and $f\, \Omega x$.

(13) $A_\varepsilon = \mathbb{S}_\varepsilon \neq \emptyset$.

(14) $\mathsf{H} - \mathbb{S}$ is non-empty and consists of predicates in one of the shapes $a\, \breve{o} k\, f$, $a\, \lambda k\, f$, $f \wedge g$, $f \vee g$, $f \Rightarrow g$, $f^{\lozenge}$, and $a \triangle$.

(15) $A_\emptyset - \mathbb{S}$ is non-empty and consists of sentences in one of the shapes $a\, \breve{o} k\, f$, $a\, \lambda k\, f$, $f \wedge g$, $f \vee g$, $f \Rightarrow g$, and $f^{\lozenge}$.

For instance, in order to prove $A_\varepsilon = \mathbb{S}_\varepsilon$ in (13) by contradiction, suppose $a \in A_\varepsilon - \mathbb{S}$. Then, since A is a union of the descendants $\mathbb{S}_0, \mathbb{S}_1, \ldots$ of $\mathbb{S}$ and $\mathbb{S}_0 = \mathbb{S}$, a belongs to $\mathbb{S}_n$ for some $n \geq 1$, and so a is an image of some operation in Λ. This contradicts the facts (1) - (8).

We will use the following abbreviation for quantifiers:

$$\underline{p} = \neg(\leftarrow p], \qquad \overline{p} = (p \rightarrow), \qquad \text{for each } p \in \mathbb{P},$$
$$\forall = \underline{0}, \qquad \exists = \overline{0}, \qquad \text{for } 0 = \min\ \mathbb{P}.$$

We will use the symbols $\forall$ and $\exists$ for certain other notions, but hopefully there will be no confusion. For each $X \in \mathcal{P}\mathbb{P}$, we denote by X° the complement $\mathbb{P} - X$. Then $\mathfrak{P}$ is closed under the three set-theoretical operations $\cap, \cup, \circ$ on $\mathcal{P}\mathbb{P}$.

3.3 MPC worlds

Let $(\mathsf{A}, \mathsf{T}, \sigma, \mathbb{S}, \mathbb{C}, \mathbb{X}, \Gamma)$ be the MPC language defined in §3.2. Here we define the domain $\mathcal{W}$ of the actual worlds for A. Define

$$\Lambda' = \Lambda \cap \Gamma = \{\lambda\mathsf{k}, \wedge, \vee, \Rightarrow, \Diamond, \triangle, \sqcap, \sqcup, \Box \mid \lambda \in \{\breve{\mathfrak{o}}\} \cup \mathfrak{Q},\ \mathsf{k} \in \mathsf{K}\},$$

and let T' be the operational subalgebra of T obtained by reducing the OS of T from Λ to Λ'.

First we take an arbitrary non-empty set S, and define

$$W = (\mathsf{S} \to \mathbb{T}) \cup \mathsf{S} \cup \bigcup_{\mathsf{P} \in \mathcal{P}\mathsf{K}} \big((\mathsf{P} \to \mathsf{S}) \to \mathbb{T}\big),$$

where $(\emptyset \to \mathsf{S}) \to \mathbb{T} = \mathbb{T}$ for $\emptyset \in \mathcal{P}\mathsf{K}$, because $\emptyset \to \mathsf{S} = \{\emptyset\}$ by the set-theoretical definition of $\mathsf{Y} \to \mathsf{Z}$ and $\{\emptyset\} \to \mathbb{T} = \mathbb{T}$ by convention. We call S the **base** of W.

Next we define the sort mapping ρ of W into $\mathsf{T}' = \{\delta, \varepsilon\} \cup \mathcal{P}\mathsf{K}$ so that the t-parts W_t $(\mathsf{t} \in \mathsf{T}')$ satisfy

$$W_\delta = \mathsf{S} \to \mathbb{T}, \qquad W_\varepsilon = \mathsf{S}, \qquad W_\mathsf{P} = (\mathsf{P} \to \mathsf{S}) \to \mathbb{T}$$

for each $\mathsf{P} \in \mathcal{P}\mathsf{K}$. In particular $W_\emptyset = \mathbb{T}$.

Next we define a family of operations on W indexed by Λ'. The definition depends on two parameters. The one is an arbitrary $\mathbb{P}$-measure $\mathsf{X} \mapsto |\mathsf{X}|$ on S. The other is an arbitrary reflexive relation $\exists$ on S, which we call the **basic relation** of W. In order to define the operations, we first extend $\exists$ to the relation between $(\mathsf{S} \to \mathbb{T}) \cup \mathsf{S}$ and S by

$$\mathsf{a} \exists \mathsf{b} \iff \mathsf{ab} = 1 \tag{3.1}$$

for each $\mathsf{a} \in \mathsf{S} \to \mathbb{T}$ and each $\mathsf{b} \in \mathsf{S}$. It is unnecessary to furthermore extend $\exists$ to a relation on $(\mathsf{S} \to \mathbb{T}) \cup \mathsf{S}$. Next, when $\mathsf{s} \in \mathsf{S}$ and $\mathsf{k} \in \mathsf{P} \in \mathcal{P}\mathsf{K}$, we define for each $\theta \in (\mathsf{P} - \{\mathsf{k}\}) \to \mathsf{S}$ the element $(\mathsf{k}/\mathsf{s})\theta \in \mathsf{P} \to \mathsf{S}$ by

$$\big((\mathsf{k}/\mathsf{s})\theta\big)\mathsf{l} = \begin{cases} \theta\mathsf{l} & \text{when } \mathsf{l} \in \mathsf{P} - \{\mathsf{k}\}, \\ \mathsf{s} & \text{when } \mathsf{l} = \mathsf{k}. \end{cases} \tag{3.2}$$

Next we define

$$\neg(\neg\mathfrak{p}) = \mathfrak{p}$$

for each $\mathfrak{p} \in \mathfrak{P}$. Thus, if $\lambda \in \mathfrak{P}$ then $\neg\lambda \in \neg\mathfrak{P}$, while if $\lambda \in \neg\mathfrak{P}$ then $\neg\lambda \in \mathfrak{P}$. Finally we define the seven kinds of operations on W as follows.

1. The family of binary operations $\breve{\mathfrak{o}}\mathsf{k}$ $(\mathsf{k} \in \mathsf{K})$.

$$\mathrm{Dom}\,\breve{\mathfrak{o}}\mathsf{k} = \mathsf{S} \times \bigcup_{\mathsf{k} \in \mathsf{P} \in \mathcal{P}\mathsf{K}} \big((\mathsf{P} \to \mathsf{S}) \to \mathbb{T}\big).$$

For each $\mathsf{s} \in \mathsf{S}$ and each $\mathsf{f} \in (\mathsf{P} \to \mathsf{S}) \to \mathbb{T}$ with $\mathsf{k} \in \mathsf{P} \in \mathcal{P}\mathsf{K}$, we define $\mathsf{s}\,\breve{\mathfrak{o}}\mathsf{k}\,\mathsf{f}$ to be the element of $\big((\mathsf{P} - \{\mathsf{k}\}) \to \mathsf{S}\big) \to \mathbb{T}$ such that

$$(\mathsf{s}\,\breve{\mathfrak{o}}\mathsf{k}\,\mathsf{f})\theta = \mathsf{f}\big((\mathsf{k}/\mathsf{s})\theta\big)$$

for each $\theta \in (\mathsf{P} - \{\mathsf{k}\}) \to \mathsf{S}$.

2. The family of binary operations λk $\big((\lambda, k) \in \mathfrak{Q} \times K\big)$.

$$\operatorname{Dom} \lambda k = \big((S \to \mathbb{T}) \cup S\big) \times \bigcup_{k \in P \in \mathcal{P}K} \big((P \to S) \to \mathbb{T}\big).$$

For each $a \in (S \to \mathbb{T}) \cup S$ and each $f \in (P \to S) \to \mathbb{T}$ with $k \in P \in \mathcal{P}K$, we define $a \lambda k\, f$ to be the element of $\big((P - \{k\}) \to S\big) \to \mathbb{T}$ such that

$$(a \lambda k\, f)\theta = 1 \iff \begin{cases} \big|\big\{s \in S \mid a \exists s,\ f\big((k/s)\theta\big) = 0\big\}\big| \in \neg\lambda & \text{when } \lambda \in \neg\mathfrak{P}, \\ \big|\big\{s \in S \mid a \exists s,\ f\big((k/s)\theta\big) = 1\big\}\big| \in \lambda & \text{when } \lambda \in \mathfrak{P} \end{cases}$$

for each $\theta \in (P - \{k\}) \to S$. Notice $f\big((k/s)\theta\big) = (s\, \breve{o} k\, f)\theta$.

3. The three binary operations $\wedge, \vee, \Rightarrow$.

$$\operatorname{Dom} \wedge = \operatorname{Dom} \vee = \operatorname{Dom} \Rightarrow = \left(\bigcup_{P \in \mathcal{P}K} \big((P \to S) \to \mathbb{T}\big) \right)^2.$$

For each $f \in (P \to S) \to \mathbb{T}$ and each $g \in (Q \to S) \to \mathbb{T}$ with $P, Q \in \mathcal{P}K$, we define $f \wedge g$, $f \vee g$, $f \Rightarrow g$ to be the elements of $\big((P \cup Q) \to S\big) \to \mathbb{T}$ such that

$$\begin{aligned} (f \wedge g)\theta &= f(\theta|_P) \wedge g(\theta|_Q), \\ (f \vee g)\theta &= f(\theta|_P) \vee g(\theta|_Q), \\ (f \Rightarrow g)\theta &= f(\theta|_P) \Rightarrow g(\theta|_Q) \end{aligned}$$

for each $\theta \in (P \cup Q) \to S$, where $\wedge, \vee, \Rightarrow$ on the right-hand sides of the equations are the meet, join, and implication on the Boolean lattice $\mathbb{T}$ defined by

$$a \wedge b = \inf\{a, b\}, \qquad a \vee b = \sup\{a, b\}, \qquad a \Rightarrow b = \sup\{1 - a, b\}$$

for all $a, b \in \mathbb{T}$.

4. The unary operation $\Diamond$.

$$\operatorname{Dom} \Diamond = \bigcup_{P \in \mathcal{P}K} \big((P \to S) \to \mathbb{T}\big).$$

For each $f \in (P \to S) \to \mathbb{T}$ with $P \in \mathcal{P}K$, we define $f^\Diamond$ to be the element of $(P \to S) \to \mathbb{T}$ such that

$$(f^\Diamond)\theta = (f\theta)^\Diamond$$

for each $\theta \in P \to S$, where $\Diamond$ on the right-hand side of the equation is the complement on the Boolean lattice $\mathbb{T}$ defined by

$$a^\Diamond = 1 - a$$

for all $a \in \mathbb{T}$.

5. The unary operation $\triangle$.

$$\mathrm{Dom}\,\triangle = (\mathsf{S} \to \mathbb{T}) \cup \mathsf{S}.$$

For each $a \in (\mathsf{S} \to \mathbb{T}) \cup \mathsf{S}$, we define $a\triangle$ to be the element of $(\{\pi\} \to \mathsf{S}) \to \mathbb{T}$ such that

$$(a\triangle)\theta = 1 \iff a \ni \theta\pi$$

for each $\theta \in \{\pi\} \to \mathsf{S}$.

6. The two binary operations $\sqcap, \sqcup$.

$$\mathrm{Dom}\,\sqcap = \mathrm{Dom}\,\sqcup = \big((\mathsf{S} \to \mathbb{T}) \cup \mathsf{S}\big)^2.$$

For each $(a, b) \in \big((\mathsf{S} \to \mathbb{T}) \cup \mathsf{S}\big)^2$, we define $a \sqcap b$ and $a \sqcup b$ to be the elements of $\mathsf{S} \to \mathbb{T}$ such that

$$\begin{aligned} a \sqcap b \ni s &\iff a \ni s \text{ and } b \ni s, \\ a \sqcup b \ni s &\iff a \ni s \text{ or } b \ni s \end{aligned}$$

for each $s \in \mathsf{S}$.

7. The unary operation $\square$.

$$\mathrm{Dom}\,\square = (\mathsf{S} \to \mathbb{T}) \cup \mathsf{S}.$$

For each $a \in (\mathsf{S} \to \mathbb{T}) \cup \mathsf{S}$, we define $a^\square$ to be the element of $\mathsf{S} \to \mathbb{T}$ such that

$$a^\square \ni s \iff a \not\ni s$$

for each $s \in \mathsf{S}$.

We let W be the algebra equipped with the above seven kinds of operations. Then (W, T', ρ) becomes a sorted algebra and satisfies $W_t \neq \emptyset$ for all $t \in \mathsf{T}'$. Therefore, W is a cognizable world for A.

We call the sorted algebras constructed as above the **MPC worlds** congizable by the MPC language $(\mathsf{A}, \mathsf{T}, \sigma, \mathbb{S}, \mathbb{C}, \mathbb{X}, \Gamma)$ and denote by $\mathcal{W}$ the set of all such worlds. We choose $\mathcal{W}$ as the domain of the actual worlds for A.

3.4 Interpretations of the nominalizers

Let $(\mathsf{A}, \mathsf{T}, \sigma, \mathbb{S}, \mathbb{C}, \mathbb{X}, \Gamma)$ be the MPC language defined in §3.2, and let $\mathcal{W}$ be the domain of the actual worlds for A defined in §3.3. Following §2.2, here we define the interpretations λ_W of the variable operations $\lambda \in \Lambda \cap \Gamma\mathbb{X}$ on the MPC worlds $W \in \mathcal{W}$, and thereby complete the definition of MPCL.

Since $\Lambda \cap \Gamma\mathbb{X}$ consists of the nominalizers, $\lambda = \Omega x$ for some $x \in \mathbb{X}_\varepsilon$, and so the domain T_λ of λ on T is equal to $\{\emptyset\}$ and $\lambda\emptyset = \delta$. Moreover $W_\delta = \mathsf{S} \to \mathbb{T} = W_{\sigma x} \to W_\emptyset$. Thus, λ_W is a mapping of $W_{\sigma x} \to W_\emptyset$ into itself, and so we define λ_W to be the identity mapping of $W_{\sigma x} \to W_\emptyset$. Then the domain of the operation β_λ on $W^{V_{\mathbb{X},W}}$ corresponding to the index λ is equal to $V_{\mathbb{X},W} \to W_\emptyset = V_{\mathbb{X},W} \to \mathbb{T}$, and for each $\varphi \in V_{\mathbb{X},W} \to \mathbb{T}$, we have $\beta_\lambda \varphi \in V_{\mathbb{X},W} \to W_\delta = V_{\mathbb{X},W} \to (S \to \mathbb{T})$ with $(\beta_\lambda \varphi)\nu = \varphi\big((x/\Box)\nu\big)$ for each $\nu \in V_{\mathbb{X},W}$, hence $\big((\beta_\lambda \varphi)\nu\big)s = \varphi\big((x/s)\nu\big)$ for each $s \in S$.

Since $\lambda = \Omega x$ ($x \in \mathbb{X}_\varepsilon$) and we will denote $\beta_\lambda \varphi$ by $\varphi\,\Omega x$, we conclude that the domain of the nominalizer Ωx on $W^{V_{\mathbb{X},W}}$ is equal to $V_{\mathbb{X},W} \to \mathbb{T}$, the image $\varphi\,\Omega x$ of $\varphi \in V_{\mathbb{X},W} \to \mathbb{T}$ belongs to $V_{\mathbb{X},W} \to (S \to \mathbb{T})$, so $(\varphi\,\Omega x)\nu \in S \to \mathbb{T}$ for each $\nu \in V_{\mathbb{X},W}$, and the following holds for each $s \in S$:

$$\big((\varphi\,\Omega x)\nu\big)s = \varphi\big((x/s)\nu\big). \tag{3.3}$$

This may be expressed as follows by using (3.1):

$$\big((\varphi\,\Omega x)\nu\big) \ni s \iff \varphi\big((x/s)\nu\big) = 1. \tag{3.4}$$

This completes the definition of the logical system MPCL.

3.5 Predicate logical space

Let $A, \mathcal{W}, (\lambda_W)_{\lambda,W}$ be the logical system MPCL defined above. Then, $\emptyset \in \mathcal{P}K \subseteq T$, $A_\emptyset \neq \emptyset$ by §3.2 (15), and $W_\emptyset = \mathbb{T}$ for each $W \in \mathcal{W}$. Therefore, $A, \mathcal{W}, (\lambda_W)_{\lambda,W}$ is a logical system with a truth $\emptyset$, so it yields the sentence logical space $(A_\emptyset, \mathcal{F}_\mathcal{W})$, and we have announced at the end of §2.3 that the purpose of a completeness theorem for $A, \mathcal{W}, (\lambda_W)_{\lambda,W}$ is to present an $\mathcal{F}_\mathcal{W}$-complete deduction pair on $A_\emptyset$. However as for MPCL, another larger logical space on the set $H = \bigcup_{P \in \mathcal{P}K} A_P$ of the predicates of A is more worth studying.

Let W be an MPC world in $\mathcal{W}$, Φ be a $\mathbb{C}$-denotation into W, and ν be an $\mathbb{X}$-denotation into W. Then the semantic mapping $\Phi^* \in A \to W^{V_{\mathbb{X},W}}$ is sort-consistent, and the projection by ν is a sort-consistent mapping of $W^{V_{\mathbb{X},W}}$ into W. Therefore if $f \in H$, then $(\Phi^* f)\nu$ belongs to $W_{K_f} = (K_f \to W_\varepsilon) \to \mathbb{T}$, and so if furthermore $\theta \in K \to W_\varepsilon$, then $\big((\Phi^* f)\nu\big)(\theta|_{K_f})$ belongs to $\mathbb{T}$. Thus the quadruple W, Φ, ν, θ yields the mapping $f \mapsto \big((\Phi^* f)\nu\big)(\theta|_{K_f})$ of H into $\mathbb{T}$. Let $\mathcal{G}_\mathcal{W}$ denote the set of those mappings obtained from all possible such quadruples. Then $(H, \mathcal{G}_\mathcal{W})$ is a logical space, which we call the **predicate logical space**. It is an extension of the sentence logical space $(A_\emptyset, \mathcal{F}_\mathcal{W})$ in the sense that $A_\emptyset \subseteq H$ and $\mathcal{F}_\mathcal{W} = \big\{\varphi|_{A_\emptyset} \bigm| \varphi \in \mathcal{G}_\mathcal{W}\big\}$ hold.

3.6 Design concept for MPCL

As mentioned in the introduction, our mathematical model is not that of natural languages, but the MCL language A is a model of substances which are supposed to exist in human brain, and the MCL worlds W congizable by A are models of the real worlds which human cognizes. However some of the substances in the human brain are supposed to be expressed by natural languages. Therefore, the OS's of A and W are designed so that plenty of elements of A and W translate verbatim into expressions of Japanese language, and probably into other languages as well under some transformations on the OS's of A and W. Such a design is of vital importance from the viewpoint of our mathematical psychology. Therefore it may be in order to compare A and W here to natural languages, particularly to Japanese.

First, constant nominals in $\mathbb{C} \cap G = \mathbb{C}_\delta \cup \mathbb{C}_\varepsilon$ translate into nouns, and constant predicates in $\mathbb{C} \cap H = \bigcup_{P \in \mathcal{P}K} \mathbb{C}_P$ translate into verbs, predicate adjectives, and predicate nominal adjectives, while variables in $\mathbb{X}$ translate into nothing. Members of the OS

$$\Lambda = \{\lambda k, \wedge, \vee, \Rightarrow, \Diamond, \triangle, \sqcap, \sqcup, \square, \Omega x \mid \lambda \in \{\breve{o}\} \cup \mathfrak{Q},\ k \in K,\ x \in \mathbb{X}_\varepsilon\}$$

of A also have their translations unless variables occur in their arguments. The cases k in $\breve{o}k$ ($k \in K$) translate into postpositions called teniwoha, such as the case markers "ga" (nominative), "wo" (accusative), and "ni" (dative). In particular, the nominative case π translates into "ga," "wa" (topic marker), and so on. The operations λk with $\lambda \in \mathfrak{Q}$ translate into combinations of Japanese quantifiers and teniwoha. Recall that the set $\mathfrak{Q}$ of the quantifiers is divided into $\neg\mathfrak{P}$ and $\mathfrak{P}$, and $\mathfrak{P}$ is the set of the unions of a finite number of intervals of $\mathbb{P}$ in one of the shapes $(p \rightarrow), (p, q]$, and $(\leftarrow q]$. For instance, the quantifiers $\overline{p} = (p \rightarrow)$ translate into words meaning "more than p," and $\forall = \neg(\leftarrow, 0]$ translates into words meaning "all." The operations $\wedge, \vee, \Rightarrow, \Diamond$ translate into words meaning "and," "or," "then," "not" respectively, and the operation $\triangle$ translates into the copulas "dearu," "da," or "desu" meaning "be." The operation $\sqcap$ translates into nothing or into the particle "no" which joins appositional nominals as in "gaka no Gogh" meaning "painter Gogh." The operation $\sqcup$ translates into words meaning "or" used in listing parallel nominals. A translation of the operation $\square$ is illustrated below. The nominalizers Ωx ($x \in \mathbb{X}_\varepsilon$) translate into the relative pronoun "that," the conjunction "that," and so on, although those corresponding to the relative pronoun "that" have poor translations into Japanese. The translation of an element $a \in A - \mathbb{S}$ is verbatim with a few exceptions in the sense that it is usually obtained by replacing the primes and operations occurring in a by their above-mentioned counterparts in Japanese.

By way of illustration, if we translate $a, b \in \mathbb{C}_\varepsilon$ into "pētā" (Peter, a proper nown) and "usagi" (rabbit), then, since π translates into the topic marker "wa" (no English equivalent) and $\triangle$ translates into the copula "desu" (is), $a\,\breve{o}\pi b\triangle$ translates verbatim into "pētā wa usagi desu" [Peter *wa* rabbit is] (Peter is a rabbit). Hereafter, counterparts written by English words but in Japanese word order will be shown in square brackets, with verbs and nouns in singular forms and without articles, because Japanese uses no plural forms or articles. Also, Japanese particles which have no English equivalents will be printed in italics. If furthermore we translate $c \in \mathbb{C}_\varepsilon$ into "mame" (bean), $\omega \in K - \{\pi\}$ into the accusative case marker "wo" (no English equivalent), $f \in \mathbb{C}_{\{\pi,\omega,\dots\}}$ into "taberu" (eat), and retranslate π into the nominative case marker "ga" (no English equivalent), then $a\,\breve{o}\pi\,(c\,\breve{o}\omega\,f)$ translates verbatim into "pētā ga mame wo taberu" [Peter *ga* bean *wo* eat] (Peter eats a bean), while $a\,\breve{o}\pi\,(c\,\forall\omega\,f)$ translates verbatim into "pētā ga mame subete wo taberu" [Peter *ga* bean all *wo* eat] (Peter eats all beans). If furthermore we translate $\mathsf{d} \in \mathbb{C}_\varepsilon$ into "yasai" (vegetable), then $\big(\mathsf{a}\,\breve{o}\pi\,(\mathsf{x}\,\breve{o}\omega\,\mathsf{f})\big)\Omega\mathsf{x} \sqcap \mathsf{d}$ translates verbatim into "pētā ga taberu yasai" [Peter *ga* eat vegetable](vegetables that Peter eats), while $\big(\big(\mathsf{a}\,\breve{o}\pi\,(\mathsf{x}\,\breve{o}\omega\,\mathsf{f})\big)\Omega\mathsf{x}\big)^{\square} \sqcap \mathsf{d}$ translates verbatim into "pētā ga taberu igai no yasai" [Peter *ga* eat other than *no* vegetable] (vegetables other than that Peter eats), where $\mathsf{x}\,\breve{o}\omega$ and $\Omega\mathsf{x}$ translate into nothing and $\sqcap$ translates into "no" following before-mentioned rules. On the other hand, $\big(\mathsf{a}\,\breve{o}\pi\,(\mathsf{c}\,\breve{o}\omega\,\mathsf{f})\big)\Omega\mathsf{x}$ translates verbatim into "pētā ga mame wo taberu koto" [Peter *ga* bean *wo* eat that] (that Peter eats a bean). All the above Japanese translations are grammatical.

If α is a binary operation of an algebra, we are free to denote its images by $\alpha(\mathsf{a}, \mathsf{b})$, $\mathsf{a}\,\alpha\,\mathsf{b}$, $(\mathsf{a}, \mathsf{b})\alpha$, and so on. Therefore, we could design our MPC language so that $\mathsf{a}\,\lambda\pi\,(\mathsf{b}\,\breve{o}\omega\,\mathsf{f})$ was denoted by $\pi\lambda(\mathsf{a}, \mathsf{f}\,\omega\breve{o}\,\mathsf{b})$ instead. Under such a design, if we translate a, b, and f into "boy," "Mary," and "love(s)," then $\pi\forall(\mathsf{a}, \mathsf{f}\,\omega\breve{o}\,\mathsf{b})$ and $\pi\exists(\mathsf{a}, \mathsf{f}\,\omega\breve{o}\,\mathsf{b})$ translate verbatim into "Every boy loves Mary" and "A boy loves Mary," because π and ω have no English counterparts. If we translate a into "John," then $\pi\breve{o}(\mathsf{a}, \mathsf{f}\,\omega\breve{o}\,\mathsf{b})$ translates verbatim into "John loves Mary." The reader might compare these with their PL counterparts $\forall\mathsf{x}\big(\mathsf{boy}(\mathsf{x}) \Rightarrow \mathsf{love}(\mathsf{x}, \text{Mary})\big)$, $\exists\mathsf{x}\big(\mathsf{boy}(\mathsf{x}) \wedge \mathsf{love}(\mathsf{x}, \text{Mary})\big)$, $\mathsf{love}(\text{John}, \text{Mary})$, which hardly translate verbatim into English unless we ignore the operations $\Rightarrow$ and $\wedge$. Thus, under some transformations on its OS, the MPC language may translate verbatim into English as well as into Japanese.

Suppose we know that there are at most ten boys in the class and that more than eight boys love Mary. Then we certainly think and say that almost all boys love Mary. Thus the sentence "Almost all boys love Mary" may be considered an elliptical expression of the knowledge "There are at most ten boys in the class, and at least eight boys love Mary" or something like that. For this reason, we do not think that logical systems need to be furnished with quantifiers meaning "almost all" or other words which show ratios between quantities. Recall here that MPC language A is not a model of natural languages but a model of the human brain system of describing human notions. Since expressions of natural languages are always incomplete and deformed expressions of human notions, and since we human do not express all our notions by natural languages, not all expressions of natural languages should be translated verbatim into formal languages, and vice versa. As for quantifiers, we only need to design a logical system by which we can express and analyze notions like "More than eight boys love Mary" and "There are at most ten boys." In MPCL, these notions are expressed by the predicates $\mathsf{a}\,(8 \to)\pi\,\mathsf{b}\,\breve{o}\omega\,\mathsf{f}$ and $\mathsf{a}\,(\leftarrow 10]\pi\,\mathsf{one}\triangle$ respectively, where one is an abbreviation for $(\mathsf{x}\,\breve{o}\pi\,\mathsf{x}\triangle)\Omega\mathsf{x}$ with $\mathsf{x} \in \mathbb{X}_\varepsilon$. Every predicate $\mathsf{a}\,\lambda\pi\,\mathsf{one}\triangle$ with $\lambda \in \{\breve{o}\} \cup \mathfrak{Q}$ translates into existential sentences, which implies that $\mathsf{one}\triangle = \big((\mathsf{x}\,\breve{o}\pi\,\mathsf{x}\triangle)\Omega\mathsf{x}\big)\triangle$ in $\mathsf{a}\,\lambda\pi\,\mathsf{one}\triangle$ exceptionally translates into words "aru" (exist), "iru" (be), and so on.

For human beings, even extent or degree is an entity in the outer world, so A is supposed to have a specific case μ which marks extent or degree just as π is a subjective marker and ω is an accusative marker. We also suppose that every constant predicate $\mathsf{f} \in \mathbb{C} \cap \mathsf{H}$ has μ in its range K_f. However, μ is often silent in the sense that μ itself and some of its arguments do not always translate into natural languages. By way of illustration, if we translate $a \in \mathbb{C}_\varepsilon$ into "pētā" (Peter) and $f \in \mathbb{C}_{\{\pi,\mu,\dots\}}$ into the predicate nominal adjective "wanpaku-da" (be naughty), then $a\,\breve{o}\pi\,(b\,\breve{o}\mu\,f)$ with $b \in \mathbb{C}_\varepsilon$ translates into "pētā wa wanpaku-da" [Peter *wa* naughty] (Peter is naughty) with translations of b and μ missing. However, if we translate $c \in \mathbb{C}_\varepsilon$ into "saru" (monkey), then the predicate

$$a\,\breve{o}\pi\,\Big(\big(\big(c\,\breve{o}\pi\,(x\,\breve{o}\mu\,f)\big)\,\Omega x\big)\,\forall\mu\,f\Big)$$

obtained from $a\,\breve{o}\pi\,(b\,\breve{o}\mu\,f)$ by replacing b by $\big(c\,\breve{o}\pi\,(x\,\breve{o}\mu\,f)\big)\,\Omega x$ and $\breve{o}\mu$ by $\forall\mu$ translates into "pētā wa saru noyouni wanpaku-da" [Peter *wa* monkey as naughty] (Peter is as naughty as a monkey), which implies that the nominal $\big(c\,\breve{o}\pi\,(x\,\breve{o}\mu\,f)\big)\,\Omega x$ translates into "saru no wanpaku-sa" [monkey of naughtiness] (naughtiness of a monkey) and $\forall\mu$ translates into "noyouni" (as), which in turn implies that the nominalizer Ωx in this expression translates into the suffix "-sa" (-ness). We often say "pētā wa totemo wanpaku-da" [Peter *wa* very naughty] (Peter is very naughty), which may be regarded as an elliptical expression of various notions such as "Peter is as naughty as a monkey," "Peter is as naughty as a puppy," "Peter is as naughty as a kitten," and so on. Thus, the adverb "very" may be regarded as an incomplete and deformed expression of notions $\big(d\,\breve{o}\pi\,(x\,\breve{o}\mu\,f)\big)\,\Omega x\big)\,\forall\mu$ with d varying through $\mathbb{C}_\varepsilon$. For this reason, logical systems need not be furnished with a specific category of elements which translate verbatim into adverbs of extent or degree. For certain reasons which we will not show here, the same remark applies to other kind of adverbs.

The above account of elements and operations of A except the nominalizers Ωx also applies to W. In fact, the existence of verbatim translations of elements and operations of W into Japanese is far more surprising than that of A, because the definition of the MPC world

$$W = (S \to \mathbb{T}) \cup S \cup \bigcup_{P \in \mathcal{P}K} \big((P \to S) \to \mathbb{T}\big)$$

is entirely mathematical and has no obvious connection with natural languages. First, the base S is a mere set although equipped with a reflexive relation $\exists$ and a $\mathbb{P}$-measure $X \mapsto |X|$. Next, $S \to \mathbb{T}$ is identified with the power set $\mathcal{P}S$. Finally, if $P \in \mathcal{P}K$ consists of n elements, then $P \to S$ is identified with S^n, and so $(P \to S) \to \mathbb{T}$ is identified with $S^n \to \mathbb{T}$, which is the set of the n-ary relations on S. Thus, the MPC world W may be identified with the entirety of a set S, its subsets, and the multiary relations on S, which are all purely mathematical notions. Therefore there are no obvious reasons why we may/should expect that elements of W will translate verbatim into natural languages.

Nevertheless plenty of elements of W translate verbatim into Japanese. Such translations are made possible by several simple mathematical devices, the first of which is based on the following theorem.

Theorem 3.1 Let $A_1, \ldots, A_n, B$ be sets and define

$$F = (A_1 \times \cdots \times A_n) \to B,$$
$$F^* = A_1 \to (A_2 \to (\cdots \to (A_n \to B) \cdots)).$$

Then for each $f \in F$, there exists a unique element $f^* \in F^*$ which satisfies $f(a_1, \ldots, a_n) = (\cdots((f^* a_1)a_2)\cdots)a_n$ for each $(a_1, \ldots, a_n) \in A_1 \times \cdots \times A_n$, and the mapping $f \mapsto f^*$ is a bijection of F onto F^*.

This theorem implies that n-ary functions f are decomposed into the series f^*, $f^* a_1$, $(f^* a_1)a_2$, $\ldots$, $(\cdots((f^* a_1)a_2)\cdots)a_n$ of $n+1$ functions, all of which are unary except the last one which is a constant. It is said that this technique of decomposing functions was invented by logicians Moses Schönfinkel and Gottlob Frege and computer scientists call it **currying** after Haskell Curry. However, we call it **linearization**.

The second device is reversing the notation for functions. Since modern mathematics has been built by Europeans, mathematical notation follows the word order of Indo-European languages. In Japanese word order, functions $f(a_1, \ldots, a_n)$ should be denoted by $(a_1, \ldots, a_n)f$, Theorem 3.1 should be stated on the bijection between the sets

$$B \leftarrow (A_1 \times \cdots \times A_n),$$
$$((\cdots(B \leftarrow A_1)\cdots) \leftarrow A_{n-1}) \leftarrow A_n,$$

and f should be decomposed into the series

$$f^*,\ a_n f^*,\ (a_{n-1}(a_n f^*)),\ \ldots,\ a_1(a_2(\cdots(a_n f^*)\cdots))$$

of $n+1$ functions. For instance, binary functions $(a, b)f$ are decomposed into the series f^*, bf^*, $a(bf^*)$ of three functions.

The third device is accompanying a_i with the number i to indicate that a_i is the i-th argument of the function $(a_1, \ldots, a_n)f$. For instance, the functions bf^* and $a(bf^*)$ above are denoted by $b2f^*$, and $a1(b2f^*)$. However, we need not use the numbers $1, 2$. We may use any symbols instead. Moreover, we need not use Roman letters a, b, f^*. We may use any symbols instead. So let us replace 1 and 2 by Japanese particles "ga" and "wo," and replace a, b, and f^* by Japanese words "pētā" (Peter), "mame" (bean), and "taberu" (eat). Then the functions f^*, $b2f^*$, and $a1(b2f^*)$ are also denoted by the following three Japanese predicates:

taberu	[eat]	(eats) ,
mame wo taberu	[bean *wo* eat]	(eats a bean),
pētā ga mame wo taberu	[Peter *ga* bean *wo* eat]	(Peter eats a bean).

Thus these predicates may be regarded as alternative expressions of the function values f^*, $b2f^*$, and $a1(b2f^*)$ respectively, which tallies with the fact that Japanese counterparts of "eats" and "eats a bean" above are grammatical elliptical sentences. It seems now reasonable to believe that every elementary Japanese predicate is a linearized expression of a function in the real world.

However, we have to half abandon linearization. This is because Japanese word order is loose except for rigid constraint that verbs, predicate adjectives, and predicate nominal adjectives which are main in a predicate must occur in the final position of the predicate. For instance, "pētā ga mame wo taberu" [Peter *ga* bean *wo* eat] and "mame wo pētā ga taberu" [bean *wo* Peter *ga* eat] are both grammatical and have the same truth value, although emphasis is different. In other words, in Japanese people's brain, the function $a1(b2f^*)$ above has an alternative expression $b2(a1f^*)$. Rigid linearization does not work well for such looseness.

Thus, in designing the MPC world W, we first identified S^n with $\{1, \dots, n\} \to S$, next replaced the ordered set $\{1, \dots, n\}$ by an arbitrary unordered set P with $\#P = n$, thereby replacing the set $S^n \to \mathbb{T}$ of the n-ary relations on S by $(P \to S) \to \mathbb{T}$. This is the way we have reached the substructure

$$\bigcup_{P \in \mathcal{P}K} \big((P \to S) \to \mathbb{T}\big)$$

of W. Linearization is hidden alive here. Let $P = \{k_1, \dots, k_n\} \in \mathcal{P}K$, and for each $f \in S^n \to \mathbb{T}$, define $f^* \in (P \to S) \to \mathbb{T}$ by

$$f^*\theta = f(\theta k_1, \dots, \theta k_n)$$

for each $\theta \in P \to S$. Then Theorem 9.3 implies that

$$f(s_1, \dots, s_n) = s_{\rho 1}\, \breve{o}k_{\rho 1}\, (s_{\rho 2}\, \breve{o}k_{\rho 2}\, (\cdots (s_{\rho n}\, \breve{o}k_{\rho n}\, f^*) \cdots))$$

holds for each $(s_1, \dots, s_n) \in S^n$ and each $\rho \in \mathfrak{S}_n$, where $\mathfrak{S}_n$ is the symmetric group on the letters $1, \dots, n$. In this sense, every element of $S^n \to \mathbb{T}$ with $n \leq \#K$ is loosely linearized by the operations $\breve{o}k$ $(k \in K)$.

Our design of the quantifiers is a simple one. Let $(x, y)f$ be a binary relation on S written in Japanese word order. Suppose $a \in S$ is "Peter," $c \in S$ is "bean," and $(x, y)f = 1$ means that y eats x [x *wo* y *ga* eat]. Linearize $(x, y)f$ into $x\, \breve{o}\omega\, (y\, \breve{o}\pi f)$ and denote $a\, \breve{o}\pi f$ by g. Then ask yourself when you can assert that Peter eats all beans. Of course, you can assert so, only when the number of beans $s \in S$ satisfying $s\, \breve{o}\omega\, g = 0$ is less than 0. Next ask when you can assert that Peter eats some beans. Of course, you can assert so, only when the number of beans $s \in S$ satisfying $s\, \breve{o}\omega\, g = 1$ is greater than 0. Next ask when you can assert that Peter eats at most p beans. Of course, you can assert so, only when the number of beans $s \in S$ satisfying $s\, \breve{o}\omega\, g = 1$ is less than or equal to p. Finally ask how you can count the number of beans $s \in S$ satisfying $s\, \breve{o}\omega\, g = 1$. Of course you need first judge, for each element $s \in S$, whether s is a bean or not, because there are many kinds of beans such as soya beans, French beans, and so on. In order that you can judge so, there must exist a relation $\exists$ on S such that $c \exists s$ holds iff s is a bean. Moreover, in order that you can count the number of elements of S, you need some quantitative system $\mathbb{P}$ and some $\mathbb{P}$-measure $X \mapsto |X|$ on S. Conversely, if such a relation and a measure exist, then you can certainly define as follows:

$$\left.\begin{array}{r}\text{[bean all } \textit{wo} \text{ pētā } \textit{ga} \text{ eat]}\\ \text{(Peter eats all beans)}\end{array}\right\} \iff \left|\left\{s \in S \mid c \ni s,\ s\,\breve{o}\omega\, g = 0\right\}\right| \leq 0,$$

$$\left.\begin{array}{r}\text{[bean some } \textit{wo} \text{ pētā } \textit{ga} \text{ eat]}\\ \text{(Peter eats some beans)}\end{array}\right\} \iff \left|\left\{s \in S \mid c \ni s,\ s\,\breve{o}\omega\, g = 1\right\}\right| > 0,$$

$$\left.\begin{array}{r}\text{[bean } p \text{ at most } \textit{wo} \text{ pētā } \textit{ga} \text{ eat]}\\ \text{(Peter eats at most } p \text{ beans)}\end{array}\right\} \iff \left|\left\{s \in S \mid c \ni s,\ s\,\breve{o}\omega\, g = 1\right\}\right| \leq p.$$

It now seems reasonable to define operations $\neg(\leftarrow 0]\omega,\ (0 \rightarrow)\omega,\ (\leftarrow p]\omega$ by

$$\begin{aligned} c\,\neg(\leftarrow 0]\omega\, g = 1 &\iff \left|\left\{s \in S \mid c \ni s,\ s\,\breve{o}\omega\, g = 0\right\}\right| \in (\leftarrow 0],\\ c\,(0 \rightarrow)\omega\, g = 1 &\iff \left|\left\{s \in S \mid c \ni s,\ s\,\breve{o}\omega\, g = 1\right\}\right| \in (0 \rightarrow),\\ c\,(\leftarrow p]\omega\, g = 1 &\iff \left|\left\{s \in S \mid c \ni s,\ s\,\breve{o}\omega\, g = 1\right\}\right| \in (\leftarrow p],\end{aligned}$$

and denote $\neg(\leftarrow 0]$ and $(0 \rightarrow)$ by $\forall$ and $\exists$. This is the way we have reached the definition of the operations λk ($\lambda \in \mathfrak{Q}$) on W. Thus, MPCL has plenty of quantifiers, all of which accompany themselves with cases and exist not only in the OS of the language A but also in that of the world W. This is in contrast to PL, which has only two quantifiers $\forall$ and $\exists$ which accompany themselves with variables and exist only in the OS of the language. Recall here again that MPC language A is not a model of natural languages but a model of the human brain system of describing human notions. As such, A may well have a seemingly unconventional set $\mathfrak{Q}$ of quantifiers.

Now the substructure $(S \rightarrow \mathbb{T}) \cup S$ of the MPC world W is identified with $\mathcal{P}S \cup S$ and is designed to be a model of the entirety E of the entities in the real world. A few remarks are also in order about how we have reached this design. Let X be a set, for instance, the set of all living beings. Then we human can think about the set $\mathcal{P}X$ of the subsets of X, such as the set of multicellular organisms, of vertebrates, and so on. We can furthermore think about the set $\mathcal{P}^2X$ of the subsets of $\mathcal{P}X$, such as the set of birds, of fishes, and so on. We can furthermore think about $\mathcal{P}^3X, \mathcal{P}^4X, \ldots$. Therefore, if S is subset of a model E' of E, it seems that E' must contain $\bigcup_{n\geq 0} \mathcal{P}^n S$, whereas our model contains only $\mathcal{P}S \cup S$. This is because the base of MPC worlds may vary. If S is the base of an MPC world W, then there exists an MPC world W_1 whose base S_1 is equal to $\mathcal{P}S \cup S$ and whose basic relation $\ni_1$ is an extension of the extended relation (3.1) between $(S \rightarrow \mathbb{T}) \cup S$ and S. In particular, W_1 contains $\mathcal{P}^2 S \cup \mathcal{P}S \cup S$. Therefore for each $n \geq 1$, there exists an MPC world W_n which contains $\bigcup_{k=0}^{n+1} \mathcal{P}^k S$. Thus, $(S \rightarrow \mathbb{T}) \cup S$ is enough for a model of E.

For further account of this sort, we refer the reader to Chapters 1, 2, 7 of [3], in particular Chapter 7, *Case Language and Japanese Language*.

4 Statement of the main result

Let $(A, T, \sigma, \mathbb{S}, \mathbb{C}, \mathbb{X}, \Gamma), \mathcal{W}, (\lambda_W)_{\lambda,W}$ be the logical system MPCL defined in §3.1-§3.4 and $(H, \mathcal{G}_{\mathcal{W}})$ be the predicate logical space defined by $A, \mathcal{W}, (\lambda_W)_{\lambda,W}$ in

§3.5. Here we present a deduction pair $(\wp, \nabla)$ on H which will be proved $\mathcal{G}_{\mathcal{W}}$-complete under certain reasonable additional conditions on the sets $\mathsf{K}, \mathbb{S}, \mathbb{X}$ of cases, primes, and variables of A. A similar $\mathcal{F}_{\mathcal{W}}$-complete deduction pair on $\mathsf{A}_{\emptyset}$ has been obtained, but we will not present it here.

The presentation of the deduction rule $\wp$ is simple. It is the familiar **modus ponens** defined by the fractional expression

$$\wp = \frac{\mathsf{f} \quad \mathsf{f} \Rightarrow \mathsf{g}}{\mathsf{g}}, \tag{4.1}$$

where $\mathsf{f}, \mathsf{g} \in \mathsf{H}$. This means that, if $\mathsf{f}_1, \dots, \mathsf{f}_n, \mathsf{g} \in \mathsf{H}$, then $\mathsf{f}_1 \cdots \mathsf{f}_n \wp \mathsf{g}$ iff $n = 2$ and f_2 is equal to $\mathsf{f}_1 \Rightarrow \mathsf{g}$. Meanings of other fractional expressions below and in §11.1 are similar.

On the other hand, the presentation of the deduction basis ∇ is hard. First we take an arbitrary variable $x_0 \in \mathbb{X}_{\varepsilon}$ and define

$$\mathsf{one} = (x_0 \,\breve{o}\pi\, x_0 \triangle)\, \Omega x_0. \tag{4.2}$$

Next we define four more logics on H also by fractional expressions as follows:

$$\begin{aligned}
\& &= \frac{\mathsf{f} \quad \mathsf{g}}{\mathsf{f} \wedge \mathsf{g}} && (\mathsf{f}, \mathsf{g} \in \mathsf{H}),\\
\perp &= \frac{\mathsf{f}}{a \,\breve{o}\mathsf{k}\, \mathsf{f}} && (\mathsf{f} \in \mathsf{H},\ a \in \mathsf{A}_{\varepsilon},\ \mathsf{k} \in \mathsf{K}_{\mathsf{f}}),\\
\top &= \frac{x \,\breve{o}\mathsf{k}\, \mathsf{f}}{\mathsf{f}} && (\mathsf{f} \in \mathsf{H},\ x \in \mathbb{X}_{\varepsilon},\ \mathsf{k} \in \mathsf{K}_{\mathsf{f}},\ x \not\ll \mathsf{f}),\\
\forall &= \frac{\mathsf{f}}{\mathsf{one}\, \forall\pi\, (\mathsf{f}\, \Omega x) \triangle} && (\mathsf{f} \in \mathsf{A}_{\emptyset},\ x \in \mathbb{X}_{\varepsilon}).
\end{aligned}$$

Next we present a subset ∂ of H in order to define ∇ by

$$\nabla = [\partial]_{\wp \cup \& \cup \perp \cup \top \cup \forall}. \tag{4.3}$$

This implies that ∇ is the union of the $\wp \cup \& \cup \perp \cup \top \cup \forall$-descendants $\partial_0, \partial_1, \dots$ of ∂, which are inductively defined as follows. First $\partial_0 = \partial$, and for $n \geq 1$, ∂_n is the set of all elements on the following list.

g $\quad (\mathsf{g} \in \mathsf{H}$ and $\mathsf{f} \Rightarrow \mathsf{g} \in \partial_{n_1}$ for some $\mathsf{f} \in \partial_{n_2}$ with $n_1 + n_2 = n - 1)$,
$\mathsf{f} \wedge \mathsf{g}$ $\quad (\mathsf{f} \in \partial_{n_1},\ \mathsf{g} \in \partial_{n_2},\ n_1 + n_2 = n - 1)$,
$a \,\breve{o}\mathsf{k}\, \mathsf{f}$ $\quad (\mathsf{f} \in \partial_{n-1},\ a \in \mathsf{A}_{\varepsilon},\ \mathsf{k} \in \mathsf{K}_{\mathsf{f}})$,
f $\quad (\mathsf{f} \in \mathsf{H}$ and $x \,\breve{o}\mathsf{k}\, \mathsf{f} \in \partial_{n-1}$ for some $x \in \mathbb{X}_{\varepsilon}$ and $\mathsf{k} \in \mathsf{K}_{\mathsf{f}}$ with $x \not\ll \mathsf{f})$,
$\mathsf{one}\, \forall\pi\, (\mathsf{f}\, \Omega x) \triangle$ $\quad (\mathsf{f} \in \partial_{n-1} \cap \mathsf{A}_{\emptyset},\ x \in \mathbb{X}_{\varepsilon})$.

In other words, an element $h \in \mathsf{H}$ belongs to ∇ if and only if there exist elements $h_1, \dots, h_n \in \mathsf{H}$ such that $h_n = h$ and, for each $i \in \{1, \dots, n\}$, either $h_i \in \partial$ or h_i satisfies one of the following five conditions.

- $h_j = h_k \Rightarrow h_i$ for some $j, k \in \{1, \dots, i-1\}$.

- $h_i = h_j \wedge h_k$ for some $j, k \in \{1, \ldots, i-1\}$.
- $h_i = a\,\breve{o}k\,h_j$ for some $a \in A_\varepsilon$, $j \in \{1, \ldots, i-1\}$, and $k \in K_{h_j}$.
- $h_j = x\,\breve{o}k\,h_i$ for some $j \in \{1, \ldots, i-1\}$, $x \in \mathbb{X}_\varepsilon$, and $k \in K_{h_i}$ with $x \not\lll h_i$.
- $h_i = one \forall\pi\,(h_j\,\Omega x)\triangle$ for some $j \in \{1, \ldots, i-1\}$, and $x \in \mathbb{X}_\varepsilon$ with $h_j \in A_\emptyset$.

The set ∂ consists of the following twenty five kinds of elements, the first of which is a familiar one:

$$\begin{aligned} f^{\Diamond} \vee f, \\ (f \wedge g) \Rightarrow f, \\ (f \wedge g) \Rightarrow g, \\ f \Rightarrow (f \vee g), \\ g \Rightarrow (f \vee g), \\ (f^{\Diamond} \vee g) \Rightarrow (f \Rightarrow g), \\ \big((f \Rightarrow h) \wedge (g \Rightarrow h)\big) \Rightarrow \big((f \vee g) \Rightarrow h\big), \\ \big((h \Rightarrow f) \wedge (h \Rightarrow g)\big) \Rightarrow \big(h \Rightarrow (f \wedge g)\big), \\ \Big((h \Rightarrow f) \wedge \big(h \Rightarrow (f \Rightarrow g)\big)\Big) \Rightarrow (h \Rightarrow g), \end{aligned} \qquad \text{(Boolean elem.)}$$

where $f, g, h \in H$. The remaining elements of ∂ are characteristic of MPCL:

$$a\,\breve{o}\pi\,a\triangle, \qquad (= \text{elem.})$$

where $a \in A_\varepsilon$.

$$(a\,\overline{\infty}\pi\,one\triangle)^{\Diamond}, \qquad (\overline{\infty} \text{ elem.})$$

where $a \in G$ and ∞ is the maximum of $\mathbb{P}$ in case it exists.

$$a\,\lambda k\,(b\,\breve{o}l\,f) \Leftrightarrow b\,\breve{o}l\,(a\,\lambda k\,f), \qquad (\mathfrak{Q}, \breve{o} \text{ elem.})$$

where $a \in G$, $b \in A_\varepsilon$, $f \in H$, $k, l \in K_f$, $k \neq l$, and $\lambda \in \{\breve{o}\} \cup \mathfrak{Q}$. Also $a \in A_\varepsilon$ in case $\lambda = \breve{o}$. The two-way arrow $\Leftrightarrow$ is a device to show a predicate $g \Rightarrow h$ and its reverse $h \Rightarrow g$ together. We will continue using this device.

$$(a_i\,\breve{o}k_i)_{i=1,\ldots,l}(f \wedge g) \Leftrightarrow \big((a_i\,\breve{o}k_i)_{i=1,\ldots,m} f \wedge (a_i\,\breve{o}k_i)_{i=n+1,\ldots,l}\, g\big), \quad (\wedge \text{ elem.})$$

$$(a_i\,\breve{o}k_i)_{i=1,\ldots,l}(f \vee g) \Leftrightarrow \big((a_i\,\breve{o}k_i)_{i=1,\ldots,m} f \vee (a_i\,\breve{o}k_i)_{i=n+1,\ldots,l}\, g\big), \quad (\vee \text{ elem.})$$

$$(a_i\,\breve{o}k_i)_{i=1,\ldots,l}(f \Rightarrow g) \Leftrightarrow \big((a_i\,\breve{o}k_i)_{i=1,\ldots,m} f \Rightarrow (a_i\,\breve{o}k_i)_{i=n+1,\ldots,l}\, g\big), \quad (\Rightarrow \text{ elem.})$$

where $a_1, \ldots, a_l \in A_\varepsilon$, $f, g \in H$, and $k_1, \ldots, k_l$ are distinct cases such that $k_1, \ldots, k_n \in K_f - K_g$, $k_{n+1}, \ldots, k_m \in K_f \cap K_g$, and $k_{m+1}, \ldots, k_l \in K_g - K_f$ $(0 \leq n \leq m \leq l)$. Also, $(a_i\,\breve{o}k_i)_{i=1,\ldots,l}(f \wedge g)$ is an abbreviation for $a_1\,\breve{o}k_1\,(a_2\,\breve{o}k_2\,(\cdots(a_l\,\breve{o}k_l\,(f \wedge g))\cdots))$, and similarly for analogous expressions.

$$\big((a_i\,\breve{o}k_i)_{i=1,\ldots,n}(f^{\Diamond})\big) \Leftrightarrow \big((a_i\,\breve{o}k_i)_{i=1,\ldots,n} f\big)^{\Diamond}, \qquad (\Diamond \text{ elem.})$$

where $\mathfrak{a}_1, \dots, \mathfrak{a}_n \in A_\varepsilon$, $\mathsf{f} \in \mathsf{H}$, and $\mathsf{k}_1, \dots, \mathsf{k}_n$ are distinct cases in K_f.

$$\mathfrak{a} \neg\mathfrak{p}\mathsf{k}\,\mathsf{f} \Leftrightarrow \mathfrak{a}\,\mathfrak{p}\mathsf{k}\,\mathsf{f}^\Diamond, \tag{$\neg$ elem.}$$
$$\mathfrak{a}\,\mathfrak{p}^\circ\mathsf{k}\,\mathsf{f} \Leftrightarrow (\mathfrak{a}\,\mathfrak{p}\mathsf{k}\,\mathsf{f})^\Diamond, \tag{$\circ$ elem.}$$

where $\mathfrak{a} \in \mathsf{G}$, $\mathsf{f} \in \mathsf{H}$, $\mathsf{k} \in \mathsf{K}_\mathsf{f}$, and $\mathfrak{p} \in \mathfrak{P}$.

$$\mathfrak{a}\,(\mathfrak{p} \cap \mathfrak{q})\mathsf{k}\,\mathsf{f} \Leftrightarrow (\mathfrak{a}\,\mathfrak{p}\mathsf{k}\,\mathsf{f} \wedge \mathfrak{a}\,\mathfrak{q}\mathsf{k}\,\mathsf{f}), \tag{$\cap$ elem.}$$
$$\mathfrak{a}\,(\mathfrak{p} \cup \mathfrak{q})\mathsf{k}\,\mathsf{f} \Leftrightarrow (\mathfrak{a}\,\mathfrak{p}\mathsf{k}\,\mathsf{f} \vee \mathfrak{a}\,\mathfrak{q}\mathsf{k}\,\mathsf{f}), \tag{$\cup$ elem.}$$

where $\mathfrak{a} \in \mathsf{G}$, $\mathsf{f} \in \mathsf{H}$, $\mathsf{k} \in \mathsf{K}_\mathsf{f}$, and $\mathfrak{p}, \mathfrak{q} \in \mathfrak{P}$.

$$\mathfrak{a}\,\overline{\mathfrak{p}}\mathsf{k}\,\mathsf{f} \Leftrightarrow \mathfrak{a}\,\overline{\mathfrak{p}}\pi\big((\mathsf{x}\,\breve{o}\mathsf{k}\,\mathsf{f})\,\Omega\mathsf{x}\big)\triangle, \tag{$\mathfrak{P}$ elem.}$$

where $\mathfrak{a} \in \mathsf{G}$, $\mathsf{f} \in \mathsf{H}$, $\mathsf{x} \in \mathbb{X}_\varepsilon$, $\mathsf{K}_\mathsf{f} = \{\mathsf{k}\}$, $\mathfrak{p} \in \mathbb{P}$, and $\mathsf{x} \not\ll \mathsf{f}$.

$$\mathfrak{a}\,\overline{\mathfrak{p}}\pi\,\mathfrak{b}\triangle \Leftrightarrow (\mathfrak{a} \sqcap \mathfrak{b})\,\overline{\mathfrak{p}}\pi\,\mathit{one}\triangle, \tag{$\triangle$ elem.}$$

where $\mathfrak{a}, \mathfrak{b} \in \mathsf{G}$, and $\mathfrak{p} \in \mathbb{P}$.

$$\big(\mathit{one}\,\forall\pi\,\big((\mathsf{f} \Rightarrow \mathsf{g})\,\Omega\mathsf{x}\big)\triangle\big) \Rightarrow \big(\mathsf{f} \Rightarrow \mathit{one}\,\forall\pi\,(\mathsf{g}\,\Omega\mathsf{x})\triangle\big), \tag{$\forall, \Rightarrow$ elem.}$$

where $\mathsf{f}, \mathsf{g} \in A_\emptyset$, $\mathsf{x} \in \mathbb{X}_\varepsilon$, and $\mathsf{x} \not\ll \mathsf{f}$.

$$\big(\mathit{one}\,\forall\pi\,\big(\big((\mathsf{x}\,\breve{o}\pi\,\mathfrak{a}\triangle) \Rightarrow (\mathsf{x}\,\breve{o}\mathsf{k}\,\mathsf{f})\big)\,\Omega\mathsf{x}\triangle\big)\big) \Rightarrow \mathfrak{a}\,\forall\mathsf{k}\,\mathsf{f}, \tag{$\forall$ elem.}$$

where $\mathsf{x} \in \mathbb{X}_\varepsilon$, $\mathfrak{a} \in \mathsf{G}$, $\mathsf{f} \in \mathsf{H}$, $\mathsf{K}_\mathsf{f} = \{\mathsf{k}\}$, and $\mathsf{x} \not\ll \mathfrak{a}, \mathsf{f}$.

$$(\mathfrak{a}\,\forall\pi\,\mathfrak{b}\triangle \wedge \mathfrak{a}\,\overline{\mathfrak{p}}\mathsf{k}\,\mathsf{f}) \Rightarrow \mathfrak{b}\,\overline{\mathfrak{p}}\mathsf{k}\,\mathsf{f}, \tag{$\forall, \mathfrak{P}$ elem.}$$

where $\mathfrak{a}, \mathfrak{b} \in \mathsf{G}$, $\mathsf{f} \in \mathsf{H}$, $\mathsf{k} \in \mathsf{K}_\mathsf{f}$, and $\mathfrak{p} \in \mathbb{P}$.

$$(\mathfrak{a} \sqcup \mathfrak{b})\,\overline{\mathfrak{p} + \mathfrak{q}}\mathsf{k}\,\mathsf{f} \Rightarrow (\mathfrak{a}\,\overline{\mathfrak{p}}\mathsf{k}\,\mathsf{f} \vee \mathfrak{b}\,\overline{\mathfrak{q}}\mathsf{k}\,\mathsf{f}), \tag{$\sqcup, +$ elem.}$$

where $\mathfrak{a}, \mathfrak{b} \in \mathsf{G}$, $\mathsf{f} \in \mathsf{H}$, $\mathsf{k} \in \mathsf{K}_\mathsf{f}$, and $\mathfrak{p}, \mathfrak{q} \in \mathbb{P}$.

$$(\mathit{one}^\Box\,\overline{\mathfrak{p}}\mathsf{k}\,\mathsf{f})^\Diamond, \tag{$\mathit{one}^\Box$ elem.}$$

where $\mathsf{f} \in \mathsf{H}$, $\mathsf{k} \in \mathsf{K}_\mathsf{f}$, and $\mathfrak{p} \in \mathbb{P}$.

$$\mathfrak{b}\,\breve{o}\pi\,\mathfrak{a}\triangle \Rightarrow \mathfrak{a}\,\exists\pi\,\mathit{one}\triangle, \tag{$\exists$ elem.}$$

where $\mathfrak{a} \in \mathsf{G}$ and $\mathfrak{b} \in A_\varepsilon$.

$$(\mathfrak{a} \sqcap \mathfrak{b})\triangle \Leftrightarrow (\mathfrak{a}\triangle \wedge \mathfrak{b}\triangle), \tag{$\sqcap$ elem.}$$
$$(\mathfrak{a} \sqcup \mathfrak{b})\triangle \Leftrightarrow (\mathfrak{a}\triangle \vee \mathfrak{b}\triangle), \tag{$\sqcup$ elem.}$$
$$(\mathfrak{a}^\Box)\triangle \Leftrightarrow (\mathfrak{a}\triangle)^\Diamond, \tag{$\Box$ elem.}$$

where $\mathfrak{a}, \mathfrak{b} \in \mathsf{G}$.

$$\mathfrak{a}\,\breve{o}\pi\,(\mathsf{f}\,\Omega\mathsf{x})\triangle \Leftrightarrow \mathsf{f}(\mathsf{x}/\mathfrak{a}), \tag{Ω elem.}$$

where $\mathsf{a} \in \mathsf{A}_\varepsilon$, $\mathsf{f} \in \mathsf{A}_\emptyset$, $\mathsf{x} \in \mathbb{X}_\varepsilon$, and x is free from a in f. The (x/a) on the right-hand side of $\Leftrightarrow$ denotes the substitution of a for x.

$$\mathsf{one}\,\forall\pi\,(\mathsf{f}\,\Omega\mathsf{x})\triangle \Rightarrow \mathsf{f}, \qquad (\forall\text{– elem.})$$

where $\mathsf{f} \in \mathsf{A}_\emptyset$ and $\mathsf{x} \in \mathbb{X}_\varepsilon$.

This completes the presentation of the deduction pair $(\wp, \nabla)$. Under all the definitions given so far, we can now state our main result.

Theorem 4.1 Assume that the range K_f is a finite set for each $\mathsf{f} \in \mathsf{H}$ and that both $\mathbb{S}_\varepsilon$ and $\mathbb{X}_\varepsilon$ are enumerable sets. Then the deduction pair $(\wp, \nabla)$ on H is $\mathcal{G}_\mathcal{W}$-complete.

We prove this theorem in §11.

Corollary 4.1.1 The set ∇ is equal to the set of the $\mathcal{G}_\mathcal{W}$-tautologies on H.

Proof Theorem 4.1 implies that $(\wp, \nabla)$ is $\mathcal{G}_\mathcal{W}$-core-complete, while (4.3) implies that ∇ is closed under $\wp$, hence the above result.

5 Related results and open problems

Theorem 4.1 is a consequence of another kind of completeness theorem on the **sequents**. Its proof is based on a method of the second author [7] for sequents in an antecedent of MPCL, with errors in [7] corrected in this paper. The second author also proved a **cut elimination theorem** for the sequents.

We may extend the set $\mathfrak{Q}$ of the quantifiers of MPCL. Recall that $\mathfrak{Q}$ is divided into $\neg\mathfrak{P}$ and $\mathfrak{P}$, and $\mathfrak{P}$ is the set of the unions of a finite number of intervals of $\mathbb{P}$ in one of the shapes $(\mathsf{p} \rightarrow), (\mathsf{p}, \mathsf{q}]$, and $(\leftarrow \mathsf{q}]$. This restriction to intervals appears unnatural. It appears natural to extend $\mathfrak{P}$ to the set $\mathfrak{P}'$ of the finite unions of all kinds of intervals. In fact in [3], MPCL with $\mathfrak{P}$ replaced by $\mathfrak{P}'$ has been investigated as well. However, no completeness theorem has been obtained as yet for $\mathfrak{P}'$ unless $\mathfrak{P} = \mathfrak{P}'$, which seems to be deeply related with the fundamental question of how human cognizes quantities in the outer world. We note that $\mathfrak{P} = \mathfrak{P}'$ if each element $\mathsf{p} \in \mathbb{P} - \{0\}$ has an immediate predecessor, for instance if $\mathbb{P} = \mathbb{Z}_{\geq 0}$.

As mentioned in the introduction, MPCL should be extended also to CL with arbitrary number of phases. We have posed a design for CL in [3], but have yet to investigate it.

The logical spaces in the most general sense are put into three classes in view of a certain property related to completeness, models, and consistency. Both the sentence logical space $(\mathsf{A}_\emptyset, \mathcal{F}_\mathcal{W})$ and the predicate logical space $(\mathsf{H}, \mathcal{G}_\mathcal{W})$ of MPCL belong to the 3rd class, which means that consistent subsets of $\mathsf{A}_\emptyset$ nor H do not necessarily possess models. Takaoka [8] has shown that even consistent sets of *closed* elements of $\mathsf{A}_\emptyset$ do not necessarily possess models. The relationship of this phenomenon with Kurt Gödel's celebrated work on incompleteness [1] should be investigated.

6 Rudiments of formal languages

Here we state rudiments of formal languages and sorted algebras to the extent necessary for the proof of Theorem 4.1. For omitted proofs and further results, we refer the reader to [2][3][4].

Theorem 6.1 Let $(A, T, \sigma, S, C, X, \Gamma)$ be a formal language, $a \in A$, W be an actual world for A, Φ be a C-denotation into W, and ν, ν' be X-denotations into W. Assume that $\nu x = \nu' x$ for each variable x which occurs free in a. Then $(\Phi^* a)\nu = (\Phi^* a)\nu'$.

This is called a **denotation theorem**. The following is a special case of the **substitution-redenotation theorem**.

Theorem 6.2 Let $(A, T, \sigma, S, C, X, \Gamma)$ be a formal language, $a, c \in A$, $x \in X$, W be an actual world for A, and Φ be a C-denotation into W. Assume that x is free from c in a and $\sigma c = \sigma x$. Then $\big(\Phi^*\big(a(x/c)\big)\big)\nu = (\Phi^* a)\big(\big(x/(\Phi^* c)\nu\big)\nu\big)$ for every X-denotation ν into W, where (x/c) is the substitution of c for x and $\big(x/(\Phi^* c)\nu\big)$ is the redenotation for x by $(\Phi^* c)\nu$.

Theorem 6.3 Let (A, T, σ, S) be a USA, $(\alpha_\lambda)_{\lambda \in \Lambda}$ be the OS of A, and n_λ be an arity of α_λ for each $\lambda \in \Lambda$. For each subset B of A and $a \in A$, let B^a denote the set of the elements of B which occur in a. Furthermore define $\Lambda^a = \{\lambda \in \Lambda \mid (\mathrm{Im}\, \alpha_\lambda)^a \neq \emptyset\}$ (if $\lambda \in \Lambda^a$, then we say that λ **occurs** in a). Then the following holds.

- For each $a \in A$, S^a and Λ^a are finite sets.
- $\Lambda^a = \begin{cases} \emptyset & \text{when } a \in S, \\ \{\lambda\} \cup \bigcup_{k=1}^{n_\lambda} \Lambda^{a_k} & \text{when } a = \alpha_\lambda(a_1, \ldots, a_{n_\lambda}). \end{cases}$
- If $a = \alpha_\lambda(a_1, \ldots, a_{n_\lambda}) \in A$, then $S^a_{\mathrm{free}} = \bigcup_{k=1}^{n_\lambda} S^{a_k}_{\mathrm{free}} - S^\lambda$.
- If $a, b \in A$ and $S^b_{\mathrm{free}} \cap \bigcup_{\lambda \in \Lambda^a} S^\lambda = \emptyset$, then every element $s \in S$ is free from b in a.
- Let $a, b, c \in A$, $s \in S$ and assume $b = a(s/c)$, where (s/c) denotes the substitution of c for s. Then $S^b_{\mathrm{free}} \subseteq S^c_{\mathrm{free}} \cup \big(S^a_{\mathrm{free}} - \{s\}\big)$ and $\Lambda^b \subseteq \Lambda^a \cup \Lambda^c$, while if $s \not\leqslant a$ then $\Lambda^b \subseteq \Lambda^a$.

Lemma 6.1 Let (A, T, σ, S) be a USA, $(\alpha_\lambda)_{\lambda \in \Lambda}$ be the OS of A, n_λ be an arity of α_λ for each $\lambda \in \Lambda$, and φ be a mapping of $\Lambda \amalg S$ into an additive semigroup M. Then there exists a mapping F of A into M which satisfies the following conditions:

$$F|_S = \varphi|_S, \qquad F\big(\alpha_\lambda(a_1, \ldots, a_{n_\lambda})\big) = \varphi\lambda + Fa_1 + \cdots + Fa_{n_\lambda}.$$

Proof By a remark following Theorem 2.1, we can define $Fa \in M$ for each $a \in A$ by induction on $\mathrm{rank}\, a$ so that the above conditions are satisfied.

7 Rudiments of logical spaces

Here we state rudiments of logical spaces to the extent necessary for the proof of Theorem 4.1. For omitted proofs and further results, we refer the reader to [3][5].

Throughout this section, we let A be a set, and denote elements of A by $x, y, \ldots$, while elements of A^* by $\alpha, \beta, \ldots$, both with or without numerical subscripts. When $\alpha = x_1 \cdots x_n \in A^*$, we will denote the subset $\{x_1, \ldots, x_n\}$ of A also by α. This **sequence convention** will be used throughout the remainder of this paper.

7.1 Fundamental theorem of completeness

Let $(A, \mathcal{F})$ be a logical space with $\mathcal{F} \subseteq A \to \mathbb{T}$. We define

$$\vec{A} = A^* \times A^*$$

and define the relation $\preccurlyeq$ on A^* by

$$\alpha \preccurlyeq \beta \iff \inf \varphi\alpha \leq \sup \varphi\beta \text{ for all } \varphi \in \mathcal{F},$$

where the infimum and supremum are taken with respect to the usual order $\leq$ on $\mathbb{T}$. We call $\preccurlyeq$ the **validity relation** of $(A, \mathcal{F})$ or the **$\mathcal{F}$-validity relation**. It is easy to show that $\preccurlyeq$ satisfies the following five familiar laws:

$$x \preccurlyeq x, \qquad \text{(repetition law)}$$

$$\left.\begin{array}{l} \alpha \preccurlyeq \beta \implies x\alpha \preccurlyeq \beta, \\ \alpha \succcurlyeq \beta \implies x\alpha \succcurlyeq \beta, \end{array}\right\} \qquad \text{(weakening law)}$$

$$\left.\begin{array}{l} xx\alpha \preccurlyeq \beta \implies x\alpha \preccurlyeq \beta, \\ xx\alpha \succcurlyeq \beta \implies x\alpha \succcurlyeq \beta, \end{array}\right\} \qquad \text{(contraction law)}$$

$$\left.\begin{array}{l} \alpha xy\beta \preccurlyeq \gamma \implies \alpha yx\beta \preccurlyeq \gamma, \\ \alpha xy\beta \succcurlyeq \gamma \implies \alpha yx\beta \succcurlyeq \gamma, \end{array}\right\} \qquad \text{(exchange law)}$$

$$\left.\begin{array}{l} \alpha \preccurlyeq x\gamma, \\ x\beta \preccurlyeq \delta \end{array}\right\} \implies \alpha\beta \preccurlyeq \delta\gamma. \qquad \text{(strong cut law)}$$

Consequently, $\preccurlyeq$ satisfies the following law:

$$\left.\begin{array}{l} \alpha \preccurlyeq x,\ x\beta \preccurlyeq \gamma \implies \alpha\beta \preccurlyeq \gamma, \\ \alpha \succcurlyeq x,\ x\beta \succcurlyeq \gamma \implies \alpha\beta \succcurlyeq \gamma. \end{array}\right\} \qquad \text{(cut law)}$$

Furthermore, we define

$$\vec{C} = \{(\alpha, \beta) \in \vec{A} \mid \alpha \preccurlyeq \beta\}.$$

Then the $\mathcal{F}$-core C defined by (2.5) satisfies

$$C = \{x \in A \mid \varepsilon \preccurlyeq x\},$$

where ε is the element of A^* of length 0.

Let (R, D) be a deduction pair on A. We say that (R, D) is **$\mathcal{F}$-sound** if it satisfies the following two conditions:

- $(\alpha, x) \in A^* \times A,\ \alpha \mathrel{R} x \implies \alpha \preccurlyeq x$,
- D is $\mathcal{F}$-sound, that is, $D \subseteq C$.

The former condition is called the **$\mathcal{F}$-soundness** of R. It is easily shown by the cut law that if R is $\mathcal{F}$-sound then C is closed under R. Furthermore, we define the **deduction relation** $\preccurlyeq_{R,D}$ on A^* by

$$\alpha \preccurlyeq_{R,D} \beta \iff [\alpha \cup D]_R \supseteq \bigcap_{y \in \beta} [\{y\} \cup D]_R,$$

and then define

$$\vec{A}_{R,D} = \{(\alpha, \beta) \in \vec{A} \mid \alpha \preccurlyeq_{R,D} \beta\}. \tag{7.1}$$

Under the above definitions, the following theorem holds.

Theorem 7.1 Let (R, D) be an $\mathcal{F}$-sound deduction pair on A, and let $(\vec{R}, \vec{D})$ be a deduction pair on $\vec{A}$. Assume that the following two conditions are satisfied:

- $\vec{C} \subseteq [\vec{D}]_{\vec{R}}$,
- $\vec{D} \subseteq \vec{A}_{R,D}$ and $\vec{A}_{R,D}$ is closed under $\vec{R}$.

Then (R, D) is $\mathcal{F}$-complete.

7.2 Boolean relations

Here we assume that A is a set, that $\wedge, \vee$, and $\Rightarrow$ are global binary operations on A, and that $\Diamond$ is a global unary operation on A.

A relation $\preccurlyeq$ on A^* is said to be **Boolean**, if it satisfies the repetition law, weakening law, contraction law, exchange law, strong cut law, and the following four laws:

$$x \wedge y \preccurlyeq x, \quad x \wedge y \preccurlyeq y, \quad xy \preccurlyeq x \wedge y, \qquad \text{(conjunction law)}$$

$$x \vee y \succcurlyeq x, \quad x \vee y \succcurlyeq y, \quad xy \succcurlyeq x \vee y, \qquad \text{(disjunction law)}$$

$$x^\Diamond \preccurlyeq x \Rightarrow y, \quad y \preccurlyeq x \Rightarrow y, \quad x \Rightarrow y \preccurlyeq x^\Diamond y, \qquad \text{(implication law)}$$

$$xx^\Diamond \preccurlyeq \varepsilon, \quad xx^\Diamond \succcurlyeq \varepsilon. \qquad \text{(negation law)}$$

Also, a relation $\preccurlyeq$ on A^* is said to be **weakly Boolean**, if it satisfies the repetition law, weakening law, contraction law, exchange law, and the following four laws:

$$\left.\begin{array}{l} xy\alpha \preccurlyeq \beta \implies x \wedge y, \alpha \preccurlyeq \beta, \\ \alpha \preccurlyeq x\beta,\ \alpha \preccurlyeq y\beta \implies \alpha \preccurlyeq x \wedge y, \beta, \end{array}\right\} \qquad \text{(strong conjunction law)}$$

$$\left.\begin{array}{l} x\alpha \preccurlyeq \beta,\ y\alpha \preccurlyeq \beta \implies x \vee y, \alpha \preccurlyeq \beta, \\ \alpha \preccurlyeq xy\beta \implies \alpha \preccurlyeq x \vee y, \beta, \end{array}\right\} \qquad \text{(strong disjunction law)}$$

$$\left.\begin{array}{l} \alpha \preccurlyeq x\beta,\ y\alpha \preccurlyeq \beta \implies x \Rightarrow y, \alpha \preccurlyeq \beta, \\ x\alpha \preccurlyeq y\beta \implies \alpha \preccurlyeq x \Rightarrow y, \beta, \end{array}\right\} \qquad \text{(strong implication law)}$$

$$\left.\begin{array}{l} \alpha \preccurlyeq x\beta \implies x^\Diamond\alpha \preccurlyeq \beta, \\ x\alpha \preccurlyeq \beta \implies \alpha \preccurlyeq x^\Diamond\beta. \end{array}\right\} \qquad \text{(strong negation law)}$$

Obviously, the largest relation on A^* is Boolean and weakly Boolean. We call it the **trivial** relation.

Theorem 7.2 Let $\preccurlyeq$ be a relation on A^*. Then the following two conditions are equivalent.

- The $\preccurlyeq$ is Boolean with respect to $\wedge, \vee, \Rightarrow, \Diamond$.
- The $\preccurlyeq$ is weakly Boolean with respect to $\wedge, \vee, \Rightarrow, \Diamond$ and satisfies the cut law.

If R is a relation on a set X, the **symmetric core** of R is defined to be the intersection $R \cap R^*$ of R and its dual R^* regarded as subsets of $X \times X$. If R is a preorder, then $R \cap R^*$ is an equivalence relation.

Theorem 7.3 Let $\preccurlyeq$ be a Boolean relation on A^* with respect to $\wedge, \vee, \Rightarrow, \Diamond$. Then the following holds for $\preccurlyeq$ and its symmetric core $\asymp$:

$$\begin{cases} \alpha x y \beta \preccurlyeq \gamma \iff \alpha, x \wedge y, \beta \preccurlyeq \gamma, \\ \alpha x y \beta \succcurlyeq \gamma \iff \alpha, x \vee y, \beta \succcurlyeq \gamma, \end{cases}$$

$$\begin{cases} x_1 \wedge \cdots \wedge x_n \asymp (\cdots (x_1 \wedge x_2) \wedge \cdots) \wedge x_n, \\ x_1 \vee \cdots \vee x_n \asymp (\cdots (x_1 \vee x_2) \vee \cdots) \vee x_n, \end{cases}$$

irrespective of the order of applying the operations $\wedge$ and $\vee$ on the left-hand side of $\asymp$,

$$\left. \begin{matrix} x_1 \preccurlyeq y_1, \\ x_2 \preccurlyeq y_2 \end{matrix} \right\} \implies \begin{cases} x_1 \wedge x_2 \preccurlyeq y_1 \wedge y_2, \\ x_1 \vee x_2 \preccurlyeq y_1 \vee y_2, \end{cases}$$

$$x\alpha \preccurlyeq y \iff \alpha \preccurlyeq x \Rightarrow y,$$

$$\begin{cases} \alpha \preccurlyeq x\beta \iff x^{\Diamond}\alpha \preccurlyeq \beta, \\ \alpha \succcurlyeq x\beta \iff x^{\Diamond}\alpha \succcurlyeq \beta, \end{cases}$$

$$\alpha \preccurlyeq \beta \iff \alpha \preccurlyeq x \wedge x^{\Diamond}, \beta \iff x \vee x^{\Diamond}, \alpha \preccurlyeq \beta.$$

Theorem 7.4 Let $(A, \mathcal{F})$ be a logical space wit $\mathcal{F} \subseteq A \to \mathbb{T}$, and assume that every member of $\mathcal{F}$ is a homomorphism with respect to $\wedge, \vee, \Rightarrow, \Diamond$ (such a logical space is said to be **Boolean** with respect to $\wedge, \vee, \Rightarrow, \Diamond$). Then the following holds.

- The validity relation of $(A, \mathcal{F})$ is Boolean with respect to $\wedge, \vee, \Rightarrow, \Diamond$. It is non-trivial unless $\mathcal{F} = \emptyset$.
- The two logics $\wp = \dfrac{x \quad x \Rightarrow y}{y}$ and $\& = \dfrac{x \quad y}{x \wedge y}$ on A are $\mathcal{F}$-sound.
- The Boolean elements of A with respect to $\wedge, \vee, \Rightarrow, \Diamond$ are $\mathcal{F}$-sound.
- The deduction pair $(\wp, C)$ consisting of $\wp$ and the $\mathcal{F}$-core C is $\mathcal{F}$-complete.

Theorem 7.5 Let (R, D) be a deduction pair on A with $R = \wp \cup \&$. Assume that $[D]_R$ contains all Boolean elements of A with respect to $\wedge, \vee, \Rightarrow, \Diamond$. Then the deduction relation $\preccurlyeq_{R,D}$ is Boolean with respect to $\wedge, \vee, \Rightarrow, \Diamond$.

8 Rudiments of quantities and measures

Here we prove rudiments of quantitative systems and measures to the extent necessary for the proof of Theorem 4.1. For further results, we refer the reader to [3]

Theorem 8.1 Let $(\mathbb{P}, +, 0, \leq)$ be a quantitative system and $\mathbb{Q}$ be a finitely generated support subalgebra of $(\mathbb{P}, +)$. Then $\mathbb{Q}$ is well-ordered with respect to $\leq$.

Proof Since $(\mathbb{Q} \cup \{0\}, +, 0, \leq)$ is a quantitative system, we may assume $\mathbb{P} = \mathbb{Q} \ni 0$. Then $\mathbb{P} = [\{q_1, \dots, q_k\}]$ with $q_1 \geq q_2 \geq \cdots \geq q_k = 0$. If $k = 1$, $\mathbb{P}$ is equal to $\{0\}$ and so well-ordered. Therefore we assume $k > 1$ and argue by induction on k. Let $\mathbb{P}' = [\{q_2, \dots, q_k\}]$. Then $\mathbb{P}'$ is well-ordered by the induction hypothesis. We only need to show that every downward closed interval $(\leftarrow r]$ of $\mathbb{P}$ is well-ordered. There are elements $n_1, \dots, n_k \in \mathbb{Z}_{\geq 0}$ such that $r = n_1 q_1 + \cdots + n_k q_k$. Define $n_0 = n_1 + \cdots + n_k$. Then $r \leq n_0 q_1$. Take an arbitrary element $p \in (\leftarrow r]$. Then $p = n q_1 + p'$ for some $n \in \mathbb{Z}_{\geq 0}$ and $p' \in \mathbb{P}'$. We may take the n so that $n \leq n_0$, because if $n_0 < n$, then $r \leq n_0 q_1 \leq n q_1 \leq n q_1 + p' = p$, so $p = r = n_1 q_1 + \cdots + n_k q_k$ and $n_1 \leq n_0$. Therefore, defining $\mathbb{P}'_n = \{n q_1 + p' \mid p' \in \mathbb{P}'\}$ for $n = 0, \dots, n_0$, we have $(\leftarrow r] \subseteq \bigcup_{n=0}^{n_0} \mathbb{P}'_n$. Since the mapping $p' \mapsto n q_1 + p'$ of $\mathbb{P}'$ onto $\mathbb{P}'_n$ is increasing, $\mathbb{P}'_n$ is well-ordered for $n = 0, \dots, n_0$. Therefore $(\leftarrow r]$ is well-ordered.

Theorem 8.2 Let S be a non-empty set, $(\mathbb{P}, +, 0, \leq)$ be a quantitative system, $0 < \acute{o} \in \mathbb{P}$, and R be a relation between $\mathcal{P}S$ and $\mathbb{P}$. Assume that R satisfies the following three conditions for all $X, Y \in \mathcal{P}S$ and all $p, q \in \mathbb{P}$:

- $X = \emptyset \iff X \, R \, 0$,
- $X \subseteq Y$ and $Y \, R \, p \implies X \, R \, p$,
- $X \, R \, p$ and $Y \, R \, q \implies (X \cup Y) \, R \, (p + q)$.

Assume furthermore that, for each $X \in \mathcal{P}S$, there exists the minimum of the subset $\{p \in \mathbb{P} \mid X \, R \, p\} \cup \{\acute{o}\}$ of $\mathbb{P}$, and let $|X|$ denote the minimum. Then the mapping $X \mapsto |X|$ is a $\mathbb{P}$-measure on S.

Proof Since $\emptyset \, R \, 0$, we have $|\emptyset| = 0$. Conversely if $|X| = 0$, then since $0 < \acute{o}$, we have $X \, R \, 0$ and so $X = \emptyset$.

Assume $X \subseteq Y$ and let $p = |Y|$. If $Y \, R \, p$, then $X \, R \, p$ and so $|X| \leq p$. If $p = \acute{o}$, then $|X| \leq p$ also.

Let $p = |X|$ and $q = |Y|$. If $X \mathrel{R} p$ and $Y \mathrel{R} q$, then $(X \cup Y) \mathrel{R} (p + q)$ and so $|X \cup Y| \leq p + q$. If $p = ó$, then $|X \cup Y| \leq p \leq p + q$. If $q = ó$, then $|X \cup Y| \leq q \leq p + q$.

9 Structure of the MPC World

Here we analyze the structure of the MPC world

$$W = (S \to \mathbb{T}) \cup S \cup \bigcup_{P \in \mathcal{P}K} \big((P \to S) \to \mathbb{T}\big)$$

defined in §3.3 to the extent necessary for the proof of Theorem 4.1. For further results, we refer the reader to [3].

We define

$$E = W_\delta \cup W_\varepsilon = (S \to \mathbb{T}) \cup S,$$

which we call the set of the **entities**. We also define

$$F = \bigcup_{P \in \mathcal{P}K} W_P = \bigcup_{P \in \mathcal{P}K} \big((P \to S) \to \mathbb{T}\big),$$

which we call the set of the **affairs**. For each $f \in F$, we denote by K_f the element $P \in \mathcal{P}K$ satisfying $f \in W_P$ and call it the **frame** of f.

9.1 Boolean structure

The results here will often be used without notices.

The set $W_\delta = S \to \mathbb{T}$ is identified with the power set $\mathcal{P}S$, and so is a Boolean lattice with respect to the order $\sqsubseteq$ defined by

$$a \sqsubseteq b \iff as \leq bs \text{ for each } s \in S.$$

The least element 0 and the greatest element 1 of W_δ are characterized by the properties that $0s = 0$ and $1s = 1$ for each $s \in S$. The sets E and W_δ are closed under the operations $\sqcap, \sqcup, \square$, and their restrictions to W_δ are equal to the meet, join, and complement on the Boolean lattice W_δ. The order $\sqsubseteq$ on W_δ is extended to the preorder $\sqsubseteq$ on E defined by

$$a \sqsubseteq b \iff \text{If } a \ni s \in S \text{ then } b \ni s, \tag{9.1}$$

so that $0 \sqsubseteq a$ and $a \sqsubseteq 1$ for all $a \in E$.

Similarly as above for each $P \in \mathcal{P}K$, $W_P = (P \to S) \to \mathbb{T}$ is a Boolean lattice with respect to the order $\leq$ defined by

$$f \leq g \iff f\theta \leq g\theta \text{ for each } \theta \in P \to S.$$

The least element 0 and the greatest element 1 of W_P are characterized by the properties that $0\theta = 0$ and $1\theta = 1$ for each $\theta \in P \to S$. The sets F and W_P ($P \in$

$\mathcal{P}\mathsf{K}$) are closed under the operations $\wedge, \vee, \Rightarrow, \Diamond$, and their restrictions to W_P are equal to the meet, join, implication, complement on the Boolean lattice W_P, and $\mathsf{f} \Rightarrow \mathsf{g} = \mathsf{f}^\Diamond \vee \mathsf{g}$ for each $(\mathsf{f}, \mathsf{g}) \in \mathsf{F}^2$. Moreover, for each $\theta \in \mathsf{P} \to \mathsf{S}$, the projection $\mathsf{f} \mapsto \mathsf{f}\theta$ of W_P into $\mathbb{T}$ is a homomorphism with respect to the operations $\wedge, \vee, \Rightarrow, \Diamond$.

Theorem 9.1 The following holds for all $\mathsf{a}, \mathsf{b} \in \mathsf{E}$:

$$(\mathsf{a} \sqcap \mathsf{b})\triangle = \mathsf{a}\triangle \wedge \mathsf{b}\triangle, \qquad (\mathsf{a} \sqcup \mathsf{b})\triangle = \mathsf{a}\triangle \vee \mathsf{b}\triangle, \qquad (\mathsf{a}^\Box)\triangle = (\mathsf{a}\triangle)^\Diamond.$$

This theorem is important, but its proof is easy and omitted.

All the orders $\leq$ on W_P $(\mathsf{P} \in \mathcal{P}\mathsf{K})$ are extended to a single relation $\lessdot$ on F defined by

$$\mathsf{f} \lessdot \mathsf{g} \iff \mathsf{f}(\theta|_{\mathsf{K}_\mathsf{f}}) \leq \mathsf{g}(\theta|_{\mathsf{K}_\mathsf{g}}) \text{ for each } \theta \in (\mathsf{K}_\mathsf{f} \cup \mathsf{K}_\mathsf{g}) \to \mathsf{S},$$

which satisfies the following:

$$\mathsf{f} \lessdot \mathsf{g} \iff \mathsf{f}(\theta|_{\mathsf{K}_\mathsf{f}}) \leq \mathsf{g}(\theta|_{\mathsf{K}_\mathsf{g}}) \text{ for each } \theta \in \mathsf{K} \to \mathsf{S}.$$

Therefore $\lessdot$ is a preorder. We denote its symmetric core by $\doteq$, namely,

$$\mathsf{f} \doteq \mathsf{g} \iff \mathsf{f}(\theta|_{\mathsf{K}_\mathsf{f}}) = \mathsf{g}(\theta|_{\mathsf{K}_\mathsf{g}}) \text{ for each } \theta \in \mathsf{K} \to \mathsf{S}.$$

Then $\doteq$ is an equivalence relation, and its restriction to W_P is the equality $=$.

For each element $\mathsf{f} \in \mathsf{F}$, we define the element $\mathsf{f}^\sharp \in W_\mathsf{K}$ by

$$\mathsf{f}^\sharp\theta = \mathsf{f}(\theta|_{\mathsf{K}_\mathsf{f}})$$

for each $\theta \in \mathsf{K} \to \mathsf{S}$, and call the mapping $\mathsf{f} \mapsto \mathsf{f}^\sharp$ the **inflation**. Then

$$\mathsf{f} \lessdot \mathsf{g} \iff \mathsf{f}^\sharp \leq \mathsf{g}^\sharp, \qquad \mathsf{f} \doteq \mathsf{g} \iff \mathsf{f}^\sharp = \mathsf{g}^\sharp,$$

and so we may extended the preorder $\lessdot$ on F to the relation $\lessdot$ on F^* by

$$\begin{aligned} &\mathsf{f}_1 \cdots \mathsf{f}_m \lessdot \mathsf{g}_1 \cdots \mathsf{g}_n \\ &\iff \inf\{\mathsf{f}_1^\sharp, \ldots, \mathsf{f}_m^\sharp\} \leq \sup\{\mathsf{g}_1^\sharp, \ldots, \mathsf{g}_n^\sharp\} \\ &\iff \inf\{\mathsf{f}_1^\sharp\theta, \ldots, \mathsf{f}_m^\sharp\theta\} \leq \sup\{\mathsf{g}_1^\sharp\theta, \ldots, \mathsf{g}_n^\sharp\theta\} \text{ for each } \theta \in \mathsf{K} \to \mathsf{S} \end{aligned} \tag{9.2}$$

for all $\mathsf{f}_1 \cdots \mathsf{f}_m, \mathsf{g}_1 \cdots \mathsf{g}_n \in \mathsf{F}^*$ with $\mathsf{f}_1, \ldots, \mathsf{f}_m, \mathsf{g}_1, \ldots, \mathsf{g}_n \in \mathsf{F}$. Let $\mathsf{P}_1, \ldots, \mathsf{P}_m$, $\mathsf{Q}_1, \ldots, \mathsf{Q}_n$ be the frames of $\mathsf{f}_1 \cdots \mathsf{f}_m$, $\mathsf{g}_1 \cdots \mathsf{g}_n$, and let R be their union. Then

$$\begin{aligned} &\mathsf{f}_1 \cdots \mathsf{f}_m \lessdot \mathsf{g}_1 \cdots \mathsf{g}_n \\ &\quad \iff \inf\{\mathsf{f}_1(\theta|_{\mathsf{P}_1}), \ldots, \mathsf{f}_m(\theta|_{\mathsf{P}_m})\} \leq \sup\{\mathsf{g}_1(\theta|_{\mathsf{Q}_1}), \ldots, \mathsf{g}_n(\theta|_{\mathsf{Q}_n})\} \\ &\qquad\qquad \text{for each } \theta \in \mathsf{R} \to \mathsf{S}. \end{aligned} \tag{9.3}$$

It also follows that the inflation $\sharp \in \mathsf{F} \to W_\mathsf{K}$ is a homomorphism with respect to the operations $\wedge, \vee, \Rightarrow, \Diamond$. Consequently, the relation $\doteq$ is consistent with the operations. If $\theta \in \mathsf{K} \to \mathsf{S}$, then since the projection by θ is also a homomorphism of W_K into $\mathbb{T}$ with respect to the operations, the mapping $\mathsf{f} \mapsto \mathsf{f}^\sharp\theta$ is a homomorphism of F into $\mathbb{T}$ with respect to the operations. Therefore, Theorem 7.4 shows that the following holds.

Theorem 9.2 The relation $\lessdot$ on F^* is Boolean with respect to the operations $\wedge, \vee, \Rightarrow, \Diamond$.

9.2 Repetition of the operations $\breve{o}k$

Let $s_1, \ldots, s_n \in S$ and $k_1, \ldots, k_n$ be distinct cases in $P \in \mathcal{P}K$. Then it follows from the definition of the operations $\breve{o}k_i$ $(i = 1, \ldots, n)$ that, for each $f \in W_P$, the element $s_1 \,\breve{o}k_1\, (s_2 \,\breve{o}k_2\, (\cdots (s_n \,\breve{o}k_n\, f) \cdots))$ exists and belongs to $W_{P-\{k_1,\ldots,k_n\}}$. We sometimes abbreviate it by $(s_i \,\breve{o}k_i)_{i=1,\ldots,n} f$ or $(s_i \,\breve{o}k_i)_i f$. Let $\theta \in (P - \{k_1, \ldots, k_n\}) \to S$. Then, generalizing (3.2), we define the element $\left(\dfrac{k_1, \ldots, k_n}{s_1, \ldots, s_n}\right) \theta$ of $P \to S$ as follows:

$$\left(\left(\frac{k_1, \ldots, k_n}{s_1, \ldots, s_n}\right) \theta\right) k = \begin{cases} \theta k & \text{when } k \in P - \{k_1, \ldots, k_n\}, \\ s_i & \text{when } k = k_i \ (i = 1, \ldots, n). \end{cases}$$

We sometimes abbreviate $\left(\dfrac{k_1, \ldots, k_n}{s_1, \ldots, s_n}\right)$ by $(k_i/s_i)_{i=1,\ldots,n}$ or $(k_i/s_i)_i$.

Lemma 9.1 Let $k_1, \ldots, k_n$ be distinct cases in $P \in \mathcal{P}K$, $P \subseteq Q \in \mathcal{P}K$, $k_{n+1}, \ldots, k_m$ be distinct cases in $Q - P$, $\theta \in (Q - \{k_1, \ldots, k_m\}) \to S$, and $s_1, \ldots, s_m \in S$. Then $P - \{k_1, \ldots, k_n\} \subseteq Q - \{k_1, \ldots, k_m\}$ and the following holds:

$$\left.\left(\left(\frac{k_1, \ldots, k_m}{s_1, \ldots, s_m}\right) \theta\right)\right|_P = \left(\frac{k_1, \ldots, k_n}{s_1, \ldots, s_n}\right) \theta|_{P-\{k_1,\ldots,k_n\}}.$$

The proof is easy and omitted.

Theorem 9.3 Let $s_1, \ldots, s_n \in S$, $k_1, \ldots, k_n$ be distinct cases in $P \in \mathcal{P}K$, and $f \in W_P$. Then the following holds for each $\theta \in (P - \{k_1, \ldots, k_n\}) \to S$:

$$(s_1 \,\breve{o}k_1\, (s_2 \,\breve{o}k_2\, (\cdots (s_n \,\breve{o}k_n\, f) \cdots)))\theta = f\left(\left(\frac{k_1, \ldots, k_n}{s_1, \ldots, s_n}\right) \theta\right).$$

Proof We may assume $n > 1$ and argue by induction on n. Define $Q = P - \{k_n\}$ and $g = s_n \,\breve{o}k_n\, f$. Then $g \in W_Q$ and $\theta \in (Q - \{k_1, \ldots, k_{n-1}\}) \to S$, so $\theta' = \left(\dfrac{k_1, \ldots, k_{n-1}}{s_1, \ldots, s_{n-1}}\right) \theta$ belongs to $Q \to S$ and $g\theta' = f\big((k_n/s_n)\theta'\big)$. Obviously $(k_n/s_n)\theta' = \left(\dfrac{k_1, \ldots, k_n}{s_1, \ldots, s_n}\right) \theta$. Thus we may complete the proof as follows:

$$\begin{aligned}
&(s_1 \,\breve{o}k_1\, (s_2 \,\breve{o}k_2\, (\cdots (s_n \,\breve{o}k_n\, f) \cdots)))\theta \\
&= (s_1 \,\breve{o}k_1\, (s_2 \,\breve{o}k_2\, (\cdots (s_{n-1} \,\breve{o}k_{n-1}\, g) \cdots)))\theta \\
&= g\left(\left(\frac{k_1, \ldots, k_{n-1}}{s_1, \ldots, s_{n-1}}\right) \theta\right) = g\theta' = f\big((k_n/s_n)\theta'\big) = f\left(\left(\frac{k_1, \ldots, k_n}{s_1, \ldots, s_n}\right) \theta\right),
\end{aligned}$$

where the second equality holds by the induction hypothesis.

Corollary 9.3.1 Let $s_1, \dots, s_n \in S$, $k_1, \dots, k_n$ be distinct cases in $P \in \mathcal{P}K$, and $f \in W_P$. Then the following holds for every $\rho \in \mathfrak{S}_n$, where $\mathfrak{S}_n$ is the symmetric group on the letters $1, \dots, n$:

$$s_1 \check{o} k_1 \, (s_2 \check{o} k_2 \, (\cdots (s_n \check{o} k_n \, f) \cdots)) = s_{\rho 1} \check{o} k_{\rho 1} \, (s_{\rho 2} \check{o} k_{\rho 2} \, (\cdots (s_{\rho n} \check{o} k_{\rho n} \, f) \cdots)).$$

Proof Theorem 9.3 yields the following for each $\theta \in (P - \{k_1, \dots, k_n\}) \to S$, hence the above result:

$$\begin{aligned} &(s_1 \check{o} k_1 \, (s_2 \check{o} k_2 \, (\cdots (s_n \check{o} k_n \, f) \cdots)))\theta = f \left(\left(\frac{k_1, \dots, k_n}{s_1, \dots, s_n} \right) \theta \right) \\ &= f \left(\left(\frac{k_{\rho 1}, \dots, k_{\rho n}}{s_{\rho 1}, \dots, s_{\rho n}} \right) \theta \right) = (s_{\rho 1} \check{o} k_{\rho 1} \, (s_{\rho 2} \check{o} k_{\rho 2} \, (\cdots (s_{\rho n} \check{o} k_{\rho n} \, f) \cdots)))\theta. \end{aligned}$$

Corollary 9.3.2 Let $k_1, \dots, k_n$ be distinct cases in $P \in \mathcal{P}K$ and $f \in W_P$. Then the following holds for each $\theta \in P \to S$:

$$f\theta = \big((\theta k_1) \check{o} k_1 \, \big((\theta k_2) \check{o} k_2 \, \big(\cdots \big((\theta k_n) \check{o} k_n \, f\big) \cdots \big)\big)\big)\theta|_{P - \{k_1, \dots, k_n\}}.$$

Proof This is a consequence of Theorem 9.3 because $\theta k_1, \dots, \theta k_n \in S$ and $\theta = \left(\frac{k_1, \dots, k_n}{\theta k_1, \dots, \theta k_n} \right) \theta|_{P - \{k_1, \dots, k_n\}}$.

Corollary 9.3.3 Let $s_1, \dots, s_n \in S$, $k_1, \dots, k_n$ be distinct cases in $P \in \mathcal{P}K$, and $f \in W_P$. Then $(s_i \check{o} k_i)_{i=1,\dots,n}(f^\Diamond) = \big((s_i \check{o} k_i)_{i=1,\dots,n} f\big)^\Diamond$.

Proof Theorem 9.3 and the definition of the operation $\Diamond$ yield the following for each $\theta \in (P - \{k_1, \dots, k_n\}) \to S$, hence the above result:

$$\begin{aligned} \big((s_i \check{o} k_i)_i (f^\Diamond)\big)\theta &= f^\Diamond\big((k_i/s_i)_i \theta\big) = \big(f\big((k_i/s_i)_i \theta\big)\big)^\Diamond = \big(\big((s_i \check{o} k_i)_i f\big)\theta\big)^\Diamond \\ &= \big((s_i \check{o} k_i)_i f\big)^\Diamond \theta. \end{aligned}$$

Lemma 9.2 Let $k_1, \dots, k_n$ be distinct cases in $P \in \mathcal{P}K$, $k_{n+1}, \dots, k_m$ be distinct cases in $K - P$, $s_1, \dots, s_m \in S$, and $f \in W_P$. Then the following holds:

$$s_1 \check{o} k_1 \, (s_2 \check{o} k_2 \, (\cdots (s_m \check{o} k_m \, f^\sharp) \cdots)) \doteq s_1 \check{o} k_1 \, (s_2 \check{o} k_2 \, (\cdots (s_n \check{o} k_n \, f) \cdots)).$$

Proof Theorem 9.3, Lemma 9.1, and remarks in §9.1 yield the following for each $\theta \in (K - \{k_1, \dots, k_m\}) \to S$, hence the above result:

$$\begin{aligned} &(s_1 \check{o} k_1 \, (s_2 \check{o} k_2 \, (\cdots (s_m \check{o} k_m \, f^\sharp) \cdots)))\theta = f^\sharp \left(\left(\frac{k_1, \dots, k_m}{s_1, \dots, s_m} \right) \theta \right) \\ &= f \left(\left(\left(\frac{k_1, \dots, k_m}{s_1, \dots, s_m} \right) \theta \right) \Big|_P \right) = f \left(\left(\frac{k_1, \dots, k_n}{s_1, \dots, s_n} \right) \theta|_{P - \{k_1, \dots, k_n\}} \right) \\ &= (s_1 \check{o} k_1 \, (s_2 \check{o} k_2 \, (\cdots (s_n \check{o} k_n \, f) \cdots)))\theta|_{P - \{k_1, \dots, k_n\}}. \end{aligned}$$

Lemma 9.3 Let $s_1, \ldots, s_n \in S$, $k_1, \ldots, k_n$ be distinct cases in $P \in \mathcal{P}K$, and $f, g \in W_P$. Then the following holds:

$$(s_i \,\breve{o} k_i)_i (f \wedge g) = (s_i \,\breve{o} k_i)_i f \wedge (s_i \,\breve{o} k_i)_i g,$$
$$(s_i \,\breve{o} k_i)_i (f \vee g) = (s_i \,\breve{o} k_i)_i f \vee (s_i \,\breve{o} k_i)_i g,$$
$$(s_i \,\breve{o} k_i)_i (f \Rightarrow g) = (s_i \,\breve{o} k_i)_i f \Rightarrow (s_i \,\breve{o} k_i)_i g.$$

Proof Let $*$ be any one of $\wedge, \vee, \Rightarrow$, and $\theta \in (P - \{k_1, \ldots, k_n\}) \to S$. Then Theorem 9.3 and the definition of the operation $*$ yield the following, hence the above results:

$$\begin{aligned}\big((s_i \,\breve{o} k_i)_i (f * g)\big)\theta &= (f * g)\big((k_i/s_i)_i \theta\big) = f\big((k_i/s_i)_i \theta\big) * g\big((k_i/s_i)_i \theta\big) \\ &= \big((s_i \,\breve{o} k_i)_i f\big)\theta * \big((s_i \,\breve{o} k_i)_i g\big)\theta = \big((s_i \,\breve{o} k_i)_i f * (s_i \,\breve{o} k_i)_i g\big)\theta.\end{aligned}$$

Theorem 9.4 Let $s_1, \ldots, s_l \in S$, $f, g \in F$, and $k_1, \ldots, k_l$ be distinct cases such that $k_1, \ldots, k_n \in K_f - K_g$, $k_{n+1}, \ldots, k_m \in K_f \cap K_g$, and $k_{m+1}, \ldots, k_l \in K_g - K_f$ $(0 \leq n \leq m \leq l)$. Then the following holds :

$$(s_i \,\breve{o} k_i)_{i=1,\ldots,l} (f \wedge g) = (s_i \,\breve{o} k_i)_{i=1,\ldots,m} f \wedge (s_i \,\breve{o} k_i)_{i=n+1,\ldots,l} g,$$
$$(s_i \,\breve{o} k_i)_{i=1,\ldots,l} (f \vee g) = (s_i \,\breve{o} k_i)_{i=1,\ldots,m} f \vee (s_i \,\breve{o} k_i)_{i=n+1,\ldots,l} g,$$
$$(s_i \,\breve{o} k_i)_{i=1,\ldots,l} (f \Rightarrow g) = (s_i \,\breve{o} k_i)_{i=1,\ldots,m} f \Rightarrow (s_i \,\breve{o} k_i)_{i=n+1,\ldots,l} g.$$

Proof Let $*$ be any one of $\wedge, \vee, \Rightarrow$. Then Lemmas 9.2, 9.3, and remarks in §9.1 yield the following, hence the above result:

$$\begin{aligned}(s_i \,\breve{o} k_i)_{i=1,\ldots,l} (f * g) &\doteq (s_i \,\breve{o} k_i)_{i=1,\ldots,l} (f * g)^\sharp = (s_i \,\breve{o} k_i)_{i=1,\ldots,l} (f^\sharp * g^\sharp) \\ &= (s_i \,\breve{o} k_i)_{i=1,\ldots,l} f^\sharp * (s_i \,\breve{o} k_i)_{i=1,\ldots,l} g^\sharp \\ &\doteq (s_i \,\breve{o} k_i)_{i=1,\ldots,m} f * (s_i \,\breve{o} k_i)_{i=n+1,\ldots,l} g.\end{aligned}$$

Lemma 9.4 Let $k_1, \ldots, k_n$ be distinct cases in $P \in \mathcal{P}K$ and $f, g \in W_P$. Then $f \leq g$ iff

$$s_1 \,\breve{o} k_1 \,(s_2 \,\breve{o} k_2 \,(\cdots (s_n \,\breve{o} k_n \, f) \cdots)) \leq s_1 \,\breve{o} k_1 \,(s_2 \,\breve{o} k_2 \,(\cdots (s_n \,\breve{o} k_n \, g) \cdots))$$

for any elements $s_1, \ldots, s_n \in S$.

Proof If $f \leq g$ and $s_1, \ldots, s_n \in S$, then Theorem 9.3 yields

$$\begin{aligned}(s_1 \,\breve{o} k_1 \,(s_2 \,\breve{o} k_2 \,(\cdots (s_n \,\breve{o} k_n \, f) \cdots)))\theta &= f\left(\left(\frac{k_1, \ldots, k_n}{s_1, \ldots, s_n}\right)\theta\right) \\ &\leq g\left(\left(\frac{k_1, \ldots, k_n}{s_1, \ldots, s_n}\right)\theta\right) = (s_1 \,\breve{o} k_1 \,(s_2 \,\breve{o} k_2 \,(\cdots (s_n \,\breve{o} k_n \, g) \cdots)))\theta\end{aligned}$$

for each $\theta \in (P - \{k_1, \ldots, k_n\}) \to S$, hence $(s_i \,\breve{o} k_i)_i f \leq (s_i \,\breve{o} k_i)_i g$. Conversely if $(s_i \,\breve{o} k_i)_i f \leq (s_i \,\breve{o} k_i)_i g$ for any elements $s_1, \ldots, s_n \in S$, then Corollary 9.3.2 yields the following for each $\theta \in P \to S$, hence $f \leq g$:

$$\begin{aligned} f\theta &= \big((\theta k_1)\,\breve{o} k_1 \,\big((\theta k_2)\,\breve{o} k_2 \,\big(\cdots \big((\theta k_n)\,\breve{o} k_n\, f\big) \cdots\big)\big)\big)\theta|_{P-\{k_1,\ldots,k_n\}} \\ &\leq \big((\theta k_1)\,\breve{o} k_1 \,\big((\theta k_2)\,\breve{o} k_2 \,\big(\cdots \big((\theta k_n)\,\breve{o} k_n\, g\big) \cdots\big)\big)\big)\theta|_{P-\{k_1,\ldots,k_n\}} = g\theta. \end{aligned}$$

Theorem 9.5 Let $f, g \in F$ and $k_1, \ldots, k_l$ be distinct cases with $k_1, \ldots, k_n \in K_f - K_g$, $k_{n+1}, \ldots, k_m \in K_f \cap K_g$, and $k_{m+1}, \ldots, k_l \in K_g - K_f$ $(0 \leq n \leq m \leq l)$. Then $f \lessdot g$ iff

$$(s_i \,\breve{o} k_i)_{i=1,\ldots,m} f \lessdot (s_i \,\breve{o} k_i)_{i=n+1,\ldots,l}\, g$$

for any elements $s_1, \ldots, s_l \in S$.

Proof This is derived from Lemmas 9.4, 9.2, Corollary 9.3.1, and remarks in §9.1. First, $f \lessdot g$ iff $f^\sharp \leq g^\sharp$. Second, $f^\sharp \leq g^\sharp$ iff $(s_i \,\breve{o} k_i)_i f^\sharp \leq (s_i \,\breve{o} k_i)_i g^\sharp$ for any elements $s_1, \ldots, s_l \in S$, where $(s_i \,\breve{o} k_i)_i$ is an abbreviation for $(s_i \,\breve{o} k_i)_{i=1,\ldots,l}$. Third, $(s_i \,\breve{o} k_i)_i f^\sharp \leq (s_i \,\breve{o} k_i)_i g^\sharp$ iff $(s_i \,\breve{o} k_i)_i f^\sharp \lessdot (s_i \,\breve{o} k_i)_i g^\sharp$. Finally,

$$(s_i \,\breve{o} k_i)_i f^\sharp \doteq (s_i \,\breve{o} k_i)_{i=1,\ldots,m} f, \qquad (s_i \,\breve{o} k_i)_i g^\sharp \doteq (s_i \,\breve{o} k_i)_{i=n+1,\ldots,l}\, g.$$

Thus this theorem holds.

Theorem 9.6 Let $f_1, \ldots, f_m, g_1, \ldots, g_n \in F$, $\alpha, \beta \in F^*$, and $k \in K$. Assume that k belongs to the frames of $f_1, \ldots, f_m, g_1, \ldots, g_n$ but does not belong to the frames of the affairs which occur in α or β. Then

$$f_1 \cdots f_m \alpha \lessdot g_1 \cdots g_n \beta$$

iff the following holds for all $s \in S$:

$$s \,\breve{o} k\, f_1, \cdots, s \,\breve{o} k\, f_m, \alpha \lessdot s \,\breve{o} k\, g_1, \cdots, s \,\breve{o} k\, g_n, \beta.$$

Proof Suppose $\alpha = f'_1 \cdots f'_{m'}$, $\beta = g'_1 \cdots g'_{n'}$ with $f'_1, \ldots, f'_{m'}, g'_1, \ldots, g'_{n'} \in F$. Let $s \in S$ and define $h = s \,\breve{o}\pi\, s\triangle$. Then k does not belong to the frames of $h \wedge h^\diamond$ or $h \vee h^\diamond$. Therefore by Theorems 9.2 and 7.3, we may assume $m' \neq 0 \neq n'$. Define

$$\begin{aligned} f &= (f_1 \wedge \cdots \wedge f_m) \wedge (f'_1 \wedge \cdots \wedge f'_{m'}), \\ g &= (g_1 \vee \cdots \vee g_n) \vee (g'_1 \vee \cdots \vee g'_{n'}), \end{aligned}$$

where the orders of applying the operations $\wedge, \vee$ within parentheses are arbitrary. Then if $m \neq 0 \neq n$, we may argue as follows by using Theorems 7.3, 9.2, 9.4, and 9.5:

$$\begin{aligned} f_1 \cdots f_m \alpha \lessdot g_1 \cdots g_n \beta \iff\; & f \lessdot g \\ \iff\; & s \,\breve{o} k\, f \lessdot s \,\breve{o} k\, g \text{ for any } s \in S, \\ s \,\breve{o} k\, f \lessdot s \,\breve{o} k\, g \iff\; & (s \,\breve{o} k\, f_1 \wedge \cdots \wedge s \,\breve{o} k\, f_m) \wedge (f'_1 \wedge \cdots \wedge f'_{m'}) \\ & \lessdot (s \,\breve{o} k\, g_1 \vee \cdots \vee s \,\breve{o} k\, g_n) \vee (g'_1 \vee \cdots \vee g'_{n'}) \\ \iff\; & s \,\breve{o} k\, f_1, \cdots, s \,\breve{o} k\, f_m, \alpha \lessdot s \,\breve{o} k\, g_1, \cdots, s \,\breve{o} k\, g_n, \beta. \end{aligned}$$

This completes the proof in case $m \neq 0 \neq n$. If $m = 0 \neq n$, argue similarly by replacing $s\,\breve{o}k\,f$ by f. If $m \neq 0 = n$, replace $s\,\breve{o}k\,g$ by g. If $m = n = 0$, there is nothing to prove.

9.3 Operations λk and $\triangle$

Theorem 9.7 Let $a, b \in E$. Then the following holds.

- Let $a \in S$. Then, $a\,\breve{o}\pi\,b\triangle = 1$ iff $b \ni a$. Therefore $a\,\breve{o}\pi\,a\triangle = 1$, and if $b \in W_\delta$, then $a\,\breve{o}\pi\,b\triangle = ba$.
- $a\,\forall\pi\,b\triangle = 1$ iff $a \sqsubseteq b$.
- $a\,\overline{p}\pi\,b\triangle = (a \sqcap b)\,\overline{p}\pi\,1\triangle$ for all $p \in \mathbb{P}$.
- If $p \in \mathbb{P}$, then $a\,\overline{p}\pi\,1\triangle = 1$ iff $\left|\left\{s \in S \mid a \ni s\right\}\right| > p$. Therefore, $a\,\overline{\infty}\pi\,1\triangle = 0$ for the maximum ∞ of $\mathbb{P}$ in case it exists.
- $a\,\exists\pi\,1\triangle = 1$ iff there exists an element $s \in S$ such that $s\,\breve{o}\pi\,a\triangle = 1$.

Proof Since $a\,\breve{o}\pi\,b\triangle = (b\triangle)(\pi/a)$ by the definition of the operation $\breve{o}\pi$, it follows from the definition of the operation $\triangle$ that $a\,\breve{o}\pi\,b\triangle = 1$ iff $b \ni a$. The second assertion is proved by the following reasoning using the definition of the operations $\forall\pi, \triangle$, and $\sqcap$ on W:

$$
\begin{aligned}
& a\,\forall\pi\,b\triangle = 1 \\
\iff\ & \left|\left\{s \in S \mid a \ni s,\ (b\triangle)(\pi/s) = 0\right\}\right| \leq 0 \\
\iff\ & \left|\{s \in S \mid a \ni s,\ b \not\ni s\}\right| \leq 0 \\
\iff\ & \{s \in S \mid a \ni s,\ b \not\ni s\} = \emptyset \\
\iff\ & \text{if } a \ni s \in S \text{ then } b \ni s \\
\iff\ & a \sqsubseteq b \qquad\qquad \text{(by (9.1))}.
\end{aligned}
$$

The third assertion is proved by the following reasoning:

$$
\begin{aligned}
& a\,\overline{p}\pi\,b\triangle = 1 \\
\iff\ & \left|\left\{s \in S \mid a \ni s,\ (b\triangle)(\pi/s) = 1\right\}\right| > p \\
\iff\ & \left|\left\{s \in S \mid a \ni s,\ b \ni s\right\}\right| > p \\
\iff\ & \left|\left\{s \in S \mid a \sqcap b \ni s\right\}\right| > p \\
\iff\ & \left|\left\{s \in S \mid a \sqcap b \ni s,\ 1 \ni s\right\}\right| > p \\
\iff\ & (a \sqcap b)\,\overline{p}\pi\,1\triangle = 1.
\end{aligned}
$$

The fourth assertion is also proved by this reasoning, because $1 \ni s$ for all $s \in S$. The fifth assertion is a consequence of the fourth and the first.

9.4 Operations λk and the relation $\lessdot$

Theorem 9.8 Let $a \in E,\ k \in P \in \mathcal{P}K,\ f \in W_P$, and $\mathfrak{p} \in \mathfrak{P}$. Then the following holds:

$$a\,\neg\mathfrak{p}k\,f = a\,\mathfrak{p}k\,f^{\Diamond}, \qquad\qquad a\,\mathfrak{p}^{\circ}k\,f = (a\,\mathfrak{p}k\,f)^{\Diamond}.$$

Proof Let $\theta \in (P-\{k\}) \to S$. Then the following holds, hence the above result:

$$\begin{aligned}
(a\,\neg\mathfrak{p}k\,f)\theta = 1 &\iff \big|\big\{s \in S \mid a \ni s,\ f\big((k/s)\theta\big) = 0\big\}\big| \in \mathfrak{p} \\
&\iff \big|\big\{s \in S \mid a \ni s,\ f^{\Diamond}\big((k/s)\theta\big) = 1\big\}\big| \in \mathfrak{p} \\
&\iff (a\,\mathfrak{p}k\,f^{\Diamond})\theta = 1, \\
(a\,\mathfrak{p}^{\circ}k\,f)\theta = 1 &\iff \big|\big\{s \in S \mid a \ni s,\ f\big((k/s)\theta\big) = 1\big\}\big| \in \mathfrak{p}^{\circ} \\
&\iff (a\,\mathfrak{p}k\,f)\theta = 0 \\
&\iff (a\,\mathfrak{p}k\,f)^{\Diamond}\theta = 1.
\end{aligned}$$

Theorem 9.9 Let $a \in E,\ k \in P \in \mathcal{P}K,\ f \in W_P$, and $\mathfrak{p}, \mathfrak{q} \in \mathfrak{P}$. Then the following holds:

$$a\,(\mathfrak{p} \cap \mathfrak{q})k\,f = a\,\mathfrak{p}k\,f \wedge a\,\mathfrak{q}k\,f, \qquad a\,(\mathfrak{p} \cup \mathfrak{q})k\,f = a\,\mathfrak{p}k\,f \vee a\,\mathfrak{q}k\,f.$$

Proof Let $\theta \in (P-\{k\}) \to S$. Then the following holds, hence the first result:

$$\begin{aligned}
\big(a\,(\mathfrak{p} \cap \mathfrak{q})k\,f\big)\theta = 1 &\iff \big|\big\{s \in S \mid a \ni s,\ f\big((k/s)\theta\big) = 1\big\}\big| \in \mathfrak{p} \cap \mathfrak{q} \\
&\iff \begin{cases} \big|\big\{s \in S \mid a \ni s,\ f\big((k/s)\theta\big) = 1\big\}\big| \in \mathfrak{p}, \\ \big|\big\{s \in S \mid a \ni s,\ f\big((k/s)\theta\big) = 1\big\}\big| \in \mathfrak{q} \end{cases} \\
&\iff (a\,\mathfrak{p}k\,f)\theta = (a\,\mathfrak{q}k\,f)\theta = 1 \\
&\iff (a\,\mathfrak{p}k\,f)\theta \wedge (a\,\mathfrak{q}k\,f)\theta = 1 \\
&\iff (a\,\mathfrak{p}k\,f \wedge a\,\mathfrak{q}k\,f)\theta = 1.
\end{aligned}$$

The rest of the proof is omitted.

Theorem 9.10 Let $a, b \in E,\ k \in P \in \mathcal{P}K,\ f \in W_P$, and $p \in \mathbb{P}$. Then the following holds:

$$a\,\forall\pi\,b\triangle,\ a\,\overline{p}k\,f \lessdot b\,\overline{p}k\,f.$$

Proof Let $\theta \in (P-\{k\}) \to S$ and assume $a\,\forall\pi\,b\triangle = (a\,\overline{p}k\,f)\theta = 1$. Then $a \sqsubseteq b$ by Theorem 9.7 and $\big|\big\{s \in S \mid a \ni s,\ f\big((k/s)\theta\big) = 1\big\}\big| > p$, and so since the $\mathbb{P}$-measure is increasing, we have $\big|\big\{s \in S \mid b \ni s,\ f\big((k/s)\theta\big) = 1\big\}\big| > p$, which means $(b\,\overline{p}k\,f)\theta = 1$. Therefore $a\,\forall\pi\,b\triangle,\ a\,\overline{p}k\,f \lessdot b\,\overline{p}k\,f$ by (9.3).

Theorem 9.11 Let $a, b \in E,\ k \in P \in \mathcal{P}K,\ f \in W_P$, and $p, q \in \mathbb{P}$. Then the following holds:

$$(a \sqcup b)\,\overline{p+q}k\,f \lessdot a\,\overline{p}k\,f,\ b\,\overline{q}k\,f.$$

Proof Let $\theta \in (P - \{k\}) \to S$. Then

$$\begin{aligned}&\bigl\{s \in S \mid a \sqcup b \ni s,\ f\bigl((k/s)\theta\bigr) = 1\bigr\}\\ &= \bigl\{s \in S \mid a \ni s,\ f\bigl((k/s)\theta\bigr) = 1\bigr\} \cup \bigl\{s \in S \mid b \ni s,\ f\bigl((k/s)\theta\bigr) = 1\bigr\}.\end{aligned}$$

Therefore, if $(a\,\overline{p}k\,f)\theta = (b\,\overline{q}k\,f)\theta = 0$, then

$$\bigl|\bigl\{s \in S \mid a \ni s,\ f\bigl((k/s)\theta\bigr) = 1\bigr\}\bigr| \le p, \quad \bigl|\bigl\{s \in S \mid b \ni s,\ f\bigl((k/s)\theta\bigr) = 1\bigr\}\bigr| \le q,$$

so $\bigl|\bigl\{s \in S \mid a \sqcup b \ni s,\ f\bigl((k/s)\theta\bigr) = 1\bigr\}\bigr| \le p + q$, hence $\bigl((a \sqcup b)\,\overline{p+q}k\,f\bigr)\theta = 0$. Therefore $(a \sqcup b)\,\overline{p+q}k\,f \lessdot a\,\overline{p}k\,f,\ b\,\overline{q}k\,f$ by (9.3).

Theorem 9.12 Let $a \in E$, $b \in S$, $k, l \in P \in \mathcal{P}K$, $k \neq l$, $f \in W_P$, and $\lambda \in \{\breve{o}\} \cup \mathfrak{Q}$. Let $a \in S$ in case $\lambda = \breve{o}$. Then the following holds:

$$a\,\lambda k\,(b\,\breve{o}l\,f) = b\,\breve{o}l\,(a\,\lambda k\,f).$$

Proof When $\lambda = \breve{o}$, this holds by Corollary 9.3.1. Let $\mathfrak{p} \in \mathfrak{P}$. Then the following holds for each $\theta \in (P - \{k, l\}) \to S$:

$$\begin{aligned}\bigl(a\,\neg\mathfrak{p}k\,(b\,\breve{o}l\,f)\bigr)\theta = 1 &\iff \bigl|\bigl\{s \in S \mid a \ni s,\ (b\,\breve{o}l\,f)\bigl((k/s)\theta\bigr) = 0\bigr\}\bigr| \in \mathfrak{p}\\ &\iff \bigl|\bigl\{s \in S \mid a \ni s,\ f\bigl((l/b)(k/s)\theta\bigr) = 0\bigr\}\bigr| \in \mathfrak{p},\\ \bigl(b\,\breve{o}l\,(a\,\neg\mathfrak{p}k\,f)\bigr)\theta = 1 &\iff (a\,\neg\mathfrak{p}k\,f)\bigl((l/b)\theta\bigr) = 1\\ &\iff \bigl|\bigl\{s \in S \mid a \ni s,\ f\bigl((k/s)(l/b)\theta\bigr) = 0\bigr\}\bigr| \in \mathfrak{p}.\end{aligned}$$

These conditions are equivalent because $(l/b)(k/s)\theta = (k/s)(l/b)\theta$. Hence the above result in case $\lambda \in \neg\mathfrak{P}$. The rest of the proof is omitted.

10 Structure of the predicate logical space

Let $(A, T, \sigma, \mathbb{S}, \mathbb{C}, \mathbb{X}, \Gamma), \mathcal{W}, (\lambda_W)_{\lambda, W}$ be the logical system MPCL defined in §3.1-§3.4. Here we analyze the structure of the predicate logical space $(H, \mathcal{G}_{\mathcal{W}})$ defined by $A, \mathcal{W}, (\lambda_W)_{\lambda, W}$ in §3.5 to the extent necessary for the proof of Theorem 4.1. For further results, we refer the reader to [3].

As in §7.1, we will denote elements of H by $f, g, \ldots$, while elements of H^* by $\alpha, \beta, \ldots$, both with or without numerical subscripts. When $\alpha = f_1 \cdots f_n \in H^*$, we will denote the subset $\{f_1, \ldots, f_n\}$ of H also by α. The element of H^* of length 0 will be denoted by a blank.

Our attention will be focused on the validity relation $\preccurlyeq$ of $(H, \mathcal{G}_{\mathcal{W}})$ defined as in §7.1 by

$$\alpha \preccurlyeq \beta \iff \inf \varphi\alpha \le \sup \varphi\beta \text{ for all } \varphi \in \mathcal{G}_{\mathcal{W}} \tag{10.1}$$

and the symmetric core $\asymp$ of the restriction of $\preccurlyeq$ to $H \times H$.

First of all, as was notice in §3.2, H is closed under the operations $\wedge, \vee, \Rightarrow, \Diamond$, whose restrictions to H is global.

Theorem 10.1 The predicate logical space $(\mathsf{H}, \mathcal{G}_{\mathcal{W}})$ is Boolean with respect to the operations $\wedge, \vee, \Rightarrow, \Diamond$ on H.

Proof $\mathcal{G}_{\mathcal{W}}$ is non-empty and consists of the mappings $f \mapsto \big((\Phi^* f)\nu\big)(\theta|_{K_f}) = \big((\Phi^* f)\nu\big)^{\sharp}\theta$ of H into $\mathbb{T}$ determined by the quadruples W, Φ, ν, θ of an MPC world $W \in \mathcal{W}$, a $\mathbb{C}$-denotation Φ into W, an $\mathbb{X}$-denotation ν into W, and $\theta \in \mathsf{K} \to W_\varepsilon$. Since the semantic mapping Φ^*, the projection by ν, the inflation $\sharp$, and the projection by θ are all homomorphisms with respect to $\wedge, \vee, \Rightarrow, \Diamond$, so is the members of $\mathcal{G}_{\mathcal{W}}$.

Lemma 10.1 Let $f_1, \ldots, f_m, g_1, \ldots, g_n \in \mathsf{H}$. Then $f_1 \cdots f_m \preccurlyeq g_1 \cdots g_n$ iff $(\Phi^* f_1)\nu \cdots (\Phi^* f_m)\nu \lessdot (\Phi^* g_1)\nu \cdots (\Phi^* g_n)\nu$ for each triple W, Φ, ν of an MPC world $W \in \mathcal{W}$, a $\mathbb{C}$-denotation Φ into W, and an $\mathbb{X}$-denotation ν into W.

Proof This is a consequence of (9.2).

Theorem 10.2 Let $f_1, \ldots, f_m, g_1, \ldots, g_n \in \mathsf{H}$, $\alpha, \beta \in \mathsf{H}^*$, and $k \in \mathsf{K}$. Assume that k belongs to the ranges of $f_1, \ldots, f_m, g_1, \ldots, g_n$ but does not belong to those of the predicates in $\alpha \cup \beta$. Then the following holds for all $a \in A_\varepsilon$:

$$\begin{aligned} & f_1 \cdots f_m \alpha \preccurlyeq g_1 \cdots g_n \beta \\ & \quad \Longrightarrow\ a\,\breve{o}k\, f_1, \cdots, a\,\breve{o}k\, f_m, \alpha \preccurlyeq a\,\breve{o}k\, g_1, \cdots, a\,\breve{o}k\, g_n, \beta. \end{aligned} \qquad \text{(gen. case+ law)}$$

Assume furthermore that a variable $x \in \mathbb{X}_\varepsilon$ does not occur free in the predicates in $\{f_1, \ldots, f_m, g_1, \ldots, g_n\} \cup \alpha \cup \beta$. Then the following holds:

$$\begin{aligned} & x\,\breve{o}k\, f_1, \cdots, x\,\breve{o}k\, f_m, \alpha \preccurlyeq x\,\breve{o}k\, g_1, \cdots, x\,\breve{o}k\, g_n, \beta \\ & \quad \Longrightarrow\ f_1 \cdots f_m \alpha \preccurlyeq g_1 \cdots g_n \beta. \end{aligned} \qquad \text{(gen. case− law)}$$

Proof The gen. case+ law is an immediate consequence of Theorem 9.6 and Lemma 10.1. In order to prove the gen. case− law, let $\alpha = f'_1 \cdots f'_{m'}$, $\beta = g'_1 \cdots g'_{n'}$ with $f'_1, \ldots, f'_{m'}, g'_1, \ldots, g'_{n'} \in \mathsf{H}$. Let W be an MPC world in $\mathcal{W}$ and Φ be a $\mathbb{C}$-denotation into W. Then the premise of the gen. case− law implies that the following holds for every $\nu \in V_{\mathbb{X},W}$, because Φ^* and the projection by ν are homomorphisms with respect to the operation $\breve{o}k$:

$$\begin{aligned} & (\Phi^* x)\nu\,\breve{o}k\,(\Phi^* f_1)\nu, \cdots, (\Phi^* x)\nu\,\breve{o}k\,(\Phi^* f_m)\nu, (\Phi^* f'_1)\nu, \cdots, (\Phi^* f'_{m'})\nu \\ & \quad \lessdot (\Phi^* x)\nu\,\breve{o}k\,(\Phi^* g_1)\nu, \cdots, (\Phi^* x)\nu\,\breve{o}k\,(\Phi^* g_n)\nu, (\Phi^* g'_1)\nu, \cdots, (\Phi^* g'_{n'})\nu. \end{aligned}$$

Take an arbitrary element $s \in W_\varepsilon$ and define $\nu' = (x/s)\nu$. Then the above holds with ν replaced by ν', and so (2.4), (2.3), and Theorem 6.1 yield the following:

$$\begin{aligned} & s\,\breve{o}k\,(\Phi^* f_1)\nu, \cdots, s\,\breve{o}k\,(\Phi^* f_m)\nu, (\Phi^* f'_1)\nu, \cdots, (\Phi^* f'_{m'})\nu \\ & \qquad \lessdot s\,\breve{o}k\,(\Phi^* g_1)\nu, \cdots, s\,\breve{o}k\,(\Phi^* g_n)\nu, (\Phi^* g'_1)\nu, \cdots, (\Phi^* g'_{n'})\nu. \end{aligned}$$

Since s is arbitrary, Theorem 9.6 shows that

$$(\Phi^* f_1)\nu, \cdots, (\Phi^* f_m)\nu, (\Phi^* f'_1)\nu, \cdots, (\Phi^* f'_{m'})\nu \\ \lessdot (\Phi^* g_1)\nu, \cdots, (\Phi^* g_n)\nu, (\Phi^* g'_1)\nu, \cdots, (\Phi^* g'_{n'})\nu$$

holds. Since W, Φ, ν are arbitrary, Lemma 10.1 shows that the conclusion of gen. case$-$ law holds.

Theorem 10.3 Let $a \in G$, $f \in H$, $x \in \mathbb{X}_\varepsilon$, $K_f = \{k\}$, and $p \in \mathbb{P}$. Assume $x \not\preccurlyeq f$. Then $a\,\overline{p}k\,f \asymp a\,\overline{p}\pi\big((x\,\breve{o}k\,f)\,\Omega x\big)\triangle$.

Proof In view of Lemma 10.1, we need to show

$$\big(\Phi^*(a\,\overline{p}k\,f)\big)\nu = \big(\Phi^*\big(a\,\overline{p}\pi\big((x\,\breve{o}k\,f)\,\Omega x\big)\triangle\big)\big)\nu$$

for each triple W, Φ, ν of an MPC world $W \in \mathcal{W}$, a $\mathbb{C}$-denotation Φ into W, and an $\mathbb{X}$-denotation ν into W. Since Φ^* is a homomorphism with respect to the operations $\overline{p}\pi, \triangle, \Omega x$, and $\breve{o}k$, and since the projections by $\mathbb{X}$-denotations are homomorphisms with respect to $\overline{p}\pi, \triangle$, and $\breve{o}k$, we have

$$\begin{aligned}
&\big(\Phi^*\big(a\,\overline{p}\pi\big((x\,\breve{o}k\,f)\,\Omega x\big)\triangle\big)\big)\nu = 1\\
&\iff (\Phi^* a)\nu\,\overline{p}\pi\Big(\big((\Phi^* x)\,\breve{o}k\,(\Phi^* f)\big)\,\Omega x\Big)\nu\triangle = 1\\
&\iff \Big|\Big\{s \in W_\varepsilon \mid (\Phi^* a)\nu \ni s,\ s\,\breve{o}\pi\Big(\big((\Phi^* x)\,\breve{o}k\,(\Phi^* f)\big)\,\Omega x\Big)\nu\triangle = 1\Big\}\Big| > p,
\end{aligned}$$

where

$$\begin{aligned}
&s\,\breve{o}\pi\Big(\big((\Phi^* x)\,\breve{o}k\,(\Phi^* f)\big)\,\Omega x\Big)\nu\triangle\\
&= \Big(\Big(\big((\Phi^* x)\,\breve{o}k\,(\Phi^* f)\big)\,\Omega x\Big)\nu\Big)s && \text{(by Theorem 9.7)}\\
&= \big((\Phi^* x)\,\breve{o}k\,(\Phi^* f)\big)\big((x/s)\nu\big) && \text{(by (3.3))}\\
&= (\Phi^* x)\big((x/s)\nu\big)\,\breve{o}k\,(\Phi^* f)\big((x/s)\nu\big)\\
&= s\,\breve{o}k\,(\Phi^* f)\nu && \text{(by (2.4), (2.3), and Theorem 6.1).}
\end{aligned}$$

Therefore

$$\begin{aligned}
&\big(\Phi^*\big(a\,\overline{p}\pi\big((x\,\breve{o}k\,f)\,\Omega x\big)\triangle\big)\big)\nu = 1\\
&\iff \Big|\Big\{s \in W_\varepsilon \mid (\Phi^* a)\nu \ni s,\ s\,\breve{o}k\,(\Phi^* f)\nu = 1\Big\}\Big| > p\\
&\iff (\Phi^* a)\nu\,\overline{p}k\,(\Phi^* f)\nu = 1\\
&\iff \big(\Phi^*(a\,\overline{p}k\,f)\big)\nu = 1,
\end{aligned}$$

because both Φ^* and the projection by ν are homomorphisms with respect to $\overline{p}k$. This completes the proof.

Theorem 10.4 Let $a \in G$, $f, g \in A_\emptyset$, and $x \in \mathbb{X}_\varepsilon$. Assume $x \not\preccurlyeq f$. Then $f, a\,\forall\pi\big((f \Rightarrow g)\,\Omega x\big)\triangle \preccurlyeq a\,\forall\pi\,(g\,\Omega x)\triangle$.

Proof In view of Lemma 10.1, we need to show

$$(\Phi^* f)\nu = \big(\Phi^*\big(a \,\forall\pi\,\big((f \Rightarrow g)\,\Omega x\big)\triangle\big)\big)\nu = 1 \implies \big(\Phi^*\big(a \,\forall\pi\,(g\,\Omega x)\triangle\big)\big)\nu = 1$$

for each triple W, Φ, ν of an MPC world $W \in \mathcal{W}$, a $\mathbb{C}$-denotation Φ into W, and an $\mathbb{X}$-denotation ν into W. Since Φ^* is a homomorphism with respect to the operations $\forall\pi, \triangle, \Omega x$, and $\Rightarrow$ and the projection by ν is a homomorphism with respect to $\forall\pi$ and $\triangle$,

$$\big(\Phi^*\big(a \,\forall\pi\,\big((f \Rightarrow g)\,\Omega x\big)\triangle\big)\big)\nu = (\Phi^* a)\nu \,\forall\pi\,\big((\Phi^* f \Rightarrow \Phi^* g)\,\Omega x\big)\nu\triangle,$$
$$\big(\Phi^*\big(a \,\forall\pi\,(g\,\Omega x)\triangle\big)\big)\nu = (\Phi^* a)\nu \,\forall\pi\,\big((\Phi^* g)\,\Omega x\big)\nu\triangle,$$

and so we have

$$\big(\Phi^*\big(a \,\forall\pi\,\big((f \Rightarrow g)\,\Omega x\big)\triangle\big)\big)\nu = 1 \iff (\Phi^* a)\nu \sqsubseteq \big((\Phi^* f \Rightarrow \Phi^* g)\,\Omega x\big)\nu,$$
$$\big(\Phi^*\big(a \,\forall\pi\,(g\,\Omega x)\triangle\big)\big)\nu = 1 \iff (\Phi^* a)\nu \sqsubseteq \big((\Phi^* g)\,\Omega x\big)\nu$$

by Theorem 9.7. Therefore assume $(\Phi^* f)\nu = \big(\Phi^*\big(a \,\forall\pi\,\big((f \Rightarrow g)\,\Omega x\big)\triangle\big)\big)\nu = 1$ and $(\Phi^* a)\nu \ni s \in W_\varepsilon$. Then $(\Phi^* f)\big((x/s)\nu\big) = 1$ by Theorem 6.1, and

$$\begin{aligned}
1 &= \Big(\big((\Phi^* f \Rightarrow \Phi^* g)\,\Omega x\big)\nu\Big)s && \text{(by (9.1))}\\
&= (\Phi^* f \Rightarrow \Phi^* g)\big((x/s)\nu\big) && \text{(by (3.3))}\\
&= (\Phi^* f)\big((x/s)\nu\big) \Rightarrow (\Phi^* g)\big((x/s)\nu\big) \\
&= (\Phi^* f)\big((x/s)\nu\big) \Rightarrow \Big(\big((\Phi^* g)\,\Omega x\big)\nu\Big)s && \text{(by (3.3))},
\end{aligned}$$

where the third equality holds because the projection by $(x/s)\nu$ is a homomorphism with respect to $\Rightarrow$. Thus $\Big(\big((\Phi^* g)\,\Omega x\big)\nu\Big)s = 1$, that is, $\big((\Phi^* g)\,\Omega x\big)\nu \ni s$. This completes the proof.

Theorem 10.5 Let $a \in A_\varepsilon$, $f \in A_\emptyset$, and $x \in \mathbb{X}_\varepsilon$. Assume that x is free from a in f. Then $a \,\breve{o}\pi\,(f\,\Omega x)\triangle \asymp f(x/a)$, where (x/a) denotes the substitution of a for x.

Proof Since the substitution (x/a) is sort-consistent, both $a \,\breve{o}\pi\,(f\,\Omega x)\triangle$ and $f(x/a)$ belong to $A_\emptyset$. Therefore, in view of Lemma 10.1, we need to show

$$\big(\Phi^*\big(a \,\breve{o}\pi\,(f\,\Omega x)\triangle\big)\big)\nu = \big(\Phi^*\big(f(x/a)\big)\big)\nu$$

for each triple W, Φ, ν of an MPC world $W \in \mathcal{W}$, a $\mathbb{C}$-denotation Φ into W, and an $\mathbb{X}$-denotation ν into W. This is accomplished by the following calculation:

$$\begin{aligned}
\big(\Phi^*\big(a \,\breve{o}\pi\,(f\,\Omega x)\triangle\big)\big)\nu &= (\Phi^* a)\nu \,\breve{o}\pi\,\Big(\big((\Phi^* f)\,\Omega x\big)\nu\Big)\triangle \\
&= \Big(\big((\Phi^* f)\,\Omega x\big)\nu\Big)\big((\Phi^* a)\nu\big) && \text{(by Theorem 9.7)}\\
&= (\Phi^* f)\Big(\big(x/(\Phi^* a)\nu\big)\nu\Big) && \text{(by (3.3))}\\
&= \big(\Phi^*\big(f(x/a)\big)\big)\nu && \text{(by Theorem 6.2)}.
\end{aligned}$$

The first equality holds because Φ^* is a homomorphism with respect to the operations $\breve{o}\pi, \triangle$, and Ωx and the projection by ν is a homomorphism with respect to $\breve{o}\pi$ and $\triangle$.

The following four theorems center around the definition (4.2).

Theorem 10.6 For any $\mathbb{C}$-denotation Φ into any MPC world $W \in \mathcal{W}$ and any $\mathbb{X}$-denotation ν into W, $(\Phi^*\mathsf{one})\nu$ is equal to the greatest element 1 of W_δ, while $\big(\Phi^*(\mathsf{one}^\square)\big)\nu$ is equal to the least element 0 of W_δ.

Proof Since $\big(\Phi^*(\mathsf{one}^\square)\big)\nu = \big((\Phi^*\mathsf{one})\nu\big)^\square$, we only need to prove $(\Phi^*\mathsf{one})\nu = 1$, which is accomplished by the following calculation for each $s \in W_\varepsilon$:

$$
\begin{aligned}
&\big((\Phi^*\mathsf{one})\nu\big)s \\
&= \Big(\big(\Phi^*\big((x_0\, \breve{\sigma}\pi\, x_0 \triangle)\, \Omega x_0\big)\big)\nu\Big)s && \text{(by (4.2))} \\
&= \Big(\Big(\big((\Phi^* x_0)\, \breve{\sigma}\pi\, (\Phi^* x_0)\triangle\big)\, \Omega x_0\Big)\nu\Big)s \\
&= \big((\Phi^* x_0)\, \breve{\sigma}\pi\, (\Phi^* x_0)\triangle\big)\big((x_0/s)\nu\big) && \text{(by (3.3))} \\
&= (\Phi^* x_0)\big((x_0/s)\nu\big)\, \breve{\sigma}\pi\, \big((\Phi^* x_0)\big((x_0/s)\nu\big)\big)\triangle \\
&= s\, \breve{\sigma}\pi\, s\triangle && \text{(by (2.4) and (2.3))} \\
&= 1 && \text{(by Theorem 9.7).}
\end{aligned}
$$

The second equality holds because Φ^* is a homomorphism with respect to the operations $\Omega x_0, \breve{\sigma}\pi$, and $\triangle$. The fourth equality holds because the projection by $(x_0/s)\nu$ is a homomorphism with respect to $\breve{\sigma}\pi$ and $\triangle$.

Theorem 10.7 Let $f \in A_\emptyset$ and $x \in \mathbb{X}_\varepsilon$. Then $\mathsf{one}\,\forall\pi\,(f\,\Omega x)\triangle \preccurlyeq f$.

Proof We need to show

$$
\big(\Phi^*\big(\mathsf{one}\,\forall\pi\,(f\,\Omega x)\triangle\big)\big)\nu = 1 \implies (\Phi^* f)\nu = 1
$$

for each triple W, Φ, ν of an MPC world $W \in \mathcal{W}$, a $\mathbb{C}$-denotation Φ into W, and an $\mathbb{X}$-denotation ν into W, which is accomplished by the following reasoning:

$$
\begin{aligned}
&\big(\Phi^*\big(\mathsf{one}\,\forall\pi\,(f\,\Omega x)\triangle\big)\big)\nu = 1 \\
&\iff 1\,\forall\pi\,\big((\Phi^* f)\,\Omega x\big)\nu\triangle = 1 && \text{(by Theorem 10.6)} \\
&\iff 1 \sqsubseteq \big((\Phi^* f)\,\Omega x\big)\nu && \text{(by Theorem 9.7)} \\
&\iff \big((\Phi^* f)\,\Omega x\big)\nu = 1 \\
&\iff \big(\big((\Phi^* f)\,\Omega x\big)\nu\big)s = 1 \text{ for all } s \in W_\varepsilon \\
&\iff (\Phi^* f)\big((x/s)\nu\big) = 1 \text{ for all } s \in W_\varepsilon && \text{(by (3.3))} \\
&\implies (\Phi^* f)\nu = 1 && \text{(since } (x/\nu x)\nu = \nu\text{).}
\end{aligned}
$$

Theorem 10.8 Let $f \in A_\emptyset$ and $x \in \mathbb{X}_\varepsilon$. Then $\preccurlyeq \mathsf{one}\,\forall\pi\,(f\,\Omega x)\triangle$ iff $\preccurlyeq f$.

Proof The validity relation $\preccurlyeq$ satisfies the cut law as noticed in §7.1 (cf. Theorems 10.1 and 7.4), and $\mathsf{one}\,\forall\pi\,(f\,\Omega x)\triangle \preccurlyeq f$ by Theorem 10.7. Therefore if $\preccurlyeq \mathsf{one}\,\forall\pi\,(f\,\Omega x)\triangle$, then $\preccurlyeq f$ by the cut law. Conversely assume $\preccurlyeq f$. Then

$(\Phi^* f)\big((x/s)\nu\big) = 1$ for each quadruple W, Φ, ν, s of an MPC world $W \in \mathcal{W}$, a $\mathbb{C}$-denotation Φ into W, and an $\mathbb{X}$-denotation ν into W, and $s \in W_\varepsilon$. Therefore, by the reasoning in the proof of Theorem 10.7, we have $\big(\Phi^*\big(\mathsf{one}\,\forall\pi\,(f\,\Omega x)\triangle\big)\big)\nu = 1$ for all triples W, Φ, ν, hence $\preccurlyeq \mathsf{one}\,\forall\pi\,(f\,\Omega x)\triangle$.

Theorem 10.9 Let $x \in \mathbb{X}_\varepsilon$, $a \in G$, $f \in H$, and $K_f = \{k\}$. Assume $x \not\ll a, f$. Then $\mathsf{one}\,\forall\pi\,\big(\big((x\,\breve{o}\pi\,a\triangle)\Rightarrow(x\,\breve{o}k\,f)\big)\,\Omega x\big)\triangle \asymp a\,\forall k\,f$.

Proof Let W, Φ, ν be an arbitrary triple of an MPC world $W \in \mathcal{W}$, a $\mathbb{C}$-denotation Φ into W, and an $\mathbb{X}$-denotation ν into W. Then

$$\begin{aligned}
&\big(\Phi^*\big(\mathsf{one}\,\forall\pi\,\big(\big((x\,\breve{o}\pi\,a\triangle)\Rightarrow(x\,\breve{o}k\,f)\big)\,\Omega x\big)\triangle\big)\big)\nu = 1\\
&\iff \big(\Phi^*\big((x\,\breve{o}\pi\,a\triangle)\Rightarrow(x\,\breve{o}k\,f)\big)\big)\big((x/s)\nu\big) = 1 \text{ for all } s \in W_\varepsilon
\end{aligned}$$

by the reasoning in the proof of Theorem 10.7. Both Φ^* and the projection by $(x/s)\nu$ are homomorphisms with respect to the operations $\Rightarrow, \breve{o}\pi, \triangle$, and $\breve{o}k$. Also $(\Phi^* x)\big((x/s)\nu\big) = s$ by (2.4) and (2.3). Also $(\Phi^* a)\big((x/s)\nu\big) = (\Phi^* a)\nu$ and $(\Phi^* f)\big((x/s)\nu\big) = (\Phi^* f)\nu$ by Theorem 6.1. Therefore,

$$\begin{aligned}
&\big(\Phi^*\big((x\,\breve{o}\pi\,a\triangle)\Rightarrow(x\,\breve{o}k\,f)\big)\big)\big((x/s)\nu\big) = 1 \text{ for all } s \in W_\varepsilon\\
&\iff \big(s\,\breve{o}\pi\,(\Phi^* a)\nu\triangle\big)\Rightarrow\big(s\,\breve{o}k\,(\Phi^* f)\nu\big) = 1 \text{ for all } s \in W_\varepsilon\\
&\iff \text{if } s \in W_\varepsilon \text{ and } s\,\breve{o}\pi\,(\Phi^* a)\nu\triangle = 1 \text{ then } s\,\breve{o}k\,(\Phi^* f)\nu = 1\\
&\iff \text{if } (\Phi^* a)\nu \ni s \in W_\varepsilon \text{ then } \big((\Phi^* f)\nu\big)(k/s) = 1\\
&\iff \big\{s \in W_\varepsilon \mid (\Phi^* a)\nu \ni s,\ \big((\Phi^* f)\nu\big)(k/s) = 0\big\} = \emptyset\\
&\iff \big|\big\{s \in W_\varepsilon \mid (\Phi^* a)\nu \ni s,\ \big((\Phi^* f)\nu\big)(k/s) = 0\big\}\big| \leq 0\\
&\iff (\Phi^* a)\nu\,\forall k\,(\Phi^* f)\nu = 1\\
&\iff \big(\Phi^*(a\,\forall k\,f)\big)\nu = 1.
\end{aligned}$$

This completes the proof.

11 Proof of the main result

Let $(A, T, \sigma, \mathbb{S}, \mathbb{C}, \mathbb{X}, \Gamma), \mathcal{W}, (\lambda_W)_{\lambda,W}$ be the logical system MPCL defined in §3.1-§3.4. Here we prove Theorem 4.1 on the predicate logical space $(H, \mathcal{G}_W)$ defined by $A, \mathcal{W}, (\lambda_W)_{\lambda,W}$ in §3.5.

As in §10, we will denote elements of H by $f, g, \ldots$, while elements of H^* by $\alpha, \beta, \ldots$, both with or without numerical subscripts. When $\alpha = f_1 \cdots f_n \in H^*$, we will denote the subset $\{f_1, \ldots, f_n\}$ of H also by α. The element of H^* of length 0 will be denoted by a blank.

11.1 A sequent deduction pair

Our proof of Theorem 4.1 is based on Theorem 7.1 and so, extending the notation of §7.1, we define

$$\vec{H} = H^* \times H^*$$

and denote the elements (α, β) of $\vec{H}$ by $\alpha \to \beta$ or $\beta \leftarrow \alpha$. We call elements of $\vec{H}$ so denoted the **sequents**. In particular, a sequent consisting of sentences is called a **sentence sequent**. Let $\preccurlyeq$ be the validity relation (10.1) of $(H, \mathcal{G}_{\mathcal{W}})$ and define

$$\vec{C} = \{\alpha \to \beta \in \vec{H} \mid \alpha \preccurlyeq \beta\}.$$

Then the $\mathcal{G}_{\mathcal{W}}$-core C satisfies

$$C = \{f \in H \mid \ \preccurlyeq f\} = \{f \in H \mid \ \to f \in \vec{C}\}.$$

In view of Theorem 7.1, here we present a deduction pair $(\vec{R}, \vec{D})$ on $\vec{H}$ for which we will prove $\vec{C} = [\vec{D}]_{\vec{R}}$ in §11.3. In the course of the presentation, we will prove the following.

Lemma 11.1 $\vec{C}$ is closed under $\vec{R}$ and $\vec{D} \subseteq \vec{C}$. Therefore $[\vec{D}]_{\vec{R}} \subseteq \vec{C}$.

First, the deduction rule $\vec{R}$ is the union of the nineteen logics on $\vec{H}$ defined by fractional expressions as follows (cf. (4.1)). The first sixteen logics are familiar ones and in one to one correspondence with the weakening law, contraction law, exchange law, cut law, strong conjunction law, strong disjunction law, strong implication law, and strong negation law.

$$\frac{\alpha \to \beta}{f\alpha \to \beta}, \qquad \frac{\alpha \leftarrow \beta}{f\alpha \leftarrow \beta}, \qquad \text{(weakening)}$$

$$\frac{ff\alpha \to \beta}{f\alpha \to \beta}, \qquad \frac{ff\alpha \leftarrow \beta}{f\alpha \leftarrow \beta}, \qquad \text{(contraction)}$$

$$\frac{\alpha fg\beta \to \gamma}{\alpha gf\beta \to \gamma}, \qquad \frac{\alpha fg\beta \leftarrow \gamma}{\alpha gf\beta \leftarrow \gamma}, \qquad \text{(exchange)}$$

$$\frac{\alpha \to f \quad f\beta \to \gamma}{\alpha\beta \to \gamma}, \qquad \frac{\alpha \leftarrow f \quad f\beta \leftarrow \gamma}{\alpha\beta \leftarrow \gamma}, \qquad \text{(cut)}$$

$$\frac{fg\alpha \to \beta}{f \wedge g, \alpha \to \beta}, \qquad \frac{\alpha \to f\beta \quad \alpha \to g\beta}{\alpha \to f \wedge g, \beta}, \qquad \text{(conjunction)}$$

$$\frac{f\alpha \to \beta \quad g\alpha \to \beta}{f \vee g, \alpha \to \beta}, \qquad \frac{\alpha \to fg\beta}{\alpha \to f \vee g, \beta}, \qquad \text{(disjunction)}$$

$$\frac{\alpha \to f\beta \quad g\alpha \to \beta}{f \Rightarrow g, \alpha \to \beta}, \qquad \frac{f\alpha \to g\beta}{\alpha \to f \Rightarrow g, \beta}, \qquad \text{(implication)}$$

$$\frac{\alpha \to f\beta}{f^{\Diamond}\alpha \to \beta}, \qquad \frac{f\alpha \to \beta}{\alpha \to f^{\Diamond}\beta}. \tag{negation}$$

The remaining three logics are in one to one correspondence with the logics $\perp, \top, \forall$ on H used in §4 to define ∇ by (4.3).

$$\frac{\to f}{\to a\,\breve{o}k\,f}, \tag{case+}$$

where $a \in A_\varepsilon$ and $k \in K_f$.

$$\frac{\to x\,\breve{o}k\,f}{\to f}, \tag{case−}$$

where $x \in \mathbb{X}_\varepsilon$, $k \in K_f$, and $x \not\ll f$.

$$\frac{\to f}{\to \mathsf{one}\,\forall\pi\,(f\,\Omega x)\triangle}, \tag{∀+}$$

where $f \in A_\emptyset$, $x \in \mathbb{X}_\varepsilon$, and one was defined by (4.2).

This completes the presentation of the nineteen constituents of the deduction rule $\vec{R}$. Since the validity relation $\preccurlyeq$ is Boolean with respect to $\wedge, \vee, \Rightarrow, \Diamond$ by Theorems 10.1 and 7.4, it follows from Theorem 7.2 that $\vec{C}$ is closed under the first sixteen logics, weakening to negation. Also, it follows from Theorems 10.2 and 10.8 that $\vec{C}$ is closed under the remaining three logics case+, case−, and ∀+. Thus $\vec{C}$ is closed under $\vec{R}$.

Next, the deduction basis $\vec{D}$ is the set of the following twenty five kinds of sequents, the first of which is a familiar one:

$$f \to f, \tag{repetition seq.}$$

where $f \in \mathsf{H}$. These belong to $\vec{C}$ because $\preccurlyeq$ satisfies the repetition law. The remaining twenty four kinds of sequents are in one to one correspondence with the twenty four kinds of elements of ∂ presented in §4 other than the Boolean elements, that is, the = elements to ∀− elements.

$$\to a\,\breve{o}\pi\,a\triangle, \tag{= seq.}$$

where $a \in A_\varepsilon$. Theorem 9.7 and Lemma 10.1 show that these belong to $\vec{C}$.

$$a\,\overline{\infty}\pi\,\mathsf{one}\triangle \to\ , \tag{$\overline{\infty}$ seq.}$$

where $a \in G$ and ∞ is the maximum of $\mathbb{P}$ in case it exists. Theorem 9.7, Lemma 10.1, and Theorem 10.6 show that the $\overline{\infty}$ sequents belong to $\vec{C}$.

$$a\lambda k\,(b\,\breve{o}l\,f) \rightleftarrows b\,\breve{o}l\,(a\lambda k\,f), \tag{$\mathfrak{Q}, \breve{o}$ seq.}$$

where $a \in G$, $b \in A_\varepsilon$, $f \in \mathsf{H}$, $k, l \in K_f$, $k \neq l$, $\lambda \in \{\breve{o}\} \cup \mathfrak{Q}$. Also $a \in A_\varepsilon$ in case $\lambda = \breve{o}$. The two-way arrow $\rightleftarrows$ is a device to show a sequent $\alpha \to \beta$ and its reverse $\alpha \leftharpoondown \beta$ together. We will continue using this device. Theorem 9.12 and Lemma 10.1 show that the $\mathfrak{Q}, \breve{o}$ sequents belong to $\vec{C}$.

$$(a_i\,\breve{o}k_i)_{i=1,\dots,l}(f \wedge g) \rightleftarrows (a_i\,\breve{o}k_i)_{i=1,\dots,m} f \wedge (a_i\,\breve{o}k_i)_{i=n+1,\dots,l}\, g, \qquad (\wedge \text{ seq.})$$
$$(a_i\,\breve{o}k_i)_{i=1,\dots,l}(f \vee g) \rightleftarrows (a_i\,\breve{o}k_i)_{i=1,\dots,m} f \vee (a_i\,\breve{o}k_i)_{i=n+1,\dots,l}\, g, \qquad (\vee \text{ seq.})$$
$$(a_i\,\breve{o}k_i)_{i=1,\dots,l}(f \Rightarrow g) \rightleftarrows (a_i\,\breve{o}k_i)_{i=1,\dots,m} f \Rightarrow (a_i\,\breve{o}k_i)_{i=n+1,\dots,l}\, g, \qquad (\Rightarrow \text{ seq.})$$

where $a_1, \dots, a_l \in A_\varepsilon$, $f, g \in H$, and $k_1, \dots, k_l$ are distinct cases such that $k_1, \dots, k_n \in K_f - K_g$, $k_{n+1}, \dots, k_m \in K_f \cap K_g$, and $k_{m+1}, \dots, k_l \in K_g - K_f$ $(0 \le n \le m \le l)$. Theorem 9.4 and Lemma 10.1 show that these three kinds of sequents belong to $\vec{C}$.

$$(a_i\,\breve{o}k_i)_{i=1,\dots,n}(f^{\Diamond}) \rightleftarrows \big((a_i\,\breve{o}k_i)_{i=1,\dots,n} f\big)^{\Diamond}, \qquad (\Diamond \text{ seq.})$$

where $a_1, \dots, a_n \in A_\varepsilon$, $f \in H$, and $k_1, \dots, k_n$ are distinct cases in K_f. The $\Diamond$ sequents belong to $\vec{C}$ by Corollary 9.3.3 and Lemma 10.1.

$$a \neg \mathfrak{p} k\, f \rightleftarrows a\, \mathfrak{p} k\, f^{\Diamond}, \qquad (\neg \text{ seq.})$$
$$a\, \mathfrak{p}^{\circ} k\, f \rightleftarrows (a\, \mathfrak{p} k\, f)^{\Diamond}, \qquad (\circ \text{ seq.})$$

where $a \in G$, $f \in H$, $k \in K_f$, and $\mathfrak{p} \in \mathfrak{P}$. These two kinds of sequents belong to $\vec{C}$ by Theorem 9.8 and Lemma 10.1.

$$a\,(\mathfrak{p} \cap \mathfrak{q}) k\, f \rightleftarrows a\, \mathfrak{p} k\, f \wedge a\, \mathfrak{q} k\, f, \qquad (\cap \text{ seq.})$$
$$a\,(\mathfrak{p} \cup \mathfrak{q}) k\, f \rightleftarrows a\, \mathfrak{p} k\, f \vee a\, \mathfrak{q} k\, f, \qquad (\cup \text{ seq.})$$

where $a \in G$, $f \in H$, $k \in K_f$, and $\mathfrak{p}, \mathfrak{q} \in \mathfrak{P}$. These two kinds of sequents belong to $\vec{C}$ by Theorem 9.9 and Lemma 10.1.

$$a\, \overline{p} k\, f \rightleftarrows a\, \overline{p} \pi \big((x\, \breve{o} k\, f)\, \Omega x\big) \triangle, \qquad (\mathfrak{P} \text{ seq.})$$

where $a \in G$, $f \in H$, $x \in \mathbb{X}_\varepsilon$, $K_f = \{k\}$, $p \in \mathbb{P}$, and $x \not\ll f$. These belong to $\vec{C}$ by Theorem 10.3.

$$a\, \overline{p} \pi\, b \triangle \rightleftarrows (a \sqcap b)\, \overline{p} \pi\, one \triangle, \qquad (\triangle \text{ seq.})$$

where $a, b \in G$, and $p \in \mathbb{P}$. These belong to $\vec{C}$ by Theorem 9.7, Lemma 10.1, and Theorem 10.6.

$$f,\ one \forall \pi \big((f \Rightarrow g)\, \Omega x\big) \triangle \rightarrow one \forall \pi\, (g\, \Omega x) \triangle, \qquad (\forall, \Rightarrow \text{ seq.})$$

where $f, g \in A_\emptyset$, $x \in \mathbb{X}_\varepsilon$, and $x \not\ll f$. These belong to $\vec{C}$ by Theorem 10.4.

$$one \forall \pi \big(\big((x\, \breve{o} \pi\, a \triangle) \Rightarrow (x\, \breve{o} k\, f)\big)\, \Omega x\big) \triangle \rightarrow a \forall k\, f, \qquad (\forall \text{ seq.})$$

where $x \in \mathbb{X}_\varepsilon$, $a \in G$, $f \in H$, $K_f = \{k\}$, and $x \not\ll a, f$. These belong to $\vec{C}$ by Theorem 10.9.

$$a \forall \pi\, b \triangle,\, a\, \overline{p} k\, f \rightarrow b\, \overline{p} k\, f, \qquad (\forall, \mathfrak{P} \text{ seq.})$$

where $\mathsf{a}, \mathsf{b} \in \mathsf{G},\ \mathsf{f} \in \mathsf{H},\ \mathsf{k} \in \mathsf{K}_\mathsf{f}$, and $\mathsf{p} \in \mathbb{P}$. These belong to $\vec{\mathsf{C}}$ by Theorem 9.10 and Lemma 10.1.

$$(\mathsf{a} \sqcup \mathsf{b})\,\overline{\mathsf{p}+\mathsf{q}}\mathsf{k}\,\mathsf{f} \to \mathsf{a}\,\overline{\mathsf{p}}\mathsf{k}\,\mathsf{f}, \mathsf{b}\,\overline{\mathsf{q}}\mathsf{k}\,\mathsf{f}, \tag{$\sqcup,+$ seq.}$$

where $\mathsf{a}, \mathsf{b} \in \mathsf{G},\ \mathsf{f} \in \mathsf{H},\ \mathsf{k} \in \mathsf{K}_\mathsf{f}$, and $\mathsf{p}, \mathsf{q} \in \mathbb{P}$. These belong to $\vec{\mathsf{C}}$ by Theorem 9.11 and Lemma 10.1.

$$\mathsf{one}^\square\,\overline{\mathsf{p}}\mathsf{k}\,\mathsf{f} \to\ , \tag{$\mathsf{one}^\square$ seq.}$$

where $\mathsf{f} \in \mathsf{H},\ \mathsf{k} \in \mathsf{K}_\mathsf{f}$, and $\mathsf{p} \in \mathbb{P}$. These belong to $\vec{\mathsf{C}}$ by Theorem 10.6, Lemma 10.1, and the definition of the operation $\overline{\mathsf{p}}\mathsf{k}$ on the MPC worlds.

$$\mathsf{b}\,\breve{o}\pi\,\mathsf{a}\triangle \to \mathsf{a}\,\exists\pi\,\mathsf{one}\triangle, \tag{$\exists$ seq.}$$

where $\mathsf{a} \in \mathsf{G}$ and $\mathsf{b} \in \mathsf{A}_\varepsilon$. These belong to $\vec{\mathsf{C}}$ by Theorem 9.7 and Lemma 10.1, and Theorem 10.6.

$$(\mathsf{a} \sqcap \mathsf{b})\triangle \rightleftarrows \mathsf{a}\triangle \wedge \mathsf{b}\triangle, \tag{$\sqcap$ seq.}$$
$$(\mathsf{a} \sqcup \mathsf{b})\triangle \rightleftarrows \mathsf{a}\triangle \vee \mathsf{b}\triangle, \tag{$\sqcup$ seq.}$$
$$(\mathsf{a}^\square)\triangle \rightleftarrows (\mathsf{a}\triangle)^\Diamond, \tag{$\square$ seq.}$$

where $\mathsf{a}, \mathsf{b} \in \mathsf{G}$. These belong to $\vec{\mathsf{C}}$ by Theorem 9.1 and Lemma 10.1.

$$\mathsf{a}\,\breve{o}\pi\,(\mathsf{f}\,\Omega\mathsf{x})\triangle \rightleftarrows \mathsf{f}(\mathsf{x}/\mathsf{a}), \tag{Ω seq.}$$

where $\mathsf{a} \in \mathsf{A}_\varepsilon,\ \mathsf{f} \in \mathsf{A}_\emptyset,\ \mathsf{x} \in \mathbb{X}_\varepsilon$, and x is free from a in f. These belong to $\vec{\mathsf{C}}$ by Theorem 10.5.

$$\mathsf{one}\,\forall\pi\,(\mathsf{f}\,\Omega\mathsf{x})\triangle \to \mathsf{f}, \tag{$\forall-$ seq.}$$

where $\mathsf{f} \in \mathsf{A}_\emptyset$ and $\mathsf{x} \in \mathbb{X}_\varepsilon$. These belong to $\vec{\mathsf{C}}$ by Theorem 10.7.

This completes the presentation of the deduction pair $(\vec{\mathsf{R}}, \vec{\mathsf{D}})$ on $\vec{\mathsf{H}}$ and the proof of Lemma 11.1.

We also obtain the following result on the predicate logical space $(\mathsf{H}, \mathcal{G}_\mathcal{W})$ and the deduction pair $(\wp, \nabla)$ and the logic $\&$ on H presented in §4.

Lemma 11.2 The deduction pair $(\wp \cup \&, \nabla)$ on H is $\mathcal{G}_\mathcal{W}$-sound.

Proof We have shown in Theorem 10.1 that the logical space $(\mathsf{H}, \mathcal{G}_\mathcal{W})$ is Boolean with respect to $\wedge, \vee, \Rightarrow, \Diamond$. Therefore by Theorem 7.4, the validity relation $\preccurlyeq$ of $(\mathsf{H}, \mathcal{G}_\mathcal{W})$ is Boolean, $\mathsf{R} = \wp \cup \&$ is $\mathcal{G}_\mathcal{W}$-sound, and the Boolean elements of H are $\mathcal{G}_\mathcal{W}$-sound. Consequently, the $\mathcal{G}_\mathcal{W}$-core C is closed under R. Since $\vec{\mathsf{C}}$ is closed under the logics case+, case−, and $\forall+$ on $\vec{\mathsf{H}}$, it follows that C is also closed under the logics $\perp, \top$, and $\forall$ on H. Therefore, in view of (4.3), it remains to show that the elements of ∂ other than the Boolean elements also belong to C. However, it is almost equivalent to the proved fact $\vec{\mathsf{D}} \subseteq \vec{\mathsf{C}}$.

For instance, since the $=$ sequents $\rightarrow \mathsf{a}\,\breve{\mathsf{o}}\pi\,\mathsf{a}\triangle$ belong to $\vec{\mathsf{C}}$, the $=$ elements $\mathsf{a}\,\breve{\mathsf{o}}\pi\,\mathsf{a}\triangle$ belong to C. Also, since the $\overline{\infty}$ sequents $\mathsf{a}\,\overline{\infty}\pi\,\mathsf{one}\triangle \rightarrow$ belong to $\vec{\mathsf{C}}$, we have $\mathsf{a}\,\overline{\infty}\pi\,\mathsf{one}\triangle \preccurlyeq$, hence $\preccurlyeq (\mathsf{a}\,\overline{\infty}\pi\,\mathsf{one}\triangle)^{\Diamond}$ by Theorem 7.3. Thus the $\overline{\infty}$ elements belong to C. The same argument applies to the $\mathsf{one}^{\square}$ elements.

Furthermore, since the $\mathfrak{Q}, \breve{\mathsf{o}}$ sequents $\mathsf{a}\,\lambda\mathsf{k}\,(\mathsf{b}\,\breve{\mathsf{o}}\mathsf{l}\,\mathsf{f}) \rightleftarrows \mathsf{b}\,\breve{\mathsf{o}}\mathsf{l}\,(\mathsf{a}\,\lambda\mathsf{k}\,\mathsf{f})$ belong to $\vec{\mathsf{C}}$, we have $\mathsf{a}\,\lambda\mathsf{k}\,(\mathsf{b}\,\breve{\mathsf{o}}\mathsf{l}\,\mathsf{f}) \preccurlyeq \mathsf{b}\,\breve{\mathsf{o}}\mathsf{l}\,(\mathsf{a}\,\lambda\mathsf{k}\,\mathsf{f})$ and $\mathsf{a}\,\lambda\mathsf{k}\,(\mathsf{b}\,\breve{\mathsf{o}}\mathsf{l}\,\mathsf{f}) \succcurlyeq \mathsf{b}\,\breve{\mathsf{o}}\mathsf{l}\,(\mathsf{a}\,\lambda\mathsf{k}\,\mathsf{f})$, hence

$$\preccurlyeq \mathsf{a}\,\lambda\mathsf{k}\,(\mathsf{b}\,\breve{\mathsf{o}}\mathsf{l}\,\mathsf{f}) \Rightarrow \mathsf{b}\,\breve{\mathsf{o}}\mathsf{l}\,(\mathsf{a}\,\lambda\mathsf{k}\,\mathsf{f}), \qquad \preccurlyeq \mathsf{b}\,\breve{\mathsf{o}}\mathsf{l}\,(\mathsf{a}\,\lambda\mathsf{k}\,\mathsf{f}) \Rightarrow \mathsf{a}\,\lambda\mathsf{k}\,(\mathsf{b}\,\breve{\mathsf{o}}\mathsf{l}\,\mathsf{f})$$

by Theorem 7.3. Thus the $\mathfrak{Q}, \breve{\mathsf{o}}$ elements belong to C. The same argument applies to the remaining elements of ∂ other than the $\forall, \Rightarrow$ elements, the $\forall, \mathfrak{P}$ elements, and the $\sqcup, +$ elements, which are shown to belong to C as follows.

Since the $\forall, \Rightarrow$ sequents $\mathsf{f},\ \mathsf{one}\,\forall\pi\,\big((\mathsf{f} \Rightarrow \mathsf{g})\,\Omega\mathsf{x}\big)\triangle \rightarrow \mathsf{one}\,\forall\pi\,(\mathsf{g}\,\Omega\mathsf{x})\triangle$ belong to $\vec{\mathsf{C}}$, we have $\mathsf{f},\ \mathsf{one}\,\forall\pi\,\big((\mathsf{f} \Rightarrow \mathsf{g})\,\Omega\mathsf{x}\big)\triangle \preccurlyeq \mathsf{one}\,\forall\pi\,(\mathsf{g}\,\Omega\mathsf{x})\triangle$, hence

$$\preccurlyeq \big(\mathsf{one}\,\forall\pi\,\big((\mathsf{f} \Rightarrow \mathsf{g})\,\Omega\mathsf{x}\big)\triangle\big) \Rightarrow \big(\mathsf{f} \Rightarrow \mathsf{one}\,\forall\pi\,(\mathsf{g}\,\Omega\mathsf{x})\triangle\big)$$

by Theorem 7.3. Thus the $\forall, \Rightarrow$ elements belong to C.

Since the $\forall, \mathfrak{P}$ sequents $\mathsf{a}\,\forall\pi\,\mathsf{b}\triangle, \mathsf{a}\,\overline{\mathsf{p}}\mathsf{k}\,\mathsf{f} \rightarrow \mathsf{b}\,\overline{\mathsf{p}}\mathsf{k}\,\mathsf{f}$ belong to $\vec{\mathsf{C}}$, we have $\mathsf{a}\,\forall\pi\,\mathsf{b}\triangle, \mathsf{a}\,\overline{\mathsf{p}}\mathsf{k}\,\mathsf{f} \preccurlyeq \mathsf{b}\,\overline{\mathsf{p}}\mathsf{k}\,\mathsf{f}$, hence $\preccurlyeq (\mathsf{a}\,\forall\pi\,\mathsf{b}\triangle \wedge \mathsf{a}\,\overline{\mathsf{p}}\mathsf{k}\,\mathsf{f}) \Rightarrow \mathsf{b}\,\overline{\mathsf{p}}\mathsf{k}\,\mathsf{f}$ by Theorem 7.3. Thus the $\forall, \mathfrak{P}$ elements belong to C.

Since the $\sqcup, +$ sequents $(\mathsf{a} \sqcup \mathsf{b})\,\overline{\mathsf{p} + \mathsf{q}}\mathsf{k}\,\mathsf{f} \rightarrow \mathsf{a}\,\overline{\mathsf{p}}\mathsf{k}\,\mathsf{f}, \mathsf{b}\,\overline{\mathsf{q}}\mathsf{k}\,\mathsf{f}$ belong to $\vec{\mathsf{C}}$, we have $(\mathsf{a} \sqcup \mathsf{b})\,\overline{\mathsf{p} + \mathsf{q}}\mathsf{k}\,\mathsf{f} \preccurlyeq \mathsf{a}\,\overline{\mathsf{p}}\mathsf{k}\,\mathsf{f}, \mathsf{b}\,\overline{\mathsf{q}}\mathsf{k}\,\mathsf{f}$ belong to $\vec{\mathsf{C}}$, hence

$$\preccurlyeq (\mathsf{a} \sqcup \mathsf{b})\,\overline{\mathsf{p} + \mathsf{q}}\mathsf{k}\,\mathsf{f} \Rightarrow (\mathsf{a}\,\overline{\mathsf{p}}\mathsf{k}\,\mathsf{f} \vee \mathsf{b}\,\overline{\mathsf{q}}\mathsf{k}\,\mathsf{f})$$

by Theorem 7.3. Thus the $\sqcup, +$ elements belong to C.

Remark 11.1 Logics $\perp, \top$, and $\forall$ on H are not in general $\mathcal{G}_{\mathcal{W}}$-sound, although the $\mathcal{G}_{\mathcal{W}}$-core is closed under them.

11.2 Laws derived from the sequent deduction pair

Let $(\vec{\mathsf{R}}, \vec{\mathsf{D}})$ be the deduction pair on $\vec{\mathsf{H}}$ defined in §11.1. Then $[\vec{\mathsf{D}}]_{\vec{\mathsf{R}}}$ is a subset of $\vec{\mathsf{H}} = \mathsf{H}^* \times \mathsf{H}^*$ and so is regarded as a relation on H^*. Let $\preccurlyeq_*$ denote it and let $\asymp_*$ be the symmetric core of the restriction of $\preccurlyeq_*$ to $\mathsf{H} \times \mathsf{H}$. Then since $[\vec{\mathsf{D}}]_{\vec{\mathsf{R}}}$ is closed under $\vec{\mathsf{R}}$ and contains $\vec{\mathsf{D}}$, it first follows that $\preccurlyeq_*$ satisfies the repetition law, weakening law, contraction law, exchange law, cut law, strong conjunction law, strong disjunction law, strong implication law, and strong negation law. Therefore $\preccurlyeq_*$ is a Boolean relation by Theorem 7.2, and consequently $\asymp_*$ is an equivalence relation. Since $\vec{\mathsf{D}} \subseteq [\vec{\mathsf{D}}]_{\vec{\mathsf{R}}}$, it also follows that $\preccurlyeq_*$ and $\asymp_*$ satisfy the following twenty seven laws.

$$\preccurlyeq_* \mathsf{f} \implies \preccurlyeq_* \mathsf{a}\,\breve{\mathsf{o}}\mathsf{k}\,\mathsf{f}, \qquad \text{(case+ law)}$$

where $\mathsf{a} \in \mathsf{A}_\varepsilon$ and $\mathsf{k} \in \mathsf{K}_\mathsf{f}$.

$$\preccurlyeq_* \mathsf{x}\,\breve{\mathsf{o}}\mathsf{k}\,\mathsf{f} \implies \preccurlyeq_* \mathsf{f}, \qquad \text{(case– law)}$$

where $x \in \mathbb{X}_\varepsilon,\ k \in K_f$, and $x \not\ll f$.

$$\preccurlyeq_* f \implies \preccurlyeq_* \mathrm{one}\,\forall\pi\,(f\,\Omega x)\triangle, \tag{$\forall+$ law}$$

where $f \in A_\emptyset$ and $x \in \mathbb{X}_\varepsilon$.

$$\preccurlyeq_* a\,\breve{o}\pi\,a\triangle, \tag{$=$ law}$$

where $a \in A_\varepsilon$.

$$a\,\overline{\infty}\pi\,\mathrm{one}\triangle \preccurlyeq_* , \tag{$\overline{\infty}$ law}$$

where $a \in G$ and ∞ is the maximum of $\mathbb{P}$ in case it exists.

$$a\lambda k\,(b\,\breve{o}l\,f) \asymp_* b\,\breve{o}l\,(a\lambda k\,f), \tag{$\mathfrak{Q}, \breve{o}$ law}$$

where $a \in G,\ b \in A_\varepsilon,\ f \in H,\ k, l \in K_f,\ k \neq l$, and $\lambda \in \{\breve{o}\} \cup \mathfrak{Q}$. Also $a \in A_\varepsilon$ in case $\lambda = \breve{o}$.

$$(a_i\,\breve{o}k_i)_{i=1,\dots,l}(f \wedge g) \asymp_* (a_i\,\breve{o}k_i)_{i=1,\dots,m} f \wedge (a_i\,\breve{o}k_i)_{i=n+1,\dots,l}\, g, \tag{$\wedge$ law}$$
$$(a_i\,\breve{o}k_i)_{i=1,\dots,l}(f \vee g) \asymp_* (a_i\,\breve{o}k_i)_{i=1,\dots,m} f \vee (a_i\,\breve{o}k_i)_{i=n+1,\dots,l}\, g, \tag{$\vee$ law}$$
$$(a_i\,\breve{o}k_i)_{i=1,\dots,l}(f \Rightarrow g) \asymp_* (a_i\,\breve{o}k_i)_{i=1,\dots,m} f \Rightarrow (a_i\,\breve{o}k_i)_{i=n+1,\dots,l}\, g, \tag{$\Rightarrow$ law}$$

where $a_1, \dots, a_l \in A_\varepsilon,\ f, g \in H$, and $k_1, \dots, k_l$ are distinct cases such that $k_1, \dots, k_n \in K_f - K_g,\ k_{n+1}, \dots, k_m \in K_f \cap K_g$, and $k_{m+1}, \dots, k_l \in K_g - K_f$ $(0 \le n \le m \le l)$.

$$(a_i\,\breve{o}k_i)_{i=1,\dots,n}(f^\Diamond) \asymp_* \big((a_i\,\breve{o}k_i)_{i=1,\dots,n} f\big)^\Diamond, \tag{$\Diamond$ law}$$

where $a_1, \dots, a_n \in A_\varepsilon,\ f \in H$, and $k_1, \dots, k_n$ are distinct cases in K_f.

$$a\neg\mathfrak{p}k\,f \asymp_* a\,\mathfrak{p}k\,f^\Diamond, \tag{$\neg$ law}$$
$$a\,\mathfrak{p}^\circ k\,f \asymp_* (a\,\mathfrak{p}k\,f)^\Diamond, \tag{$\circ$ law}$$

where $a \in G,\ f \in H,\ k \in K_f$, and $\mathfrak{p} \in \mathfrak{P}$.

$$a\,(\mathfrak{p} \cap \mathfrak{q})k\,f \asymp_* a\,\mathfrak{p}k\,f \wedge a\,\mathfrak{q}k\,f, \tag{$\cap$ law}$$
$$a\,(\mathfrak{p} \cup \mathfrak{q})k\,f \asymp_* a\,\mathfrak{p}k\,f \vee a\,\mathfrak{q}k\,f, \tag{$\cup$ law}$$

where $a \in G,\ f \in H,\ k \in K_f$, and $\mathfrak{p}, \mathfrak{q} \in \mathfrak{P}$.

$$a\,\overline{p}k\,f \asymp_* a\,\overline{p}\pi\big((x\,\breve{o}k\,f)\,\Omega x\big)\triangle, \tag{$\mathfrak{P}$ law}$$

where $a \in G,\ f \in H,\ x \in \mathbb{X}_\varepsilon,\ K_f = \{k\},\ p \in \mathbb{P}$, and $x \not\ll f$.

$$a\,\overline{p}\pi\,b\triangle \asymp_* (a \sqcap b)\,\overline{p}\pi\,\mathrm{one}\triangle, \tag{$\triangle$ law}$$

where $a, b \in G$, and $p \in \mathbb{P}$.

$$f,\ \mathrm{one}\,\forall\pi\big((f \Rightarrow g)\,\Omega x\big)\triangle \preccurlyeq_* \mathrm{one}\,\forall\pi\,(g\,\Omega x)\triangle, \tag{$\forall, \Rightarrow$ law}$$

where $f, g \in A_\emptyset$, $x \in \mathbb{X}_\varepsilon$, and $x \not\ll f$.

$$\mathsf{one}\,\forall\pi\,\big(\big((x\,\breve{o}\pi\,a\triangle)\Rightarrow(x\,\breve{o}k\,f)\big)\,\Omega x\big)\triangle \preccurlyeq_* a\,\forall k\,f, \tag{$\forall$ law}$$

where $x \in \mathbb{X}_\varepsilon$, $a \in G$, $f \in H$, $K_f = \{k\}$, and $x \not\ll a, f$.

$$a\,\forall\pi\,b\triangle, a\,\overline{p}k\,f \preccurlyeq_* b\,\overline{p}k\,f, \tag{$\forall, \mathfrak{P}$ law}$$

where $a, b \in G$, $f \in H$, $k \in K_f$, and $p \in \mathbb{P}$.

$$(a \sqcup b)\,\overline{p+q}k\,f \preccurlyeq_* a\,\overline{p}k\,f,\ b\,\overline{q}k\,f, \tag{$\sqcup, +$ law}$$

where $a, b \in G$, $f \in H$, $k \in K_f$, and $p, q \in \mathbb{P}$.

$$\mathsf{one}^\Box\,\overline{p}k\,f \preccurlyeq_*\ , \tag{$\mathsf{one}^\Box$ law}$$

where $f \in H$, $k \in K_f$, and $p \in \mathbb{P}$.

$$b\,\breve{o}\pi\,a\triangle \preccurlyeq_* a\,\exists\pi\,\mathsf{one}\triangle, \tag{$\exists$ law}$$

where $a \in G$ and $b \in A_\varepsilon$.

$$(a \sqcap b)\triangle \asymp_* a\triangle \wedge b\triangle, \tag{$\sqcap$ law}$$

$$(a \sqcup b)\triangle \asymp_* a\triangle \vee b\triangle, \tag{$\sqcup$ law}$$

$$(a^\Box)\triangle \asymp_* (a\triangle)^\Diamond, \tag{$\Box$ law}$$

where $a, b \in G$.

$$a\,\breve{o}\pi\,(f\,\Omega x)\triangle \asymp_* f(x/a), \tag{Ω law}$$

where $a \in A_\varepsilon$, $f \in A_\emptyset$, $x \in \mathbb{X}_\varepsilon$, and x is free from a in f.

$$\mathsf{one}\,\forall\pi\,(f\,\Omega x)\triangle \preccurlyeq_* f, \tag{$\forall-$ law}$$

where $f \in A_\emptyset$ and $x \in \mathbb{X}_\varepsilon$.

This completes the verbatim translations of the presentation of the sequent deduction pair $(\vec{R}, \vec{D})$ into the laws which $\preccurlyeq_* = [\vec{D}]_{\vec{R}}$ satisfies, from which we will derive more useful laws in the following lemmas.

Lemma 11.3 Let $a_1, \ldots, a_n \in A_\varepsilon$, $f_1, \ldots, f_m \in H$, and $k_1, \ldots, k_n$ be distinct cases in $K_{f_1} \cap \cdots \cap K_{f_m}$. Then the following holds irrespective of the order of applying the operations $\wedge, \vee$:

$$(a_i\,\breve{o}k_i)_{i=1,\ldots,n}(f_1 \wedge \cdots \wedge f_m) \asymp_* (a_i\,\breve{o}k_i)_{i=1,\ldots,n}f_1 \wedge \cdots \wedge (a_i\,\breve{o}k_i)_{i=1,\ldots,n}f_m, \tag{gen. $\wedge$ law}$$

$$(a_i\,\breve{o}k_i)_{i=1,\ldots,n}(f_1 \vee \cdots \vee f_m) \asymp_* (a_i\,\breve{o}k_i)_{i=1,\ldots,n}f_1 \vee \cdots \vee (a_i\,\breve{o}k_i)_{i=1,\ldots,n}f_m. \tag{gen. $\vee$ law}$$

Proof We may assume $m > 1$ and argue by induction on m. As for the generalized $\vee$ law, suppose $\vee$ has been applied in such an order that $f_1 \vee \cdots \vee f_m = (f_1 \vee \cdots \vee f_j) \vee (f_{j+1} \vee \cdots \vee f_m)$ holds. Then

$$(a_i\, \breve{o} k_i)_i (f_1 \vee \cdots \vee f_m) \asymp_* (a_i\, \breve{o} k_i)_i (f_1 \vee \cdots \vee f_j) \vee (a_i\, \breve{o} k_i)_i (f_{j+1} \vee \cdots \vee f_m)$$

by the $\vee$ law, where $(a_i\, \breve{o} k_i)_i$ is an abbreviation for $(a_i\, \breve{o} k_i)_{i=1,\dots,n}$. Also,

$$(a_i\, \breve{o} k_i)_i (f_1 \vee \cdots \vee f_j) \asymp_* (a_i\, \breve{o} k_i)_i f_1 \vee \cdots \vee (a_i\, \breve{o} k_i)_i f_j,$$
$$(a_i\, \breve{o} k_i)_i (f_{j+1} \vee \cdots \vee f_m) \asymp_* (a_i\, \breve{o} k_i)_i f_{j+1} \vee \cdots \vee (a_i\, \breve{o} k_i)_i f_m$$

by the induction hypothesis. Applying Theorem 7.3 to the above three displayed $\asymp_*$ relations, we see that the generalized $\vee$ law holds. The generalized $\wedge$ law may be proved similarly.

Lemma 11.4 Let $a \in G$, $f \in H$, $k \in K_f$, and $\mathfrak{p}_1, \dots, \mathfrak{p}_n \in \mathfrak{P}$. Then the following holds irrespective of the order of applying the operations $\wedge, \vee$:

$$a\,(\mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_n) k f \asymp_* a\, \mathfrak{p}_1 k f \wedge \cdots \wedge a\, \mathfrak{p}_n k f, \qquad \text{(gen. } \cap \text{ law)}$$
$$a\,(\mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_n) k f \asymp_* a\, \mathfrak{p}_1 k f \vee \cdots \vee a\, \mathfrak{p}_n k f. \qquad \text{(gen. } \cup \text{ law)}$$

Proof We may assume $n > 1$ and argue by induction on n. As for the gen. $\cup$ law, $a\,(\mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_n) k f \asymp_* a\,(\mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_{n-1}) k f \vee a\, \mathfrak{p}_n k f$ by the $\cup$ law, and $a\,(\mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_{n-1}) k f \asymp_* a\, \mathfrak{p}_1 k f \vee \cdots \vee a\, \mathfrak{p}_{n-1} k f$ by the induction hypothesis. Applying Theorem 7.3 to these $\asymp_*$ relations, we see that the gen. $\cup$ law holds. The gen. $\cap$ law may be proved similarly.

Lemma 11.5 Let $a_1, \dots, a_n \in G$, $f \in H$, $k \in K_f$, and $p_1, \dots, p_n \in \mathbb{P}$. Then the following holds irrespective of the order of applying the operation $\sqcup$:

$$(a_1 \sqcup \cdots \sqcup a_n)\, \overline{p_1 + \cdots + p_n} k f \preccurlyeq_* a_1\, \overline{p_1} k f, \cdots, a_n\, \overline{p_n} k f. \quad \text{(gen. } \sqcup, + \text{ law)}$$

Proof We may assume $n > 1$ and argue by induction on n. Suppose $\sqcup$ has been applied in such an order that $a_1 \sqcup \cdots \sqcup a_n = (a_1 \sqcup \cdots \sqcup a_i) \sqcup (a_{i+1} \sqcup \cdots \sqcup a_n)$ holds, then

$$\begin{aligned}&(a_1 \sqcup \cdots \sqcup a_n)\, \overline{p_1 + \cdots + p_n} k f \\ &\quad \preccurlyeq_* (a_1 \sqcup \cdots \sqcup a_i)\, \overline{p_1 + \cdots + p_i} k f,\ (a_{i+1} \sqcup \cdots \sqcup a_n)\, \overline{p_{i+1} + \cdots + p_n} k f\end{aligned}$$

by the $\sqcup, +$ law, and

$$(a_1 \sqcup \cdots \sqcup a_i)\, \overline{p_1 + \cdots + p_i} k f \preccurlyeq_* a_1\, \overline{p_1} k f, \cdots, a_i\, \overline{p_i} k f,$$
$$(a_{i+1} \sqcup \cdots \sqcup a_n)\, \overline{p_{i+1} + \cdots + p_n} k f \preccurlyeq_* a_{i+1}\, \overline{p_{i+1}} k f, \cdots, a_n\, \overline{p_n} k f$$

by the induction hypothesis. Applying the cut law twice to the above three displayed $\preccurlyeq_*$ relations, we see that the gen. $\sqcup, +$ law holds.

Lemma 11.6 The following holds for $\mathsf{a}_1, \ldots, \mathsf{a}_n \in \mathsf{G}$, irrespective of the order of applying the operations $\sqcap, \sqcup, \wedge, \vee$:

$$(\mathsf{a}_1 \sqcap \cdots \sqcap \mathsf{a}_n)\triangle \asymp_* \mathsf{a}_1\triangle \wedge \cdots \wedge \mathsf{a}_n\triangle, \qquad \text{(gen. } \sqcap \text{ law)}$$
$$(\mathsf{a}_1 \sqcup \cdots \sqcup \mathsf{a}_n)\triangle \asymp_* \mathsf{a}_1\triangle \vee \cdots \vee \mathsf{a}_n\triangle. \qquad \text{(gen. } \sqcup \text{ law)}$$

Proof We may assume $n > 1$ and argue by induction on n. As to the gen. $\sqcup$ law, suppose $\sqcup$ has been applied in such an order that $\mathsf{a}_1 \sqcup \cdots \sqcup \mathsf{a}_n = (\mathsf{a}_1 \sqcup \cdots \sqcup \mathsf{a}_i) \sqcup (\mathsf{a}_{i+1} \sqcup \cdots \sqcup \mathsf{a}_n)$ holds. Then

$$(\mathsf{a}_1 \sqcup \cdots \sqcup \mathsf{a}_n)\triangle \asymp_* (\mathsf{a}_1 \sqcup \cdots \sqcup \mathsf{a}_i)\triangle \vee (\mathsf{a}_{i+1} \sqcup \cdots \sqcup \mathsf{a}_n)\triangle,$$
$$(\mathsf{a}_1 \sqcup \cdots \sqcup \mathsf{a}_i)\triangle \asymp_* \mathsf{a}_1\triangle \vee \cdots \vee \mathsf{a}_i\triangle,$$
$$(\mathsf{a}_{i+1} \sqcup \cdots \sqcup \mathsf{a}_n)\triangle \asymp_* \mathsf{a}_{i+1}\triangle \vee \cdots \vee \mathsf{a}_n\triangle$$

by the $\sqcup$ law and the induction hypothesis. Applying Theorem 7.3 to the above displayed $\asymp_*$ relations, we see that the gen. $\sqcup$ law holds. The gen. $\sqcap$ law may be proved similarly.

Lemma 11.7 Let $\mathsf{f}_1, \ldots, \mathsf{f}_m, \mathsf{g}_1, \ldots, \mathsf{g}_n \in \mathsf{H}$, $\alpha, \beta \in \mathsf{H}^*$, $\mathsf{a} \in \mathsf{A}_\varepsilon$, and $\mathsf{k} \in \mathsf{K}$. Assume that k belongs to the ranges of $\mathsf{f}_1, \ldots, \mathsf{f}_m, \mathsf{g}_1, \ldots, \mathsf{g}_n$ but does not belong to those of the predicates in $\alpha \cup \beta$. Then the following holds (cf. Theorems 10.2):

$$\begin{aligned} &\mathsf{f}_1 \cdots \mathsf{f}_m \alpha \preccurlyeq_* \mathsf{g}_1 \cdots \mathsf{g}_n \beta \\ &\quad\Longrightarrow\ \mathsf{a}\,\breve{\mathsf{o}}\mathsf{k}\,\mathsf{f}_1, \cdots, \mathsf{a}\,\breve{\mathsf{o}}\mathsf{k}\,\mathsf{f}_m, \alpha \preccurlyeq_* \mathsf{a}\,\breve{\mathsf{o}}\mathsf{k}\,\mathsf{g}_1, \cdots, \mathsf{a}\,\breve{\mathsf{o}}\mathsf{k}\,\mathsf{g}_n, \beta. \end{aligned} \qquad \text{(gen. case+ law)}$$

Proof We may assume that either $m \geq 1$ or $n \geq 1$. Let $\alpha = \mathsf{f}'_1 \cdots \mathsf{f}'_{m'}$, $\beta = \mathsf{g}'_1 \cdots \mathsf{g}'_{n'}$ with $\mathsf{f}'_1, \ldots, \mathsf{f}'_{m'}, \mathsf{g}'_1, \ldots, \mathsf{g}'_{n'} \in \mathsf{H}$. Notice here that we may assume $m' \neq 0 \neq n'$ as in the proof of Theorem 9.6 by Theorem 7.3. Define

$$\mathsf{h} = (\mathsf{f}_1 \wedge \cdots \wedge \mathsf{f}_m) \wedge (\mathsf{g}_1 \vee \cdots \vee \mathsf{g}_n)^\Diamond,$$
$$\mathsf{h}' = (\mathsf{f}'_1 \wedge \cdots \wedge \mathsf{f}'_{m'})^\Diamond \vee (\mathsf{g}'_1 \vee \cdots \vee \mathsf{g}'_{n'}),$$

where the orders of applying the operations $\wedge, \vee$ within parentheses are arbitrary. Then Theorem 7.3 shows that the premise of the gen. case+ law implies $\preccurlyeq_* \mathsf{h} \Rightarrow \mathsf{h}'$, and so $\preccurlyeq_* \mathsf{a}\,\breve{\mathsf{o}}\mathsf{k}\,(\mathsf{h} \Rightarrow \mathsf{h}')$ by the case+ law. Furthermore, since $\mathsf{k} \in \mathsf{K}_\mathsf{h} - \mathsf{K}_{\mathsf{h}'}$, $\mathsf{a}\,\breve{\mathsf{o}}\mathsf{k}\,(\mathsf{h} \Rightarrow \mathsf{h}') \asymp_* \mathsf{a}\,\breve{\mathsf{o}}\mathsf{k}\,\mathsf{h} \Rightarrow \mathsf{h}'$ by the $\Rightarrow$ law. Thus

$$\mathsf{a}\,\breve{\mathsf{o}}\mathsf{k}\,\mathsf{h} \preccurlyeq_* \mathsf{h}'$$

by the cut law and Theorem 7.3. Furthermore,

$$\begin{aligned} &\big(\mathsf{a}\,\breve{\mathsf{o}}\mathsf{k}\,(\mathsf{f}_1 \wedge \cdots \wedge \mathsf{f}_m)\big) \wedge \big(\mathsf{a}\,\breve{\mathsf{o}}\mathsf{k}\,(\mathsf{g}_1 \vee \cdots \vee \mathsf{g}_n)\big)^\Diamond \\ &\asymp_* \big(\mathsf{a}\,\breve{\mathsf{o}}\mathsf{k}\,(\mathsf{f}_1 \wedge \cdots \wedge \mathsf{f}_m)\big) \wedge \big(\mathsf{a}\,\breve{\mathsf{o}}\mathsf{k}\,(\mathsf{g}_1 \vee \cdots \vee \mathsf{g}_n)^\Diamond\big) \asymp_* \mathsf{a}\,\breve{\mathsf{o}}\mathsf{k}\,\mathsf{h} \end{aligned}$$

by the $\Diamond$ law, Theorem 7.3 and the $\wedge$ law. Thus

$$\mathsf{a}\,\breve{\mathsf{o}}\mathsf{k}\,(\mathsf{f}_1 \wedge \cdots \wedge \mathsf{f}_m) \preccurlyeq_* \mathsf{a}\,\breve{\mathsf{o}}\mathsf{k}\,(\mathsf{g}_1 \vee \cdots \vee \mathsf{g}_n),\ \mathsf{h}'$$

by the cut law and Theorem 7.3. Furthermore,

$$\mathsf{a}\,\breve{o}\mathsf{k}\,\mathsf{f}_1 \wedge \cdots \wedge \mathsf{a}\,\breve{o}\mathsf{k}\,\mathsf{f}_m \asymp_* \mathsf{a}\,\breve{o}\mathsf{k}\,(\mathsf{f}_1 \wedge \cdots \wedge \mathsf{f}_m),$$
$$\mathsf{a}\,\breve{o}\mathsf{k}\,(\mathsf{g}_1 \vee \cdots \vee \mathsf{g}_n) \asymp_* \mathsf{a}\,\breve{o}\mathsf{k}\,\mathsf{g}_1 \vee \cdots \vee \mathsf{a}\,\breve{o}\mathsf{k}\,\mathsf{g}_n$$

by the gen. $\wedge$ law and gen. $\vee$ law. Applying the cut law to the above three displayed relations, we have $\mathsf{a}\,\breve{o}\mathsf{k}\,\mathsf{f}_1 \wedge \cdots \wedge \mathsf{a}\,\breve{o}\mathsf{k}\,\mathsf{f}_m \preccurlyeq_* \mathsf{a}\,\breve{o}\mathsf{k}\,\mathsf{g}_1 \vee \cdots \vee \mathsf{a}\,\breve{o}\mathsf{k}\,\mathsf{g}_n,\ \mathsf{h}'$, hence the conclusion of the gen. case+ law by Theorem 7.3.

Lemma 11.8 Let $\mathsf{f}_1, \ldots, \mathsf{f}_m, \mathsf{g}_1, \ldots, \mathsf{g}_n \in \mathsf{H}$, $\alpha, \beta \in \mathsf{H}^*$, $\mathsf{x} \in \mathbb{X}_\varepsilon$, and $\mathsf{k} \in \mathsf{K}$. Assume that k belongs to the ranges of $\mathsf{f}_1, \ldots, \mathsf{f}_m$, $\mathsf{g}_1, \ldots, \mathsf{g}_n$ but does not belong to those of the predicates in $\alpha \cup \beta$ and that x does not occur free in the predicates in $\{\mathsf{f}_1, \ldots, \mathsf{f}_m,\ \mathsf{g}_1, \ldots, \mathsf{g}_n\} \cup \alpha \cup \beta$. Then the following holds (cf. Theorems 10.2):

$$\begin{aligned} &\mathsf{x}\,\breve{o}\mathsf{k}\,\mathsf{f}_1, \cdots, \mathsf{x}\,\breve{o}\mathsf{k}\,\mathsf{f}_m, \alpha \preccurlyeq_* \mathsf{x}\,\breve{o}\mathsf{k}\,\mathsf{g}_1, \cdots, \mathsf{x}\,\breve{o}\mathsf{k}\,\mathsf{g}_n, \beta \\ &\quad\Longrightarrow\ \mathsf{f}_1 \cdots \mathsf{f}_m \alpha \preccurlyeq_* \mathsf{g}_1 \cdots \mathsf{g}_n \beta. \end{aligned} \qquad \text{(gen. case− law)}$$

Proof We may assume that either $\mathsf{m} \geq 1$ or $\mathsf{n} \geq 1$. Let $\alpha = \mathsf{f}'_1 \cdots \mathsf{f}'_{\mathsf{m}'}$, $\beta = \mathsf{g}'_1 \cdots \mathsf{g}'_{\mathsf{n}'}$ with $\mathsf{f}'_1, \ldots, \mathsf{f}'_{\mathsf{m}'},\ \mathsf{g}'_1, \ldots, \mathsf{g}'_{\mathsf{n}'} \in \mathsf{H}$. Define $\mathsf{e} = \mathsf{one}\forall\pi\,\mathsf{one}\triangle$. Then k does not belong to the range of $\mathsf{e} \wedge \mathsf{e}^\Diamond$ nor of $\mathsf{e} \vee \mathsf{e}^\Diamond$, and x does not occur free in $\mathsf{e} \wedge \mathsf{e}^\Diamond$ nor in $\mathsf{e} \vee \mathsf{e}^\Diamond$. Therefore by Theorem 7.3, we may assume $\mathsf{m}' \neq 0 \neq \mathsf{n}'$. Define $\mathsf{h}' = (\mathsf{f}'_1 \wedge \cdots \wedge \mathsf{f}'_{\mathsf{m}'})^\Diamond \vee (\mathsf{g}'_1 \vee \cdots \vee \mathsf{g}'_{\mathsf{n}'})$, where the orders of applying the operations $\wedge, \vee$ within parentheses are arbitrary. Then by Theorem 7.3 the premise of the gen. case− law implies

$$\mathsf{x}\,\breve{o}\mathsf{k}\,\mathsf{f}_1 \wedge \cdots \wedge \mathsf{x}\,\breve{o}\mathsf{k}\,\mathsf{f}_m \preccurlyeq_* \mathsf{x}\,\breve{o}\mathsf{k}\,\mathsf{g}_1 \vee \cdots \vee \mathsf{x}\,\breve{o}\mathsf{k}\,\mathsf{g}_n,\ \mathsf{h}'.$$

Furthermore,

$$\mathsf{x}\,\breve{o}\mathsf{k}\,(\mathsf{f}_1 \wedge \cdots \wedge \mathsf{f}_m) \asymp_* \mathsf{x}\,\breve{o}\mathsf{k}\,\mathsf{f}_1 \wedge \cdots \wedge \mathsf{x}\,\breve{o}\mathsf{k}\,\mathsf{f}_m,$$
$$\mathsf{x}\,\breve{o}\mathsf{k}\,\mathsf{g}_1 \vee \cdots \vee \mathsf{x}\,\breve{o}\mathsf{k}\,\mathsf{g}_n \asymp_* \mathsf{x}\,\breve{o}\mathsf{k}\,(\mathsf{g}_1 \vee \cdots \vee \mathsf{g}_n)$$

by the gen. $\wedge$ law and the gen. $\vee$ law. Applying the cut law to the above three displayed relations and using Theorem 7.3, we have

$$\big(\mathsf{x}\,\breve{o}\mathsf{k}\,(\mathsf{f}_1 \wedge \cdots \wedge \mathsf{f}_m)\big) \wedge \big(\mathsf{x}\,\breve{o}\mathsf{k}\,(\mathsf{g}_1 \vee \cdots \vee \mathsf{g}_n)\big)^\Diamond \preccurlyeq_* \mathsf{h}'.$$

Define $\mathsf{h} = (\mathsf{f}_1 \wedge \cdots \wedge \mathsf{f}_m) \wedge (\mathsf{g}_1 \vee \cdots \vee \mathsf{g}_n)^\Diamond$, where the orders of applying the operations $\wedge, \vee$ within parentheses are arbitrary. Then

$$\begin{aligned} \mathsf{x}\,\breve{o}\mathsf{k}\,\mathsf{h} &\asymp_* \big(\mathsf{x}\,\breve{o}\mathsf{k}\,(\mathsf{f}_1 \wedge \cdots \wedge \mathsf{f}_m)\big) \wedge \big(\mathsf{x}\,\breve{o}\mathsf{k}\,(\mathsf{g}_1 \vee \cdots \vee \mathsf{g}_n)^\Diamond\big) \\ &\asymp_* \big(\mathsf{x}\,\breve{o}\mathsf{k}\,(\mathsf{f}_1 \wedge \cdots \wedge \mathsf{f}_m)\big) \wedge \big(\mathsf{x}\,\breve{o}\mathsf{k}\,(\mathsf{g}_1 \vee \cdots \vee \mathsf{g}_n)\big)^\Diamond \end{aligned}$$

by the $\wedge$ law, $\Diamond$ law, and Theorem 7.3. Therefore $\mathsf{x}\,\breve{o}\mathsf{k}\,\mathsf{h} \preccurlyeq_* \mathsf{h}'$ by the cut law, hence $\preccurlyeq_* \mathsf{x}\,\breve{o}\mathsf{k}\,\mathsf{h} \Rightarrow \mathsf{h}'$. Furthermore, $\mathsf{x}\,\breve{o}\mathsf{k}\,\mathsf{h} \Rightarrow \mathsf{h}' \asymp_* \mathsf{x}\,\breve{o}\mathsf{k}\,(\mathsf{h} \Rightarrow \mathsf{h}')$ by the $\Rightarrow$ law. Therefore $\preccurlyeq_* \mathsf{x}\,\breve{o}\mathsf{k}\,(\mathsf{h} \Rightarrow \mathsf{h}')$ by the cut law, and since $\mathsf{x} \not\ll \mathsf{h} \Rightarrow \mathsf{h}'$, we have $\preccurlyeq_* \mathsf{h} \Rightarrow \mathsf{h}'$ by the case− law, hence the conclusion of the gen. case− law.

Lemma 11.9 Let $x \in \mathbb{X}_\varepsilon$, $a, b_1, \ldots, b_n \in G$, $\alpha, \beta \in (A_\emptyset)^*$, $f \in H$, $k \in K_f$, $p, q_1, \ldots, q_n \in \mathbb{P}$, and assume that x does not occur free in the elements of $\{a, b_1, \ldots, b_n\} \cup \alpha \cup \beta$ and that $p \geq \sum_{i=1}^{n} q_i$ holds, where if $n = 0$ then $\sum_{i=1}^{n} q_i = 0$ by definition. Then the following holds:

$$\begin{aligned} & x\,\breve{o}\pi\, a\triangle, \alpha \preccurlyeq_* x\,\breve{o}\pi\, b_1\triangle, \cdots, x\,\breve{o}\pi\, b_n\triangle, \beta \\ & \Longrightarrow\ a\,\overline{p}k\,f, \alpha \preccurlyeq_* b_1\,\overline{q_1}k\,f, \cdots, b_n\,\overline{q_n}k\,f, \beta. \end{aligned} \qquad \text{(pigeonhole principle)}$$

Proof First assume $n = 0$. Let $b = \mathrm{one}^\Box$ and $q = 0$. Then $x \not\lessdot b$, $p \geq q$, and the premise $x\,\breve{o}\pi\, a\triangle, \alpha \preccurlyeq_* \beta$ of the pigeonhole principle yields $x\,\breve{o}\pi\, a\triangle, \alpha \preccurlyeq_* x\,\breve{o}\pi\, b\triangle, \beta$ by the weakening law. If we show $a\,\overline{p}k\,f, \alpha \preccurlyeq_* b\,\overline{q}k\,f, \beta$, then since $b\,\overline{q}k\,f \preccurlyeq_*$ by the $\mathrm{one}^\Box$ law, we have $a\,\overline{p}k\,f, \alpha \preccurlyeq_* \beta$ as desired by the cut law.

Therefore, we may assume $n \geq 1$. Let $\alpha = g_1 \cdots g_l$, $\beta = h_1 \cdots h_m$ with $g_1, \ldots, g_l$, $h_1, \ldots, h_m \in A_\emptyset$. We may assume $l \neq 0 \neq m$ by Theorem 7.3 as in the proof of Lemma 11.8. Define

$$e = (g_1 \wedge \cdots \wedge g_l) \wedge (h_1 \vee \cdots \vee h_m)^\Diamond,$$

where the orders of applying the operations $\wedge, \vee$ within parentheses are arbitrary. Then $e \in A_\emptyset$, and the premise of the pigeonhole principle implies

$$x\,\breve{o}\pi\, a\triangle, e \preccurlyeq_* x\,\breve{o}\pi\, b_1\triangle \vee \cdots \vee x\,\breve{o}\pi\, b_n\triangle,$$

and $x\,\breve{o}\pi\, b_1\triangle \vee \cdots \vee x\,\breve{o}\pi\, b_n\triangle \asymp_* x\,\breve{o}\pi\,(b_1\triangle \vee \cdots \vee b_n\triangle)$ by the gen. $\vee$ law. Furthermore, since $b_1\triangle \vee \cdots \vee b_n\triangle \asymp_* (b_1 \sqcup \cdots \sqcup b_n)\triangle$ by the gen. $\sqcup$ law, $x\,\breve{o}\pi\,(b_1\triangle \vee \cdots \vee b_n\triangle) \asymp_* x\,\breve{o}\pi\,(b_1 \sqcup \cdots \sqcup b_n)\triangle$ by the gen. case+ law. Therefore, for $b = b_1 \sqcup \cdots \sqcup b_n$, the premise of the pigeonhole principle implies $\preccurlyeq_* e \Rightarrow \big((x\,\breve{o}\pi\, a\triangle) \Rightarrow (x\,\breve{o}\pi\, b\triangle)\big)$, and applying the $\forall$+ law, we have

$$\preccurlyeq_* \mathrm{one}\,\forall\pi\,\Big(\big(e \Rightarrow \big((x\,\breve{o}\pi\, a\triangle) \Rightarrow (x\,\breve{o}\pi\, b\triangle)\big)\big)\,\Omega x\Big)\triangle.$$

Furthermore, since $x \not\lessdot e$, we have

$$\begin{gathered} e,\ \mathrm{one}\,\forall\pi\,\Big(\big(e \Rightarrow \big((x\,\breve{o}\pi\, a\triangle) \Rightarrow (x\,\breve{o}\pi\, b\triangle)\big)\big)\,\Omega x\Big)\triangle \\ \preccurlyeq_* \mathrm{one}\,\forall\pi\,\Big(\big((x\,\breve{o}\pi\, a\triangle) \Rightarrow (x\,\breve{o}\pi\, b\triangle)\big)\,\Omega x\Big)\triangle \end{gathered}$$

by the $\forall, \Rightarrow$ law, and since x does not occur free in a nor in $b\triangle$, we have

$$\mathrm{one}\,\forall\pi\,\Big(\big((x\,\breve{o}\pi\, a\triangle) \Rightarrow (x\,\breve{o}\pi\, b\triangle)\big)\,\Omega x\Big)\triangle \preccurlyeq_* a\,\forall\pi\, b\triangle$$

by the $\forall$ law, and we have

$$a\,\forall\pi\, b\triangle,\ a\,\overline{p}k\,f \preccurlyeq_* b\,\overline{p}k\,f$$

by the $\forall, \mathfrak{P}$ law, and since $\overline{p} \subseteq \overline{q_1 + \cdots + q_n}$ by the assumption $p \geq q_1 + \cdots + q_n$, we have

$$b\,\overline{p}k\,f \preccurlyeq_* b\,\overline{q_1 + \cdots + q_n}k\,f$$

by the $\cap$ law and the conjunction law, and we have

$$b\,\overline{q_1 + \cdots + q_n}k\,f \preccurlyeq_* b_1\,\overline{q_1}k\,f, \cdots, b_n\,\overline{q_n}k\,f$$

by the gen. $\sqcup, +$ law. Applying the cut law to the above six displayed $\preccurlyeq_*$ relations, we finally obtain

$$e,\ a\,\overline{p}k\,f \preccurlyeq_* b_1\,\overline{q_1}k\,f, \cdots, b_n\,\overline{q_n}k\,f,$$

hence the conclusion of the pigeonhole principle.

Lemma 11.10 Let $a_1, \ldots, a_n \in A_\varepsilon$, $f \in H$, and $k_1, \ldots, k_n$ be distinct cases in K_f. Then the following holds for every $\rho \in \mathfrak{S}_n$, where $\mathfrak{S}_n$ is the symmetric group on the letters $1, \ldots, n$:

$$(a_i\,\breve{o}k_i)_{i=1,\ldots,n} f \asymp_* (a_{\rho i}\,\breve{o}k_{\rho i})_{i=1,\ldots,n} f. \qquad \text{(permutation law)}$$

Proof We only need to consider the case where ρ is a transposition, in which case, the result follows from the $\mathfrak{Q}, \breve{o}$ law and the gen. case+ law.

11.3 Core-completeness of the sequent deduction pair

We have proved $[\vec{D}]_{\vec{R}} \subseteq \vec{C}$ in Lemma 11.1. Here we conversely prove $\vec{C} \subseteq [\vec{D}]_{\vec{R}}$ or equivalently that if $\alpha \preccurlyeq \beta$ then $\alpha \preccurlyeq_* \beta$. In order to prove it by contradiction, we assume that there exists a sequent $\alpha \to \beta$ such that $\alpha \preccurlyeq \beta$ and $\alpha \not\preccurlyeq_* \beta$. We call such a sequent a **counter sequent**. Also, a sequent $\alpha \to \beta$ is **singular** if $\alpha \not\preccurlyeq_* \beta$.

Lemma 11.11 There exists a sentence counter sequent.

Proof We are assuming by way of contradiction that a counter sequent exists. Let $f_1 \cdots f_m \alpha \to g_1 \cdots g_n \beta$ be a counter sequent and assume that a case k belongs to the ranges of $f_1, \ldots, f_m, g_1, \ldots, g_n$ but does not belong to those of the predicates in $\alpha \cup \beta$. Since we are assuming that $\mathbb{X}_\varepsilon$ is enumerable, Theorem 6.3 shows that there exists a variable $x \in \mathbb{X}_\varepsilon$ which does not occur free in the predicates in $\{f_1, \ldots, f_m\} \cup \{g_1, \ldots, g_n\} \cup \alpha \cup \beta$. Since $\preccurlyeq$ satisfies the gen. case+ law by Theorem 10.2 and $\preccurlyeq_*$ satisfies the gen. case$-$ law by Lemma 11.8, it follows that $x\,\breve{o}k\,f_1, \cdots, x\,\breve{o}k\,f_m, \alpha \to x\,\breve{o}k\,g_1, \cdots, x\,\breve{o}k\,g_n, \beta$ is a counter sequent. Since we are assuming that the ranges of predicates are finite sets, we conclude that there exists a sentence counter sequent.

Lemma 11.12 There exists a sentence counter sequent $\alpha_0 \to \beta_0$ which satisfies the following condition G.

G : If Ωx occurs in a sentence in $\alpha_0 \cup \beta_0$ for some $x \in \mathbb{X}_\varepsilon$, then the x does not occur free in the sentences in $\alpha_0 \cup \beta_0$. Also Ωx_0 in (4.2) occurs in a sentence in $\alpha_0 \cup \beta_0$.

Proof There exists a sentence counter sequent $\alpha \to \beta$ by Lemma 11.11. Since $\alpha \preccurlyeq \beta$ but $\preccurlyeq$ is a non-trivial Boolean relation by Theorems 10.1 and 7.4, either α or β is non-empty. Let $\alpha = f_1 \cdots f_m$, $\beta = g_1 \cdots g_n$, and define

$$h = (f_1 \wedge \cdots \wedge f_m)^{\Diamond} \vee (g_1 \vee \cdots \vee g_n).$$

Then h is a sentence. Since $\preccurlyeq_*$ is also a Boolean relation by the discussions in §11.2, Theorem 7.3 shows that $\to h$ is a counter sequent.

For each $f \in A_\emptyset$ and $x \in \mathbb{X}_\varepsilon$, let $\forall x\, f$ denote the sentence $\mathsf{one}\,\forall\pi(f\,\Omega x)\triangle$ for the time being. If $f \in A_\emptyset$ satisfies $\preccurlyeq f$, then we have $\preccurlyeq \forall x\, f$ for all $x \in \mathbb{X}_\varepsilon$ by Theorem 10.8. Conversely, if $f \in A_\emptyset$ and $x \in \mathbb{X}_\varepsilon$ satisfy $\preccurlyeq_* \forall x\, f$, then since $\forall x\, f \preccurlyeq_* f$ by the $\forall-$ law, we have $\preccurlyeq_* f$ by the cut law. Therefore, if $\to f$ is a sentence counter sequent, so is $\to \forall x\, f$ for all $x \in \mathbb{X}_\varepsilon$. Furthermore, Theorem 6.3 shows that $\mathbb{X}^{\forall x\, f}_{\mathrm{free}} = \mathbb{X}^{f}_{\mathrm{free}} - \{x\}$ for each $f \in A_\emptyset$ and $x \in \mathbb{X}_\varepsilon$.

There exists a non-empty subset $\{x_1, \ldots, x_k\}$ of $\mathbb{X}_\varepsilon$ which contains every element of $\mathbb{X}_\varepsilon$ which occurs free in h. By the above, $\to \forall x_k \cdots \forall x_1\, h$ is a sentence counter sequent, and no element of $\mathbb{X}_\varepsilon$ occurs free in $\forall x_k \cdots \forall x_1\, h$. Therefore, if we let $\to \forall x_k \cdots \forall x_1\, h$ be $\alpha_0 \to \beta_0$, then it obviously satisfies G.

In a long series of lemmas, we will prove $\alpha_0 \not\preccurlyeq \beta_0$, which conclude the proof of $\vec{\mathsf{C}} \subseteq [\vec{\mathsf{D}}]_{\vec{\mathsf{R}}}$ by contradiction.

Lemma 11.13 There exist subsets $\mathbb{X}'_\varepsilon$ and $\mathbb{X}''_\varepsilon$ of $\mathbb{X}_\varepsilon$ which satisfy the following five conditions.

- $\mathbb{X}_\varepsilon = \mathbb{X}'_\varepsilon \amalg \mathbb{X}''_\varepsilon$.
- If $x \in \mathbb{X}'_\varepsilon$, then Ωx does not occur in the sentences in $\alpha_0 \cup \beta_0$.
- Elements of $\mathbb{X}''_\varepsilon$ do not occur free in the sentences in $\alpha_0 \cup \beta_0$.
- $\mathbb{X}'_\varepsilon$ is an enumerable set, while $\mathbb{X}''_\varepsilon$ is a finite set.
- The variable $x_0 \in \mathbb{X}_\varepsilon$ in (4.2) belongs to $\mathbb{X}''_\varepsilon$.

Proof Let $\mathbb{X}''_\varepsilon$ be the set of the variables $x \in \mathbb{X}_\varepsilon$ such that Ωx occurs in a sentence in $\alpha_0 \cup \beta_0$, and define $\mathbb{X}'_\varepsilon = \mathbb{X}_\varepsilon - \mathbb{X}''_\varepsilon$. Then $\mathbb{X}''_\varepsilon$ is a finite set by Theorem 6.3. Since $\alpha_0 \to \beta_0$ satisfies the condition G and we are assuming that $\mathbb{X}_\varepsilon$ is an enumerable set, $\mathbb{X}'_\varepsilon$ and $\mathbb{X}''_\varepsilon$ satisfy the five conditions.

Now we define

$$\mathbb{S}'_\varepsilon = \mathbb{C}_\varepsilon \cup \mathbb{X}'_\varepsilon = \mathbb{S}_\varepsilon - \mathbb{X}''_\varepsilon.$$

Then since $\mathbb{X}''_\varepsilon$ is a finite set by the above lemma and we are assuming that $\mathbb{S}_\varepsilon$ is an enumerable set, it follows that $\mathbb{S}'_\varepsilon$ is also enumerable.

We will say that an element $a \in A$ is **good** if it satisfies the following two conditions.

- If $x \in \mathbb{X}'_\varepsilon$, then Ωx does not occur in a.

- Elements of $\mathbb{X}''_\varepsilon$ do not occur free in a.

Furthermore, we say that a sequent is **good** if it consists of good predicates. Then elements of $\alpha_0 \cup \beta_0$ are good sentences, and $\alpha_0 \to \beta_0$ is a good sentence counter sequent.

From now on, "SS," "GSS," and "GSSS" are abbreviations for "Singular Sequent," "Good Singular Sequent," and "Good Sentence Singular Sequent" respectively. Thus $\alpha_0 \to \beta_0$ is a GSSS.

Lemma 11.14 The following holds.

- Let λ be an n-ary operation in the OS of A (hence $n \leq 2$) other than the nominalizers, and let $(a_1, \ldots, a_n) \in \mathrm{Dom}\,\lambda$. Then $\lambda(a_1, \ldots, a_n)$ is good iff $a_1, \ldots, a_n$ are good.

- Let $a \in A_\varepsilon$, $f \in A_\emptyset$, $x \in \mathbb{X}_\varepsilon$ and assume that $a\,\check{o}\pi(f\,\Omega x)\triangle$ is good. Then $f(x/a)$ is also good and x is free from a in f.

Proof Theorem 6.3 implies that Ωx occurs in $\lambda(a_1, \ldots, a_n)$ iff it occurs in some of $a_1, \ldots, a_n$. It also implies that $x \in \mathbb{X}_\varepsilon$ occurs free in $\lambda(a_1, \ldots, a_n)$ iff it occurs free in some of $a_1, \ldots, a_n$. Therefore the first assertion holds.

The second assertion is also proved by Theorem 6.3. Suppose $a\,\check{o}\pi(f\,\Omega x)\triangle$ is good. Then a and $f\,\Omega x$ are also good by the above. Hence if $y \in \mathbb{X}'_\varepsilon$, then Ωy does not occur in $f\,\Omega x$, and so it does not occur in f either. Since $A_\varepsilon = \mathbb{S}_\varepsilon$, no operations occur in a. Therefore, Ωy does not occur in $f(x/a)$. Next, elements of $\mathbb{X}''_\varepsilon$ do not occur free in a, and elements of $\mathbb{X}''_\varepsilon$ other than x do not occur free in f. Therefore, elements of $\mathbb{X}''_\varepsilon$ do not occur free in $f(x/a)$. Thus, $f(x/a)$ is good. If a variable $y \in \mathbb{X}_\varepsilon$ occurs free in a, then $y \in \mathbb{X}'_\varepsilon$, and so Ωy does not occur in f. Therefore, x is free from a in f.

Lemma 11.15 Let $\alpha \to \beta$ be an SS. Then the following holds.

- $\alpha \cap \beta = \emptyset$.

- If $f \in \alpha$, then $f\alpha \to \beta$ is an SS. If $g \in \beta$, then $\alpha \to g\beta$ is an SS.

- If $\alpha = f_1 \cdots f_l$, $\beta = g_1 \cdots g_m$ and elements $f'_1, \ldots f'_l, g'_1, \ldots, g'_m$ of H satisfy $f_i \preccurlyeq_* f'_i$ $(i = 1, \ldots, l)$ and $g'_j \preccurlyeq_* g_j$ $(j = 1, \ldots, m)$, then $f'_1 \cdots f'_l \to g'_1 \cdots g'_m$ is an SS.

Also the following holds for $n = 1, 2, \ldots$, irrespective of the order of applying the operations $\wedge, \vee$.

- If $f_1 \wedge \cdots \wedge f_n \in \alpha$, then $f_1 \cdots f_n\alpha \to \beta$ is an SS.

- If $f_1 \wedge \cdots \wedge f_n \in \beta$, then $\alpha \to f_i\beta$ is an SS for some $i \in \{1, \ldots, n\}$.

- If $f_1 \vee \cdots \vee f_n \in \alpha$, then $f_i\alpha \to \beta$ is an SS for some $i \in \{1, \ldots, n\}$.

- If $f_1 \vee \cdots \vee f_n \in \beta$, then $\alpha \to f_1 \cdots f_n\beta$ is an SS.

Also the following holds.

- If $f \Rightarrow g \in \alpha$, then either $\alpha \to f\beta$ or $g\alpha \to \beta$ is an SS.
- If $f \Rightarrow g \in \beta$, then $f\alpha \to g\beta$ is an SS.
- If $f^{\Diamond} \in \alpha$, then $\alpha \to f\beta$ is an SS.
- If $f^{\Diamond} \in \beta$, then $f\alpha \to \beta$ is an SS.

Proof If $\alpha \cap \beta \neq \emptyset$, then $\alpha \preccurlyeq_* \beta$ by the repetition law, weakening law, and exchange law. Therefore $\alpha \cap \beta = \emptyset$.

If $f \in \alpha$ and $f\alpha \preccurlyeq_* \beta$, then $\alpha \preccurlyeq_* \beta$ by the exchange law and contraction law. Therefore if $f \in \alpha$, then $f\alpha \to \beta$ is an SS.

As for the third assertion, if $f'_1 \cdots f'_l \preccurlyeq_* g'_1 \cdots g'_m$, then $\alpha \preccurlyeq_* \beta$ by the cut law and exchange law. Therefore $f'_1 \cdots f'_l \to g'_1 \cdots g'_m$ is an SS.

Suppose $f_1 \wedge \cdots \wedge f_n \in \alpha$. Then $f_1 \wedge \cdots \wedge f_n, \alpha \to \beta$ is an SS as shown above. If $f_1 \cdots f_n \alpha \preccurlyeq_* \beta$, then $f_1 \wedge \cdots \wedge f_n, \alpha \preccurlyeq_* \beta$ by the strong conjunction law. Therefore, $f_1 \cdots f_n \alpha \to \beta$ is an SS.

The rest of the proof is similar and omitted.

Lemma 11.16 Let $\alpha \to \beta$ be a GSS and $f \in H$. Also, let $a_1, \ldots, a_n \in A_\varepsilon$, $k_1, \ldots, k_n$ be distinct cases in K_f, and $\rho \in \mathfrak{S}_n$. Then the following holds.

- If $(a_i \breve{o} k_i)_{i=1,\ldots,n} f \in \alpha$, then $(a_{\rho i} \breve{o} k_{\rho i})_{i=1,\ldots,n} f, \alpha \to \beta$ is a GSS.
- If $(a_i \breve{o} k_i)_{i=1,\ldots,n} f \in \beta$, then $\alpha \to (a_{\rho i} \breve{o} k_{\rho i})_{i=1,\ldots,n} f, \beta$ is a GSS.

Proof The sequents in question are good by Lemma 11.14. Since $(a_i \breve{o} k_i)_i f \asymp^* (a_{\rho i} \breve{o} k_{\rho i})_i f$ by the permutation law, the sequents in question are singular by Lemma 11.15.

Lemma 11.17 Let $\alpha \to \beta$ be a GSS and $f, g \in H$. Also, let $a_1, \ldots, a_l \in A_\varepsilon$ and $k_1, \ldots, k_l$ be distinct cases such that $k_1, \ldots, k_n \in K_f - K_g$, $k_{n+1}, \ldots, k_m \in K_f \cap K_g$, and $k_{m+1}, \ldots, k_l \in K_g - K_f$ $(0 \leq n \leq m \leq l)$. Then the following holds.

- If $(a_i \breve{o} k_i)_{i=1,\ldots,l}(f \wedge g) \in \alpha$, then
  $(a_i \breve{o} k_i)_{i=1,\ldots,m} f, (a_i \breve{o} k_i)_{i=n+1,\ldots,l} g, \alpha \to \beta$ is a GSS.
- If $(a_i \breve{o} k_i)_{i=1,\ldots,l}(f \wedge g) \in \beta$, then
  either $\alpha \to (a_i \breve{o} k_i)_{i=1,\ldots,m} f, \beta$ or $\alpha \to (a_i \breve{o} k_i)_{i=n+1,\ldots,l} g, \beta$ is a GSS.
- If $(a_i \breve{o} k_i)_{i=1,\ldots,l}(f \vee g) \in \alpha$, then
  either $(a_i \breve{o} k_i)_{i=1,\ldots,m} f, \alpha \to \beta$ or $(a_i \breve{o} k_i)_{i=n+1,\ldots,l} g, \alpha \to \beta$ is a GSS.
- If $(a_i \breve{o} k_i)_{i=1,\ldots,l}(f \vee g) \in \beta$, then
  $\alpha \to (a_i \breve{o} k_i)_{i=1,\ldots,m} f, (a_i \breve{o} k_i)_{i=n+1,\ldots,l} g, \beta$ is a GSS.

- If $(a_i \,\breve{o} k_i)_{i=1,\ldots,l}(f \Rightarrow g) \in \alpha$, then
  either $\alpha \to (a_i \,\breve{o} k_i)_{i=1,\ldots,m} f, \beta$ or $(a_i \,\breve{o} k_i)_{i=n+1,\ldots,l} g, \alpha \to \beta$ is a GSS.

- If $(a_i \,\breve{o} k_i)_{i=1,\ldots,l}(f \Rightarrow g) \in \beta$, then
  $(a_i \,\breve{o} k_i)_{i=1,\ldots,m} f, \alpha \to (a_i \,\breve{o} k_i)_{i=n+1,\ldots,l} g, \beta$ is a GSS.

Proof The sequents in question are good by Lemma 11.14, and singular by the $\bigwedge$ law, $\bigvee$ law, $\Rightarrow$ law, and Lemma 11.15.

Lemma 11.18 Let $\alpha \to \beta$ a GSS and $f \in H$. Also, let $a_1, \ldots, a_n \in A_\varepsilon$ and $k_1, \ldots, k_n$ be distinct cases in K_f. Then the following holds.

- If $(a_i \,\breve{o} k_i)_{i=1,\ldots,n}(f^\Diamond) \in \alpha$, then $\alpha \to (a_i \,\breve{o} k_i)_{i=1,\ldots,n} f, \beta$ is a GSS.

- If $(a_i \,\breve{o} k_i)_{i=1,\ldots,n}(f^\Diamond) \in \beta$, then $(a_i \,\breve{o} k_i)_{i=1,\ldots,n} f, \alpha \to \beta$ is a GSS.

Proof The sequents in question are good by Lemma 11.14 and singular by the $\Diamond$ law and Lemma 11.15.

Lemma 11.19 Let $\alpha \to \beta$ be a GSS, $a \in G$, $f \in H$, $k \in K_f$, and $\mathfrak{p} \in \mathfrak{P}$. Also, let $a_1, \ldots, a_n \in A_\varepsilon$ and $k_1, \ldots, k_n$ be distinct cases of $K_f - \{k\}$. Then the following holds.

- If $(a_i \,\breve{o} k_i)_{i=1,\ldots,n}(a \neg\mathfrak{p} k\, f) \in \alpha$, then
  $(a_i \,\breve{o} k_i)_{i=1,\ldots,n}(a\, \mathfrak{p} k\, f^\Diamond), \alpha \to \beta$ is a GSS.

- If $(a_i \,\breve{o} k_i)_{i=1,\ldots,n}(a \neg\mathfrak{p} k\, f) \in \beta$, then
  $\alpha \to (a_i \,\breve{o} k_i)_{i=1,\ldots,n}(a\, \mathfrak{p} k\, f^\Diamond), \beta$ is a GSS.

Proof The sequents in question are good by Lemma 11.14, and singular by Lemma 11.15, because $(a_i \,\breve{o} k_i)_i(a \neg\mathfrak{p} k\, f) \asymp_* (a_i \,\breve{o} k_i)_i(a\, \mathfrak{p} k\, f^\Diamond)$ by the $\neg$ law and the gen. case+ law.

Lemma 11.20 Let $\alpha \to \beta$ be a GSS, $a \in G$, $f \in H$, $k \in K_f$, and $\mathfrak{p}_1, \ldots, \mathfrak{p}_m \in \mathfrak{P}$. Also, let $a_1, \ldots, a_n \in A_\varepsilon$ and $k_1, \ldots, k_n$ be distinct cases in $K_f - \{k\}$. Then the following holds.

- If $(a_i \,\breve{o} k_i)_{i=1,\ldots,n}(a\, (\mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_m) k\, f) \in \alpha$, then
  $(a_i \,\breve{o} k_i)_{i=1,\ldots,n}(a\, \mathfrak{p}_j k\, f), \alpha \to \beta$ is a GSS for some $j \in \{1, \ldots, m\}$.

- If $(a_i \,\breve{o} k_i)_{i=1,\ldots,n}(a\, (\mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_m) k\, f) \in \beta$, then
  $\alpha \to (a_i \,\breve{o} k_i)_{i=1,\ldots,n}(a\, \mathfrak{p}_1 k\, f), \cdots, (a_i \,\breve{o} k_i)_{i=1,\ldots,n}(a\, \mathfrak{p}_m k\, f), \beta$ is a GSS.

Proof The sequents in question are good by Lemma 11.14 and singular by the gen. $\cup$ law, gen. case+ law, gen. $\bigvee$ law, and Lemma 11.15.

Lemma 11.21 Let $\alpha \to \beta$ be a GSS, $a \in G$, $f \in H$, $k \in K_f$, and $\mathfrak{p}, \mathfrak{q} \in \mathfrak{P}$. Also, let $a_1, \ldots, a_n \in A_\varepsilon$ and $k_1, \ldots, k_n$ be distinct cases of $K_f - \{k\}$. Then the following holds.

- If $(\mathsf{a_i}\,\breve{o}\mathsf{k_i})_{i=1,\ldots,n}(\mathsf{a}\,(\mathfrak{p}\cap\mathsf{q})\mathsf{k}\,\mathsf{f})\in\alpha$, then $(\mathsf{a_i}\,\breve{o}\mathsf{k_i})_{i=1,\ldots,n}(\mathsf{a}\,\mathfrak{p}\mathsf{k}\,\mathsf{f}),\ (\mathsf{a_i}\,\breve{o}\mathsf{k_i})_{i=1,\ldots,n}(\mathsf{a}\,\mathsf{q}\mathsf{k}\,\mathsf{f}),\alpha\to\beta$ is a GSS.
- If $(\mathsf{a_i}\,\breve{o}\mathsf{k_i})_{i=1,\ldots,n}(\mathsf{a}\,(\mathfrak{p}\cap\mathsf{q})\mathsf{k}\,\mathsf{f})\in\beta$, then either $\alpha\to(\mathsf{a_i}\,\breve{o}\mathsf{k_i})_{i=1,\ldots,n}(\mathsf{a}\,\mathfrak{p}\mathsf{k}\,\mathsf{f}),\beta$ or $\alpha\to(\mathsf{a_i}\,\breve{o}\mathsf{k_i})_{i=1,\ldots,n}(\mathsf{a}\,\mathsf{q}\mathsf{k}\,\mathsf{f}),\beta$ is a GSS.

Proof The sequents in question are good by Lemma 11.14 and singular by the $\cap$ law, gen. case+ law, $\wedge$ law, and Lemma 11.15.

Lemma 11.22 Let $\alpha\to\beta$ be a GSS, $\mathsf{a}\in\mathsf{G}$, $\mathsf{f}\in\mathsf{H}$, $\mathsf{k}\in\mathsf{K_f}$, and $\mathfrak{p}\in\mathfrak{P}$. Also, let $\mathsf{a}_1,\ldots,\mathsf{a}_n\in A_\varepsilon$ and $\mathsf{k}_1,\ldots,\mathsf{k}_n$ be distinct cases in $\mathsf{K_f}-\{\mathsf{k}\}$. Then the following holds.

- If $(\mathsf{a_i}\,\breve{o}\mathsf{k_i})_{i=1,\ldots,n}(\mathsf{a}\,\mathfrak{p}^\circ\mathsf{k}\,\mathsf{f})\in\alpha$, then $\alpha\to(\mathsf{a_i}\,\breve{o}\mathsf{k_i})_{i=1,\ldots,n}(\mathsf{a}\,\mathfrak{p}\mathsf{k}\,\mathsf{f}),\beta$ is a GSS.
- If $(\mathsf{a_i}\,\breve{o}\mathsf{k_i})_{i=1,\ldots,n}(\mathsf{a}\,\mathfrak{p}^\circ\mathsf{k}\,\mathsf{f})\in\beta$, then $(\mathsf{a_i}\,\breve{o}\mathsf{k_i})_{i=1,\ldots,n}(\mathsf{a}\,\mathfrak{p}\mathsf{k}\,\mathsf{f}),\alpha\to\beta$ is a GSS.

Proof The sequents in question are good by Lemma 11.14 and singular by the $\circ$ law, gen. case+ law, $\Diamond$ law, and Lemma 11.15.

Lemma 11.23 Let $\alpha\to\beta$ be a GSS, $\mathsf{a}\in\mathsf{G}$, $\mathsf{f}\in\mathsf{H}$, $\mathsf{k}\in\mathsf{K_f}$, and $\mathsf{p}\in\mathbb{P}$. Also, let $\mathsf{a}_1,\ldots,\mathsf{a}_n\in A_\varepsilon$ and $\mathsf{k}_1,\ldots,\mathsf{k}_n$ be the set of distinct cases in $\mathsf{K_f}-\{\mathsf{k}\}$. Then the following holds.

- If $(\mathsf{a_i}\,\breve{o}\mathsf{k_i})_{i=1,\ldots,n}(\mathsf{a}\,\overline{\mathsf{p}}\mathsf{k}\,\mathsf{f})\in\alpha$, then $\big(\mathsf{a}\sqcap\big(\mathsf{x}\,\breve{o}\mathsf{k}\,(\mathsf{a_i}\,\breve{o}\mathsf{k_i})_{i=1,\ldots,n}\mathsf{f}\big)\,\Omega\mathsf{x}\big)\,\overline{\mathsf{p}}\pi\,\mathsf{one}\triangle,\alpha\to\beta$ is a GSS for all $\mathsf{x}\in\mathbb{X}''_\varepsilon$.
- If $(\mathsf{a_i}\,\breve{o}\mathsf{k_i})_{i=1,\ldots,n}(\mathsf{a}\,\overline{\mathsf{p}}\mathsf{k}\,\mathsf{f})\in\beta$, then $\alpha\to\big(\mathsf{a}\sqcap\big(\mathsf{x}\,\breve{o}\mathsf{k}\,(\mathsf{a_i}\,\breve{o}\mathsf{k_i})_{i=1,\ldots,n}\mathsf{f}\big)\,\Omega\mathsf{x}\big)\,\overline{\mathsf{p}}\pi\,\mathsf{one}\triangle,\beta$ is a GSS for all $\mathsf{x}\in\mathbb{X}''_\varepsilon$.

Proof That the sequents in question are good is derived from Lemma 11.14 and Theorem 6.3 as follows. First, $\mathsf{a}_1,\ldots,\mathsf{a}_n,\mathsf{a},\mathsf{f}$ are all good, and therefore so is $(\mathsf{a_i}\,\breve{o}\mathsf{k_i})_{i=1,\ldots,n}\mathsf{f}$. Although $\mathsf{x}\in\mathbb{X}''_\varepsilon$ occurs free in $\mathsf{x}\,\breve{o}\mathsf{k}\,(\mathsf{a_i}\,\breve{o}\mathsf{k_i})_{i=1,\ldots,n}\mathsf{f}$, $\big(\mathsf{x}\,\breve{o}\mathsf{k}\,(\mathsf{a_i}\,\breve{o}\mathsf{k_i})_{i=1,\ldots,n}\mathsf{f}\big)\,\Omega\mathsf{x}$ is good. Also $\mathsf{one}=(\mathsf{x}_0\,\breve{o}\pi\,\mathsf{x}_0\triangle)\,\Omega\mathsf{x}_0$ is good. Thus $\big(\mathsf{a}\sqcap\big(\mathsf{x}\,\breve{o}\mathsf{k}\,(\mathsf{a_i}\,\breve{o}\mathsf{k_i})_{i=1,\ldots,n}\mathsf{f}\big)\,\Omega\mathsf{x}\big)\,\overline{\mathsf{p}}\pi\,\mathsf{one}\triangle$ is good.

In order to show that the sequents in question are singular, define $\mathsf{g}=(\mathsf{a_i}\,\breve{o}\mathsf{k_i})_{i=1,\ldots,n}\mathsf{f}$. Then, as in the proof of the permutation law, it follows from the $\mathfrak{Q},\breve{o}$ law and the gen. case+ law that

$$(\mathsf{a_i}\,\breve{o}\mathsf{k_i})_{i=1,\ldots,n}(\mathsf{a}\,\overline{\mathsf{p}}\mathsf{k}\,\mathsf{f})\asymp_*\mathsf{a}\,\overline{\mathsf{p}}\mathsf{k}\,\mathsf{g}$$

holds. Also, since $\mathsf{K_g}=\{\mathsf{k}\}$ and x does not occur free in g, we have

$$\mathsf{a}\,\overline{\mathsf{p}}\mathsf{k}\,\mathsf{g}\asymp_*\mathsf{a}\,\overline{\mathsf{p}}\pi\big((\mathsf{x}\,\breve{o}\mathsf{k}\,\mathsf{g})\,\Omega\mathsf{x}\big)\triangle$$

by the $\mathfrak{P}$ law. Also we have

$$\mathsf{a}\,\overline{\mathsf{p}}\pi\,\big((\mathsf{x}\,\breve{o}\mathsf{k}\,\mathsf{g})\,\Omega\mathsf{x}\big)\triangle \asymp_* \big(\mathsf{a}\sqcap(\mathsf{x}\,\breve{o}\mathsf{k}\,\mathsf{g})\,\Omega\mathsf{x}\big)\,\overline{\mathsf{p}}\pi\,\mathsf{one}\triangle$$

by the $\triangle$ law. Combining the above three $\asymp_*$ equations, we have

$$(\mathsf{a}_i\,\breve{o}\mathsf{k}_i)_{i=1,\dots,n}(\mathsf{a}\,\overline{\mathsf{p}}\mathsf{k}\,\mathsf{f}) \asymp_* \big(\mathsf{a}\sqcap\big(\mathsf{x}\,\breve{o}\mathsf{k}\,(\mathsf{a}_i\,\breve{o}\mathsf{k}_i)_{i=1,\dots,n}\mathsf{f}\big)\,\Omega\mathsf{x}\big)\,\overline{\mathsf{p}}\pi\,\mathsf{one}\triangle.$$

Thus, the sequents in question are singular by Lemma 11.15.

Lemma 11.24 Let $\alpha \to \beta$ be a GSSS, $\mathsf{a}, \mathsf{b}_1, \dots, \mathsf{b}_n \in \mathsf{G}$, and $\mathsf{p}, \mathsf{q}_1, \dots, \mathsf{q}_n \in \mathbb{P}$. Assume $\mathsf{a}\,\overline{\mathsf{p}}\pi\,\mathsf{one}\triangle \in \alpha$, $\mathsf{b}_1\,\overline{\mathsf{q}_1}\pi\,\mathsf{one}\triangle, \dots, \mathsf{b}_n\,\overline{\mathsf{q}_n}\pi\,\mathsf{one}\triangle \in \beta$, and $\mathsf{p} \geq \sum_{i=1}^{n} \mathsf{q}_i$, where if $n = 0$ then $\sum_{i=1}^{n} \mathsf{q}_i = 0$ by definition. Then $\mathsf{x}\,\breve{o}\pi\,\mathsf{a}\triangle, \alpha \to \mathsf{x}\,\breve{o}\pi\,\mathsf{b}_1\triangle, \cdots, \mathsf{x}\,\breve{o}\pi\,\mathsf{b}_n\triangle, \beta$ is a GSSS for all $\mathsf{x} \in \mathbb{X}'_\varepsilon$ which do not occur free in the sentences in $\alpha \cup \beta$.

Proof The sequent in question is good by Lemma 11.14 and singular by the pigeonhole principle and Lemma 11.15 because x does not occur free in $\mathsf{a}, \mathsf{b}_1, \dots, \mathsf{b}_n$ by Theorem 6.3.

Lemma 11.25 Let $\alpha \to \beta$ be a GSS and $\mathsf{a} \in \mathsf{G}$. Assume $\mathsf{a}\,\exists\pi\,\mathsf{one}\triangle \in \beta$. Then $\alpha \to \mathsf{b}\,\breve{o}\pi\,\mathsf{a}\triangle, \beta$ is a GSS for all $\mathsf{b} \in \mathbb{S}'_\varepsilon$.

Proof The sequent in question is good by Lemma 11.14 and singular by the $\exists$ law and Lemma 11.15.

Lemma 11.26 Let $\alpha \to \beta$ be a GSS, $\mathsf{a}, \mathsf{b} \in \mathsf{G}$, and $\mathsf{c} \in \mathsf{A}_\varepsilon$. Then the following holds.

- If $\mathsf{c}\,\breve{o}\pi\,(\mathsf{a}\sqcap\mathsf{b})\triangle \in \alpha$, then $\mathsf{c}\,\breve{o}\pi\,\mathsf{a}\triangle$, $\mathsf{c}\,\breve{o}\pi\,\mathsf{b}\triangle, \alpha \to \beta$ is a GSS.
- If $\mathsf{c}\,\breve{o}\pi\,(\mathsf{a}\sqcap\mathsf{b})\triangle \in \beta$, then either $\alpha \to \mathsf{c}\,\breve{o}\pi\,\mathsf{a}\triangle, \beta$ or $\alpha \to \mathsf{c}\,\breve{o}\pi\,\mathsf{b}\triangle, \beta$ is a GSS.
- If $\mathsf{c}\,\breve{o}\pi\,(\mathsf{a}\sqcup\mathsf{b})\triangle \in \alpha$, then either $\mathsf{c}\,\breve{o}\pi\,\mathsf{a}\triangle, \alpha \to \beta$ or $\mathsf{c}\,\breve{o}\pi\,\mathsf{b}\triangle, \alpha \to \beta$ is a GSS.
- If $\mathsf{c}\,\breve{o}\pi\,(\mathsf{a}\sqcup\mathsf{b})\triangle \in \beta$, then $\alpha \to \mathsf{c}\,\breve{o}\pi\,\mathsf{a}\triangle$, $\mathsf{c}\,\breve{o}\pi\,\mathsf{b}\triangle, \beta$ is a GSS.

Proof The sequents in question are good by Lemma 11.14 and singular by the $\sqcap$ law, $\sqcup$ law, gen. case+ law, $\wedge$ law, $\vee$ law, and Lemma 11.15.

Lemma 11.27 Let $\alpha \to \beta$ be a GSS $\mathsf{a} \in \mathsf{G}$, and $\mathsf{b} \in \mathsf{A}_\varepsilon$. Then the following holds.

- If $\mathsf{b}\,\breve{o}\pi\,(\mathsf{a}^\square)\triangle \in \alpha$, then $\alpha \to \mathsf{b}\,\breve{o}\pi\,\mathsf{a}\triangle, \beta$ is a GSS.
- If $\mathsf{b}\,\breve{o}\pi\,(\mathsf{a}^\square)\triangle \in \beta$, then $\mathsf{b}\,\breve{o}\pi\,\mathsf{a}\triangle, \alpha \to \beta$ is a GSS.

Proof The sequents in question are good by Lemma 11.14 and singular by the $\Box$ law, gen. case+ law, $\Diamond$ law, and Lemma 11.15.

Lemma 11.28 Let $\alpha \to \beta$ be a GSS, $a \in A_\varepsilon$, $f \in A_\emptyset$, and $x \in \mathbb{X}_\varepsilon$. Then the following holds.

- If $a\,\breve{o}\pi(f\,\Omega x)\triangle \in \alpha$, then $f(x/a), \alpha \to \beta$ is a GSS.
- If $a\,\breve{o}\pi(f\,\Omega x)\triangle \in \beta$, then $\alpha \to f(x/a), \beta$ is a GSS.

Proof The sequents in question are good and x is free from a in f by Lemma 11.14. Therefore, they are singular by the Ω law and Lemma 11.15.

Let $\mathfrak{p} \in \mathfrak{P}$. Then $\mathfrak{p}$ is the direct sum of its connected components $\mathfrak{p}_1, \dots, \mathfrak{p}_n$, which are intervals of $\mathbb{P}$ in one of the shapes $(p, q]$, $(\leftarrow p]$, and $(p \to)$ or equal to $\mathbb{P}$. We call $\neg\mathfrak{p}_1, \dots, \neg\mathfrak{p}_n$ the connected components of $\neg\mathfrak{p} \in \neg\mathfrak{P}$, and call the end(s) of $\mathfrak{p}_i$ the end(s) of $\neg\mathfrak{p}_i$ also. We say that an element $p \in \mathbb{P}$ occurs in an element $f \in H$, if p is equal to an end of a connected component of $\lambda \in \mathfrak{Q}$ such that the operation λk occurs in f for some $k \in K$. Furthermore, we say that p occurs in a subset X of H, if p occurs in a predicate $f \in X$.

Lemma 11.29 There exists a series $(\alpha_n \to \beta_n)_{n=1,2,\dots}$ of GSSS's which satisfies the following thirty three conditions, where "$n \equiv i$" is an abbreviation for "$n \equiv i \mod 32$" for $i \in \{1, \dots, 32\}$. For the condition 24, we let $\mathbb{S}'_\varepsilon = \{a_1, a_2, \dots\}$, because $\mathbb{S}'_\varepsilon$ is an enumerable set.

(0) $\alpha_{n-1} \subseteq \alpha_n$ and $\beta_{n-1} \subseteq \beta_n$ hold, and if an element of $\mathbb{P} - \{0\}$ occurs in $\alpha_n \cup \beta_n$, then it also occurs in $\alpha_{n-1} \cup \beta_{n-1}$ $(n = 1, 2, \dots)$.

(1) If $n \equiv 1$ and $(a_i\,\breve{o}k_i)_{i=1,\dots,l}f \in \alpha_{n-1}$, then $(a_{\rho i}\,\breve{o}k_{\rho i})_{i=1,\dots,l}f \in \alpha_n$ for all $\rho \in \mathfrak{S}_l$.

(2) If $n \equiv 2$ and $(a_i\,\breve{o}k_i)_{i=1,\dots,l}f \in \beta_{n-1}$, then $(a_{\rho i}\,\breve{o}k_{\rho i})_{i=1,\dots,l}f \in \beta_n$ for all $\rho \in \mathfrak{S}_l$.

(3) If $n \equiv 3$, $(a_i\,\breve{o}k_i)_{i=1,\dots,l}(f \wedge g) \in \alpha_{n-1}$, and the range condition

$$\begin{aligned} K_f - K_g &= \{k_1, \dots, k_\nu\}, \\ K_f \cap K_g &= \{k_{\nu+1}, \dots, k_m\}, \\ K_g - K_f &= \{k_{m+1}, \dots, k_l\} \end{aligned} \tag{11.1}$$

is satisfied, then $(a_i\,\breve{o}k_i)_{i=1,\dots,m}f$, $(a_i\,\breve{o}k_i)_{i=\nu+1,\dots,l}g \in \alpha_n$.

(4) If $n \equiv 4$, $(a_i\,\breve{o}k_i)_{i=1,\dots,l}(f \wedge g) \in \beta_{n-1}$, and (11.1) is satisfied, then $(a_i\,\breve{o}k_i)_{i=1,\dots,m}f \in \beta_n$ or $(a_i\,\breve{o}k_i)_{i=\nu+1,\dots,l}g \in \beta_n$.

(5) If $n \equiv 5$, $(a_i\,\breve{o}k_i)_{i=1,\dots,l}(f \vee g) \in \alpha_{n-1}$, and (11.1) is satisfied, then $(a_i\,\breve{o}k_i)_{i=1,\dots,m}f \in \alpha_n$ or $(a_i\,\breve{o}k_i)_{i=\nu+1,\dots,l}g \in \alpha_n$.

(6) If $n \equiv 6$, $(a_i\,\breve{o}k_i)_{i=1,\dots,l}(f \vee g) \in \beta_{n-1}$, and (11.1) is satisfied, then $(a_i\,\breve{o}k_i)_{i=1,\dots,m}f,\ (a_i\,\breve{o}k_i)_{i=\nu+1,\dots,l}g \in \beta_n$.

(7) If $n \equiv 7$, $(a_i\,\breve{o}k_i)_{i=1,\dots,l}(f \Rightarrow g) \in \alpha_{n-1}$, and (11.1) is satisfied, then $(a_i\,\breve{o}k_i)_{i=1,\dots,m}f \in \beta_n$ or $(a_i\,\breve{o}k_i)_{i=\nu+1,\dots,l}g \in \alpha_n$.

(8) If $n \equiv 8$, $(a_i\,\breve{o}k_i)_{i=1,\dots,l}(f \Rightarrow g) \in \beta_{n-1}$, and (11.1) is satisfied, then $(a_i\,\breve{o}k_i)_{i=1,\dots,m}f \in \alpha_n,\ (a_i\,\breve{o}k_i)_{i=\nu+1,\dots,l}g \in \beta_n$.

(9) If $n \equiv 9$ and $(a_i\,\breve{o}k_i)_{i=1,\dots,l}(f^{\Diamond}) \in \alpha_{n-1}$, then $(a_i\,\breve{o}k_i)_{i=1,\dots,l}f \in \beta_n$.

(10) If $n \equiv 10$ and $(a_i\,\breve{o}k_i)_{i=1,\dots,l}(f^{\Diamond}) \in \beta_{n-1}$, then $(a_i\,\breve{o}k_i)_{i=1,\dots,l}f \in \alpha_n$.

(11) If $n \equiv 11$ and $(a_i\,\breve{o}k_i)_{i=1,\dots,l}(a\,\neg\mathfrak{p}k\,f) \in \alpha_{n-1}$ with $\mathfrak{p} \in \mathfrak{P}$, then $(a_i\,\breve{o}k_i)_{i=1,\dots,l}(a\,\mathfrak{p}k\,f^{\Diamond}) \in \alpha_n$.

(12) If $n \equiv 12$ and $(a_i\,\breve{o}k_i)_{i=1,\dots,l}(a\,\neg\mathfrak{p}k\,f) \in \beta_{n-1}$ with $\mathfrak{p} \in \mathfrak{P}$, then $(a_i\,\breve{o}k_i)_{i=1,\dots,l}(a\,\mathfrak{p}k\,f^{\Diamond}) \in \beta_n$.

(13) If $n \equiv 13$ and $(a_i\,\breve{o}k_i)_{i=1,\dots,l}(a\,\mathfrak{p}k\,f) \in \alpha_{n-1}$ with $\mathfrak{p} \in \mathfrak{P}$ having the connected components $\mathfrak{p}_1, \dots, \mathfrak{p}_m$ $(m \geq 2)$, then $(a_i\,\breve{o}k_i)_{i=1,\dots,l}(a\,\mathfrak{p}_j k\,f) \in \alpha_n$ for some $j \in \{1, \dots, m\}$.

(14) If $n \equiv 14$ and $(a_i\,\breve{o}k_i)_{i=1,\dots,l}(a\,\mathfrak{p}k\,f) \in \beta_{n-1}$ with $\mathfrak{p} \in \mathfrak{P}$ having the connected components $\mathfrak{p}_1, \dots, \mathfrak{p}_m$ $(m \geq 2)$, then $(a_i\,\breve{o}k_i)_{i=1,\dots,l}(a\,\mathfrak{p}_j k\,f) \in \beta_n$ for all $j \in \{1, \dots, m\}$.

(15) If $n \equiv 15$ and $(a_i\,\breve{o}k_i)_{i=1,\dots,l}(a\,\mathbb{P}k\,f) \in \alpha_{n-1}$, then $(a_i\,\breve{o}k_i)_{i=1,\dots,l}(a\,\overline{0}k\,f) \in \alpha_n$ or $(a_i\,\breve{o}k_i)_{i=1,\dots,l}(a\,(\leftarrow 0]k\,f) \in \alpha_n$.

(16) If $n \equiv 16$ and $(a_i\,\breve{o}k_i)_{i=1,\dots,l}(a\,\mathbb{P}k\,f) \in \beta_{n-1}$, then $(a_i\,\breve{o}k_i)_{i=1,\dots,l}(a\,\overline{0}k\,f),\ (a_i\,\breve{o}k_i)_{i=1,\dots,l}(a\,(\leftarrow 0]k\,f) \in \beta_n$.

(17) If $n \equiv 17$ and $(a_i\,\breve{o}k_i)_{i=1,\dots,l}(a\,(p,q]k\,f) \in \alpha_{n-1}$, then $(a_i\,\breve{o}k_i)_{i=1,\dots,l}(a\,\overline{p}k\,f),\ (a_i\,\breve{o}k_i)_{i=1,\dots,l}(a\,(\leftarrow q]k\,f) \in \alpha_n$.

(18) If $n \equiv 18$ and $(a_i\,\breve{o}k_i)_{i=1,\dots,l}(a\,(p,q]k\,f) \in \beta_{n-1}$, then $(a_i\,\breve{o}k_i)_{i=1,\dots,l}(a\,\overline{p}k\,f) \in \beta_n$ or $(a_i\,\breve{o}k_i)_{i=1,\dots,l}(a\,(\leftarrow q]k\,f) \in \beta_n$.

(19) If $n \equiv 19$ and $(a_i\,\breve{o}k_i)_{i=1,\dots,l}(a\,(\leftarrow p]k\,f) \in \alpha_{n-1}$, then $(a_i\,\breve{o}k_i)_{i=1,\dots,l}(a\,\overline{p}k\,f) \in \beta_n$.

(20) If $n \equiv 20$ and $(a_i\,\breve{o}k_i)_{i=1,\dots,l}(a\,(\leftarrow p]k\,f) \in \beta_{n-1}$, then $(a_i\,\breve{o}k_i)_{i=1,\dots,l}(a\,\overline{p}k\,f) \in \alpha_n$.

(21) If $n \equiv 21$, $(a_i\,\breve{o}k_i)_{i=1,\dots,l}(a\,\overline{p}k\,f) \in \alpha_{n-1}$, and $x \in \mathbb{X}''_{\varepsilon}$, then $\big(a \sqcap \big(x\,\breve{o}k\,(a_i\,\breve{o}k_i)_{i=1,\dots,l}f\big)\,\Omega x\big)\,\overline{p}\pi one\triangle \in \alpha_n$.

(22) If $n \equiv 22$, $(a_i\,\breve{o}k_i)_{i=1,\dots,l}(a\,\overline{p}k\,f) \in \beta_{n-1}$, and $x \in \mathbb{X}''_{\varepsilon}$, then $\big(a \sqcap \big(x\,\breve{o}k\,(a_i\,\breve{o}k_i)_{i=1,\dots,l}f\big)\,\Omega x\big)\,\overline{p}\pi one\triangle \in \beta_n$.

(23) If $n \equiv 23$, $a\,\overline{p}\pi\,\mathit{one}\triangle \in \alpha_{n-1}$, $b_1\,\overline{q_1}\pi\,\mathit{one}\triangle, \ldots, b_m\,\overline{q_m}\pi\,\mathit{one}\triangle \in \beta_{n-1}$, and $p \geq \sum_{i=1}^{m} q_i$, then $x\,\breve{o}\pi\,a\triangle \in \alpha_n$ and $x\,\breve{o}\pi\,b_1\triangle, \ldots, x\,\breve{o}\pi\,b_m\triangle \in \beta_n$ for some $x \in \mathbb{X}'_\varepsilon$.

(24) If $n \equiv 24$ and $a\,\exists\pi\,\mathit{one}\triangle \in \beta_{n-1}$, then $a_i\,\breve{o}\pi\,a\triangle \in \beta_n$ for all $i \in \{1, \ldots, n\}$, where $\mathbb{S}'_\varepsilon = \{a_1, a_2, \ldots\}$.

(25) If $n \equiv 25$ and $c\,\breve{o}\pi\,(a \sqcap b)\triangle \in \alpha_{n-1}$, then $c\,\breve{o}\pi\,a\triangle, c\,\breve{o}\pi\,b\triangle \in \alpha_n$.

(26) If $n \equiv 26$ and $c\,\breve{o}\pi\,(a \sqcap b)\triangle \in \beta_{n-1}$, then $c\,\breve{o}\pi\,a\triangle \in \beta_n$ or $c\,\breve{o}\pi\,b\triangle \in \beta_n$.

(27) If $n \equiv 27$ and $c\,\breve{o}\pi\,(a \sqcup b)\triangle \in \alpha_{n-1}$, then $c\,\breve{o}\pi\,a\triangle \in \alpha_n$ or $c\,\breve{o}\pi\,b\triangle \in \alpha_n$.

(28) If $n \equiv 28$ and $c\,\breve{o}\pi\,(a \sqcup b)\triangle \in \beta_{n-1}$, then $c\,\breve{o}\pi\,a\triangle, c\,\breve{o}\pi\,b\triangle \in \beta_n$

(29) If $n \equiv 29$ and $b\,\breve{o}\pi\,(a^{\square})\triangle \in \alpha_{n-1}$, then $b\,\breve{o}\pi\,a\triangle \in \beta_n$.

(30) If $n \equiv 30$ and $b\,\breve{o}\pi\,(a^{\square})\triangle \in \beta_{n-1}$, then $b\,\breve{o}\pi\,a\triangle \in \alpha_n$.

(31) If $n \equiv 31$ and $a\,\breve{o}\pi\,(f\,\Omega x)\triangle \in \alpha_{n-1}$, then $f(x/a) \in \alpha_n$.

(32) If $n \equiv 32$ and $a\,\breve{o}\pi\,(f\,\Omega x)\triangle \in \beta_{n-1}$, then $f(x/a) \in \beta_n$.

Proof We inductively define GSSS's $\alpha_n \to \beta_n$ ($n = 1, 2, \ldots$) starting from the GSSS $\alpha_0 \to \beta_0$. Suppose $n \geq 1$ and the GSSS $\alpha_{n-1} \to \beta_{n-1}$ has been defined. Then we enlarge $\alpha_{n-1} \to \beta_{n-1}$ to a GSSS $\alpha_n \to \beta_n$ by extending α_{n-1} or β_{n-1} or both with good sentences.

If $n \equiv 1$, extend α_{n-1} with all the sentences $(a_i\,\breve{o}k_i)_{i=1,\ldots,l}f$ such that $(a_{\rho i}\,\breve{o}k_{\rho i})_{i=1,\ldots,l}f \in \alpha_{n-1}$ for some $\rho \in \mathfrak{S}_l$. Then the resulting sequent $\alpha_n \to \beta_n$ satisfies the conditions 0, 1 and is a GSSS by Lemma 11.16.

If $n \equiv 2$, extend β_{n-1} with all the sentences $(a_i\,\breve{o}k_i)_{i=1,\ldots,l}f$ such that $(a_{\rho i}\,\breve{o}k_{\rho i})_{i=1,\ldots,l}f \in \beta_{n-1}$ for some $\rho \in \mathfrak{S}_l$. Then the resulting sequent $\alpha_n \to \beta_n$ satisfies the conditions 0, 2 and is a GSSS by Lemma 11.16.

If $n \equiv 3$, extend α_{n-1} with the paired sentences $(a_i\,\breve{o}k_i)_{i=1,\ldots,m}f$ and $(a_i\,\breve{o}k_i)_{i=\nu+1,\ldots,l}g$ made of all the sentences $(a_i\,\breve{o}k_i)_{i=1,\ldots,l}(f \wedge g) \in \alpha_{n-1}$ satisfying (11.1). Then the resulting sequent $\alpha_n \to \beta_n$ satisfies the conditions 0, 3 and is a GSSS by Lemma 11.17.

If a sentence $(a_i\,\breve{o}k_i)_{i=1,\ldots,l}(f \wedge g)$ satisfies (11.1), we call it a $\wedge$-sentence, and call sentences $(a_i\,\breve{o}k_i)_{i=1,\ldots,m}f$ and $(a_i\,\breve{o}k_i)_{i=\nu+1,\ldots,l}g$ its constituents.

If $n \equiv 4$, let $\{h_1, \ldots, h_k\}$ be the set of the $\wedge$-sentences which belong to β_{n-1}, and inductively make GSSS's $\alpha_{n,i} \to \beta_{n,i}$ ($i = 0, 1, \ldots, k$) as follows, and let $\alpha_n \to \beta_n = \alpha_{n,k} \to \beta_{n,k}$. First, let $\alpha_{n,0} \to \beta_{n,0} = \alpha_{n-1} \to \beta_{n-1}$. Next for $i \geq 1$, there is a constituent h'_i of h_i such that $\alpha_{n,i-1} \to h'_i\beta_{n,i-1}$ is a GSSS by Lemma 11.17, so let it be $\alpha_{n,i} \to \beta_{n,i}$. Then $\alpha_n \to \beta_n$ is a GSSS and satisfies the conditions 0, 4.

If $n \equiv 6$, extend β_{n-1} with the paired sentences $(a_i\,\breve{o}k_i)_{i=1,\ldots,m}f$ and $(a_i\,\breve{o}k_i)_{i=\nu+1,\ldots,l}g$ made of all the sentences $(a_i\,\breve{o}k_i)_{i=1,\ldots,l}(f \vee g) \in \beta_{n-1}$ satisfying (11.1). Then the resulting sequent $\alpha_n \to \beta_n$ satisfies the conditions 0, 6 and is a GSSS by Lemma 11.17.

If a sentence $(a_i\,\breve{o}k_i)_{i=1,\dots,l}(f \vee g)$ satisfies (11.1), we call it a $\vee$-sentence, and call sentences $(a_i\,\breve{o}k_i)_{i=1,\dots,m} f$ and $(a_i\,\breve{o}k_i)_{i=\nu+1,\dots,l} g$ its constituents.

If $n \equiv 5$, let $\{h_1, \dots, h_k\}$ be the set of the $\vee$-sentences contained in α_{n-1}, and inductively make GSSS's $\alpha_{n,i} \to \beta_{n,i}$ $(i = 0, 1, \dots, k)$ as follows, and let $\alpha_n \to \beta_n = \alpha_{n,k} \to \beta_{n,k}$. First, let $\alpha_{n,0} \to \beta_{n,0} = \alpha_{n-1} \to \beta_{n-1}$. Next for $i \geq 1$, there is a constituent h_i' of h_i such that $h_i' \alpha_{n,i-1} \to \beta_{n,i-1}$ is a GSS by Lemma 11.17, so let it be $\alpha_{n,i} \to \beta_{n,i}$. Then $\alpha_n \to \beta_n$ is a GSSS and satisfies the conditions 0, 5.

If $n \equiv 8$, for each sentence $(a_i\,\breve{o}k_i)_{i=1,\dots,l}(f \Rightarrow g) \in \beta_{n-1}$ satisfying (11.1), add $(a_i\,\breve{o}k_i)_{i=1,\dots,m} f$ to α_{n-1} and add $(a_i\,\breve{o}k_i)_{i=\nu+1,\dots,l} g$ to β_{n-1}. Let $\alpha_n \to \beta_n$ be the resulting sequent. Then it satisfies the conditions 0, 8 and is a GSSS by Lemma 11.17.

If a sentence $(a_i\,\breve{o}k_i)_{i=1,\dots,l}(f \Rightarrow g)$ satisfies (11.1), we call it a $\Rightarrow$-sentence, and call sentences $(a_i\,\breve{o}k_i)_{i=1,\dots,m} f$ and $(a_i\,\breve{o}k_i)_{i=\nu+1,\dots,l} g$ its constituents.

If $n \equiv 7$, let $\{h_1, \dots, h_k\}$ be the set of the $\Rightarrow$-sentences which belong to α_{n-1}, and inductively make GSSS's $\alpha_{n,i} \to \beta_{n,i}$ $(i = 0, 1, \dots, k)$ as follows, and let $\alpha_n \to \beta_n = \alpha_{n,k} \to \beta_{n,k}$. First, let $\alpha_{n,0} \to \beta_{n,0} = \alpha_{n-1} \to \beta_{n-1}$. Next for $i \geq 1$, let h_i', h_i'' be the constituents of h_i. Then either $\alpha_{n,i-1} \to h_i' \beta_{n,i-1}$ or $h_i'' \alpha_{n,i-1} \to \beta_{n,i-1}$ is a GSSS by Lemma 11.17, so let $\alpha_{n,i} \to \beta_{n,i}$ be the one which is a GSSS. Then $\alpha_n \to \beta_n$ is a GSSS and satisfies the conditions 0, 7.

If $n \equiv 9$, extend β_{n-1} with all the sentences $(a_i\,\breve{o}k_i)_{i=1,\dots,l} f$ such that $(a_i\,\breve{o}k_i)_{i=1,\dots,l}(f^{\diamond}) \in \alpha_{n-1}$. Then the resulting sequent $\alpha_n \to \beta_n$ satisfies the conditions 0, 9 and is a GSSS by Lemma 11.18.

If $n \equiv 10$, extend α_{n-1} with all the sentences $(a_i\,\breve{o}k_i)_{i=1,\dots,l} f$ such that $(a_i\,\breve{o}k_i)_{i=1,\dots,l}(f^{\diamond}) \in \beta_{n-1}$. Then the resulting sequent $\alpha_n \to \beta_n$ satisfies the conditions 0, 10 and is a GSSS by Lemma 11.18.

If $n \equiv 11$, extend α_{n-1} with all the sentences $(a_i\,\breve{o}k_i)_{i=1,\dots,l}(a\,\mathfrak{p}k\,f^{\diamond})$ such that $\mathfrak{p} \in \mathfrak{P}$ and $(a_i\,\breve{o}k_i)_{i=1,\dots,l}(a\neg\mathfrak{p}k\,f) \in \alpha_{n-1}$. Then the resulting sequent $\alpha_n \to \beta_n$ satisfies the conditions 0, 11 and is a GSSS by Lemma 11.19.

If $n \equiv 12$, extend β_{n-1} with all the sentences $(a_i\,\breve{o}k_i)_{i=1,\dots,l}(a\,\mathfrak{p}k\,f^{\diamond})$ such that $\mathfrak{p} \in \mathfrak{P}$ and $(a_i\,\breve{o}k_i)_{i=1,\dots,l}(a\neg\mathfrak{p}k\,f) \in \beta_{n-1}$. Then the resulting sequent $\alpha_n \to \beta_n$ satisfies the conditions 0, 12 and is a GSSS by Lemma 11.19.

If $n \equiv 14$, extend β_{n-1} with all the sentences $(a_i\,\breve{o}k_i)_{i=1,\dots,l}(a\,\mathfrak{q}k\,f)$ such that $\mathfrak{q}$ is a connected component of a disconnected element $\mathfrak{p} \in \mathfrak{P}$ such that $(a_i\,\breve{o}k_i)_{i=1,\dots,l}(a\,\mathfrak{p}k\,f) \in \beta_{n-1}$. Then the resulting sequent $\alpha_n \to \beta_n$ satisfies the conditions 0, 14 and is a GSSS by Lemma 11.20.

A sentence $(a_i\,\breve{o}k_i)_{i=1,\dots,l}(a\,\mathfrak{p}k\,f)$ with $\mathfrak{p} \in \mathfrak{P}$ disconnected will be called a disconnected sentence, and for each connected component $\mathfrak{q}$ of $\mathfrak{p}$, the sentence $(a_i\,\breve{o}k_i)_{i=1,\dots,l}(a\,\mathfrak{q}k\,f)$ will be called a constituent of $(a_i\,\breve{o}k_i)_{i=1,\dots,l}(a\,\mathfrak{p}k\,f)$.

If $n \equiv 13$, let $\{h_1, \dots, h_m\}$ be the set of the disconnected sentences contained in α_{n-1}, inductively make GSSS's $\alpha_{n,i} \to \beta_{n,i}$ $(i = 0, 1, \dots, m)$ as follows, and let $\alpha_n \to \beta_n = \alpha_{n,m} \to \beta_{n,m}$. First, let $\alpha_{n,0} \to \beta_{n,0} = \alpha_{n-1} \to \beta_{n-1}$. Next for $i \geq 1$, there is a constituent h_i' of h_i such that $h_i' \alpha_{n,i-1} \to \beta_{n,i-1}$ is a GSSS by Lemma 11.20, so let it be $\alpha_{n,i} \to \beta_{n,i}$. Then $\alpha_n \to \beta_n$ is a GSSS and satisfies the conditions 0, 13.

If $n \equiv 16$, extend β_{n-1} with all the paired sentences $(a_i\,\breve{o}k_i)_{i=1,\dots,l}(a\,\overline{0}k\,f)$ and $(a_i\,\breve{o}k_i)_{i=1,\dots,l}(a\,(\leftarrow 0]k\,f)$ such that $(a_i\,\breve{o}k_i)_{i=1,\dots,l}(a\,\mathbb{P}k\,f) \in \beta_{n-1}$. Then the resulting sequent $\alpha_n \to \beta_n$ satisfies the conditions 0, 16 and is a GSSS by Lemma 11.20.

We call a sentence $(a_i\,\breve{o}k_i)_{i=1,\dots,l}(a\,\mathbb{P}k\,f)$ a $\mathbb{P}$-sentence, and call sentences $(a_i\,\breve{o}k_i)_{i=1,\dots,l}(a\,\overline{0}k\,f)$ and $(a_i\,\breve{o}k_i)_{i=1,\dots,l}(a\,(\leftarrow 0]k\,f)$ its constituents.

If $n \equiv 15$, let $\{h_1, \dots, h_m\}$ be the set of the $\mathbb{P}$-sentences contained in α_{n-1}, inductively make GSSS's $\alpha_{n,i} \to \beta_{n,i}$ $(i = 0, 1, \dots, m)$ as follows, and let $\alpha_n \to \beta_n = \alpha_{n,m} \to \beta_{n,m}$. First, let $\alpha_{n,0} \to \beta_{n,0} = \alpha_{n-1} \to \beta_{n-1}$. Next for $i \geq 1$, there is a constituent h'_i of h_i such that $h'_i\alpha_{n,i-1} \to \beta_{n,i-1}$ is a GSSS by Lemma 11.20, so let it be $\alpha_{n,i} \to \beta_{n,i}$. Then $\alpha_n \to \beta_n$ is a GSSS and satisfies the conditions 0, 15.

If $n \equiv 17$, extend α_{n-1} with all the paired sentences $(a_i\,\breve{o}k_i)_{i=1,\dots,l}(a\,\overline{p}k\,f)$ and $(a_i\,\breve{o}k_i)_{i=1,\dots,l}(a\,(\leftarrow q]k\,f)$ such that $(a_i\,\breve{o}k_i)_{i=1,\dots,l}(a\,(p,q]k\,f) \in \alpha_{n-1}$. Then the resulting sequent $\alpha_n \to \beta_n$ satisfies the conditions 0, 17 and is a GSSS by Lemma 11.21.

We call a sentence $(a_i\,\breve{o}k_i)_{i=1,\dots,l}(a\,(p,q]k\,f)$ a proper sentence, and call sentences $(a_i\,\breve{o}k_i)_{i=1,\dots,l}(a\,\overline{p}k\,f)$ and $(a_i\,\breve{o}k_i)_{i=1,\dots,l}(a\,(\leftarrow q]k\,f)$ its constituents.

If $n \equiv 18$, let $\{h_1, \dots, h_m\}$ be the set of the proper sentences contained in β_{n-1}, and inductively make GSSS's $\alpha_{n,i} \to \beta_{n,i}$ $(i = 0, 1, \dots, m)$ as follows, and let $\alpha_n \to \beta_n = \alpha_{n,m} \to \beta_{n,m}$. First, let $\alpha_{n,0} \to \beta_{n,0} = \alpha_{n-1} \to \beta_{n-1}$. Next for $i \geq 1$, there is a constituent h'_i of h_i such that $\alpha_{n,i-1} \to h'_i\beta_{n,i-1}$ is a GSSS by Lemma 11.21, so let it be $\alpha_{n,i} \to \beta_{n,i}$. Then $\alpha_n \to \beta_n$ is a GSSS and satisfies the conditions 0, 18.

If $n \equiv 19$, extend β_{n-1} with all the sentences $(a_i\,\breve{o}k_i)_i(a\,\overline{p}k\,f)$ such that $(a_i\,\breve{o}k_i)_i(a\,(\leftarrow p]k\,f) \in \alpha_{n-1}$. Then the resulting sequent $\alpha_n \to \beta_n$ satisfies the conditions 0, 19 and is a GSSS by Lemma 11.22.

If $n \equiv 20$, extend α_{n-1} with all the sentences $(a_i\,\breve{o}k_i)_i(a\,\overline{p}k\,f)$ such that $(a_i\,\breve{o}k_i)_i(a\,(\leftarrow p]k\,f) \in \beta_{n-1}$. Then the resulting sequent $\alpha_n \to \beta_n$ satisfies the conditions 0, 20 and is a GSSS by Lemma 11.22.

If $n \equiv 21$, extend α_{n-1} with all the sentences

$$\left(a \sqcap \left(x\,\breve{o}k\,(a_i\,\breve{o}k_i)_{i=1,\dots,l}f\right)\Omega x\right)\overline{p}\pi\,\mathrm{one}\triangle$$

such that $(a_i\,\breve{o}k_i)_{i=1,\dots,l}(a\,\overline{p}k\,f) \in \alpha_{n-1}$ and $x \in \mathbb{X}''_\varepsilon$ (such an extension is possible because $\#\mathbb{X}''_\varepsilon < \infty$). Then the resulting sequent $\alpha_n \to \beta_n$ satisfies the condition 0, 21 and is a GSSS by Lemma 11.23.

We call a sentence $(a_i\,\breve{o}k_i)_{i=1,\dots,l}(a\,\overline{p}k\,f)$ an upper sentence, and call sentences $\left(a \sqcap \left(x\,\breve{o}k\,(a_i\,\breve{o}k_i)_{i=1,\dots,l}f\right)\Omega x\right)\overline{p}\pi\,\mathrm{one}\triangle$ its one-representations.

If $n \equiv 22$, extend β_{n-1} with all the one-representations of the upper sentences contained in β_{n-1}. Then the resulting sequent $\alpha_n \to \beta_n$ satisfies the condition 0, 22 and is a GSSS by Lemma 11.23.

If $n \equiv 23$, let $M = \{\mu_1, \dots, \mu_l\}$ be the set of all tuples

$$(a\,\overline{p}\pi\,\mathrm{one}\triangle, b_1\,\overline{q_1}\pi\,\mathrm{one}\triangle, \dots, b_m\,\overline{q_m}\pi\,\mathrm{one}\triangle)$$

of a sentence $a\,\overline{p}\pi\,\mathrm{one}\triangle \in \alpha_{n-1}$ and *distinct* sentences $b_1\,\overline{q_1}\pi\,\mathrm{one}\triangle, \dots,$ $b_m\,\overline{q_m}\pi\,\mathrm{one}\triangle \in \beta_{n-1}$ such that $p \geq \sum_{i=1}^m q_i$. The number of such tuples is certainly finite, because $b_1\, \overline{q_1}\pi\, \mathsf{one}\triangle, \dots, b_m\, \overline{q_m}\pi\, \mathsf{one}\triangle$ are distinct. Inductively make GSSS's $\alpha_{n,i} \to \beta_{n,i}$ $(i = 0, 1, \dots, l)$ as follows, and let $\alpha_n \to \beta_n = \alpha_{n,l} \to \beta_{n,l}$. First, let $\alpha_{n,0} \to \beta_{n,0} = \alpha_{n-1} \to \beta_{n-1}$. Next for $i \geq 1$, suppose μ_i is a tuple of $a\, \overline{p}\pi\, \mathsf{one}\triangle \in \alpha_{n-1}$ and $b_1\, \overline{q_1}\pi\, \mathsf{one}\triangle, \dots, b_m\, \overline{q_m}\pi\, \mathsf{one}\triangle \in \beta_{n-1}$, and take a variable $x \in \mathbb{X}'_\varepsilon$ which does not occur free in the sentences in $\alpha_{n,i-1} \cup \beta_{n,i-1}$ (such a variable exists because $\mathbb{X}'_\varepsilon$ is enumerable). Then

$$x\, \breve{o}\pi\, a\triangle,\ \alpha_{n,i-1} \to x\, \breve{o}\pi\, b_1\triangle,\ \cdots,\ x\, \breve{o}\pi\, b_m\triangle, \beta_{n,i-1}$$

is a GSSS by Lemma 11.24, so let it be $\alpha_{n,i} \to \beta_{n,i}$. Then $\alpha_n \to \beta_n$ is a GSSS and satisfies the conditions 0, 23.

If $n \equiv 24$, extend β_{n-1} with all the n-tupled sentences $a_1\, \breve{o}\pi\, a\triangle, \dots, a_n\, \breve{o}\pi\, a\triangle$ such that $a\, \exists\pi\, \mathsf{one}\triangle \in \beta_{n-1}$. Then the resulting sequent $\alpha_n \to \beta_n$ satisfies the conditions 0, 24 and is a GSSS by Lemma 11.25.

If $n \equiv 25$, then extend α_{n-1} with all the paired sentences $c\, \breve{o}\pi\, a\triangle$ and $c\, \breve{o}\pi\, b\triangle$ such that $c\, \breve{o}\pi\, (a \sqcap b)\triangle \in \alpha_{n-1}$. Then the resulting sequent $\alpha_n \to \beta_n$ satisfies the conditions 0, 25 and is a GSSS by Lemma 11.26.

We call a sentence $c\, \breve{o}\pi\, (a \sqcap b)\triangle$ a $\sqcap$-sentence, and call sentences $c\, \breve{o}\pi\, a\triangle$ and $c\, \breve{o}\pi\, b\triangle$ its constituents.

If $n \equiv 26$, let $\{h_1, \dots, h_m\}$ be the set of the $\sqcap$-sentences contained in β_{n-1}, and inductively make GSSS's $\alpha_{n,i} \to \beta_{n,i}$ $(i = 0, 1, \dots, m)$ as follows, and let $\alpha_n \to \beta_n = \alpha_{n,m} \to \beta_{n,m}$. First, let $\alpha_{n,0} \to \beta_{n,0} = \alpha_{n-1} \to \beta_{n-1}$. Next for $i \geq 1$, there is a constituent h'_i of h_i such that $\alpha_{n,i-1} \to h'_i \beta_{n,i-1}$ is a GSSS by Lemma 11.26, so let it be $\alpha_{n,i} \to \beta_{n,i}$. Then $\alpha_n \to \beta_n$ is a GSSS and satisfies the conditions 0, 26.

If $n \equiv 28$, extend β_{n-1} with all the paired sentences $c\, \breve{o}\pi\, a\triangle$ and $c\, \breve{o}\pi\, b\triangle$ such that $c\, \breve{o}\pi\, (a \sqcup b) \in \beta_{n-1}$. Then the resulting sequent $\alpha_n \to \beta_n$ satisfies the conditions 0, 28 and is a GSSS by Lemma 11.26.

We call a sentence $c\, \breve{o}\pi\, (a \sqcup b)\triangle$ a $\sqcup$-sentence, and call sentences $c\, \breve{o}\pi\, a\triangle$ and $c\, \breve{o}\pi\, b\triangle$ its constituents.

If $n \equiv 27$, let $\{h_1, \dots, h_m\}$ be the set of the $\sqcup$-sentences contained in α_{n-1}, and inductively make GSSS's $\alpha_{n,i} \to \beta_{n,i}$ $(i = 0, 1, \dots, m)$ as follows, and let $\alpha_n \to \beta_n = \alpha_{n,m} \to \beta_{n,m}$. First, let $\alpha_{n,0} \to \beta_{n,0} = \alpha_{n-1} \to \beta_{n-1}$. Next for $i \geq 1$, since there is a constituent h'_i of h_i such that $h'_i \alpha_{n,i-1} \to \beta_{n,i-1}$ is a GSSS by Lemma 11.26, let it be $\alpha_{n,i} \to \beta_{n,i}$. Then $\alpha_n \to \beta_n$ is a GSSS and satisfies the conditions 0, 27.

If $n \equiv 29$, extend β_{n-1} with all the sentences $b\, \breve{o}\pi\, a\triangle$ such that $b\, \breve{o}\pi\, (a^{\square})\triangle \in \alpha_{n-1}$. Then the resulting sequent $\alpha_n \to \beta_n$ satisfies the conditions 0, 29 and is a GSSS by Lemma 11.27.

If $n \equiv 30$, extend α_{n-1} with all the sentences $b\, \breve{o}\pi\, a\triangle$ such that $b\, \breve{o}\pi\, (a^{\square})\triangle \in \beta_{n-1}$. Then the resulting sequent $\alpha_n \to \beta_n$ satisfies the conditions 0, 30 and is a GSSS by Lemma 11.27.

If $n \equiv 31$, extend α_{n-1} with all the sentences $f(x/a)$ such that $a\, \breve{o}\pi\, (f\, \Omega x)\triangle \in \alpha_{n-1}$. Then the resulting sequent $\alpha_n \to \beta_n$ satisfies the conditions 0, 31 and is a GSSS by Lemma 11.28.

If $n \equiv 32$, extend β_{n-1} with all the sentences $f(x/a)$ such that $a\,\breve{o}\pi(f\,\Omega x)\triangle \in \beta_{n-1}$. Then the resulting sequent $\alpha_n \to \beta_n$ satisfies the conditions 0, 32 and is a GSSS by Lemma 11.28.

This completes the inductive definition of the series $(\alpha_n \to \beta_n)_{n=1,2,\ldots}$ of GSSS's which satisfy the above thirty three conditions.

Lemma 11.30 Let $P = \bigcup_{n\geq 1} \alpha_n$ and $Q = \bigcup_{n\geq 1} \beta_n$. Then $P \cup Q$ consists of good sentences, $P \cap Q = \emptyset$, and the following thirty five conditions hold.

(0) If an element of $\mathbb{P} - \{0\}$ occurs in $P \cup Q$, then it also occurs in $\alpha_0 \cup \beta_0$.

(1) If $(a_i\,\breve{o}k_i)_{i=1,\ldots,l} f \in P$, then $(a_{\rho i}\,\breve{o}k_{\rho i})_{i=1,\ldots,l} f \in P$ for all $\rho \in \mathfrak{S}_l$.

(2) If $(a_i\,\breve{o}k_i)_{i=1,\ldots,l} f \in Q$, then $(a_{\rho i}\,\breve{o}k_{\rho i})_{i=1,\ldots,l} f \in Q$ for all $\rho \in \mathfrak{S}_l$.

(3) If $(a_i\,\breve{o}k_i)_{i=1,\ldots,l}(f \wedge g) \in P$ satisfies (11.1), then $(a_i\,\breve{o}k_i)_{i=1,\ldots,m} f$, $(a_i\,\breve{o}k_i)_{i=\nu+1,\ldots,l} g \in P$.

(4) If $(a_i\,\breve{o}k_i)_{i=1,\ldots,l}(f \wedge g) \in Q$ satisfies (11.1), then either $(a_i\,\breve{o}k_i)_{i=1,\ldots,m} f \in Q$ or $(a_i\,\breve{o}k_i)_{i=\nu+1,\ldots,l} g \in Q$.

(5) If $(a_i\,\breve{o}k_i)_{i=1,\ldots,l}(f \vee g) \in P$ satisfies (11.1), then either $(a_i\,\breve{o}k_i)_{i=1,\ldots,m} f \in P$ or $(a_i\,\breve{o}k_i)_{i=\nu+1,\ldots,l} g \in P$.

(6) If $(a_i\,\breve{o}k_i)_{i=1,\ldots,l}(f \vee g) \in Q$ satisfies (11.1), then $(a_i\,\breve{o}k_i)_{i=1,\ldots,m} f$, $(a_i\,\breve{o}k_i)_{i=\nu+1,\ldots,l} g \in Q$.

(7) If $(a_i\,\breve{o}k_i)_{i=1,\ldots,l}(f \Rightarrow g) \in P$ satisfies (11.1), then either $(a_i\,\breve{o}k_i)_{i=1,\ldots,m} f \in Q$ or $(a_i\,\breve{o}k_i)_{i=\nu+1,\ldots,l} g \in P$.

(8) If $(a_i\,\breve{o}k_i)_{i=1,\ldots,l}(f \Rightarrow g) \in Q$ satisfies (11.1), then $(a_i\,\breve{o}k_i)_{i=1,\ldots,m} f \in P$ and $(a_i\,\breve{o}k_i)_{i=\nu+1,\ldots,l} g \in Q$.

(9) If $(a_i\,\breve{o}k_i)_{i=1,\ldots,l}(f^{\Diamond}) \in P$, then $(a_i\,\breve{o}k_i)_{i=1,\ldots,l} f \in Q$.

(10) If $(a_i\,\breve{o}k_i)_{i=1,\ldots,l}(f^{\Diamond}) \in Q$, then $(a_i\,\breve{o}k_i)_{i=1,\ldots,l} f \in P$.

(11) If $(a_i\,\breve{o}k_i)_{i=1,\ldots,l}(a\,\neg\mathfrak{p}k\,f) \in P$ with $\mathfrak{p} \in \mathfrak{P}$, then $(a_i\,\breve{o}k_i)_{i=1,\ldots,l}(a\,\mathfrak{p}k\,f^{\Diamond}) \in P$.

(12) If $(a_i\,\breve{o}k_i)_{i=1,\ldots,l}(a\,\neg\mathfrak{p}k\,f) \in Q$ with $\mathfrak{p} \in \mathfrak{P}$, then $(a_i\,\breve{o}k_i)_{i=1,\ldots,l}(a\,\mathfrak{p}k\,f^{\Diamond}) \in Q$.

(13) If $(a_i\,\breve{o}k_i)_{i=1,\ldots,l}(a\,\mathfrak{p}k\,f) \in P$ with $\mathfrak{p} \in \mathfrak{P}$ having the connected components $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$ $(m \geq 2)$, then $(a_i\,\breve{o}k_i)_{i=1,\ldots,l}(a\,\mathfrak{p}_j k\,f) \in P$ for some $j \in \{1, \ldots, m\}$.

(14) If $(a_i\,\breve{o}k_i)_{i=1,\ldots,l}(a\,\mathfrak{p}k\,f) \in Q$ with $\mathfrak{p} \in \mathfrak{P}$ having the connected components $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$ $(m \geq 2)$, then $(a_i\,\breve{o}k_i)_{i=1,\ldots,l}(a\,\mathfrak{p}_j k\,f) \in Q$ for all $j \in \{1, \ldots, m\}$.

(15) If $(a_i\,\breve{o}k_i)_{i=1,\ldots,l}(a\,\mathbb{P}k\,f) \in P$, then either $(a_i\,\breve{o}k_i)_{i=1,\ldots,l}(a\,\overline{0}k\,f) \in P$ or $(a_i\,\breve{o}k_i)_{i=1,\ldots,l}(a\,(\leftarrow 0]k\,f) \in P$.

(16) If $(a_i\,\breve{o}k_i)_{i=1,\dots,l}(a\,\mathbb{P}k\,f) \in Q$, then
$(a_i\,\breve{o}k_i)_{i=1,\dots,l}(a\,\overline{0}k\,f)$, $(a_i\,\breve{o}k_i)_{i=1,\dots,l}(a\,(\leftarrow 0]k\,f) \in Q$.

(17) If $(a_i\,\breve{o}k_i)_{i=1,\dots,l}(a\,(p,q]k\,f) \in P$, then
$(a_i\,\breve{o}k_i)_{i=1,\dots,l}(a\,\overline{p}k\,f)$, $(a_i\,\breve{o}k_i)_{i=1,\dots,l}(a\,(\leftarrow q]k\,f) \in P$.

(18) If $(a_i\,\breve{o}k_i)_{i=1,\dots,l}(a\,(p,q]k\,f) \in Q$, then
either $(a_i\,\breve{o}k_i)_{i=1,\dots,l}(a\,\overline{p}k\,f) \in Q$ or $(a_i\,\breve{o}k_i)_{i=1,\dots,l}(a\,(\leftarrow q]k\,f) \in Q$.

(19) If $(a_i\,\breve{o}k_i)_{i=1,\dots,l}(a\,(\leftarrow p]k\,f) \in P$, then $(a_i\,\breve{o}k_i)_{i=1,\dots,l}(a\,\overline{p}k\,f) \in Q$.

(20) If $(a_i\,\breve{o}k_i)_{i=1,\dots,l}(a\,(\leftarrow p]k\,f) \in Q$, then $(a_i\,\breve{o}k_i)_{i=1,\dots,l}(a\,\overline{p}k\,f) \in P$.

(21) If $(a_i\,\breve{o}k_i)_{i=1,\dots,l}(a\,\overline{p}k\,f) \in P$ and $x \in \mathbb{X}''_\varepsilon$, then
$\big(a \sqcap \big(x\,\breve{o}k\,(a_i\,\breve{o}k_i)_{i=1,\dots,l}f\big)\,\Omega x\big)\,\overline{p}\pi\,one\triangle \in P$.

(22) If $(a_i\,\breve{o}k_i)_{i=1,\dots,l}(a\,\overline{p}k\,f) \in Q$ and $x \in \mathbb{X}''_\varepsilon$, then
$\big(a \sqcap \big(x\,\breve{o}k\,(a_i\,\breve{o}k_i)_{i=1,\dots,l}f\big)\,\Omega x\big)\,\overline{p}\pi\,one\triangle \in Q$.

(23) If $a\,\overline{p}\pi\,one\triangle \in P$, $b_1\,\overline{q_1}\pi\,one\triangle, \dots, b_m\,\overline{q_m}\pi\,one\triangle \in Q$, and $p \geq \sum_{i=1}^m q_i$,
then $x\,\breve{o}\pi\,a\triangle \in P$ and $x\,\breve{o}\pi\,b_1\triangle, \cdots, x\,\breve{o}\pi\,b_m\triangle \in Q$ for some $x \in \mathbb{X}'_\varepsilon$.

(24) If $a\,\exists\pi\,one\triangle \in Q$, then $b\,\breve{o}\pi\,a\triangle \in Q$ for all elements $b \in \mathbb{S}'_\varepsilon$.

(25) If $c\,\breve{o}\pi\,(a \sqcap b)\triangle \in P$, then $c\,\breve{o}\pi\,a\triangle$, $c\,\breve{o}\pi\,b\triangle \in P$.

(26) If $c\,\breve{o}\pi\,(a \sqcap b)\triangle \in Q$, then either $c\,\breve{o}\pi\,a\triangle \in Q$ or $c\,\breve{o}\pi\,b\triangle \in Q$.

(27) If $c\,\breve{o}\pi\,(a \sqcup b)\triangle \in P$, then either $c\,\breve{o}\pi\,a\triangle \in P$ or $c\,\breve{o}\pi\,b\triangle \in P$.

(28) If $c\,\breve{o}\pi\,(a \sqcup b)\triangle \in Q$, then $c\,\breve{o}\pi\,a\triangle$, $c\,\breve{o}\pi\,b\triangle \in Q$.

(29) If $b\,\breve{o}\pi\,(a^{\square})\triangle \in P$, then $b\,\breve{o}\pi\,a\triangle \in Q$.

(30) If $b\,\breve{o}\pi\,(a^{\square})\triangle \in Q$, then $b\,\breve{o}\pi\,a\triangle \in P$.

(31) If $a\,\breve{o}\pi\,(f\,\Omega x)\triangle \in P$, then $f(x/a) \in P$.

(32) If $a\,\breve{o}\pi\,(f\,\Omega x)\triangle \in Q$, then $f(x/a) \in Q$.

(33) If $a \in A_\varepsilon$, then $a\,\breve{o}\pi\,a\triangle \notin Q$.

(34) If $a \in G$, then $a\,\overline{\infty}\pi\,one\triangle \notin P$.

Proof If $f \in P$ and $g \in Q$ and $i \in \{1, \dots, 32\}$, then since $(\alpha_n \to \beta_n)_{n=1,2,\dots}$ is increasing, there exist infinitely many positive integers n such that $f \in \alpha_n$, $g \in \beta_n$, and $n \equiv i \mod 32$. Therefore, by Lemmas 11.15 and 11.29, we see that $P \cup Q$ consists of good sentences, $P \cap Q = \emptyset$, and the first thirty three conditions hold with special attention to (24). As for (33) if $a \in A_\varepsilon$, then $\preccurlyeq_*\ a\,\breve{o}\pi\,a\triangle$ by the $=$ law, and so since $\preccurlyeq_*$ satisfies the weakening law and exchange law and $\alpha_n \to \beta_n$ is singular, we have $a\,\breve{o}\pi\,a\triangle \notin \beta_n$ for $n = 1, 2, \dots$, hence $a\,\breve{o}\pi\,a\triangle \notin Q$. (34) is similarly proved by the $\overline{\infty}$ law.

Using the sets P, Q in Lemma 11.30 of sentences, we construct an MPC world

$$W = (\mathsf{S} \to \mathbb{T}) \cup \mathsf{S} \cup \bigcup_{\mathsf{O} \in \mathcal{P}\mathsf{K}} \big((\mathsf{O} \to \mathsf{S}) \to \mathbb{T}\big)$$

congizable by A as follows. First, since $\mathbb{S}'_\varepsilon$ is enumerable and in particular non-empty, we may choose $\mathbb{S}'_\varepsilon$ as the base S of W:

$$\mathsf{S} = \mathbb{S}'_\varepsilon = \mathbb{C}_\varepsilon \cup \mathbb{X}'_\varepsilon = \mathbb{S}_\varepsilon - \mathbb{X}''_\varepsilon. \tag{11.2}$$

Then, we may define the basic relation $\exists$ on S by

$$\mathsf{b} \ni \mathsf{a} \iff \mathsf{a}\,\breve{\mathsf{o}}\pi\,\mathsf{b}\triangle \notin \mathsf{Q} \tag{11.3}$$

for $\mathsf{a}, \mathsf{b} \in \mathsf{S}$. The $\ni$ is certainly reflexive by Lemma 11.30 (33).

In order to define a $\mathbb{P}$-measure $\mathsf{X} \mapsto |\mathsf{X}|$ on S, first let $\mathbb{P}'$ be the set of the elements of $\mathbb{P}$ which occur in $\mathsf{P} \cup \mathsf{Q}$. Then $\mathbb{P}'$ is a finite set by Lemma 11.30 (0). Let $\acute{\mathsf{p}}$ be the supremum of $\mathbb{P}'$ in $\mathbb{P}$. If $\mathbb{P}' = \emptyset$, then $\acute{\mathsf{p}} = 0$ by definition. Next, we define an element $\acute{\mathsf{o}}$ of $\mathbb{P}$ as follows. If $\acute{\mathsf{p}}$ is not equal to the maximum ∞ of $\mathbb{P}$, we let $\acute{\mathsf{o}}$ be an arbitrary element of $\mathbb{P}$ such that $\acute{\mathsf{p}} < \acute{\mathsf{o}}$. If $\acute{\mathsf{p}} = \infty$, we define $\acute{\mathsf{o}} = \infty$. Since $\#\mathbb{P} > 1$, we have $0 < \acute{\mathsf{o}}$ in either case. Next, for each $\mathsf{a} \in \mathsf{G}$, we may define

$$\mathsf{S}^{\mathsf{a}} = \{\mathsf{s} \in \mathsf{S} \mid \mathsf{s}\,\breve{\mathsf{o}}\pi\,\mathsf{a}\triangle \notin \mathsf{Q}\} \tag{11.4}$$

by virtue of (11.2). In particular for $\mathsf{a} \in \mathbb{S}'_\varepsilon$, $\mathsf{S}^{\mathsf{a}} = \{\mathsf{s} \in \mathsf{S} \mid \mathsf{a} \ni \mathsf{s}\}$ by (11.2)(11.3). Next if, for an element $\mathsf{X} \in \mathcal{P}\mathsf{S}$ and an element $\mathsf{p} \in \mathbb{P}$, there exist elements $\mathsf{b}_1, \ldots, \mathsf{b}_\mathsf{m} \in \mathsf{G}$ and elements $\mathsf{q}_1, \ldots, \mathsf{q}_\mathsf{m} \in \mathbb{P}$ which satisfy the conditions

(a) $\mathsf{X} \subseteq \bigcup_{\mathsf{i}=1}^{\mathsf{m}} \mathsf{S}^{\mathsf{b}_\mathsf{i}}$,

(b) $\mathsf{p} = \sum_{\mathsf{i}=1}^{\mathsf{m}} \mathsf{q}_\mathsf{i}$,

(c) $\mathsf{b}_\mathsf{i}\,\overline{\mathsf{q}_\mathsf{i}}\pi\,\mathsf{one}\triangle \in \mathsf{Q}$ $(\mathsf{i} = 1, \ldots, \mathsf{m})$,

then we write $\mathsf{X}\,\mathsf{R}\,\mathsf{p}$. When $\mathsf{m} = 0$, $\bigcup_{\mathsf{i}=1}^{\mathsf{m}} \mathsf{S}^{\mathsf{b}_\mathsf{i}} = \emptyset$ and $\sum_{\mathsf{i}=1}^{\mathsf{m}} \mathsf{q}_\mathsf{i} = 0$ by definition, and the condition (c) is vacant. The relation R between $\mathcal{P}\mathsf{S}$ and $\mathbb{P}$ thus defined satisfies the following conditions:

(1) $\mathsf{X} = \emptyset \iff \mathsf{X}\,\mathsf{R}\,0$,

(2) $\mathsf{X} \subseteq \mathsf{Y}$ and $\mathsf{Y}\,\mathsf{R}\,\mathsf{p} \implies \mathsf{X}\,\mathsf{R}\,\mathsf{p}$,

(3) $\mathsf{X}\,\mathsf{R}\,\mathsf{p}$ and $\mathsf{Y}\,\mathsf{R}\,\mathsf{q} \implies (\mathsf{X} \cup \mathsf{Y})\,\mathsf{R}\,(\mathsf{p} + \mathsf{q})$.

That R satisfies (2)(3) and $\emptyset\,\mathsf{R}\,0$ is an immediate consequence of the definition of R and the above remark on (a)(b)(c) with $\mathsf{m} = 0$. In order to complete the proof of (1), suppose $\mathsf{X}\,\mathsf{R}\,0$. Then either $\mathsf{X} = \emptyset$ or there exist elements $\mathsf{b}_1, \ldots, \mathsf{b}_\mathsf{m} \in \mathsf{G}$ and elements $\mathsf{q}_1, \ldots, \mathsf{q}_\mathsf{m} \in \mathbb{P}$ which satisfy $\mathsf{X} \subseteq \bigcup_{\mathsf{i}=1}^{\mathsf{m}} \mathsf{S}^{\mathsf{b}_\mathsf{i}}$, $0 = \sum_{\mathsf{i}=1}^{\mathsf{m}} \mathsf{q}_\mathsf{i}$, and $\mathsf{b}_\mathsf{i}\,\overline{\mathsf{q}_\mathsf{i}}\pi\,\mathsf{one}\triangle \in \mathsf{Q}$ $(\mathsf{i} = 1, \ldots, \mathsf{m})$, where $\mathsf{m} > 0$. In the latter case, we have

$q_i = 0$, so $b_i \,\exists\pi\, \mathrm{one}\triangle \in Q$, and Lemma 11.30 (24) yields that $s \,\breve{o}\pi\, b_i \triangle \in Q$ for all elements $s \in \mathbb{S}'_\varepsilon = S$, hence $S^{b_i} = \emptyset$ $(i = 1, \dots, m)$ and thus $X = \emptyset$ as desired. Furthermore, for each element $X \in \mathcal{P}S$ and the element $ó \in \mathbb{P}$ defined above, the subset $\{p \in \mathbb{P} \mid X \mathrel{R} p\} \cup \{ó\}$ of $\mathbb{P}$ has its minimum, because $\{p \in \mathbb{P} \mid X \mathrel{R} p\}$ is contained in the closure $[\mathbb{P}' \cup \{0\}]$ of $\mathbb{P}' \cup \{0\}$ in $\mathbb{P}$ and so, by Theorem 8.1, has its minimum unless it is empty. Therefore by Theorem 8.2, we may define the $\mathbb{P}$-measure $X \mapsto |X|$ on S by

$$|X| = \min\bigl(\{p \in \mathbb{P} \mid X \mathrel{R} p\} \cup \{ó\}\bigr). \tag{11.5}$$

We have thus defined the basic relation and the $\mathbb{P}$-measure on S. Using these, we may now let W be an MPC world congizable by A equipped with the OS and the sort mapping $\rho \in W \to T'$ described in §3.3.

Next, we define a $\mathbb{C}$-denotation Φ and an $\mathbb{X}$-denotation ν into W. The definition of Φ is as follows:

(Φ1) For each $a \in \mathbb{C}_\delta$, Φa is the element of $S \to \mathbb{T}$ such that

$$(\Phi a)s = 1 \iff s \in S^a$$

for each $s \in S$.

(Φ2) For each $a \in \mathbb{C}_\varepsilon$, since $\mathbb{C}_\varepsilon \subseteq S$ by (11.2), we define $\Phi a = a$.

(Φ3) For each $f \in \mathbb{C} \cap H$, Φf is the element of $(K_f \to S) \to \mathbb{T}$ such that, if $K_f = \{k_1, \dots, k_n\}$ with $k_1, \dots, k_n$ distinct, then

$$(\Phi f)\theta = 1 \iff \bigl((\theta k_i)\, \breve{o} k_i\bigr)_{i=1,\dots,n} f \notin Q$$

for each $\theta \in K_f \to S$. Since $\theta k_i \in A_\varepsilon$ by (11.2), this definition makes sense. By virtue of Lemma 11.30 (2), this definition does not depend on the numbering $k_1, \dots, k_n$ of elements of K_f.

Then Φ is certainly a $\mathbb{C}$-denotation into W. The definition of the $\mathbb{X}$-denotation ν is similar to that of Φ as follows but with similar remarks omitted:

(ν1) For each $x \in \mathbb{X}_\delta$, νx is the element of $S \to \mathbb{T}$ such that

$$(\nu x)s = 1 \iff s \in S^x$$

for each $s \in S$.

(ν2) For each $x \in \mathbb{X}'_\varepsilon$, we define $\nu x = x$. For each $x \in \mathbb{X}''_\varepsilon$, we let νx be an arbitrary element of S.

(ν3) For each $f \in \mathbb{X} \cap H$, νf is the element of $(K_f \to S) \to \mathbb{T}$ such that, if $K_f = \{k_1, \dots, k_n\}$ with $k_1, \dots, k_n$ distinct, then

$$(\nu f)\theta = 1 \iff \bigl((\theta k_i)\, \breve{o} k_i\bigr)_{i=1,\dots,n} f \notin Q$$

for each $\theta \in K_f \to S$.

Then ν is certainly an $\mathbb{X}$-denotation into W.

Lemma 11.31 If $\mathfrak{a} \in \mathbb{S}'_\varepsilon$, then $(\Phi^*\mathfrak{a})\nu = \mathfrak{a}$. If $\mathfrak{a} \in \mathbb{S}_\delta \cup \mathbb{S}'_\varepsilon$ and $s \in S$, then $(\Phi^*\mathfrak{a})\nu \ni s$ iff $s \in S^\mathfrak{a}$.

Proof The former part is a consequence of (2.4)(Φ2)(ν2). The latter part for $\mathfrak{a} \in \mathbb{S}'_\varepsilon$ is a consequence of the former part and the remark following (11.4). The latter part for $\mathfrak{a} \in \mathbb{S}_\delta$ is a consequence of (2.4)(3.1)(Φ1)(ν1).

Lemma 11.32 Let Λ be the index set of the OS of A:

$$\Lambda = \{\lambda k, \wedge, \vee, \Rightarrow, \Diamond, \triangle, \sqcap, \sqcup, \Box, \Omega x \mid \lambda \in \{\breve{o}\} \cup \mathfrak{Q},\ k \in K,\ x \in \mathbb{X}_\varepsilon\}.$$

Then there exists a mapping I of $\Lambda \amalg A$ into $\mathbb{Z}_{\geq 0}$ which satisfies the following conditions.

(1) If $\mu \in \Lambda$ and $(\mathfrak{a}_1, \ldots, \mathfrak{a}_n) \in \operatorname{Dom}\mu$, then $\mathrm{I}\big(\mu(\mathfrak{a}_1, \ldots, \mathfrak{a}_n)\big) = \mathrm{I}\mu + \mathrm{I}\mathfrak{a}_1 + \cdots + \mathrm{I}\mathfrak{a}_n$.

(2) If $\mathfrak{a} \in \{\breve{o}k, \triangle \mid k \in K\} \amalg \mathbb{S}$, then $\mathrm{I}(\mathfrak{a}) = 0$.

(3) If $\mathfrak{a} \in \{\wedge, \vee, \Rightarrow, \Diamond, \sqcap, \sqcup, \Box, \Omega x \mid x \in \mathbb{X}_\varepsilon\}$, then $\mathrm{I}\mathfrak{a} = 1$.

(4) If $p \in \mathbb{P}$, then $\mathrm{I}(\overline{p}k) = 4$ for each $k \in K$.

(5) If $p \in \mathbb{P}$, then $\mathrm{I}((\leftarrow p]k) = 5$ for each $k \in K$.

(6) If $\mathfrak{p}$ is a connected quantifier in $\mathfrak{P}$ other than those in the shapes $\overline{p}$ and $(\leftarrow p]$, then $\mathrm{I}(\mathfrak{p}k) = 6$ for each $k \in K$.

(7) If $\mathfrak{p}$ is a disconnected quantifier in $\mathfrak{P}$, then $\mathrm{I}(\mathfrak{p}k) = 7$ for each $k \in K$.

(8) If λ is a quantifier in $\neg\mathfrak{P}$, then $\mathrm{I}(\lambda k) = 9$ for each $k \in K$.

Proof Since $\overline{p} \neq (\leftarrow q]$ for $p, q \in \mathbb{P}$, we can define $\mathrm{I} \in \Lambda \amalg \mathbb{S} \to \mathbb{Z}_{\geq 0}$ so that the conditions (2) - (8) hold. By Lemma 6.1, we can extend I to an element of $\Lambda \amalg A \to \mathbb{Z}_{\geq 0}$ so that the condition (1) also holds.

Using Lemma 11.32, we define the mapping J of $\Lambda \amalg A$ into $\mathbb{Z}_{\geq 0}$ by

$$\mathrm{J}b = \begin{cases} \mathrm{I}\mathfrak{a} + 1 & \text{if } b = \mathfrak{a}\,\overline{p}\pi\,\mathit{one}\triangle \text{ for some } \mathfrak{a} \in G \text{ and } p \in \mathbb{P}, \\ \mathrm{I}b & \text{otherwise.} \end{cases} \tag{11.6}$$

This is well-defined by a remark following Theorem 2.1. We call the non-negative integer $\mathrm{J}b$ thus defined for each $b \in A$ the **index** of b.

Lemma 11.33 If $b \in A$, then $\mathrm{I}b \geq \mathrm{J}b$. If $\mathfrak{a} \in A$, $x \in \mathbb{X}_\varepsilon$, and $b \in A_\varepsilon$, then $\mathrm{I}\big(\mathfrak{a}(x/b)\big) = \mathrm{I}\mathfrak{a}$.

Proof Since $\mathit{one} = (x_0\, \breve{o}\pi\, x_0 \triangle)\, \Omega x_0$, we have $I(\mathit{one}) = I(x_0) + I(\breve{o}\pi) + I(x_0) + I\triangle + I(\Omega x_0) = 1$, and so $I(a\, \overline{p}\pi\, \mathit{one} \triangle) = Ia + I(\overline{p}\pi) + I(\mathit{one}) + I\triangle = Ia + 5 > Ia + 1 = J(a\, \overline{p}\pi\, \mathit{one} \triangle)$, hence the former part of the lemma.

The latter part is proved by induction on the rank r of a. If $r = 0$, then $a \in \mathbb{S}$, and so $a(x/b) \in \mathbb{S}$ by (2.1) and §3.2 (13), hence $I\big(a(x/b)\big) = 0 = Ia$. Suppose $r > 0$. Then $a = \mu(a_1, \ldots, a_n)$ for some $\mu \in \Lambda$ and $a_1, \ldots, a_n \in A$. If $\mu = \Omega x$, then $a(x/b) = a$ by (2.2), so there is nothing to prove. Suppose $\mu \neq \Omega x$. Then $a(x/b) = \mu(a_1(x/b), \ldots, a_n(x/b))$ by (2.2), so $I\big(a(x/b)\big) = I\mu + I\big(a_1(x/b)\big) + \cdots + I\big(a_n(x/b)\big)$. Since $I\big(a_i(x/b)\big) = Ia_i$ $(i = 1, \ldots, n)$ by the induction hypothesis, we have $I\big(a(x/b)\big) = I\mu + Ia_1 + \cdots + Ia_n = Ia$.

Lemma 11.34 If $h \in P$, then $(\Phi^* h)\nu = 1$, while if $h \in Q$, then$(\Phi^* h)\nu = 0$.

The proof is long. Before beginning it, we notice that Lemma 11.34 conclude the proof of $\vec{C} \subseteq [\vec{D}]_{\vec{R}}$ by contradiction, because $\alpha_0 \preccurlyeq \beta_0$ by Lemma 11.12, whereas Lemmas 11.34, 10.1, and (9.3) imply that $\alpha_0 \not\preccurlyeq \beta_0$ to the contrary.

Proof Let $h \in P \cup Q$. Then, by a remark following Theorem 2.1, there are elements $a_1, \ldots, a_l \in A_\varepsilon$ $(l \geq 0)$, distinct elements $k_1, \ldots, k_l \in K_h$, and an element $h' \in H$ which satisfy the conditions

$$h = (a_i\, \breve{o}k_i)_{i=1,\ldots,l} h', \qquad h' \notin \textstyle\bigcup_{k \in K} \mathrm{Im}\, \breve{o}k, \tag{11.7}$$

and such a tuple $a_1, \ldots, a_l$, $k_1, \ldots, k_l$, h' is uniquely determined by h. Since h is good by Lemma 11.30 and $A_\varepsilon = \mathbb{S}_\varepsilon$ by §3.2 (13),

$$a_1, \ldots, a_l \in \mathbb{S}'_\varepsilon \tag{11.8}$$

by Lemma 11.14. Since h is a sentence by Lemma 11.30,

$$K_{h'} = \{k_1, \ldots, k_l\}. \tag{11.9}$$

Since $h' \notin \bigcup_{k \in K} \mathrm{Im}\, \breve{o}k$, either $h' \in \mathbb{S} \cap H$ or h' is in one of the shapes

$$a\,\lambda k\, f \ \ (\lambda \in \mathfrak{Q}), \qquad f \wedge g, \qquad f \vee g, \qquad f \Rightarrow g, \qquad f^{\Diamond}, \qquad c\triangle, \tag{11.10}$$

by §3.2 (14), where either $c \in \mathbb{S} \cap G = \mathbb{S}_\delta \cup \mathbb{S}_\varepsilon$ or c is in one of the shapes

$$a \sqcap b, \qquad a \sqcup b, \qquad a^{\Box}, \qquad f\, \Omega x \tag{11.11}$$

by §3.2 (12) (13). If $c \in \mathbb{S}_\varepsilon$, then $c \in \mathbb{S}'_\varepsilon$ by Lemma 11.14 because h is good.

We first consider the following two special cases.

The case where $h' \in \mathbb{S} \cap H$. By (11.9), (11.8), and (11.2), we may define $\theta \in K_{h'} \to S$ by $\theta k_i = a_i$ $(i = 1, \ldots, l)$, and we have

$$\begin{aligned}
(\Phi^* h)\nu &= \big(\Phi^*\big((a_i \,\breve{o} k_i)_i h'\big)\big)\nu && \text{(by (11.7))} \\
&= \big((\Phi^* a_i)\nu \,\breve{o} k_i\big)_i (\Phi^* h')\nu \\
&= (a_i \,\breve{o} k_i)_i (\Phi^* h')\nu && \text{(by (11.8) and Lemma 11.31)} \\
&= \big((\theta k_i)\,\breve{o} k_i\big)_i (\Phi^* h')\nu && \text{(by the definition of } \theta) \\
&= \big((\Phi^* h')\nu\big)\theta && \text{(by Corollary 9.3.2)} \\
&= \begin{cases} (\Phi h')\theta & \text{if } h' \in \mathbb{C} \cap H, \\ (\nu h')\theta & \text{if } h' \in \mathbb{X} \cap H \end{cases} && \text{(by (2.4)),}
\end{aligned}$$

where the second equality holds because Φ^* and the projection by ν is a homomorphism with respect to the operations $\breve{o} k_i$ $(i = 1, \ldots, l)$. Hence

$$\begin{aligned}
&(\Phi^* h)\nu = 1 \\
&\iff \big((\theta k_i)\,\breve{o} k_i\big)_i h' \notin Q && \text{(by } (\Phi 3)(\nu 3)) \\
&\iff \big(a_i \,\breve{o} k_i\big)_i h' \notin Q && \text{(by the definition of } \theta) \\
&\iff h \notin Q && \text{(by (11.7)).}
\end{aligned}$$

Therefore, if $h \in Q$ then $(\Phi^* h)\nu = 0$, while if $h \in P$ then $(\Phi^* h)\nu = 1$, because $P \cap Q = \emptyset$ by Lemma 11.30.

The case where $h' = c\triangle$ for some $c \in \mathbb{S} \cap G$. In this case, we have $l = 1$ and $k_1 = \pi$ by (11.9), $a_1 \in \mathbb{S}'_\varepsilon = S$ by (11.8) and (11.2), $h = a_1 \,\breve{o}\pi c\triangle$ by (11.7), and $c \in \mathbb{S}_\delta \cup \mathbb{S}'_\varepsilon$. Therefore,

$$(\Phi^* h)\nu = (\Phi^* a_1)\nu \,\breve{o}\pi\, (\Phi^* c)\nu\triangle = a_1 \,\breve{o}\pi\, (\Phi^* c)\nu\triangle$$

by Lemma 11.31, hence

$$\begin{aligned}
(\Phi^* h)\nu = 1 &\iff (\Phi^* c)\nu \ni a_1 && \text{(by Theorem 9.7)} \\
&\iff a_1 \in S^c && \text{(by Lemma 11.31)} \\
&\iff a_1 \,\breve{o}\pi c\triangle \notin Q && \text{(by (11.4))} \\
&\iff h \notin Q.
\end{aligned}$$

Therefore, if $h \in Q$ then $(\Phi^* h)\nu = 0$, while if $h \in P$ then $(\Phi^* h)\nu = 1$.

The general case. We argue by induction on the index Jh of h.

If $Jh = 0$, then by (11.6), Lemma 11.32, and the discussion on the shapes of h', either $h' \in \mathbb{S} \cap H$ or $h' = c\triangle$ for some $c \in \mathbb{S} \cap G$, and in either case, Lemma 11.34 has been proved above. Therefore, we assume $Jh \geq 1$, $h' \notin \mathbb{S} \cap H$, and $h' \neq c\triangle$ for any $c \in \mathbb{S} \cap G$. Then h' is in one of the shapes (11.10) and if $h' = c\triangle$, c is in one of the shapes (11.11). We will consider those cases one by one, redividing them into twelve cases.

We first consider the case where h' is equal to $a\lambda k f$ ($\lambda \in \mathfrak{Q}$) on the list (11.10), and further divide it into the three subcases where $\lambda \in \neg\mathfrak{P}$, where $\lambda \in \mathfrak{P}$ but λ is disconnected, and where $\lambda \in \mathfrak{P}$ and λ is connected. In the last case, λ is an interval in one of the four shapes $\mathbb{P}$, $(p, q]$, $(\leftarrow p]$, and $\overline{p} = (p \rightarrow)$.

We will often use the following argument. Recalling that $P \cap Q = \emptyset$ by Lemma 11.30, we say that two elements f, g of $P \cup Q$ is equivalent or write $f \sim g$, if either $f, g \in P$ or $f, g \in Q$. Then if $h \sim \grave{h}$ and $(\Phi^* h)\nu = (\Phi^* \grave{h})\nu$ and $Jh > J\grave{h}$, or if $h \nsim \grave{h}$ and $(\Phi^* h)\nu \neq (\Phi^* \grave{h})\nu$ and $Jh > J\grave{h}$, then by the induction hypothesis, Lemma 11.34 holds for the h.

Case 1: $h' = a\neg\mathfrak{p}kf$ ($\mathfrak{p} \in \mathfrak{P}$)**.** Here $h = (a_i\, \breve{o}k_i)_i(a\neg\mathfrak{p}kf)$ by (11.7), and $h \sim \grave{h} = (a_i\, \breve{o}k_i)_i(a\,\mathfrak{p}k f^{\Diamond})$ by Lemma 11.30 (11) (12). Also, $(\Phi^* h)\nu = (\Phi^* \grave{h})\nu$ by Theorem 9.8. Furthermore,

$$\begin{aligned} Jh = Ih = I(a\neg\mathfrak{p}kf) &= Ia + I(\neg\mathfrak{p}k) + If = Ia + 9 + If \\ &> Ia + 8 + If = Ia + 7 + (If + 1) = Ia + 7 + I(f^{\Diamond}) \\ &\geq Ia + I(\mathfrak{p}k) + I(f^{\Diamond}) = I(a\,\mathfrak{p}kf^{\Diamond}) = I\grave{h} = J\grave{h} \end{aligned}$$

by (11.6) and Lemma 11.32. Therefore, Lemma 11.34 holds in this case.

Case 2: $h' = a\,\mathfrak{p}kf$ ($\mathfrak{p} \in \mathfrak{P}$, **disconnected**)**.** Here $h = (a_i\, \breve{o}k_i)_i(a\,\mathfrak{p}kf)$. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_m$ be the connected components of $\mathfrak{p}$, and define $h_j = (a_i\, \breve{o}k_i)_i(a\,\mathfrak{p}_j kf)$ ($j = 1, \dots, m$). Then

$$(\Phi^* h)\nu = (\Phi^* h_1)\nu \vee \cdots \vee (\Phi^* h_m)\nu$$

by Theorem 9.9 and Lemma 9.3, where the order of applying the operation $\vee$ on $\mathbb{T}$ on the right-hand side is arbitrary because it is associative. Furthermore,

$$\begin{aligned} Jh = Ih = I(a\,\mathfrak{p}kf) &= Ia + I(\mathfrak{p}k) + If = Ia + 7 + If \\ &> Ia + 6 + If \geq Ia + I(\mathfrak{p}_j k) + If = I(a\,\mathfrak{p}_j kf) = Ih_j \geq Jh_j \end{aligned}$$

for each $j \in \{1, \dots, m\}$ by (11.6), Lemmas 11.32, and 11.33. By Lemma 11.30 (13) (14), if $h \in P$ then $h_j \in P$ for some $j \in \{1, \dots, m\}$, while if $h \in Q$ then $h_j \in Q$ for all $j \in \{1, \dots, m\}$. Therefore, Lemma 11.34 holds in this case.

Case 3: $h' = a\mathbb{P}kf$ **and** $\infty = \max\mathbb{P}$ **does not exist.** In this case, $h = (a_i\, \breve{o}k_i)_i(a\mathbb{P}kf)$. Define $h_1 = (a_i\, \breve{o}k_i)_i(a\,\overline{0}kf)$, $h_2 = (a_i\, \breve{o}k_i)_i(a\,(\leftarrow 0]kf)$. Then, since $\mathbb{P} = \overline{0} \cup (\leftarrow 0]$,

$$(\Phi^* h)\nu = (\Phi^* h_1)\nu \vee (\Phi^* h_2)\nu$$

by Theorem 9.9 and Lemma 9.3. Since $\mathbb{P} \neq \overline{p}$ and $\mathbb{P} \neq (\leftarrow p]$ for all $p \in \mathbb{P}$,

$$\begin{aligned} Jh = Ih = I(a\mathbb{P}kf) &= Ia + I(\mathbb{P}k) + If = Ia + 6 + If \\ &> \begin{cases} Ia + 4 + If = Ia + I(\overline{0}k) + If = I(a\,\overline{0}kf) = Ih_1 \geq Jh_1, \\ Ia + 5 + If = Ia + I((\leftarrow 0]k) + If = I(a\,(\leftarrow 0]kf) = Ih_2 = Jh_2 \end{cases} \end{aligned}$$

by (11.6), Lemmas 11.32, and 11.33. By Lemma 11.30 (15) (16), if $h \in P$ then either h_1 or h_2 belongs to P, while if $h \in Q$ then $h_1, h_2 \in Q$. Therefore, Lemma 11.34 holds in this case.

Case 4: $h' = a\,(p,q]k\,f$ $(p, q \in \mathbb{P},\ q \neq \infty)$**.** Here $h = (a_i\,\breve{o}k_i)_i(a\,(p,q]k\,f)$. Define $h_1 = (a_i\,\breve{o}k_i)_i(a\,\overline{p}k\,f)$, $h_2 = (a_i\,\breve{o}k_i)_i(a\,(\leftarrow q]k\,f)$. Then since $(p,q] = \overline{p} \cap (\leftarrow q]$,

$$(\Phi^* h)\nu = (\Phi^* h_1)\nu \wedge (\Phi^* h_2)\nu$$

by Theorem 9.9 and Lemma 9.3. Since $(p,q] \neq \overline{r}$ and $(p,q] \neq (\leftarrow r]$ for all $r \in \mathbb{P}$,

$$\begin{aligned} Jh = Ih &= I(a\,(p,q]k\,f) = Ia + I((p,q]k) + If = Ia + 6 + If \\ &> \begin{cases} Ia + 4 + If = Ia + I(\overline{0}k) + If = I(a\,\overline{p}k\,f) = Ih_1 \geq Jh_1, \\ Ia + 5 + If = Ia + I((\leftarrow 0]k) + If = I(a\,(\leftarrow q]k\,f) = Ih_2 = Jh_2 \end{cases} \end{aligned}$$

by (11.6), Lemmas 11.32, and 11.33. By Lemma 11.30 (17) (18), if $h \in P$ then $h_1, h_2 \in P$, while if $h \in Q$ then either h_1 or h_2 belongs to Q. Therefore, Lemma 11.34 holds in this case.

Case 5: $h' = a\,(\leftarrow p]k\,f$ $(p \in \mathbb{P})$**.** This case includes the case where $h' = a\,\mathbb{P}k\,f$ and $\infty = \max\mathbb{P}$ exists, which was excluded from the case (3). Here $h = (a_i\,\breve{o}k_i)_i(a\,(\leftarrow p]k\,f) \not\sim \grave{h} = (a_i\,\breve{o}k_i)_i(a\,\overline{p}k\,f)$ by Lemma 11.30 (19) (20). Also, $(\Phi^* h)\nu \neq (\Phi^* \grave{h})\nu$ by Theorem 9.8 and Corollary 9.3.3. Furthermore,

$$\begin{aligned} Jh = Ih = I(a\,(\leftarrow p]k\,f) &= Ia + I\big((\leftarrow p]k\big) + If = Ia + 5 + If \\ &> Ia + 4 + If = Ia + I(\overline{p}k) + If = I(a\,\overline{p}k\,f) = I\grave{h} \geq J\grave{h} \end{aligned}$$

by (11.6), Lemmas 11.32, and 11.33. Therefore, Lemma 11.34 holds in this case.

Case 6: $h' = a\,\overline{p}k\,f \neq a\,\overline{p}\pi\,\mathit{one}\triangle$ $(p \in \mathbb{P})$**.** If $\infty = \max\mathbb{P}$ exists, then $\overline{p} = (p, \infty]$. Therefore the case $h' = a\,(p,\infty]k\,f$ excluded from the case 4 is included in this case 6 unless $h' = a\,(p,\infty]\pi\,\mathit{one}\triangle$. Here $h = (a_i\,\breve{o}k_i)_i(a\,\overline{p}k\,f)$. Recall $\mathbb{X}''_\varepsilon \neq \emptyset$ and define

$$\grave{h} = \big(a \sqcap \big(x\,\breve{o}k\,(a_i\,\breve{o}k_i)_i f\big)\,\Omega x\big)\,\overline{p}\pi\,\mathit{one}\triangle$$

for $x \in \mathbb{X}''_\varepsilon$. Then $h \sim \grave{h}$ by Lemma 11.30 (21) (22). We can prove $h \asymp \grave{h}$ similarly to Lemma 11.23 as follows. Define $g = (a_i\,\breve{o}k_i)_i f$. Then $K_g = \{k\}$, and

$$h \asymp a\,\overline{p}k\,g$$

by Theorem 9.12 and Lemma 10.1. Also, since $x \not\lll g$ by Lemma 11.14,

$$a\,\overline{p}k\,g \asymp a\,\overline{p}\pi\big((x\,\breve{o}k\,g)\,\Omega x\big)\triangle$$

by Theorem 10.3. Also,

$$\mathsf{a}\,\overline{\mathsf{p}}\pi\big((\mathsf{x}\,\breve{\mathsf{o}}\mathsf{k}\,\mathsf{g})\,\Omega\mathsf{x}\big)\triangle \asymp \big(\mathsf{a}\sqcap(\mathsf{x}\,\breve{\mathsf{o}}\mathsf{k}\,\mathsf{g})\,\Omega\mathsf{x}\big)\,\overline{\mathsf{p}}\pi\,\mathsf{one}\triangle$$

by Theorem 9.7, Lemma 10.1, and Theorem 10.6. Thus $\mathsf{h} \asymp \grave{\mathsf{h}}$, hence $(\Phi^*\mathsf{h})\nu = (\Phi^*\grave{\mathsf{h}})\nu$ by Lemma 10.1. Furthermore,

$$\begin{aligned}\mathsf{J}\mathsf{h} &= \mathsf{I}\mathsf{h} = \mathsf{I}\mathsf{a} + \mathsf{I}(\overline{\mathsf{p}}\mathsf{k}) + \mathsf{I}\mathsf{f} = \mathsf{I}\mathsf{a} + 4 + \mathsf{I}\mathsf{f} = \mathsf{I}\mathsf{a} + 4 + \mathsf{I}\mathsf{g}\\ &> \mathsf{I}\mathsf{a} + 3 + \mathsf{I}\mathsf{g} = \mathsf{I}\mathsf{a} + \mathsf{I}\sqcap + \mathsf{I}\mathsf{g} + \mathsf{I}(\Omega\mathsf{x}) + 1 = \mathsf{I}\big(\mathsf{a}\sqcap(\mathsf{x}\,\breve{\mathsf{o}}\mathsf{k}\,\mathsf{g})\,\Omega\mathsf{x}\big) + 1 = \mathsf{J}\grave{\mathsf{h}}\end{aligned}$$

by (11.6) and Lemma 11.32. Therefore, Lemma 11.34 holds in this case.

Case 7: $\mathsf{h}' = \mathsf{a}\,\overline{\mathsf{p}}\pi\,\mathsf{one}\triangle$ $(\mathsf{p} \in \mathbb{P})$**.** If $\infty = \max\mathbb{P}$ exists, then $\overline{\mathsf{p}} = (\mathsf{p}, \infty]$. Therefore, the case $\mathsf{h}' = \mathsf{a}\,(\mathsf{p}, \infty]\pi\,\mathsf{one}\triangle$ excluded from the cases 4 and 6 is included in this case 7, where $\mathsf{h} = \mathsf{h}' = \mathsf{a}\,\overline{\mathsf{p}}\pi\,\mathsf{one}\triangle$. Define

$$\mathsf{X} = \{\mathsf{s} \in \mathsf{S} \mid (\Phi^*\mathsf{a})\nu \ni \mathsf{s}\}.$$

Then

$$(\Phi^*\mathsf{h})\nu = 1 \iff \mathsf{p} < |\mathsf{X}|$$

by Theorems 10.6 and 9.7.

First, we consider the case where $\mathsf{h} = \mathsf{a}\,\overline{\mathsf{p}}\pi\,\mathsf{one}\triangle \in \mathsf{P}$, and in order to prove $(\Phi^*\mathsf{h})\nu = 1$ by contradiction, we assume $\mathsf{p} \geq |\mathsf{X}|$. Then $\mathsf{p} \neq \infty$ by Lemma 11.30 (34). Also $\mathsf{p} \in \mathbb{P}'$, and so $\mathsf{p} \leq \sup\mathbb{P}' = \acute{\mathsf{p}}$. Since we have chosen $\acute{\mathsf{o}}$ so that either $\acute{\mathsf{p}} < \acute{\mathsf{o}}$ or $\acute{\mathsf{o}} = \infty$, it follows that $\mathsf{p} < \acute{\mathsf{o}}$ holds. Therefore $|\mathsf{X}| \neq \acute{\mathsf{o}}$, and so the definition (11.5) of $|\mathsf{X}|$ shows that there exist elements $\mathsf{b}_1, \ldots, \mathsf{b}_m \in \mathsf{G}$ and elements $\mathsf{q}_1, \ldots, \mathsf{q}_m \in \mathbb{P}$ which satisfy the conditions

(i) $\mathsf{X} \subseteq \bigcup_{i=1}^{m} \mathsf{S}^{\mathsf{b}_i}$,

(ii) $\mathsf{p} \geq \sum_{i=1}^{m} \mathsf{q}_i$,

(iii) $\mathsf{b}_i\,\overline{\mathsf{q}_i}\pi\,\mathsf{one}\triangle \in \mathsf{Q}$ $(i = 1, \ldots, m)$,

where $m \geq 0$. Therefore by Lemma 11.30 (23), there exists an element $\mathsf{x} \in \mathbb{X}'_\varepsilon$ such that $\mathsf{x}\,\breve{\mathsf{o}}\pi\,\mathsf{a}\triangle \in \mathsf{P}$ and $\mathsf{x}\,\breve{\mathsf{o}}\pi\,\mathsf{b}_1\triangle, \cdots, \mathsf{x}\,\breve{\mathsf{o}}\pi\,\mathsf{b}_m\triangle \in \mathsf{Q}$. Furthermore, $\mathsf{J}\mathsf{h} = \mathsf{I}\mathsf{a} + 1 > \mathsf{I}\mathsf{a} = \mathsf{I}(\mathsf{x}\,\breve{\mathsf{o}}\pi\,\mathsf{a}\triangle) = \mathsf{J}(\mathsf{x}\,\breve{\mathsf{o}}\pi\,\mathsf{a}\triangle)$ by (11.6) and Lemma 11.32. Therefore $\big(\Phi^*(\mathsf{x}\,\breve{\mathsf{o}}\pi\,\mathsf{a}\triangle)\big)\nu = 1$ by the induction hypothesis. Since $\big(\Phi^*(\mathsf{x}\,\breve{\mathsf{o}}\pi\,\mathsf{a}\triangle)\big)\nu = (\Phi^*\mathsf{x})\nu\,\breve{\mathsf{o}}\pi\,(\Phi^*\mathsf{a})\nu\triangle = \mathsf{x}\,\breve{\mathsf{o}}\pi\,(\Phi^*\mathsf{a})\nu\triangle$ by Lemma 11.31, Theorem 9.7 yields $(\Phi^*\mathsf{a})\nu \ni \mathsf{x}$, hence $\mathsf{x} \in \mathsf{X}$. On the other hand, since $\mathsf{x}\,\breve{\mathsf{o}}\pi\,\mathsf{b}_i\triangle \in \mathsf{Q}$, the definition (11.4) of $\mathsf{S}^{\mathsf{b}_i}$ yields $\mathsf{x} \notin \mathsf{S}^{\mathsf{b}_i}$ for $i = 1, \ldots, m$. This contradicts (i).

Next, we consider the case where $\mathsf{h} = \mathsf{a}\,\overline{\mathsf{p}}\pi\,\mathsf{one}\triangle \in \mathsf{Q}$. Suppose $\mathsf{s} \in \mathsf{S} - \mathsf{S}^{\mathsf{a}}$. Then $\mathsf{s}\,\breve{\mathsf{o}}\pi\,\mathsf{a}\triangle \in \mathsf{Q}$ by (11.4). Also $\mathsf{J}\mathsf{h} = \mathsf{I}\mathsf{a} + 1 > \mathsf{I}\mathsf{a} = \mathsf{I}(\mathsf{s}\,\breve{\mathsf{o}}\pi\,\mathsf{a}\triangle) = \mathsf{J}(\mathsf{s}\,\breve{\mathsf{o}}\pi\,\mathsf{a}\triangle)$ by (11.6) and Lemma 11.32. Therefore $\big(\Phi^*(\mathsf{s}\,\breve{\mathsf{o}}\pi\,\mathsf{a}\triangle)\big)\nu = 0$ by the induction hypothesis, and so $(\Phi^*\mathsf{a})\nu \not\ni \mathsf{s}$ by Lemma 11.31 and Theorem 9.7, that is, $\mathsf{s} \in \mathsf{S} - \mathsf{X}$. Thus $\mathsf{X} \subseteq \mathsf{S}^{\mathsf{a}}$, and so $|\mathsf{X}| \leq \mathsf{p}$ by (11.5). Therefore $(\Phi^*\mathsf{h})\nu = 0$.

This completes consideration of the case where h' is equal to $\mathsf{a}\,\lambda\mathsf{k}\,\mathsf{f}$ $(\lambda \in \mathfrak{Q})$ in (11.10).

Case 8: $h' = f \wedge g$ **or** $f \vee g$ **or** $f \Rightarrow g$**.** Let $*$ denote any one of the operations $\wedge, \vee, \Rightarrow$. Then $h = (a_i \breve{o} k_i)_{i=1,\dots,l}(f * g)$. Let $\rho \in \mathfrak{S}_l$ and define $h_\rho = (a_{\rho i} \breve{o} k_{\rho i})_{i=1,\dots,l}(f * g)$. Then $h \sim h_\rho$ by Lemma 11.30 (1) (2), $Jh = Jh_\rho$ by (11.6) and Lemmas 11.32, and $(\Phi^* h)\nu = (\Phi^* h_\rho)\nu$ by Corollary 9.3.1. Therefore, we may assume that (11.1) is satisfied. Define $h_f = (a_i \breve{o} k_i)_{i=1,\dots,m} f$, and $h_g = (a_i \breve{o} k_i)_{i=\nu+1,\dots,l} g$. Then

$$(\Phi^* h)\nu = (\Phi^* h_f)\nu * (\Phi^* h_g)\nu$$

by Theorem 9.4. Also, the following holds by Lemma 11.30 (3) - (8).

- When $*$ is $\wedge$, if $h \in P$ then $h_f, h_g \in P$, while if $h \in Q$ then either $h_f \in Q$ or $h_g \in Q$.
- When $*$ is $\vee$, if $h \in P$ then either $h_f \in P$ or $h_g \in P$, while if $h \in Q$ then $h_f, h_g \in Q$.
- When $*$ is $\Rightarrow$, if $h \in P$ then either $h_f \in Q$ or $h_g \in P$, while if $h \in Q$ then $h_f \in P$ and $h_g \in Q$.

Furthermore,

$$Jh = Ih = If + I* + Ig = If + 1 + Ig > \begin{cases} If = Ih_f \geq Jh_f, \\ Ig = Ih_g \geq Jh_g \end{cases}$$

by (11.6), Lemmas 11.32, and 11.33. Therefore, Lemma 11.34 holds in this case.

Case 9: $h' = f^\Diamond$**.** Here $h = (a_i \breve{o} k_i)_i (f^\Diamond)$, and $h \nsim \grave{h} = (a_i \breve{o} k_i)_i f$ by Lemma 11.30 (9) (10). Also, $(\Phi^* h)\nu \neq (\Phi^* \grave{h})\nu$ by Corollary 9.3.3. Furthermore, $Jh = Ih = If + I\Diamond = If + 1 > If = I\grave{h} \geq J\grave{h}$ by (11.6), Lemmas 11.32, and 11.33. Therefore, Lemma 11.34 holds in this case.

It now remains to consider the case where $h' = c\triangle$ and c is in one of the shapes (11.11).

Case 10: $h' = (a \sqcap b)\triangle$ **or** $(a \sqcup b)\triangle$**.** Let $*$ denote $\sqcap$ or $\sqcup$. Then $l = 1$, $k_1 = \pi$, and $h = a_1 \breve{o}\pi (a * b)\triangle$. Define $h_a = a_1 \breve{o}\pi\, a\triangle$, $h_b = a_1 \breve{o}\pi\, b\triangle$. Then the following holds by Theorems 9.1, 9.4, and Lemma 11.30 (25) - (28).

- When $*$ is $\sqcap$, $(\Phi^* h)\nu = (\Phi^* h_a)\nu \wedge (\Phi^* h_b)\nu$, and if $h \in P$ then $h_a, h_b \in P$, while if $h \in Q$ then either $h_a \in Q$ or $h_b \in Q$.
- When $*$ is $\sqcup$, $(\Phi^* h)\nu = (\Phi^* h_a)\nu \vee (\Phi^* h_b)\nu$, and if $h \in P$ then either $h_a \in P$ or $h_b \in P$, while if $h \in Q$ then $h_a, h_b \in Q$.

Furthermore,

$$Jh = Ih = Ia + I* + Ib = Ia + 1 + Ib > \begin{cases} Ia = Ih_a = Jh_a, \\ Ib = Ih_b = Jh_b \end{cases}$$

by (11.6) and Lemma 11.32. Therefore, Lemma 11.34 holds in this case.

Case 11: $h' = a^{\square}\triangle$**.** Here $l = 1$, $k_1 = \pi$, $h = a_1 \,\breve{o}\pi\, a^{\square}\triangle$, and $h \nsim \grave{h} = a_1 \,\breve{o}\pi\, a\triangle$ by Lemma 11.30 (29) (30). Also, $(\Phi^* h)\nu \neq (\Phi^* \grave{h})\nu$ by Theorem 9.1 and Corollary 9.3.3. Furthermore,

$$Jh = Ih = Ia + I\square = Ia + 1 > Ia = I\grave{h} = J\grave{h}$$

by (11.6) and Lemma 11.32. Therefore, Lemma 11.34 holds in this case.

Case 12: $h' = (f\,\Omega x)\triangle$**.** Here $l = 1$, $k_1 = \pi$, $h = a_1 \,\breve{o}\pi (f\,\Omega x)\triangle$, and $h \sim \grave{h} = f(x/a_1)$ by Lemma 11.30 (31) (32). Also,

$$(\Phi^* h)\nu = (\Phi^* a_1)\nu \,\breve{o}\pi \big((\Phi^* f)\,\Omega x\big)\nu\triangle = a_1 \,\breve{o}\pi \big((\Phi^* f)\,\Omega x\big)\nu\triangle$$

by Lemma 11.31, so

$$(\Phi^* h)\nu = 1 \iff \big((\Phi^* f)\,\Omega x\big)\nu \ni a_1 \iff (\Phi^* f)\big((x/a_1)\nu\big) = 1$$

by Theorem 9.7 and (3.4). Since x is free from a_1 in f by Lemma 11.14,

$$(\Phi^* f)\big((x/a_1)\nu\big) = (\Phi^* f)\big(\big(x/(\Phi^* a_1)\nu\big)\nu\big) = \big(\Phi^* f(x/a_1)\big)\nu = (\Phi^* \grave{h})\nu$$

by Lemma 11.31 and Theorem 6.2. Thus $(\Phi^* h)\nu = (\Phi^* \grave{h})\nu$. Furthermore,

$$Jh = Ih = If + I(\Omega x) = If + 1 > If = I\big(f(x/a_1)\big) = I\grave{h} \geq J\grave{h}$$

by (11.6), Lemmas 11.32, and 11.33. Therefore, Lemma 11.34 holds in this case.

This completes the proof of Lemma 11.34. Thus we have proved $\vec{C} \subseteq [\vec{D}]_{\vec{R}}$. Therefore $\vec{C} = [\vec{D}]_{\vec{R}}$.

11.4 Proof of the completeness theorem for MPCL

Here we prove the Main Theorem 4.1 of this paper.

Lemma 11.35 Let (R, D) be a $\mathcal{G}_{\mathcal{W}}$-sound deduction pair on H, and assume that the deduction relation $\preccurlyeq_{R,D}$ is Boolean with respect to the operations $\wedge, \vee, \Rightarrow, \Diamond$ on H and satisfies the twenty seven laws, the case+ law to the $\forall-$ law, listed in §11.2. Then (R, D) is $\mathcal{G}_{\mathcal{W}}$-complete.

Proof Define $\vec{H}_{R,D} = \{\alpha \to \beta \in \vec{H} \mid \alpha \preccurlyeq_{R,D} \beta\}$ following (7.1). Let $(\vec{R}, \vec{D})$ be the deduction pair on $\vec{H}$ presented in §11.1. Then the assumption on $\preccurlyeq_{R,D}$ and Theorem 7.2 imply that $\vec{H}_{R,D}$ is closed under $\vec{R}$ and contains $\vec{D}$. We have shown $\vec{C} = [\vec{D}]_{\vec{R}}$ in §11.3. Therefore, (R, D) is $\mathcal{G}_{\mathcal{W}}$-complete by Theorem 7.1.

Remark 11.2 The equality $\vec{C} = [\vec{D}]_{\vec{R}}$ means that the validity relation $\preccurlyeq$ of the predicate logical space $(H, \mathcal{G}_{\mathcal{W}})$ is the smallest of the relations on H^* which are Boolean with respect to $\wedge, \vee, \Rightarrow, \Diamond$ and satisfy the twenty seven laws listed in §11.2. Lemma 11.35 may be proved by this fact, but we preferred to use Theorem 7.1.

Proof of Theorem 4.1 Define $\mathsf{R} = \wp \cup \&$. Then the deduction pair (R, ∇) on H is $\mathcal{G}_{\mathcal{W}}$-sound by Lemma 11.2 and the deduction relation $\preccurlyeq_{\mathsf{R},\nabla}$ is Boolean with respect to the operations $\wedge, \vee, \Rightarrow, \Diamond$ by Theorem 7.5. Also, since ∇ is closed under $\mathsf{R} \cup \bot \cup \top \cup \forall$ and contains ∂ by (4.3), it follows that $\preccurlyeq_{\mathsf{R},\nabla}$ satisfies the twenty seven laws listed in §11.2.

For instance as for the case+ law, assume $\preccurlyeq_{\mathsf{R},\nabla} \mathsf{f}$. Then $\mathsf{f} \in [\nabla]_{\mathsf{R}} = \nabla$, and so since ∇ is closed under $\bot$, we have $\mathsf{a}\,\breve{\mathsf{o}}\mathsf{k}\,\mathsf{f} \in \nabla = [\nabla]_{\mathsf{R}}$ for all $\mathsf{a} \in \mathsf{A}_{\varepsilon}$ and $\mathsf{k} \in \mathsf{K}_{\mathsf{f}}$, hence $\preccurlyeq_{\mathsf{R},\nabla} \mathsf{a}\,\breve{\mathsf{o}}\mathsf{k}\,\mathsf{f}$. Thus $\preccurlyeq_{\mathsf{R},\nabla}$ satisfies the case+ law. The same argument applies to the case− law and the $\forall$+ law as well.

Since $\mathsf{a}\,\breve{\mathsf{o}}\pi\,\mathsf{a}\triangle \in \partial \subseteq \nabla \subseteq [\nabla]_{\mathsf{R}}$ for all $\mathsf{a} \in \mathsf{A}_{\varepsilon}$, we have $\preccurlyeq_{\mathsf{R},\nabla} \mathsf{a}\,\breve{\mathsf{o}}\pi\,\mathsf{a}\triangle$. Thus $\preccurlyeq_{\mathsf{R},\nabla}$ satisfies the $=$ law.

Suppose the maximum ∞ of $\mathbb{P}$ exists. Then, since $(\mathsf{a}\,\overline{\infty}\pi\,\mathsf{one}\triangle)^{\Diamond} \in \partial \subseteq \nabla \subseteq [\nabla]_{\mathsf{R}}$ for all $\mathsf{a} \in \mathsf{G}$, we have $\preccurlyeq_{\mathsf{R},\nabla} (\mathsf{a}\,\overline{\infty}\pi\,\mathsf{one}\triangle)^{\Diamond}$, hence $\mathsf{a}\,\overline{\infty}\pi\,\mathsf{one}\triangle \preccurlyeq_{\mathsf{R},\nabla}$ by Theorem 7.3. Thus $\preccurlyeq_{\mathsf{R},\nabla}$ satisfies the $\overline{\infty}$ law. The same argument applies to the $\mathsf{one}^{\Box}$ law as well.

As for the $\mathfrak{Q}, \breve{\mathsf{o}}$ law, let $\mathsf{a} \in \mathsf{G}$, $\mathsf{b} \in \mathsf{A}_{\varepsilon}$, $\mathsf{f} \in \mathsf{H}$, $\mathsf{k}, \mathsf{l} \in \mathsf{K}_{\mathsf{f}}$, $\mathsf{k} \neq \mathsf{l}$, and $\lambda \in \{\breve{\mathsf{o}}\} \cup \mathfrak{Q}$. Also, let $\mathsf{a} \in \mathsf{A}_{\varepsilon}$ in case $\lambda = \breve{\mathsf{o}}$. Then, since $\mathsf{a}\,\lambda\mathsf{k}\,(\mathsf{b}\,\breve{\mathsf{o}}\mathsf{l}\,\mathsf{f}) \Rightarrow \mathsf{b}\,\breve{\mathsf{o}}\mathsf{l}\,(\mathsf{a}\,\lambda\mathsf{k}\,\mathsf{f}) \in \partial \subseteq \nabla \subseteq [\nabla]_{\mathsf{R}}$, we have $\preccurlyeq_{\mathsf{R},\nabla} \mathsf{a}\,\lambda\mathsf{k}\,(\mathsf{b}\,\breve{\mathsf{o}}\mathsf{l}\,\mathsf{f}) \Rightarrow \mathsf{b}\,\breve{\mathsf{o}}\mathsf{l}\,(\mathsf{a}\,\lambda\mathsf{k}\,\mathsf{f})$, hence $\mathsf{a}\,\lambda\mathsf{k}\,(\mathsf{b}\,\breve{\mathsf{o}}\mathsf{l}\,\mathsf{f}) \preccurlyeq_{\mathsf{R},\nabla} \mathsf{b}\,\breve{\mathsf{o}}\mathsf{l}\,(\mathsf{a}\,\lambda\mathsf{k}\,\mathsf{f})$ by Theorem 7.3. Similarly we have $\mathsf{b}\,\breve{\mathsf{o}}\mathsf{l}\,(\mathsf{a}\,\lambda\mathsf{k}\,\mathsf{f}) \preccurlyeq_{\mathsf{R},\nabla} \mathsf{a}\,\lambda\mathsf{k}\,(\mathsf{b}\,\breve{\mathsf{o}}\mathsf{l}\,\mathsf{f})$, and thus $\preccurlyeq_{\mathsf{R},\nabla}$ satisfies the $\mathfrak{Q}, \breve{\mathsf{o}}$ law. The same argument applies to the remaining laws other than the $\forall, \Rightarrow$ law, the $\forall, \mathfrak{P}$ law, and the $\sqcup, +$ law.

As for the $\forall, \Rightarrow$ law, let $\mathsf{f}, \mathsf{g} \in \mathsf{A}_{\emptyset}$, $\mathsf{x} \in \mathbb{X}_{\varepsilon}$, and assume $\mathsf{x} \not\preccurlyeq \mathsf{f}$. Then, since

$$\big(\mathsf{one}\,\forall\pi\,\big((\mathsf{f} \Rightarrow \mathsf{g})\,\Omega\mathsf{x}\big)\triangle\big) \Rightarrow \big(\mathsf{f} \Rightarrow \mathsf{one}\,\forall\pi\,(\mathsf{g}\,\Omega\mathsf{x})\triangle\big) \in \partial \subseteq \nabla \subseteq [\nabla]_{\mathsf{R}},$$

we have $\preccurlyeq_{\mathsf{R},\nabla} \big(\mathsf{one}\,\forall\pi\,\big((\mathsf{f} \Rightarrow \mathsf{g})\,\Omega\mathsf{x}\big)\triangle\big) \Rightarrow \big(\mathsf{f} \Rightarrow \mathsf{one}\,\forall\pi\,(\mathsf{g}\,\Omega\mathsf{x})\triangle\big)$, hence

$$\mathsf{one}\,\forall\pi\,\big((\mathsf{f} \Rightarrow \mathsf{g})\,\Omega\mathsf{x}\big)\triangle \preccurlyeq_{\mathsf{R},\nabla} \mathsf{f} \Rightarrow \mathsf{one}\,\forall\pi\,(\mathsf{g}\,\Omega\mathsf{x})\triangle,$$
$$\mathsf{f}, \mathsf{one}\,\forall\pi\,\big((\mathsf{f} \Rightarrow \mathsf{g})\,\Omega\mathsf{x}\big)\triangle \preccurlyeq_{\mathsf{R},\nabla} \mathsf{one}\,\forall\pi\,(\mathsf{g}\,\Omega\mathsf{x})\triangle.$$

by Theorem 7.3. Thus $\preccurlyeq_{\mathsf{R},\nabla}$ satisfies the $\forall, \Rightarrow$ law.

As for the $\forall, \mathfrak{P}$ law, let $\mathsf{a}, \mathsf{b} \in \mathsf{G}$, $\mathsf{f} \in \mathsf{H}$, $\mathsf{k} \in \mathsf{K}_{\mathsf{f}}$, and $\mathsf{p} \in \mathbb{P}$. Then, since

$$(\mathsf{a}\,\forall\pi\,\mathsf{b}\triangle \wedge \mathsf{a}\,\overline{\mathsf{p}}\mathsf{k}\,\mathsf{f}) \Rightarrow \mathsf{b}\,\overline{\mathsf{p}}\mathsf{k}\,\mathsf{f} \in \partial \subseteq \nabla \subseteq [\nabla]_{\mathsf{R}},$$

we have $\preccurlyeq_{\mathsf{R},\nabla} (\mathsf{a}\,\forall\pi\,\mathsf{b}\triangle \wedge \mathsf{a}\,\overline{\mathsf{p}}\mathsf{k}\,\mathsf{f}) \Rightarrow \mathsf{b}\,\overline{\mathsf{p}}\mathsf{k}\,\mathsf{f}$, hence

$$\mathsf{a}\,\forall\pi\,\mathsf{b}\triangle \wedge \mathsf{a}\,\overline{\mathsf{p}}\mathsf{k}\,\mathsf{f} \preccurlyeq_{\mathsf{R},\nabla} \mathsf{b}\,\overline{\mathsf{p}}\mathsf{k}\,\mathsf{f},$$
$$\mathsf{a}\,\forall\pi\,\mathsf{b}\triangle, \mathsf{a}\,\overline{\mathsf{p}}\mathsf{k}\,\mathsf{f} \preccurlyeq_{\mathsf{R},\nabla} \mathsf{b}\,\overline{\mathsf{p}}\mathsf{k}\,\mathsf{f}.$$

by Theorem 7.3. Thus $\preccurlyeq_{\mathsf{R},\nabla}$ satisfies the $\forall, \mathfrak{P}$ law.

As for the $\sqcup, +$ law, let $\mathsf{a}, \mathsf{b} \in \mathsf{G}$, $\mathsf{f} \in \mathsf{H}$, $\mathsf{k} \in \mathsf{K}_{\mathsf{f}}$, and $\mathsf{p}, \mathsf{q} \in \mathbb{P}$. Then, since

$$(\mathsf{a} \sqcup \mathsf{b})\,\overline{\mathsf{p} + \mathsf{q}}\mathsf{k}\,\mathsf{f} \Rightarrow (\mathsf{a}\,\overline{\mathsf{p}}\mathsf{k}\,\mathsf{f} \vee \mathsf{b}\,\overline{\mathsf{q}}\mathsf{k}\,\mathsf{f}) \in \partial \subseteq \nabla \subseteq [\nabla]_{\mathsf{R}},$$

we have $\preccurlyeq_{\mathsf{R},\nabla} (\mathsf{a} \sqcup \mathsf{b})\,\overline{\mathsf{p} + \mathsf{q}}\mathsf{k}\,\mathsf{f} \Rightarrow (\mathsf{a}\,\overline{\mathsf{p}}\mathsf{k}\,\mathsf{f} \vee \mathsf{b}\,\overline{\mathsf{q}}\mathsf{k}\,\mathsf{f})$, hence

$$(\mathsf{a} \sqcup \mathsf{b})\,\overline{\mathsf{p} + \mathsf{q}}\mathsf{k}\,\mathsf{f} \preccurlyeq_{\mathsf{R},\nabla} \mathsf{a}\,\overline{\mathsf{p}}\mathsf{k}\,\mathsf{f} \vee \mathsf{b}\,\overline{\mathsf{q}}\mathsf{k}\,\mathsf{f},$$
$$(\mathsf{a} \sqcup \mathsf{b})\,\overline{\mathsf{p} + \mathsf{q}}\mathsf{k}\,\mathsf{f} \preccurlyeq_{\mathsf{R},\nabla} \mathsf{a}\,\overline{\mathsf{p}}\mathsf{k}\,\mathsf{f}, \mathsf{b}\,\overline{\mathsf{q}}\mathsf{k}\,\mathsf{f}.$$

by Theorem 7.3. Thus $\preccurlyeq_{\mathsf{R},\nabla}$ satisfies the $\sqcup, +$ law.

Therefore R, ∇ is $\mathcal{G}_{\mathcal{W}}$-complete by Lemma 11.35. Consequently $\mathsf{C} = [\nabla]_{\mathsf{R}} = \nabla$, and so $(\wp, \nabla)$ is also $\mathcal{G}_{\mathcal{W}}$-complete by Theorem 7.4.

References

[1] Feferman, S., et al. (eds.), *Kurt Gödel Collected Works*, Oxford Univ. Press, 1986.

[2] Gomi, K., "Foundations of algebraic logic," preprint to be uploaded to http://faculty.ms.u-tokyo.ac.jp/users/preprint/preprint2008.html and http://homepage3.nifty.com/gomiken/english/mathpsy.htm.

[3] Gomi, K., *Mathematical Psychology* (in Japanese), personal electronic publication, http://homepage3.nifty.com/gomiken/mathpsy/index.htm, 1997.

[4] Gomi, K., "Sorted algebras and based algebras," preprint to be uploaded to http://faculty.ms.u-tokyo.ac.jp/users/preprint/preprint2008.html and http://homepage3.nifty.com/gomiken/english/mathpsy.htm.

[5] Gomi, K., "Theory of completeness for logical spaces," preprint to be uploaded to http://faculty.ms.u-tokyo.ac.jp/users/preprint/preprint2008.html and http://homepage3.nifty.com/gomiken/english/mathpsy.htm.

[6] Hilbert, D., and P. Bernays, *Grundlagen der Mathematik*, Springer, 1934, 1939.

[7] Mizumura, Y., *Completeness of the logical system MCL* (in Japanese), master's thesis, Graduate School of Mathematical Sciences, University of Tokyo, 2000, expanded and uploaded to http://homepage3.nifty.com/gomiken/mathpsy/mizmra/mcl.zip.

[8] Takaoka, Y., *MPCL is in the 3rd class*, a seminary talk recorded in [3].

[9] Thomason, R. (ed.), *Formal Philosophy*, selected papers of Richard Montague, Yale University Press, 1974.

UTMS

2008–12 Hajime Fujita, Mikio Furuta and Takahiko Yoshida: *Acyclic polarizations and localization of Riemann-Roch numbers I.*

2008–13 Rolci Cipolatti and Masahiro Yamamoto: *Inverse hyperbolic problem by a finite time of observations with arbitrary initial values.*

2008–14 Yoshifumi Matsuda: *Groups of real analytic diffeomorphisms of the circle with a finite image under the rotation number function.*

2008–15 Shoichi Kaji: *The* $(\mathfrak{g}, K)$*-module structures of the principal series representations of* $SL(4, R)$.

2008–16 G. Bayarmagnai: *The* $(\mathfrak{g}, K)$*-module structures of principal series of SU(2,2).*

2008–17 Takashi Tsuboi: *On the group of real analytic diffeomorphisms.*

2008–18 Takefumi Igarashi and Noriaki Umeda: *Nonexistence of global solutions in time for reaction-diffusion systems with inhomogeneous terms in cones.*

2008–19 Oleg Yu. Imanouilov, Gunther Uhlmann, and Masahiro Yamamoto: *Partial data for the Calderón problem in two dimensions.*

2008–20 Xuefeng Liu and Fumio Kikuchi: *Analysis and estimation of error constants for* P_0 *and* P_1 *interpolations over triangular finite elements.*

2008–21 Fumio Kikuchi, Keizo Ishii and Issei Oikawa: *Discontinuous Galerkin FEM of Hybrid displacement type – Development of polygonal elements –.*

2008–22 Toshiyuki Kobayashi and Gen Mano: *The Schrödinger model for the minimal representation of the indefinite orthogonal group* $O(p, q)$.

2008–23 Kensaku Gomi and Yasuaki Mizumura: *A Completeness Theorem for the Logical System MPCL Designed for Mathematical Psychology.*

The Graduate School of Mathematical Sciences was established in the University of Tokyo in April, 1992. Formerly there were two departments of mathematics in the University of Tokyo: one in the Faculty of Science and the other in the College of Arts and Sciences. All faculty members of these two departments have moved to the new graduate school, as well as several members of the Department of Pure and Applied Sciences in the College of Arts and Sciences. In January, 1993, the preprint series of the former two departments of mathematics were unified as the Preprint Series of the Graduate School of Mathematical Sciences, The University of Tokyo. For the information about the preprint series, please write to the preprint series office.

ADDRESS:
Graduate School of Mathematical Sciences, The University of Tokyo
3–8–1 Komaba Meguro-ku, Tokyo 153-8914, JAPAN
TEL +81-3-5465-7001 FAX +81-3-5465-7012