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1 Introduction

Harish-Chandra expansion of the matrix coefficients of standard representations of a real reductive group has been one of the fundamental themes in the real harmonic analysis on real redutive groups. The investigation and result of class-one principal series by Harish-Chandra have been considered to be rather satisfactory. However, say, to have deeper arithmetic results for automorphic forms, it seems to be necessary to have more *effective computable* results on this kind of expansion for more general class of represenations including discrete series represenations. Futher it is much better to have more explicit results not only on the leading coefficients but also on the coefficients of higher degree in each term of the asymptotic expansion, which is a kind of hypergeometric series. But this problem seems to be quite difficult currently to discuss generally.

In this paper, we find the expansion formula of the matrix coefficients of the relatively small generalized principal series representation (which we sometimes refer as P_J principal series) of the real symplectic group of the real rank 2, $Sp(2,\mathbb{R})$. The reason of the choice of this type of representations is because it has the same invariants (the Gelfand-Kirillov dimension and the Bernstein degree) as the large discrete series of $Sp(2,\mathbb{R})$. Thus we may expect that a similar result is also valid for the large discrete series.

In contrast to the class-one case where the expansion consists of the terms corresponding to elements of the Weyl group (of order 8 here), our expansion has 4 terms (*cf.* Theorem 6.1). Here 4 is the Bernstein degree of our representations. Moreover we have explicit power series expansion with respect to a (probably) good choice of local parameters at infinity (*cf.* Theorem 7.1).

In the literature there are a number of papers to compute Harish-Chandra expansions, normally for relatively small groups. But it seems to be few results to handle the case of parabolic induction with respect to a non-minimal parabolic subgroup of a group with real rank bigger than 1. Therefore, in view of the relation between the Plancherel measure and c-function (i.e., the coefficients of the asymtotic expansion), our determination of the corresponding coefficients might also be interesting.

Our method of proof is done by a very down-to-earth or 'elementary' manner. We take the advantage to start from an (Eulerian) integral expression of our matrix coefficient in terms of Gaussin hypergeometric function, obtained in a previous paper [2]. What we neesd is the classical connection formula of Kummer and some general frame work of the asymptotic behavior of the ideally analytic solution of holonomic systems in our setting (§4 and §5).

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2 P_J-principal series representations

In this section, we recall some facts about representations of $Sp(2,\mathbb{R})$ and their K-type. Notations are same as those of [2].

Let $G = Sp(2,\mathbb{R})$ be a split real semisimple Lie group of real rank 2 with a maximal compact subgroup K which is isomorphic to the unitary group $U(2)$. The group G has two standard maximal parabolic subgroups. One is associated with the short simple root $e_1 - e_2$ and called the Siegel parabolic subgroup. The other is associated with the long simple root $2e_2$ and called the Jacobi parabolic subgroup P_J. Some people call this subgroup the Klingen parabolic subgroup.

We set the Langlands decomposition of P_J is $P_J = M_J A_J N_J$, then M_J is isomorphic to $SL(2,\mathbb{R}) \times \{\pm 1\}$. Let D_l^+ is the holomorphic discrete series representation of $SL(2,\mathbb{R})$ and D_l^- is the anti-holomorphic discrete series representation of $SL(2,\mathbb{R})$. The parameter l is the Blattner parameter, which satisfies the condition that $l \in \mathbb{N}$ and $l \geq 2$. We denote the character of $\{\pm 1\}$ by ε and the complex valued linear form on $\mathfrak{a}_J = \mathrm{Lie}(A_J)$ by ν. The generalized principal series representation of $Sp(2,\mathbb{R})$ which we call P_J-principal series representation is the induced representation $\pi_{(D_l^\pm,\varepsilon),\nu} = \mathrm{Ind}_{P_J}^G(D_l^\pm \otimes \varepsilon \otimes e_1^{\nu+\rho_J} \otimes \mathrm{id}_{N_J})$. Here, id_{N_J} is the trivial representation of N_J and ρ_J is the half sum of positive roots corresponding to N_J.

A P_J-principal series representation has a special K-type of multiplicity free. We call the K-type as "the corner K-type".

If the character ε of $\{\pm 1\}$ satisfies $\varepsilon(-1) = (-1)^l$, then the corner K-type of $\pi_{(D_l^\pm,\varepsilon),\nu}$ is the one dimensional representation $\tau_{(l,l)}$ whose highest weight is (l,l) and if $\varepsilon = (-1)^{l+1}$ holds, then the corner K-type is the two dimensional representation $\tau_{(l,l-1)}$ whose highest weight is $(l, l-1)$.

Let $(\eta, V_\eta), (\tau, V_\tau)$ be in $\hat{K}$. We denote the contragredient representation of τ by τ^*. We define the space of spherical functions

$$C^\infty_{\eta,\tau}(K\backslash G/K) = \{f : G \to V_\eta \otimes V_{\tau^*} \mid \quad f \text{ is a } C^\infty \text{ function}, f(k_1 g k_2) = \eta(k_1) \otimes \tau^*(k_2)^{-1} f(g), \\ \forall g \in G, \forall k_1, \forall k_2 \in K\}.$$

In this paper, we consider the matrix coefficient $\phi \in C^\infty_{\tau_{(k,k)},\tau_{(l,l)}}(K\backslash G/K)$ of $\pi_{(D_l^\pm,\varepsilon),\nu}$ for $k \geq l$ and $k \equiv l \mod 2$ and $\varepsilon(-1) = (-1)^l$ (It goes almost same way for the case of $\varepsilon(-1) = (-1)^{l+1}$.). If $k < l$ or $k \not\equiv l \mod 2$ holds, then $\phi = 0$ (Proposition 3.4 and Lemma 4.2 in [2]). That is why we call $\tau_{(l,l)}$ the corner K-type.

We denote the standard split Cartan subgroup of G by

$$A = \{\mathrm{diag}(a_1, a_2, a_1^{-1}, a_2^{-1}) \mid a_1, a_2 \in \mathbb{R}_{>0}\}.$$

The system of partial differential equations satisfied by the A-radial part of ϕ is the holonomic system. We choose the coordinates of A as (x_1, x_2) determined by $a_1 = \exp x_1, a_2 = \exp x_2$.

We recall the system of differential equations satisfied by ϕ (Theorem 7.5 in [2]).

Theorem 2.1. *ϕ satisfies the following system of differential equations :*

$$\begin{aligned}
&\sum_{i=1}^{2}\frac{\partial^2}{\partial x_i^2}\phi+\sum_{i=1}^{2}\{2\coth 2x_i+\coth(x_1+x_2)\}\frac{\partial}{\partial x_i}\phi \\
&\qquad +\coth(x_1-x_2)\frac{\partial}{\partial x_1}\phi-\coth(x_1-x_2)\frac{\partial}{\partial x_2}\phi \\
&\qquad -(k^2+l^2)(\operatorname{sh}^{-2}x_1+\operatorname{sh}^{-2}x_2)\phi+2kl(\operatorname{ch}2x_1\cdot\operatorname{sh}^{-2}2x_1+\operatorname{ch}2x_2\cdot\operatorname{sh}^{-2}2x_2)\phi \\
&\quad =\{\nu^2+(l-1)^2-5\}\phi, \qquad (2.1)\\
&2\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2}\phi+\{2l\coth 2x_2-2k\operatorname{sh}^{-1}2x_2+\coth(x_1+x_2)-\coth(x_1-x_2)\}\frac{\partial}{\partial x_1}\phi \\
&\qquad +\{2l\coth 2x_1-2k\operatorname{sh}^{-1}2x_1+\coth(x_1+x_2)+\coth(x_1-x_2)\}\frac{\partial}{\partial x_2}\phi \\
&\qquad +2(l\coth 2x_1-k\operatorname{sh}^{-1}2x_1)(l\coth 2x_2-k\operatorname{sh}^{-1}2x_2)\phi \\
&\qquad +(l\coth 2x_2-k\operatorname{sh}^{-1}2x_2)(\coth(x_1+x_2)+\coth(x_1+x_2))\phi \\
&\qquad +(l\coth 2x_1-k\operatorname{sh}^{-1}2x_1)(\coth(x_1+x_2)-\coth(x_1+x_2))\phi=0. \qquad (2.2)
\end{aligned}$$

In the following section, we will determine characteristic roots of the system around the infinity, $a_1/a_2=0, a_2=0$ (Note that a_1/a_2 and a_2^2 correspond simple roots e_1-e_2 and $2e_2$ respectively). There are 4 characteristic roots, so the system has 4 independent solutions. A solution ϕ is a linear combination of these solutions and its coefficients are analogues of c-functions.

3 The holonomic system

We put $\delta(x_1,x_2)=(\operatorname{ch}x_1\operatorname{ch}x_2)^{(l+k)/2}(\operatorname{sh}x_1\operatorname{sh}x_2)^{(l-k)/2}$ and

$$\psi(x_1,x_2)=\delta(x_1,x_2)\phi(x_1,x_2).$$

Proposition 3.1. *ψ satisfies the following system of partial differential equations.*

$$\begin{aligned}
&\sum_{i=1}^{2}\frac{\partial}{\partial x_i^2}\psi+\sum_{i=1}^{2}\{2k\operatorname{sh}^{-1}2x_i-2(l-1)\coth 2x_i\}\frac{\partial}{\partial x_i}\psi \\
&\qquad +\frac{\operatorname{sh}2x_1}{\operatorname{sh}^2x_1-\operatorname{sh}^2x_2}\frac{\partial}{\partial x_1}\psi-\frac{\operatorname{sh}2x_2}{\operatorname{sh}^2x_1-\operatorname{sh}^2x_2}\frac{\partial}{\partial x_2}\psi=\{\nu^2-(l-2)^2\}\psi \qquad (3.1)\\
&\frac{\partial^2}{\partial x_1\partial x_2}\psi-\frac{1}{2}\frac{\operatorname{sh}2x_2}{\operatorname{sh}^2x_1-\operatorname{sh}^2x_2}\frac{\partial}{\partial x_1}\psi+\frac{1}{2}\frac{\operatorname{sh}2x_1}{\operatorname{sh}^2x_1-\operatorname{sh}^2x_2}\frac{\partial}{\partial x_2}\psi=0 \qquad (3.2)
\end{aligned}$$

Proof. This system is easily obtained from Theorem 2.1. □

We will transform this system into the system with variables

$$y_1=(a_1/a_2)^2=\exp 2(x_1-x_2),\quad y_2=a_2^2=\exp 2x_2.$$

Since $y_1y_2=\exp 2x_1$, we have

$$\begin{aligned}
\operatorname{sh}2x_1&=\frac{y_1y_2-y_1^{-1}y_2^{-1}}{2}=\frac{y_1^2y_2^2-1}{2y_1y_2},\quad \operatorname{sh}2x_2=\frac{y_2-y_2^{-1}}{2}=\frac{y_2^2-1}{2y_2},\\
\operatorname{sh}^2x_1-\operatorname{sh}^2x_2&=\frac{1}{2}(\operatorname{ch}2x_1-\operatorname{ch}2x_2)=\frac{y_1y_2+y_1^{-1}y_2^{-1}}{4}-\frac{y_2+y_2^{-1}}{4}\\
&=\frac{(y_1-1)(y_1y_2^2-1)}{4y_1y_2},\\
\coth 2x_1&=\frac{y_1y_2+y_1^{-1}y_2^{-1}}{y_1y_2-y_1^{-1}y_2^{-1}}=\frac{y_1^2y_2^2+1}{y_1^2y_2^2-1},\quad \coth 2x_2=\frac{y_2+y_2^{-1}}{y_2-y_2^{-1}}=\frac{y_2^2+1}{y_2^2-1},
\end{aligned}$$

and

$$\frac{\partial}{\partial x_1} = 2y_1 \frac{\partial}{\partial y_1}, \quad \frac{\partial}{\partial x_2} = -2y_1 \frac{\partial}{\partial y_1} + 2y_2 \frac{\partial}{\partial y_2}.$$

We regard ϕ and ψ as functions in variables y_1, y_2 below. Then, we have the system of differential equations in y_1, y_2 as follows.

Proposition 3.2. *ψ satisfies the system differential equations:*

$$\begin{aligned}
4&\left\{2\left(y_1\frac{\partial}{\partial y_1}\right)^2 - 2\left(y_1\frac{\partial}{\partial y_1}\right)\left(y_2\frac{\partial}{\partial y_2}\right) + \left(y_2\frac{\partial}{\partial y_2}\right)^2\right\}\psi \\
&+\left\{\frac{4ky_1y_2}{y_1^2y_2^2-1} - \frac{4ky_2}{y_2^2-1} - 2(l-1)\frac{y_1^2y_2^2+1}{y_1^2y_2^2-1} + 2(l-1)\frac{y_2^2+1}{y_2^2-1}\right\}\left(2y_1\frac{\partial}{\partial y_1}\right)\psi \\
&+\left\{\frac{4ky_2}{y_2^2-1} - 2(l-1)\frac{y_2^2+1}{y_2^2-1}\right\}\left(2y_2\frac{\partial}{\partial y_2}\right)\psi \\
&+4\frac{y_1^2y_2^2-1+y_1(y_2^2-1)}{(y_1-1)(y_1y_2^2-1)}\left(y_1\frac{\partial}{\partial y_1}\right)\psi - 4\frac{y_1(y_2^2-1)}{(y_1-1)(y_1y_2^2-1)}\left(y_2\frac{\partial}{\partial y_2}\right)\psi \\
&=\{\nu^2-(l-2)^2\}\psi,
\end{aligned} \tag{3.3}$$

$$\begin{aligned}
&\left(y_1\frac{\partial}{\partial y_1}\right)\left(-y_1\frac{\partial}{\partial y_1} + y_2\frac{\partial}{\partial y_2}\right)\psi \\
&\quad -\frac{1}{2}\frac{y_1(y_2^2-1)+(y_1^2y_2^2-1)}{(y_1-1)(y_1y_2^2-1)}\left(y_1\frac{\partial}{\partial y_1}\right)\psi + \frac{1}{2}\frac{y_1^2y_2^2-1}{(y_1-1)(y_1y_2^2-1)}\left(y_2\frac{\partial}{\partial y_2}\right)\psi = 0.
\end{aligned} \tag{3.4}$$

4 Characteristic indices

We set

$$\psi(y_1, y_2) = y_1^{\alpha} y_2^{\beta} + \text{higher powers},$$

that is, (α, β) is the leading exponent of ψ at $(y_1, y_2) = (0, 0)$.

Proposition 4.1. *(α, β) is one of the followings :*

$$(\frac{1}{2}, \mu_\pm), \quad (\mu_\pm, \mu_\pm).$$

Here, $\mu_\pm = \pm\frac{\nu}{2} - \frac{l-2}{2}$.

Proof. If we set $\psi(y_1, y_2) = y_1^{\alpha} y_2^{\beta} + \text{higher powers}$, then we obtain the indicial equation

$$\alpha(-\alpha+\beta) - \frac{1}{2}\frac{(-1)}{(-1)^2}\alpha + \frac{1}{2}\frac{(-1)}{(-1)^2}\beta = 0$$

from the equation (3.4). The solutions of this equation is

$$\alpha = \frac{1}{2} \ \text{ or } \ \alpha = \beta.$$

We obtain the other indicial equation from the equation (3.3) :

$$\begin{aligned}
4(2\alpha^2 - 2\alpha\beta + \beta^2) &+ \left\{-2(l-1)\frac{1}{(-1)} + 2(l-1)\frac{1}{(-1)}\right\}(2\alpha) \\
&+\left\{-2(l-1)\frac{1}{(-1)}\right\}(2\beta) + 4\frac{(-1)}{(-1)^2}\alpha = \nu^2 - (l-2)^2.
\end{aligned} \tag{4.1}$$

In both cases $\alpha = 1/2$ and $\alpha = \beta$, the equation (4.1) is equivalent to

$$4\beta^2 + 4(l-2)\beta = \nu^2 - (l-2)^2.$$

Hence $\beta = \pm\nu/2 - (l-2)/2$. $\square$

We denote ψ with the leading term $y_1^\alpha y_2^\beta$ by $\psi_{\alpha,\beta}$.

Since the multiplier $\delta(x_1, x_2)^{-1}$ is expanded as

$$\begin{aligned}
\delta(x_1,x_2)^{-1} &= (\operatorname{ch} x_1 \operatorname{ch} x_2)^{-\frac{l+k}{2}} (\operatorname{sh} x_1 \operatorname{sh} x_2)^{-\frac{l-k}{2}} \\
&= \left(\frac{a_1 + a_1^{-1}}{2} \cdot \frac{a_2 + a_2^{-1}}{2}\right)^{-\frac{l+k}{2}} \left(\frac{a_1 - a_1^{-1}}{2} \cdot \frac{a_2 - a_2^{-1}}{2}\right)^{-\frac{l-k}{2}} \\
&= 2^{2l} a_1^l a_2^l (1 + \text{higher order}) \\
&= 2^{2l} (y_1 y_2)^{\frac{l}{2}} y_2^{\frac{l}{2}} (1 + \text{higher order}) \\
&= 2^{2l} y_1^{\frac{l}{2}} y_2^l (1 + \text{higher order}),
\end{aligned}$$

we have

$$\delta(x_1,x_2)^{-1}\psi_{\alpha,\beta}(y_1,y_2) = 2^{2l} y_1^{\frac{l}{2}} y_2^l (1 + \text{higher order}) \cdot (y_1^\alpha y_2^\beta + \text{higher order}).$$

So we set

$$\phi_{\alpha+l/2,\beta+l}(y_1,y_2) = 2^{-2l}\delta(x_1,x_2)^{-1}\psi_{\alpha,\beta}(y_1,y_2). \tag{4.2}$$

The leading term of this function $\phi_{\alpha+l/2,\beta+l}$ is

$$y_1^{\frac{l}{2}} y_2^l \cdot y_1^\alpha y_2^\beta = \begin{cases} y_1^{\frac{l+1}{2}} y_2^{\mu_\pm + l} & \text{if } \alpha = \frac{1}{2},\ \beta = \mu_\pm \\ y_1^{\mu_\pm + \frac{l}{2}} y_2^{\mu_\pm + l} & \text{if } \alpha = \beta = \mu_\pm. \end{cases}$$

These exponents are same as the Siegel-Whittaker function and the Whittaker function([1]).

5 The singular boundary value problem

We would like to represent ψ as the linear combination of $\psi_{\alpha,\beta}$.

To do that, we will obtain analytic continuation of ψ from $(y_1, y_2) = (1, 1)$, which corresponds to the identity of G, to $(y_1, y_2) = (1, 0)$ at first, then $(y_1, y_2) = (1, 0)$ to $(y_1, y_2) = (0.0)$. The first part was almost done in §9 of [2]. So we discuss the latter part in this section. General references for the singular boundary value problem are [5] and [6].

5.1 Justification of the singular boundary problem

We obtain the following equation by $1/4 \times$ the equation $(3.3) + 2 \times$ the equation (3.4).

$$\begin{aligned}
&\left(y_2 \frac{\partial}{\partial y_2}\right)^2 \psi + \left\{\frac{2k(1-y_1)y_2(1+y_1y_2^2) - 2(l-1)(1-y_1^2)y_2^2}{(y_2^2-1)(y_1^2y_2^2-1)}\right\}\left(y_1 \frac{\partial}{\partial y_1}\right)\psi \\
&+ \left\{\frac{2ky_2 - (l-1)(y_2^2+1)}{y_2^2-1} + \frac{y_1y_2^2+1}{y_1y_2^2-1}\right\}\left(y_2\frac{\partial}{\partial y_2}\right)\psi = \frac{1}{4}\{\nu^2 - (l-2)^2\}\psi.
\end{aligned}$$

This differential equation has regular singularities along $y_2 = 0$ and its indicial equation is

$$\beta^2 + \left\{\frac{-(l-1)}{(-1)} + \frac{1}{(-1)}\right\}\beta = \frac{1}{4}\{\nu^2 - (l-2)^2\}.$$

Hence, we have $\beta = \pm\nu/2 - (l-2)/2 = \mu_\pm$, which is independent from y_1. Therefore, when the difference $\mu_+ - \mu_- = \nu$ is not an integer, the above differential equation has a solution

$$\psi(y_1, y_2) = a_+(y_1, y_2) y_2^{\mu_+} + a_-(y_1, y_2) y_2^{\mu_-}.$$

Functions $a_\pm(y_1, y_2)$ are real analytic function around $(y_1, y_2) = (0, 0)$. This solution is called the ideally analytic solution.

So we assume that ν is not an integer hereafter.

5.2 The equation of the singular boundary value

In the beginning, we will find the equation which is satisfied by $f_\pm(y_1) = \lim_{y_2 \to 0} a_\pm(y_1, y_2)$.

Lemma 5.1. *The function $f_\pm(y_1)$ satisfies the following ordinary differential equation.*

$$\left\{ \left(y_1 \frac{d}{dy_1} \right)^2 - \mu_\pm \left(y_1 \frac{d}{dy_1} \right) + \frac{1}{2} \frac{y_1 + 1}{y_1 - 1} \left(y_1 \frac{d}{dy_1} \right) - \frac{1}{2} \frac{\mu_\pm}{y_1 - 1} \right\} f_\pm(y_1) = 0.$$

Proof. Inserting $\psi(y_1, y_2) = a_\pm(y_1, y_2) y_2^{\mu_\pm}$ into the equation (3.4), we have

$$\begin{aligned} y_2^{\mu_\pm} \Bigg(& - \left(y_1 \frac{\partial}{\partial y_1} \right)^2 a_\pm(y_1, y_2) + \mu_\pm \left(y_1 \frac{\partial}{\partial y_1} \right) a_\pm(y_1, y_2) + y_2 \left(y_1 \frac{\partial}{\partial y_1} \right) \frac{\partial a_\pm(y_1, y_2)}{\partial y_2} \\ & - \frac{1}{2} \frac{y_1(y_2^2 - 1) + (y_1^2 y_2^2 - 1)}{(y_1 - 1)(y_1 y_2^2 - 1)} \left(y_1 \frac{\partial}{\partial y_1} \right) a_\pm(y_1, y_2) \\ & + \frac{1}{2} \frac{y_1^2 y_2^2 - 1}{(y_1 - 1)(y_1 y_2^2 - 1)} \left(\mu_\pm a_\pm(y_1, y_2) + y_2 \frac{\partial a_\pm(y_1, y_2)}{\partial y_2} \right) \Bigg) = 0. \end{aligned}$$

Dividing both sides of this equation by $y_2^{\mu_\pm}$ and taking limit $y_2 \to 0$, then we obtain

$$\begin{aligned} & - \left(y_1 \frac{d}{dy_1} \right)^2 f_\pm(y_1) + \mu_\pm \left(y_1 \frac{d}{dy_1} \right) f_\pm(y_1) \\ & \qquad - \frac{1}{2} \frac{y_1(-1) + (-1)}{(y_1 - 1)(-1)} \left(y_1 \frac{d}{dy_1} \right) f_\pm(y_1) + \frac{1}{2} \frac{(-1)\mu_\pm}{(y_1 - 1)(-1)} f_\pm(y_1) = 0. \end{aligned}$$

□

Now changing variables as $y_1 = 1/\zeta$, the equation in the previous lemma changes into the Gaussian hypergeometric equation of $\tilde{f}_\pm(\zeta) = f_\pm(y_1)$ with parameters $a = 1/2, b = \mu_\pm, c = \mu_\pm + 1/2 = a + b$:

$$\left[\zeta(1 - \zeta) \frac{d^2}{d\zeta^2} + \left\{ (\mu_\pm + \frac{1}{2}) - (\mu_\pm + \frac{1}{2} + 1)\zeta \right\} \frac{d}{d\zeta} - \frac{1}{2} \mu_\pm \right] \tilde{f}_\pm(\zeta) = 0.$$

The solution of this equation is

$$\tilde{f}_\pm(\zeta) = \mathcal{P} \left\{ \begin{array}{cccc} 0 & \infty & 1 & \\ 0 & \frac{1}{2} & 0 & : \zeta \\ \frac{1}{2} - \mu_\pm & \mu_\pm & 0 & \end{array} \right\} = \mathcal{P} \left\{ \begin{array}{cccc} 0 & \infty & 1 & \\ 0 & \frac{1}{2} & 0 & : 1 - \zeta \\ 0 & \mu_\pm & \frac{1}{2} - \mu_\pm & \end{array} \right\}$$

Therefore, the regular solution around $\zeta = 1$(this means $y_1 = 1$) is

$${}_2F_1 \left(\frac{1}{2}, \mu_\pm; 1; 1 - \zeta \right) = {}_2F_1 \left(\frac{1}{2}, \mu_\pm; 1; 1 - \frac{1}{y_1} \right)$$

up to a constant multiple. We denote this function by $f_\pm(y_1)$.

Using the connection formula of ${}_2F_1$ ([4] equation (9.5.8)):

$$\begin{aligned} {}_2F_1(a,b;c;z) &= (1-z)^{-a}\frac{\Gamma(c)\Gamma(b-a)}{\Gamma(c-a)\Gamma(b)}\,{}_2F_1\left(a,c-b;1+a-b;\frac{1}{1-z}\right) \\ &+ (1-z)^{-b}\frac{\Gamma(c)\Gamma(a-b)}{\Gamma(c-b)\Gamma(a)}\,{}_2F_1\left(c-a,b;1-a+b;\frac{1}{1-z}\right), \end{aligned}$$

and $1/\{1-(1-1/y_1)\} = y_1$, we obtain

$$\begin{aligned} f_\pm(y_1) &= {}_2F_1\left(\frac{1}{2},\mu_\pm;1;1-\frac{1}{y_1}\right) \\ &= \left(\frac{1}{y_1}\right)^{-\frac{1}{2}}\frac{\Gamma(1)\Gamma(\mu_\pm-\frac{1}{2})}{\Gamma(1-\frac{1}{2})\Gamma(\mu_\pm)}\,{}_2F_1\left(\frac{1}{2},1-\mu_\pm;\frac{3}{2}-\mu_\pm;y_1\right) \\ &+ \left(\frac{1}{y_1}\right)^{-\mu_\pm}\frac{\Gamma(1)\Gamma(\frac{1}{2}-\mu_\pm)}{\Gamma(1-\mu_\pm)\Gamma(\frac{1}{2})}\,{}_2F_1\left(\frac{1}{2},\mu_\pm;\frac{1}{2}+\mu_\pm;y_1\right) \\ &= \frac{\Gamma(\mu_\pm-\frac{1}{2})}{\sqrt{\pi}\Gamma(\mu_\pm)}y_1^{\frac{1}{2}}\,{}_2F_1\left(\frac{1}{2},1-\mu_\pm;\frac{3}{2}-\mu_\pm;y_1\right) \\ &+ \frac{\Gamma(\frac{1}{2}-\mu_\pm)}{\sqrt{\pi}\Gamma(1-\mu_\pm)}y_1^{\mu_\pm}\,{}_2F_1\left(\frac{1}{2},\mu_\pm;\frac{1}{2}+\mu_\pm;y_1\right) \end{aligned}$$

Note that our hypothesis $\nu \notin \mathbb{Z}$ guarantees that the Gamma functions in numerators have no poles.

The function $a_\pm(y_1,y_2)y_2^{\mu_\pm} = f_\pm(y_1)y_2^{\mu_\pm}(1+O(y_2))$ is a linear combination of $\psi_{\alpha,\beta}$. Comparing the leading term, we have

$$\psi_{\frac{1}{2},\mu_\pm}(y_1,y_2) = y_1^{\frac{1}{2}}y_2^{\mu_\pm}\,{}_2F_1\left(\frac{1}{2},1-\mu_\pm;\frac{3}{2}-\mu_\pm;y_1\right) + \text{ (higher order term)},$$
$$\psi_{\mu_\pm,\mu_\pm}(y_1,y_2) = y_1^{\mu_\pm}y_2^{\mu_\pm}\,{}_2F_1\left(\frac{1}{2},\mu_\pm;\frac{1}{2}+\mu_\pm;y_1\right) + \text{ (higher order term)}$$

and

$$a_\pm(y_1,y_2)y_2^{\mu_\pm} = \frac{\Gamma(\mu_\pm-\frac{1}{2})}{\sqrt{\pi}\Gamma(\mu_\pm)}\psi_{\frac{1}{2},\mu_\pm}(y_1,y_2) + \frac{\Gamma(\frac{1}{2}-\mu_\pm)}{\sqrt{\pi}\Gamma(1-\mu_\pm)}\psi_{\mu_\pm,\mu_\pm}(y_1,y_2).$$

6 The Harish-Chandra expansion

The matrix coefficient ϕ corresponding to the corner K-type of $\pi_{(D_l^\pm,\varepsilon),\nu}$ was proved to be represented as

$$\phi(x_1,x_2) = \delta(x_1,x_2)^{-1}F_{10}\left(\begin{matrix}\mu_+ & \mu_- & \frac{1}{2} & \frac{1}{2} \\ 1 & C & & \end{matrix};-\operatorname{sh}^2 x_1,-\operatorname{sh}^2 x_2\right),$$

where F_{10} is a hypergeometric function

$$F_{10}\left(\begin{matrix}a & b & c_1 & c_2 \\ d & e & & \end{matrix};x_1,x_2\right) = \sum_{m_i\geq 0}\frac{(a)_{m_1+m_2}(b)_{m_1+m_2}(c_1)_{m_1}(c_2)_{m_2}}{m_1!m_2!(d)_{m_1+m_2}(e)_{m_1+m_2}}x_1^{m_1}x_2^{m_2}$$

and $(\lambda)_n = \dfrac{\Gamma(\lambda+n)}{\Gamma(\lambda)}$ (Theorem 8.1 in [2]).

Theorem 6.1. *Assume $\nu \notin \mathbb{Z}$. We set $C = \dfrac{3+k-l}{2} \in \dfrac{1}{2}\mathbb{Z} - \mathbb{Z}$ and $\mu_\pm$ as above. The A-radial part of the matrix coefficient of the P_J-principal series representation with respect to the corner K-type $\tau_{(l,l)}$*

$$\delta(x_1,x_2)^{-1} F_{10}\left(\begin{matrix} \mu_+ & \mu_- & \frac{1}{2} & \frac{1}{2} \\ 1 & C & & \end{matrix} ; -\operatorname{sh}^2 x_1, -\operatorname{sh}^2 x_2\right)$$

has the following expansion around $y_1 = y_2 = 0$:

$$\begin{aligned} &\frac{4^{\mu_+ + l}\Gamma(-\nu)\Gamma(C)}{\sqrt{\pi}\Gamma(\mu_-)\Gamma(C-\mu_+)} \left\{ \frac{\Gamma(\mu_+ - \frac{1}{2})}{\Gamma(\mu_+)} \phi_{(l+1)/2, \mu_+ + l} + \frac{\Gamma(\frac{1}{2} - \mu_+)}{\Gamma(1-\mu_+)} \phi_{\mu_+ + l/2, \mu_+ + l} \right\} \\ + &\frac{4^{\mu_- + l}\Gamma(\nu)\Gamma(C)}{\sqrt{\pi}\Gamma(\mu_+)\Gamma(C-\mu_-)} \left\{ \frac{\Gamma(\mu_- - \frac{1}{2})}{\Gamma(\mu_-)} \phi_{(l+1)/2, \mu_- + l} + \frac{\Gamma(\frac{1}{2} - \mu_-)}{\Gamma(1-\mu_-)} \phi_{\mu_- + l/2, \mu_- + l} \right\}. \end{aligned}$$

Proof. By setting $B_1 = B_2 = \dfrac{1}{2}, B = B_1 + B_2 = 1$ and $\mu_\pm = \pm\frac{\nu}{2} - \frac{l-2}{2}$, we have

$$\begin{aligned} &F_{10}\left(\begin{matrix} \mu_+ & \mu_- & \frac{1}{2} & \frac{1}{2} \\ 1 & C & & \end{matrix} ; \eta_1, \eta_2\right) \\ &= \frac{\Gamma(-\nu)\Gamma(C)}{\Gamma(\mu_-)\Gamma(C-\mu_+)}(-\eta_2)^{-\mu_+} F_2(\mu_+; \frac{1}{2}, \mu_+ - C + 1; 1, \nu+1; 1-\eta_1/\eta_2, 1/\eta_2) \\ &+ \frac{\Gamma(\nu)\Gamma(C)}{\Gamma(\mu_+)\Gamma(C-\mu_-)}(-\eta_2)^{-\mu_-} F_2(\mu_-; \frac{1}{2}, \mu_- - C + 1; 1, -\nu+1; 1-\eta_1/\eta_2, 1/\eta_2) \end{aligned} \tag{6.1}$$

from the equation (9.8) in [2]. Though we required the condition $B \notin \mathbb{Z}$ in Theorem 9.2 in [2], that condition should be corrected as $B \notin \{0, -1, -2, ...\}$(see [3]). So we can apply the theorem in the current problem.

We would like to know the asymptotic behavior of the matrix coefficient as $y_1, y_2 \to 0$. Since we put $y_1 = (a_1/a_2)^2, y_2 = (a_2)^2$ and $a_i = \exp x_i$ $(i = 1, 2)$ in Section 3, the limit $y_1, y_2 \to 0$ corresponds to $x_1, x_2 \to -\infty$.

As $x_i \to -\infty$ (that is, $a_i \to 0$),

$$\eta_i = -\operatorname{sh}^2 x_i = -\frac{\exp(-2x_i)}{4}(1 + O(\exp(2x_i))) = -\frac{1}{4a_i^2}(1 + O(a_i^2)) \quad (i = 1, 2).$$

Then we have

$$\begin{aligned} &1 - \frac{\eta_1}{\eta_2} = 1 - \frac{a_2^2}{a_1^2}(1 + O(a_1^2))(1 + O(a_2^2)) = 1 - \frac{1}{y_1}(1 + O(y_1 y_2))(1 + O(y_2)), \\ &\frac{1}{\eta_2} = -4a_2^2(1 + O(a_2^2)) = -4y_2(1 + O(y_2)). \end{aligned}$$

Using results of §5, the equation (6.1) is asymptotically written as

$$
\begin{aligned}
&F_{10}\left(\begin{matrix}\mu_+ & \mu_- & \frac{1}{2} & \frac{1}{2} \\ 1 & C & & \end{matrix}; -\sinh^2 x_1, -\sinh^2 x_2\right) \\
&\sim \frac{\Gamma(-\nu)\Gamma(C)}{\Gamma(\mu_-)\Gamma(C-\mu_+)}(4y_2)^{\mu_+} F_2(\mu_+; \frac{1}{2}, \mu_+ - C + 1; 1, \nu+1; 1-y_1^{-1}, -4y_2) \\
&\qquad + \frac{\Gamma(\nu)\Gamma(C)}{\Gamma(\mu_+)\Gamma(C-\mu_-)}(4y_2)^{\mu_-} F_2(\mu_-; \frac{1}{2}, \mu_- - C + 1; 1, -\nu+1; 1-y_1^{-1}, -4y_2) \\
&\sim \frac{4^{\mu_+}\Gamma(-\nu)\Gamma(C)}{\Gamma(\mu_-)\Gamma(C-\mu_+)} y_2^{\mu_+}\, {}_2F_1(\mu_+, \frac{1}{2}; 1; 1-y_1^{-1}) \\
&\qquad + \frac{4^{\mu_-}\Gamma(\nu)\Gamma(C)}{\Gamma(\mu_+)\Gamma(C-\mu_-)} y_2^{\mu_-}\, {}_2F_1(\mu_-, \frac{1}{2}; 1; 1-y_1^{-1}). \\
&= \frac{4^{\mu_+}\Gamma(-\nu)\Gamma(C)}{\Gamma(\mu_-)\Gamma(C-\mu_+)} y_2^{\mu_+} f_+(y_1) + \frac{4^{\mu_-}\Gamma(\nu)\Gamma(C)}{\Gamma(\mu_+)\Gamma(C-\mu_-)} y_2^{\mu_-} f_-(y_1) \\
&\sim \frac{4^{\mu_+}\Gamma(-\nu)\Gamma(C)}{\Gamma(\mu_-)\Gamma(C-\mu_+)} a_+(y_1, y_2) y_2^{\mu_+} + \frac{4^{\mu_-}\Gamma(\nu)\Gamma(C)}{\Gamma(\mu_+)\Gamma(C-\mu_-)} a_-(y_1, y_2) y_2^{\mu_-} \\
&= \frac{4^{\mu_+}\Gamma(-\nu)\Gamma(C)}{\Gamma(\mu_-)\Gamma(C-\mu_+)} \left(\frac{\Gamma(\mu_+ - \frac{1}{2})}{\sqrt{\pi}\Gamma(\mu_+)}\psi_{\frac{1}{2},\mu_+}(y_1, y_2) + \frac{\Gamma(\frac{1}{2} - \mu_+)}{\sqrt{\pi}\Gamma(1-\mu_+)}\psi_{\mu_+,\mu_+}(y_1, y_2)\right) \\
&\qquad + \frac{4^{\mu_-}\Gamma(\nu)\Gamma(C)}{\Gamma(\mu_+)\Gamma(C-\mu_-)} \left(\frac{\Gamma(\mu_- - \frac{1}{2})}{\sqrt{\pi}\Gamma(\mu_-)}\psi_{\frac{1}{2},\mu_-}(y_1, y_2) + \frac{\Gamma(\frac{1}{2} - \mu_-)}{\sqrt{\pi}\Gamma(1-\mu_-)}\psi_{\mu_-,\mu_-}(y_1, y_2)\right).
\end{aligned}
$$

Since $y_2 = a_2^2 > 0$, the blanch of the complex power $y_2^{\mu_\pm}$ is determined.

Multiplying $\delta(x_1, x_2)^{-1}$ on both sides of the above equation, we have the result by the equation (4.2). $\square$

From the above theorem, we find that analogues of c-functions are

$$
\begin{aligned}
c_1(\nu) &= \frac{4^{\mu_+ + l}\Gamma(-\nu)\Gamma(C)\Gamma(\mu_+ - \frac{1}{2})}{\sqrt{\pi}\Gamma(\mu_-)\Gamma(C-\mu_+)\Gamma(\mu_+)} \\
&= \frac{2^{\nu+l+2}\Gamma(-\nu)\Gamma(\frac{k-l+3}{2})\Gamma(\frac{\nu-l+1}{2})}{\sqrt{\pi}\Gamma(\frac{-\nu-l+2}{2})\Gamma(\frac{-\nu+k+1}{2})\Gamma(\frac{\nu-l+2}{2})}, \\
c_2(\nu) &= \frac{4^{\mu_- + l}\Gamma(\nu)\Gamma(C)\Gamma(\mu_- - \frac{1}{2})}{\sqrt{\pi}\Gamma(\mu_+)\Gamma(C-\mu_-)\Gamma(\mu_-)} \\
&= \frac{2^{-\nu+l+2}\Gamma(\nu)\Gamma(\frac{k-l+3}{2})\Gamma(\frac{-\nu-l+1}{2})}{\sqrt{\pi}\Gamma(\frac{\nu-l+2}{2})\Gamma(\frac{\nu+k+1}{2})\Gamma(\frac{-\nu-l+2}{2})} = c_1(-\nu), \\
c_3(\nu) &= \frac{4^{\mu_+ + l}\Gamma(-\nu)\Gamma(C)\Gamma(\frac{1}{2} - \mu_+)}{\sqrt{\pi}\Gamma(\mu_-)\Gamma(C-\mu_+)\Gamma(1-\mu_+)} \\
&= \frac{2^{\nu+l+2}\Gamma(-\nu)\Gamma(\frac{k-l+3}{2})\Gamma(\frac{-\nu+l-1}{2})}{\sqrt{\pi}\Gamma(\frac{-\nu-l+2}{2})\Gamma(\frac{-\nu+k+1}{2})\Gamma(\frac{-\nu+l}{2})}, \\
c_4(\nu) &= \frac{4^{\mu_- + l}\Gamma(\nu)\Gamma(C)\Gamma(\frac{1}{2} - \mu_-)}{\sqrt{\pi}\Gamma(\mu_+)\Gamma(C-\mu_-)\Gamma(1-\mu_-)} \\
&= \frac{2^{-\nu+l+2}\Gamma(\nu)\Gamma(\frac{k-l+3}{2})\Gamma(\frac{\nu+l-1}{2})}{\sqrt{\pi}\Gamma(\frac{\nu-l+2}{2})\Gamma(\frac{\nu+k+1}{2})\Gamma(\frac{\nu+l}{2})} = c_3(-\nu).
\end{aligned}
$$

7 Power series expansion of the fundamental system of solutions

To obtain the Harish-Chandra expansion of the matrix coefficient, only the asymptotic behavior of the solutions of the system (3.3), (3.4) was required.

In this section, we have explicit power series expression of $\psi_{\alpha,\beta}$.

Theorem 7.1. *Let u_1, u_2 be*

$$u_1 = \frac{\operatorname{sh}^2 x_2}{\operatorname{sh}^2 x_1}, \qquad u_2 = -\frac{1}{\operatorname{sh}^2 x_2}. \tag{7.1}$$

Then, $\psi_{\alpha,\beta}(y_1, y_2) = 4^{-\beta}\Psi_{\alpha,\beta}(u_1, u_2)$, where

$$\begin{aligned}
&\Psi_{\frac{1}{2},\mu_\pm}(u_1,u_2)\\
&= u_1^{\frac{1}{2}} u_2^{\mu_\pm} \sum_{m,n\geq 0} \frac{(-\mu_\pm - n + 1)_m \left(\frac{1}{2}\right)_m}{\left(-\mu_\pm - n + \frac{3}{2}\right)_m m!} \cdot \frac{(\mu_\pm - \frac{1}{2})_n(\mu_\pm - \frac{1}{2} + \frac{l-k}{2})_n}{(2\mu_\pm + l - 1)_n n!} u_1^m u_2^n\\
&= u_1^{\frac{1}{2}} u_2^{\mu_\pm} \sum_{n=0}^{\infty} \frac{(\mu_\pm - \frac{1}{2})_n(\mu_\pm - \frac{1}{2} + \frac{l-k}{2})_n}{(2\mu_\pm + l - 1)_n n!} {}_2F_1\left(\frac{1}{2}, -\mu_\pm - n + 1; -\mu_\pm - n + \frac{3}{2}; u_1\right) u_2^n
\end{aligned} \tag{7.2}$$

and

$$\begin{aligned}
&\Psi_{\mu_\pm,\mu_\pm}(u_1,u_2)\\
&= u_1^{\mu_\pm} u_2^{\mu_\pm} \sum_{n=0}^{\infty}\sum_{j=0}^{\infty} \frac{(\frac{1}{2})_j(\mu_\pm + n)_j}{j!(\mu_\pm + n + \frac{1}{2})_j} \cdot \frac{(\mu_\pm)_n(\mu_\pm - 1 + \frac{l-k}{2})_n}{(2\mu_\pm + l - 1)_n n!} u_1^{n+j} u_2^n\\
&= u_1^{\mu_\pm} u_2^{\mu_\pm} \sum_{n=0}^{\infty} \frac{(\mu_\pm)_n(\mu_\pm - 1 + \frac{l-k}{2})_n}{(2\mu_\pm + l - 1)_n n!} {}_2F_1\left(\frac{1}{2}, \mu_\pm + n; \mu_\pm + n + \frac{1}{2}; u_1\right) u_1^n u_2^n.
\end{aligned} \tag{7.3}$$

Here

$$(a)_n = a(a+1)\cdots(a+n-1) \quad (n \neq 0), \qquad (a)_0 = 1.$$

As is seen in the proof of Theorem 6.1, we have $u_1 \sim y_1$ and $u_2 \sim 4y_2$ as $y_1, y_2 \to 0$.

At first, we will express P and Q with new variables u_1, u_2.

Proposition 7.2. *We denote the differential operator in the equation (3.3) by P and that in (3.4) by Q. P and Q are written with variables u_1, u_2 as :*

$$\begin{aligned}
P &= 4(2 - u_2 - u_1u_2)\vartheta_1^2 + 4(1 - u_2)\vartheta_2^2 - 8(1 - u_2)\vartheta_1\vartheta_2\\
&\quad + \frac{4}{u_1 - 1}\left\{1 + u_1 - \frac{l-k}{2}u_2 + (l-k-2)u_1u_2 - \frac{l-k}{2}u_1^2u_2\right\}\vartheta_1\\
&\quad - \frac{4}{u_1 - 1}\left\{l - 1 - (l-2)u_1 - \frac{l-k}{2}u_2 + \frac{l-k-2}{2}u_1u_2\right\}\vartheta_2 + (l-2)^2 - \nu^2, &(7.4)\\
Q &= -u_1u_2^2\left\{\vartheta_1^2 - \vartheta_1\vartheta_2 + \frac{1}{2}\frac{u_1+1}{u_1-1}\vartheta_1 - \frac{1}{2}\frac{1}{u_1-1}\vartheta_2\right\}. &(7.5)
\end{aligned}$$

Here, we set $\vartheta_i = u_i\dfrac{\partial}{\partial u_i}$ $(i = 1, 2)$.

Indicial equations of P and Q are

$$8\alpha^2 + 4\beta^2 - 8\alpha\beta - 4\alpha + 4(l-1)\beta + (l-2)^2 - \nu_1^2 = 0, \quad \alpha^2 - \alpha\beta - \frac{1}{2}\alpha + \frac{1}{2}\beta = 0$$

respectively and the solutions of these equations are

$$(\alpha,\beta)=(\frac{1}{2},\mu_\pm),(\mu_\pm,\mu_\pm),$$

where $\mu_\pm=\pm\nu/2-(l-2)/2$.

This is a kind of the modified Appell's F_2 system.

Now we put the analytic kernel of P,Q as

$$\Psi_{\alpha,\beta}(u_1,u_2)=u_1^\alpha u_2^\beta\sum_{m,n\geq 0}a_{m,n}u_1^m u_2^n.$$

We normalize this solution as $a_{0,0}=1$.

Comparing the leading term of $\Psi_{\alpha,\beta}$ with that of $\psi_{\alpha,\beta}$ as $y_1,y_2\to 0$, we can see that

$$\Psi_{\alpha,\beta}(u_1,u_2)=4^\beta\psi_{\alpha,\beta}(y_1,y_2).$$

Note that $\mu_\pm\notin\frac{1}{2}\mathbb{Z}$, since we assume that $\nu\notin\mathbb{Z}$ in Section 5.

It is easy to prove the following lemma.

Lemma 7.3. *If $Q\Psi_{\alpha,\beta}(u_1,u_2)=0$ holds, then $a_{m,n}$ satisfy the following recurrence equations.*

$$\begin{aligned}&(\alpha+m-1)(\alpha-\beta+m-n-\frac{1}{2})a_{m-1,n}\\&\qquad+(-\alpha+\beta-m+n)(\alpha+m-\frac{1}{2})a_{m,n}=0\qquad(m\neq 0)\end{aligned}\tag{7.6}$$

$$(-\alpha+\beta+n)(\alpha-\frac{1}{2})a_{0,n}=0\tag{7.7}$$

7.1 Case 1 : $\alpha=\frac{1}{2}$

In this subsection we determine $a_{m,n}$ for the case $(\alpha,\beta)=(1/2,\mu_\pm)$.

In this case, any $a_{0,n}$ satisfies (7.7). On the other hand, we have

$$\begin{aligned}a_{m,n}&=\frac{(\beta-m+n)(\beta-m+n+1)\cdots(\beta+n-1)\cdot(m-\frac{1}{2})(m-\frac{3}{2})\cdots\frac{1}{2}}{(\beta-m+n-\frac{1}{2})(\beta-m+n+\frac{1}{2})\cdots(\beta+n-\frac{3}{2})\cdot m(m-1)\cdots 1}a_{0,n}\\&=\frac{(-\beta-n+1)_m\left(\frac{1}{2}\right)_m}{\left(-\beta-n+\frac{3}{2}\right)_m m!}a_{0,n}\\&=\frac{(-\mu_\pm-n+1)_m\left(\frac{1}{2}\right)_m}{\left(-\mu_\pm-n+\frac{3}{2}\right)_m m!}a_{0,n}\end{aligned}\tag{7.8}$$

from (7.6), where

Therefore we have only to determine $a_{0,n}$ $(n\geq 0)$ from $P\Psi_{\alpha,\beta}(u_1,u_2)=0$.

Setting $\phi_m(u_2)=\sum_{n=0}^\infty a_{m,n}u_2^n$, we have

$$\Psi_{\alpha,\beta}(u_1,u_2)=u_1^\alpha u_2^\beta\sum_{m,n\geq 0}a_{m,n}u_1^m u_2^n=u_1^\alpha u_2^\beta\sum_{m=0}^\infty\phi_m(u_2)u_1^m.$$

So we have only to determine $\phi_0(u_2)$.

Lemma 7.4. *$\phi_0(u_2)$ is a solution of the Gaussian hypergeometric equation* :

$$u_2(1-u_2)\phi_0''(u_2)+\{r-(p+q+1)u_2\}\phi_0'(u_2)-pq\phi_0(u_2)=0.$$

In particular, we have $\phi_0(u_2) = {}_2F_1(p, q; r; u_2)$ *and*

$$a_{0,n} = \frac{(p)_n (q)_n}{(r)_n n!}, \tag{7.9}$$

with parameters

$$p = \mu_\pm - \frac{1}{2}, \qquad q = \mu_\pm - \frac{1}{2} + \frac{l-k}{2}, \qquad r = 2\mu_\pm + l - 1.$$

Proof. We obtain the Gaussian hypergeometric equation from

$$u_1^{-\alpha} P \Psi_{\alpha,\beta}(u_1, u_2)|_{u_1=0} = 0$$

using

$$\begin{aligned}
\vartheta_1 \Psi_{\alpha,\beta} &= u_1^\alpha u_2^\beta \sum_{m=0}^\infty (\alpha+m)\phi_m(u_2) u_1^m, \\
\vartheta_1^2 \Psi_{\alpha,\beta} &= u_1^\alpha u_2^\beta \sum_{m=0}^\infty (\alpha+m)^2 \phi_m(u_2) u_1^m, \\
\vartheta_2 \Psi_{\alpha,\beta} &= \beta\Psi_{\alpha,\beta} + u_1^\alpha u_2^{\beta+1} \sum_{m=0}^\infty \phi'_m(u_2) u_1^m = u_1^\alpha u_2^\beta \sum_{m=0}^\infty \left\{\beta\phi_m(u_2) + u_2\phi'_m(u_2)\right\} u_1^m, \\
\vartheta_2^2 \Psi_{\alpha,\beta} &= u_1^\alpha u_2^\beta \sum_{m=0}^\infty \left\{\beta^2\phi_m(u_2) + (2\beta+1)u_2\phi'_m(u_2) + u_2^2\phi''_m(u_2)\right\} u_1^m, \\
\vartheta_1\vartheta_2 \Psi_{\alpha,\beta} &= u_1^\alpha u_2^\beta \sum_{m=0}^\infty (\alpha+m)\left\{\beta\phi_m(u_2) + u_2\phi'_m(u_2)\right\} u_1^m
\end{aligned}$$

and indicial equations above.

The latter part is obvious from the assumption that $a_{0,0} = 1$. $\square$

Now, we have

$$a_{m,n} = \frac{(-\mu_\pm - n + 1)_m \left(\frac{1}{2}\right)_m}{\left(-\mu_\pm - n + \frac{3}{2}\right)_m m!} \cdot \frac{(\mu_\pm - \frac{1}{2})_n (\mu_\pm - \frac{1}{2} + \frac{l-k}{2})_n}{(2\mu_\pm + l - 1)_n n!}$$

from equations (7.8), (7.9) and we have shown the former half of Theorem 7.1.

7.2 Case 2 : $\alpha \neq \frac{1}{2}$

In this subsection we assume that $\alpha \neq 1/2$, that is, $\alpha = \beta = \mu_\pm$. From this condition and the equation (7.7), we have $a_{0,n} = 0$ $(n \geq 1)$. At the same time we have

$$(\alpha + m - 1)(m - n - \frac{1}{2})a_{m-1,n} + (-m+n)(\alpha + m - \frac{1}{2})a_{m,n} = 0$$

from (7.6).

Note that above coefficients of $a_{m-1,n}, a_{m,n}$ except $(-m+n)$ are not zero by the assumption $\alpha = \mu_\pm \notin \frac{1}{2}\mathbb{Z}$.

If $m = n$, then we have $a_{m-1,m} = 0$.

If $m < n$, then we have $a_{m,n} = Ca_{0,n} = 0$ (C is a constant which depends on m and n).

If $m > n$, then we have

$$\begin{aligned}
a_{m,n} &= \frac{(m-n-\frac{1}{2})(m-n-\frac{3}{2})\cdots\frac{1}{2}\cdot(\alpha+m-1)(\alpha+m-2)\cdots(\alpha+n)}{(m-n)(m-n-1)\cdots 1\cdot(\alpha+m-\frac{1}{2})(\alpha+m-\frac{3}{2})\cdots(\alpha+n+\frac{1}{2})}a_{n,n} \\
&= \frac{(\frac{1}{2})_{m-n}(\alpha+n)_{m-n}}{(m-n)!(\alpha+n+\frac{1}{2})_{m-n}}a_{n,n} \\
&= \frac{(\frac{1}{2})_{m-n}(\mu_\pm+n)_{m-n}}{(m-n)!(\mu_\pm+n+\frac{1}{2})_{m-n}}a_{n,n} \qquad (7.10)
\end{aligned}$$

from (7.6).

Therefore it remains to determine $a_{n,n}$ $(n \geq 0)$ from $P\Psi_{\alpha,\beta}(u_1,u_2) = 0$.

Since

$$\Psi_{\mu_\pm,\mu_\pm}(u_1,u_2) = u_1^{\mu_\pm}u_2^{\mu_\pm}\sum_{n=0}^{\infty}\sum_{j=0}^{\infty}a_{n+j,n}u_1^{n+j}u_2^{n} = (u_1u_2)^{\mu_\pm}\sum_{j=0}^{\infty}u_1^{j}\sum_{n=0}^{\infty}a_{n+j,n}(u_1u_2)^{n}$$

holds, we have

$$\Psi_{\mu_\pm,\mu_\pm}(u_1,u_2) = t^{\mu_\pm}\sum_{j=0}^{\infty}\varphi_j(t)s^j$$

where $s = u_1, t = u_1u_2, \varphi_j(t) = \sum_{n=0}^{\infty}a_{n+j,n}t^n$. We have only to determine $\varphi_0(t)$.

Lemma 7.5. *$\varphi_0(t)$ is a solution of the Gaussian hypergeometric equation :*

$$t(1-t)\varphi_0''(t) + \{c-(a+b+1)t\}\varphi_0'(t) - ab\varphi_0(t) = 0.$$

Hence we have $\varphi_0(t) = {}_2F_1(a,b;c;t)$ and

$$a_{n,n} = \frac{(a)_n(b)_n}{(c)_n n!}, \qquad (7.11)$$

with parameters

$$a = \mu_\pm, \qquad b = \mu_\pm - 1 + \frac{l-k}{2}, \qquad c = 2\mu_\pm + l - 1.$$

Proof. We denote $\vartheta_s = s\frac{\partial}{\partial s}, \vartheta_t = t\frac{\partial}{\partial t}$. Then we have

$$\begin{aligned}
\vartheta_1 &= u_1\left(\frac{\partial s}{\partial u_1}\frac{\partial}{\partial s} + \frac{\partial t}{\partial u_1}\frac{\partial}{\partial t}\right) = u_1\left(\frac{\partial}{\partial s} + u_2\frac{\partial}{\partial t}\right) = \vartheta_s + \vartheta_t, \\
\vartheta_2 &= u_2\left(\frac{\partial s}{\partial u_2}\frac{\partial}{\partial s} + \frac{\partial t}{\partial u_2}\frac{\partial}{\partial t}\right) = \vartheta_t.
\end{aligned}$$

Using above equations, we obtain the Gaussian hypergeometric equation from

$$P\Psi_{\mu_\pm,\mu_\pm}(s,t)|_{s=0} = 0$$

in the similar way as Proposition 7.4. □

Now, we have

$$a_{m,n} = \begin{cases} 0 & (m < n) \\ \frac{(\frac{1}{2})_{m-n}(\mu_\pm+n)_{m-n}}{(m-n)!(\mu_\pm+n+\frac{1}{2})_{m-n}}\cdot\frac{(\mu_\pm)_n(\mu_\pm-1+\frac{l-k}{2})_n}{(2\mu_\pm+l-1)_n n!} & (m \geq n) \end{cases}$$

from equations (7.10), (7.11) and we have shown the latter half of Theorem 7.1.

Remark 7.6. It is an interesting problem to compare our power series solutions with the confluenced ones which were discussed in [1]

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