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Asymptotic Behavior of distributions of the sum of i.i.d. random variables with fat tail I

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1 Introduction

Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let X_n, $n = 1, 2, \ldots,$ be independent identically distributed random variables, and $F : \mathbf{R} \to [0, 1]$ and $\bar{F} : \mathbf{R} \to [0, 1]$ be given by

$$F(x) = P(X_1 \leqq x), \quad \text{and} \quad \bar{F}(x) = P(X_1 > x), \qquad x \in \mathbf{R}.$$

We assume the following.
(A-1) $\bar{F}(x)$ is a regular varying function of index $-\alpha$, for some $\alpha > 2$, as $x \to \infty$, i.e., if we let

$$L(x) = x^{\alpha} \bar{F}(x), \qquad x \in \mathbf{R},$$

then $L(x) > 0$ for any $x > 0$, and for any $a > 0$

$$\frac{L(ax)}{L(x)} \to 1, \qquad x \to \infty.$$

Also we assume the following.

(A2) $x^{\alpha-\delta} F(-x) \to 0$, as $x \to \infty$ for any $\delta > 0$.

Since $\alpha > 2$, we see that $E[|X_1|^2] < \infty$. We assume furthermore for simplicity that

(A3) $E[X_1] = 0$.

Our first main theorem is the following.

Theorem 1 *Assume the assumptions* (A-1), (A-2) *and* (A-3), *and let* $\beta : \mathbf{N} \to (0, \infty)$ *be such that*

$$\frac{\beta(n)}{(\log n)^{1/2}} \to \infty, \qquad n \to \infty.$$

Then we have

$$\sup_{s \geqq n^{1/2}\beta(n)} \left| \frac{P(\sum_{k=1}^{n} X_k > s)}{n\bar{F}(s)} - 1 \right| \to 0, \qquad n \to \infty.$$

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We remark that the following has been shown essentially by Feller [1].

Theorem 2 (Feller) *Under the assumptions* (A-1),(A-2) and (A-3), *we have for any* $N \geqq 1$

$$|\frac{P(\sum_{k=1}^N X_k > x)}{N\bar{F}(x)} - 1| \to 0, \qquad x \to \infty.$$

Note that the number of random variables is fixed in Feller's result, but is increasing in our result.

Let us assume the following furthermore.

(A-4) There is an $x_0 > 0$ such that $\bar{F}$ is twice continuously differentiable on (x_0, ∞) and that

$$x^2 \frac{d^2}{dx^2} \log \bar{F}(x) \to \alpha, \qquad x \to \infty.$$

Then we have the following.

Theorem 3 *Assume the assumptions* (A-1), (A-2), (A-3) *and* (A-4) *and let* $\beta : \mathbf{N} \to (0, \infty)$ *be such that*

$$\frac{\beta(n)}{(\log n)^{1/2}} \to \infty, \qquad n \to \infty.$$

Let $v = E[X_1^2]$. *Then we have*

$$\sup_{s \geqq n^{1/2}\beta(n)} \frac{s^2}{n} |\frac{P(\sum_{k=1}^n X_k > s)}{n\bar{F}(s)} - (1 + \frac{\alpha(\alpha+1)vn}{2s^2})| \to 0, \qquad n \to \infty.$$

2 Preparations

Proposition 4 *Let* Y *be a random variable and* $\gamma \in (1, 2]$, *and assume that*

$$E[|Y|^\gamma] < \infty \quad \textit{and} \quad E[Y] = 0.$$

Then for any $s \in \mathbf{R} \setminus \{0\}$ *and* $b > 0$

$$E[\exp(sY 1_{\{|Y| \leqq b\}})] \leqq 1 + |s|^\gamma (1 + (\frac{1}{|s|b})^{\gamma - 1}) \exp(|s|b) E[|Y|^\gamma].$$

Proof. First, note that

$$|\exp(x) - 1| \leqq 1 \vee \exp(x),$$

and

$$|\exp(x) - 1| = |\int_0^x e^y dy| \leqq |x|(1 \vee \exp(x)), \qquad x \in \mathbf{R}.$$

So we have

$$|\exp(x) - (1 + x)| = |\int_0^x (e^y - 1) dy| \leqq (|x| \wedge |x|^2)(1 \vee \exp(x))$$

for any $x \in \mathbf{R}$. Therefore we see that

$$|\exp(x) - (1 + x)| \leqq |x|^\gamma \exp(|x|), \qquad x \in \mathbf{R}$$

for $\gamma \in (1,2]$.

Therefore we have

$$|E[\exp(sY1_{\{|Y|\leqq b\}})] - (1 + E[sY1_{\{|Y|\leqq b\}}])|$$
$$\leqq |s|^\gamma \exp(|s|b)E[|Y|^\gamma].$$

Since

$$|E[sY1_{\{|Y|\leqq b\}}]| = |sE[Y, |Y| > b]| \leqq |s|b^{-(\gamma-1)}E[|Y|^\gamma],$$

we have our assertion. ∎

Proposition 5 *Let X be a random variable and $\gamma \in (1,2]$. We assume that*

$$E[|X|^\gamma] < \infty \text{ and } E[X] = 0.$$

Then for any $t > 0$ and $n \geqq 1$

$$n \log E[\exp(\pm\frac{1}{tn^{1/\gamma}}X1_{\{|X|\leqq tn^{1/\gamma}\}})] \leqq \frac{6}{t^\gamma}E[|X|^\gamma].$$

Proof. Let $Y = (1/t)X$, $s = \pm n^{-1/\gamma}$, $b = n^{1/\gamma}$, and apply Proposition 4. Since $\log(1+x) \leqq x$, $x \geqq 0$, we have our assertion. ∎

Now let X_n, $n = 1, 2, \ldots$, be independent identically distributed random variables and $\gamma \in (1,2]$. Throughout this section we assume that

$$E[|X_1|^\gamma] < \infty \text{ and } E[X_1] = 0.$$

Proposition 6 *For any $s, t > 0$ and $\varepsilon > 0$*

$$P(|\sum_{k=1}^n X_k 1_{\{|X_k|\leqq tn^{1/\gamma}\}}| \geqq sn^{1/\gamma}) \leqq 2\exp(\frac{6}{t^\gamma}E[|X_1|^\gamma])\exp(-\frac{s}{t}).$$

Proof. We see that

$$P(\pm\sum_{k=1}^n X_k 1_{\{|X_k|\leqq tn^{1/\gamma}\}} \geqq sn^{1/\gamma})$$
$$\leqq \exp(-\frac{s}{t})E[\exp(\frac{\pm 1}{tn^{1/\gamma}}\sum_{k=1}^n X_k 1_{\{|X_k|\leqq tn^{1/\gamma}\}})]$$
$$\leqq \exp(-\frac{s}{t})E[\exp(\frac{\pm 1}{tn^{1/\gamma}}X_1 1_{\{|X_1|\leqq tn^{1/\gamma}\}})]^n.$$

Then by Proposition 5 we have our assertion.

Let $F : \mathbf{R} \to [0,1]$ and $\bar{F} : \mathbf{R} \to [0,1]$ be given by

$$F(x) = P(X_1 \leqq x), \qquad x \in \mathbf{R}$$

and

$$\bar{F}(x) = P(X_1 > x), \qquad x \in \mathbf{R}.$$

Then we have the following.

Proposition 7 (1) *For any* $t, s > 0$, *and* $n \geqq 2$,

$$P(|\sum_{k=2}^{n} X_k 1_{\{|X_k| \leqq tn^{1/\gamma}\}}| > sn^{1/\gamma}) \leqq 2\exp(\frac{6}{t^\gamma} E[|X_1|^\gamma]) \exp(-\frac{s}{t}).$$

(2) *For any* $s, t > 0$, $\varepsilon \in (0,1)$ *with* $t < (1-\varepsilon)s$,

$$|P(\sum_{k=1}^{n} X_k > sn^{1/\gamma}) - nP(X_1 + \sum_{k=2}^{n} X_k 1_{\{|X_k| \leqq tn^{1/\gamma}\}} > sn^{1/\gamma},\ |\sum_{k=2}^{n} X_k 1_{\{|X_k| \leqq tn^{1/\gamma}\}}| \leqq \varepsilon sn^{1/\gamma})|$$

$$\leqq 2n(n-1)(F(-tn^{1/\gamma}) + \bar{F}(tn^{1/\gamma}))^2 + 2\exp(\frac{6}{t^\gamma} E[|X_1|^\gamma]) \exp(-\frac{s}{t})$$

$$+2n(F(-tn^{1/\gamma}) + \bar{F}(tn^{1/\gamma})) \exp(\frac{6}{t^\gamma} E[|X_1|^\gamma]) \exp(-\frac{\varepsilon s}{2t})$$

Proof. Note that

$$P(|\sum_{k=2}^{n} X_k 1_{\{|X_k| \leqq tn^{1/\gamma}\}}| > sn^{1/\gamma})$$

$$= P(|\sum_{k=1}^{n-1} X_k 1_{\{|X_k| \leqq \tilde{t}(n-1)^{1/\gamma}\}}| > \tilde{s}(n-1)^{1/\gamma}),$$

where

$$\tilde{t} = t(\frac{n}{n-1})^{1/\gamma}, \qquad \tilde{s} = s(\frac{n}{n-1})^{1/\gamma}.$$

So we have the assertion (1) from Proposition 6 .

Let us denote

$$\tilde{F}(x) = P(|X_1| > x) \leqq F(-x) + \bar{F}(x), \qquad x > 0.$$

Note that

$$P(\sum_{k=1}^{n} X_k > sn^{1/\gamma}) = \sum_{m=0}^{n} I_m,$$

where

$$I_m = P(\sum_{k=1}^{n} X_k > sn^{1/\gamma}, \sum_{k=1}^{n} 1_{\{|X_k| > tn^{1/\gamma}\}} = m), \qquad m = 0, 1, \ldots, n.$$

Then we have

$$I_m = \binom{n}{m} P(\sum_{k=1}^{n} X_k > sn^{1/\gamma},\ |X_i| > tn^{1/\gamma},\ i = 1, \ldots, m,\ |X_j| \leqq tn^{1/\gamma},\ j = m+1, \ldots, n),$$

for $m = 0, 1, \ldots, n$. So we see that

$$\sum_{m=2}^{n} I_m \leqq \sum_{m=2}^{n} \frac{n(n-1)}{m(m-1)} \binom{n-2}{m-2} \tilde{F}(tn^{1/\gamma})^m (1 - \tilde{F}(tn^{1/\gamma}))^{n-m} \leqq \frac{n(n-1)}{2} \tilde{F}(tn^{1/\gamma})^2. \quad (1)$$

Also, by Proposition 6, we have

$$I_0 \leqq 2\exp(-\frac{s}{t}) \exp(\frac{6}{t^\gamma} E[|X_1|^\gamma]). \qquad (2)$$

Let

$$A_1 = \{|X_1| > tn^{1/\gamma}\}, \qquad A_2 = \{|X_k| \leqq tn^{1/\gamma}, k = 2, 3, \dots, n\},$$

$$B_1 = \{X_1 + \sum_{k=2}^{n} X_k 1_{\{|X_k| \leqq tn^{1/\gamma}\}} > sn^{1/\gamma}\},$$

and

$$B_2 = \{|\sum_{k=2}^{n} X_k 1_{\{|X_k| \leqq tn^{1/\gamma}\}}| \leqq \varepsilon sn^{1/\gamma}\}.$$

Note that $B_1 \cap B_2 \subset A_1$, since $t < (1-\varepsilon)s$. So we see that

$$\begin{aligned} &|P(B_1 \cap A_1 \cap A_2) - P(B_1 \cap B_2)| \\ &\leqq P(B_1 \cap B_2^c \cap A_1 \cap A_2) + P(B_1 \cap B_2 \cap A_1 \cap A_2^c) \\ &\leqq P(A_1)P(B_2^c) + P(A_1)P(A_2^c). \end{aligned} \tag{3}$$

Note that

$$P(A_2^c) \leqq \sum_{k=2}^{n} P(|X_k| > tn^{1/\gamma}) = (n-1)\tilde{F}(tn^{1/\gamma}).$$

Also, by the assertion (1) we have

$$P(B_2^c) \leqq 2\exp(\frac{6}{t^\gamma}E[|X_1|^\gamma])\exp(-\frac{\varepsilon s}{2t}).$$

Since $I_1 = nP(B_1 \cap A_1 \cap A_2)$, we have the assertion from Equations (1), (2) and (3).

This completes the proof. ∎

3 Proof of Theorem 1

Now let us prove Theorem 1. Let $\beta : \mathbf{N} \to (0, \infty)$ be such that

$$\frac{\beta(n)}{(\log n)^{1/2}} \to \infty, \qquad n \to \infty.$$

Assume that Theorem 1 is not valid. Then there is a sequence of positive numbers $\{s'_n\}_{n=1}^\infty$ such that $s'_n \geqq n^{1/2}\beta(n)$, $n = 1, 2, \dots$, and

$$\overline{\lim_{n\to\infty}} |\frac{P(\sum_{k=1}^n X_k > s'_n)}{n\bar{F}(s'_n)} - 1| > 0.$$

Let $s_n = n^{-1/2}s'_n \geqq \beta(n)$. Let us take an $r \in ((\alpha+2)/(2\alpha), 1)$ and fix it. Let t_n, $n = 1, 2, \dots$, be a sequence of positive mumbers given by

$$t_n = (\log n)^{-1/2} + (\log n)^{-1}s_n^{(1+r)/2}, \qquad n \geqq 2.$$

Then we have the following.

$$t_n s_n \geqq \frac{\beta(n)}{(\log n)^{1/2}} \to \infty, \quad n \to \infty, \tag{4}$$

$$\frac{2s_n}{t_n} \geqq ((\log n)^{1/2} s_n) \wedge ((\log n) s_n^{(1-r)/2}), \quad n \geqq 2, \tag{5}$$

$$\frac{2s_n}{(\log n)t_n} \geqq (\frac{\beta(n)}{(\log n)^{1/2}}) \wedge (s_n^{(1-r)/2}) \to \infty, \quad n \to \infty, \tag{6}$$

and

$$\frac{(t_n n^{1/2})^2}{(s'_n)^{1+r}} \geqq \frac{(\log n)^{-2} s_n^{1+r} n}{s_n^{1+r} n^{(1+r)/2}} \to \infty, \quad n \to \infty. \tag{7}$$

Therefore by Equation (7), we have

$$\frac{(F(-t_n n^{1/2}) + \bar{F}(t_n n^{1/2}))^2}{\bar{F}(s'_n)^{2r}} \to 0, \quad n \to \infty.$$

Since $2r - 1 > 2/\alpha$, we have

$$(s'_n)^2 \frac{(F(-t_n n^{1/2}) + \bar{F}(t_n n^{1/2}))^2}{\bar{F}(s'_n)} \to 0, \quad n \to \infty. \tag{8}$$

Also, by Equations (4), (5) and (6) we see that for any $m \geqq 1$

$$\begin{aligned} &(ns'_n)^m \exp(\frac{m}{t_n^2} - \frac{1}{m}\frac{s_n}{t_n}) \\ &= \exp(\frac{m}{t_n^2}(1 - \frac{1}{3m^2} t_n s_n)) n^{3m/2} \exp(-(\log n)\frac{1}{3m}\frac{s_n}{(\log n)t_n}) \\ &\quad \times (s_n)^m \exp(-\frac{1}{3m}\frac{s_n}{t_n}) \to 0, \qquad n \to \infty. \end{aligned}$$

Let us take an $\varepsilon \in (0,1)$ and fix it. For $n \geqq 1$, let

$$c_n(\varepsilon) = P(X_1 + \sum_{k=2}^n X_k 1_{\{|X_k| \leqq t_n n^{1/2}\}} > s_n n^{1/2},\ |\sum_{k=2}^n X_k 1_{\{|X_k| \leqq t_n n^{1/2}\}}| \leqq \varepsilon s_n n^{1/2}).$$

Note that $t_n < (1-\varepsilon)s_n$ for sufficiently large n. Taking $\gamma = 2$ in Proposition 7 (2) we see that by Equation (8)

$$\frac{(s'_n)^2}{n} \frac{1}{n\bar{F}(s'_n)} |P(\sum_{k=1}^n X_k > s'_n) - n c_n(\varepsilon)| \to 0, \qquad n \to \infty. \tag{9}$$

Note that

$$\bar{F}(s'_n(1+\varepsilon)) - P(|\sum_{k=2}^n X_k 1_{\{|X_k| \leqq t_n n^{1/2}\}}| > \varepsilon s_n n^{1/2}) \leqq c_n(\varepsilon) \leqq \bar{F}(s'_n(1-\varepsilon)).$$

By Proposition 7 (1), we have

$$\frac{1}{\bar{F}(s'_n)} P(|\sum_{k=2}^n X_k 1_{\{X_k \leqq t_n n^{1/2}\}}| > \varepsilon s_n n^{1/2}) \to 0, \qquad n \to \infty.$$

Thus we have

$$(1+\varepsilon)^{-\alpha} = \varliminf_{n\to\infty} \frac{\bar{F}(s_n'(1+\varepsilon))}{\bar{F}(s_n')} \leqq \varliminf_{n\to\infty} \frac{c_n(\varepsilon)}{\bar{F}(s_n')} \leqq \varliminf_{n\to\infty} \frac{P(\sum_{k=1}^n X_k > s_n')}{n\bar{F}(s_n')}$$

and

$$\varlimsup_{n\to\infty} \frac{P(\sum_{k=1}^n X_k > s_n')}{n\bar{F}(s_n')} \leqq \varlimsup_{n\to\infty} \frac{c_n(\varepsilon)}{\bar{F}(s_n')} \leqq \varlimsup_{n\to\infty} \frac{\bar{F}(s_n'(1-\varepsilon))}{\bar{F}(s_n')} = (1-\varepsilon)^{-\alpha}$$

Since $\varepsilon \in (0,1)$ is arbitrary, we see that

$$\lim_{n\to\infty} \frac{P(\sum_{k=1}^n X_k > s_n')}{n\bar{F}(s_n')} = 1.$$

This contradicts our assumption.

This completes the proof of Theorem 1.

4 Some estimates

In this section, we assume that (A-1) and (A-4).

Let $g : (x_0, \infty) \to \mathbf{R}$ and $H : [-1/2, 1/2] \times (2x_0, \infty) \to (0, \infty)$ be given by

$$g(x) = x^2 \frac{d^2}{dx^2}(\log \bar{F})(x) - \alpha, \qquad x > x_0,$$

and

$$H(y;x) = \frac{\bar{F}(x(1+y))}{\bar{F}(x)}, \qquad y \in [-1/2, 1/2],\ x > 2x_0.$$

We prove the following in this section.

Proposition 8 *There are functions $a : (2x_0, \infty) \to \mathbf{R}$, $c : (2x_0, \infty) \to [0, \infty)$ and a constant $C > 0$ such that $a(x) \to 0$ and $c(x) \to 0$, as $x \to \infty$, and that*

$$|H(y;x) - \{1 + a(x)y - \alpha y + \frac{\alpha(\alpha+1)y^2}{2}\}| \leqq C(c(x)y^2 + |y|^3), \qquad y \in [-1/2, 1/2],\ x > 2x_0.$$

First we prove the following.

Proposition 9 (1) *For any $x > x_0$,*

$$\frac{d}{dx}\log(x^\alpha \bar{F}(x)) = -\int_x^\infty \frac{g(z)}{z^2}.$$

(2) *For any $y \in [-1/2, 1/2]$ and $x > 2x_0$,*

$$H(y;x) = (1+y)^{-\alpha} \exp(-\int_0^y dy' \int_{1+y'}^\infty \frac{g(xz)}{z^2} dz).$$

Proof. Note that

$$g(x) = x^2 \frac{d^2}{dx^2}(\log(x^\alpha \bar{F}(x)))$$

and $g(x) \to 0$ as $x \to \infty$. Then we see that

$$\frac{d}{dy}(\log(y^\alpha \bar{F}(y))) - \frac{d}{dx}(\log(x^\alpha \bar{F}(x))) = \int_x^y \frac{g(z)}{z^2} dz, \tag{10}$$

and so we see that

$$c_0 = \lim_{y \to \infty} \frac{d}{dy}(\log(y^\alpha \bar{F}(y)))$$

exists. Note that

$$\exp(\int_x^{2x} \frac{d}{dy}(\log(y^\alpha \bar{F}(y)))dy) = \frac{L(2x)}{L(x)} \to 1, \qquad x \to \infty.$$

So we see that $c_0 = 0$. Therefore letting $y \to \infty$ in Equation (10) we have the assertion (1).

By the assertion (1), we have

$$\frac{d}{dy} \log((1+y)^\alpha H(y;x)) = -x \int_{x(1+y)}^\infty \frac{g(z)}{z^2} dz = -\int_{1+y}^\infty \frac{g(xz)}{z^2} dz.$$

Since $H(0;x) = 1$, we have the assertion (2).

∎

Proposition 10 *Let $\tilde{a} : (2x_0, \infty) \to \mathbf{R}$ and $\tilde{c} : (2x_0, \infty) \to \mathbf{R}$ be given by*

$$\tilde{a}(x) = \frac{d}{dy}((1+y)^\alpha H(y,x))|_{y=0},$$

and

$$\tilde{c}(x) = \sup_{y \in [-1/2, 1/2]} |\frac{d^2}{dy^2}((1+y)^\alpha H(y,x))|.$$

Then $\tilde{a}(x) \to 0$ and $\tilde{c}(x) \to 0$, as $x \to \infty$, and that

$$|H(y;x) - (1+y)^{-\alpha} - \tilde{a}(x)y(1+y)^{-\alpha}| \leqq 2^\alpha \tilde{c}(x) y^2, \qquad y \in [-1/2, 1/2], \ x > 2x_0.$$

Proof. By Proposition 9 We have

$$\frac{d}{dy}((1+y)^\alpha H(y;x)) = -(1+y)^\alpha H(y;x) \int_{1+y}^\infty \frac{g(xz)}{z^2} dz$$

and so

$$\tilde{a}(x) = -\int_1^\infty \frac{g(xz)}{z^2} dz$$

Similarly, we have

$$\frac{d^2}{dy^2}((1+y)^\alpha H(y;x)) = (1+y)^\alpha H(y;x)\{(\int_{1+y}^\infty \frac{g(xz)}{z^2} dz)^2 - (1+y)^{-2} g(x(1+y))\}$$

Thereforewe have

$$\tilde{c}(x) \leqq 2^{\alpha}\{(\int_{1/2}^{\infty}\frac{|g(xz)|}{z^2}dz)^2 + 4\sup\{|g(x(1+y))|;\ y \in [-1/2,1/2]\}\}\exp(\int_{1/2}^{\infty}\frac{|g(xz)|}{z^2}dz)$$

These imply that $\tilde{a}(x) \to 0$, $\tilde{c}(x) \to 0$, as $x \to \infty$. Also we have

$$|(1+y)^{\alpha}H(y;x) - (1+\tilde{a}(x))| \leqq \tilde{c}(x)y^2, \qquad x \geqq 2x_0,\ y \in [-1/2.1/2].$$

This implies our assetion. ∎

Now Proposition 8 is an easy corollary to Proposition 10.

5 Proof of Thoerem 3.

In this section, we assume that X_n, $n = 1,2,\dots$, are i.i.d. random variables, $\alpha > 2$ and (A-1) - (A-4) are satisfied. Let $p = (\alpha+2)/2$ and $\beta = (\alpha+p)/2$. Then we see that $E[|X_1|^p] < \infty$ and there is a $C_0 > 1$ such that

$$F(-x) + \bar{F}(x) \leqq C_0 x^{-\beta}, \qquad x \geqq 1.$$

Proposition 11 *Let* $b(x) = E[X_1, |X_1| \leqq x] = -E[X_1, |X_1| > x], \quad x > 0$. *Then we have the following.*
(1) $|b(x)| \leqq E[|X_1|^p]^{1/p}(F(-x)+\bar{F}(x))^{1-1/p} \leqq C_0 x^{-\beta(p-1)/p}E[|X_1|^p]^{1/p}, \qquad x \geqq 1.$

(2) *There is a constant* $C_1 > 1$ *only dependent on* p *such that*

$$E[|\sum_{k=1}^n X_k 1_{\{|X_k|\leqq x\}}|^p]^{1/p} \leqq C_1 n^{1/2}(E[|X_1|^p]^{1/p} + |b(x)|) + n|b(x)|$$

$$\leqq C_1 E[|X_1|^p]^{1/p}(1+C_0)(n^{1/2} + nx^{-\beta(p-1)/p})$$

for any $n = 1,2,\dots,$ *and* $x \geqq 1$.

Proof. The assertion (1) is an easy consequence of Hölder's inequality. So we prove the assertion (2). Since $E[X_k 1_{\{|X_k|\leqq x\}} - b(x)] = 0$, $k = 1,2,\dots$, we see by Burkholder-Davis-Gundy's theorem that there is a constant $C_1 > 0$ depending on p only such that

$$E[|\sum_{k=1}^n (X_k 1_{\{|X_k|\leqq x\}} - b(x))|^p]^{1/p} \leqq C_1 E[|\sum_{k=1}^n (X_k 1_{\{|X_k|\leqq x\}} - b(x))^2|^{p/2}]^{1/p}$$

Then by Hölder's inequality, we have

$$E[|\sum_{k=1}^n (X_k 1_{\{|X_k|\leqq x\}} - b(x))|^p]^{1/p} \leqq C_1 E[n^{p/2-1}\sum_{k=1}^n |X_k 1_{\{|X_k|\leqq x\}} - b(x)|^p]^{1/p}$$

$$= C_1 n^{1/2} E[|X_1 1_{\{|X_1|\leqq x\}} - b(x)|^p]^{1/p} \leqq C_1 n^{1/2}(E[|X_1 1_{\{|X_1|\leqq x\}}|^p]^{1/p} + |b(x)|)$$

This implies our assertion. ∎

Remind Proposition 8 and let

$$R(y;x) = H(y;x) - \{1 + a(x)y - \alpha y + \frac{\alpha(\alpha+1)y^2}{2}\}, \qquad y \in [-1/2, 1/2],\ x > 2x_0.$$

Then there is a $C_2 > 0$ such that

$$|R(y;x)| \leqq C_2(c(x)y^2 + y^3), \qquad y \in [-1/2, 1/2],\ x > 2x_0.$$

Let

$$Y_n(t) = \sum_{k=2}^{n} X_k 1_{\{|X_k| \leqq tn^{1/2}\}}, \qquad n \geqq 2,\ t > 0.$$

Proposition 12 *Let $r \in ((\alpha+2)/(2\alpha), 1)$. Then for any $\varepsilon \in (0, 1/2)$*

$$\varlimsup_{n\to\infty} \sup\{s^2 E[|H(-\frac{1}{sn^{1/2}}Y_n(t), sn^{1/2}),\ |Y_n(t)| \leqq \varepsilon sn^{1/2}] - (1 + \frac{\alpha(\alpha+1)}{2s^2 n}v)|;$$
$$s \geqq (\log n)^{1/2},\ t \geqq (\log n)^{-1}s^{(1+r)/2}\} = 0.$$

Proof. Let $s \geqq (\log n)^{1/2}$, $t \geqq (\log n)^{-1}s^{(1+r)/2}$, and $n \geqq 3$. Then $tn^{1/2} \geqq 1$. We see that

$$\begin{aligned}
&E[H(-\frac{1}{sn^{1/2}}Y_n(t), sn^{1/2}),\ |Y_n(t)| \leqq \varepsilon sn^{1/2}] - (1 + \frac{\alpha(\alpha+1)}{2s^2}v) \\
&= \frac{a(sn^{1/2}) - \alpha}{sn^{1/2}}E[Y_n(t)] + \frac{\alpha(\alpha+1)}{2s^2 n}(E[Y_n(t)^2] - nv) \\
&-E[1 + \frac{a(sn^{1/2}) - \alpha}{sn^{1/2}}Y_n(t) + \frac{\alpha(\alpha+1)}{2s^2 n}Y_n(t)^2, |(sn^{1/2})^{-1}Y_n(t)| > \varepsilon] \\
&+E[R(-\frac{1}{sn^{1/2}}Y_n(t), sn^{1/2}), |(sn^{1/2})^{-1}Y_n(t)| \leqq \varepsilon].
\end{aligned}$$

Note that

$$\begin{aligned}
s|E[Y_n(t)]| &= ns|b(tn^{1/2})| \leqq C_0 E[|X_1|^p]^{1/p} s(tn^{1/2})^{-\beta(p-1)/p} \\
&\leqq C_0 E[|X_1|^p]^{1/p}(n^{1/2}(\log n)^{-1})^{-\beta(p-1)/p} s^{1-(1+r)\beta(p-1)/2p}, \\
E[Y_k(t)^2] - nv &= n(E[(X_1 1_{|X_1| \leqq tn^{1/2}})^2] - b(tn^{1/2})^2) + E[Y_n(t)]^2 - nv \\
&= -nE[X_1^2, |X_1| > tn^{1/2}] + n(n-1)b(tn^{1/2})^2,
\end{aligned}$$

and

$$\begin{aligned}
&n^{-p/2}E[|Y_n(t)|^p] \\
&\leqq C_1^p(1 + C_0)^p E[|X_1|^p](1 + t^{-\beta(p-1)/p)} n^{1/2(1-\beta(p-1)/p)})^p.
\end{aligned}$$

Also, note that

$$r\beta(p-1)/p > 1 + \frac{3(\alpha-2)}{8} > 1.$$

So we see that

$$\frac{1}{n}|E[Y_k(t)^2] - nv| \leqq E[X_1^2, |X_1| > tn^{1/2}] + C_0(n-1)n^{-\beta(p-1)/p}(\log n)^{2\beta(p-1)/p}E[|X_1|^p]^{2/p},$$

$$s^2(sn^{1/2})^{-k}E[|Y_n(t)|^k, |(sn^{1/2})^{-1}Y_n(t)| > \varepsilon]$$
$$\leqq s^{2-p}\varepsilon^{-p+k}n^{-p/2}E[|Y_n(t)|^p], \qquad k = 0, 1, 2,$$

and

$$s^2E[|R(-\frac{1}{sn^{1/2}}Y_n(t), sn^{1/2})|, |(sn^{1/2})^{-1}Y_n(t)| \leqq \varepsilon],$$
$$\leqq C_2(|c(sn^{1/2})|n^{-1}E[Y_n(t)^2] + \varepsilon^{-p}s^{2-p}n^{-p/2}E[|Y_n(t)|^p].$$

Combining them, we have our assertion. ∎

Now we prove Theorem 3. We use the same idea in Section 3. Let $\beta : \mathbf{N} \to (0, \infty)$ be such that

$$\frac{\beta(n)}{(\log n)^{1/2}} \to \infty, \qquad n \to \infty.$$

Assume that Theorem 3 is not valid. Then there is a sequence of positive numbers $\{s'_n\}_{n=1}^\infty$ such that $s'_n \geqq n^{1/2}\beta(n)$, $n = 1, 2, \ldots$, and

$$\overline{\lim_{n\to\infty}} \frac{(s'_n)^2}{n}|\frac{P(\sum_{k=1}^n X_k > s'_n)}{n\bar{F}(s'_n)} - (1 + \frac{\alpha(\alpha+1)vn}{(s'_n)^2})| > 0.$$

Let $s_n = n^{-1/2}s'_n \geqq \beta(n)$. Let us take an $r \in ((\alpha+2)/(2\alpha), 1)$ and fix it. Let t_n, $n = 1, 2, \ldots$, be a sequence of positive numbers given by

$$t_n = (\log n)^{-1/2} + (\log n)^{-1}s_n^{(1+r)/2}, \qquad n \geqq 2.$$

Then from Proposition 12, we see that

$$s_n^2|\frac{nP(X_1 + Y_n(t) > sn^{1/2},\ |Y_n(t)| \leqq \varepsilon sn^{1/2})}{n\bar{F}(s_n n^{1/2})} - (1 + \frac{\alpha(\alpha+1)vn}{s_n^2 n})|$$
$$= s_n^2|E[H(-\frac{1}{sn^{1/2}}Y_n(t), sn^{1/2}),\ |Y_n(t)| \leqq \varepsilon sn^{1/2}] - (1 + \frac{\alpha(\alpha+1)vn}{s_n^2 n})| \to 0$$

as $n \to \infty$. Then Theorem 3 follows from this, Proposition 7(2), Equations (4), (5), (6), (7) and (8).

This completes the proof of Theorem 3.

6 Remarks

Let X_n, $n = 1, 2, \ldots$, be independent identically distributed random variables. and $F : \mathbf{R} \to [0, 1]$ and $\bar{F} : \mathbf{R} \to [0, 1]$ be given by

$$F(x) = P(X_1 \leq x), \qquad x \in \mathbf{R}$$

and

$$\bar{F}(x) = P(X_1 > x), \qquad x \in \mathbf{R}.$$

Let us assume the following.

(B-1) $\bar{F}(x)$ is a regualar varying function of index $-\alpha$, $\alpha \in (0, 2]$, as $x \to \infty$, i.e., if we let

$$L(x) = x^\alpha \bar{F}(x), \qquad x \in \mathbf{R},$$

$L(x) > 0$ for any $x > 0$, and for any $a > 0$

$$\frac{L(ax)}{L(x)} \to 1, \qquad x \to \infty.$$

(B-2) $x^{\alpha-\delta}F(-x) \to 0, \qquad x \to \infty$ for any $\delta > 0$.

In the case $\alpha > 1$, we see that $E[|X_1|] < \infty$. Let us assume furthermore that

(B-3) $E[X_1] = 0$ if $\alpha > 1$.

Then we have the following.

Theorem 13 *Assume the assumptions* (B-1), (B-2) *and* (B-3). *Then for any* $\gamma \in (0, \alpha)$

$$\sup_{s \geqq n^{1/\gamma}} |\frac{P(\sum_{k=1}^{n} X_k > s)}{n\bar{F}(s)} - 1| \to 0, \qquad n \to \infty.$$

The proof is quite similar to that of Theorem 1.

The case that $\alpha \in (0, 1]$, we need the following propositions instead of Propositions 4 and 5.

Proposition 14 *Let* Y *be a random variable and* $\gamma \in (0, 1]$. *Assume that*

$$E[|Y|^\gamma] < \infty$$

Then for any $s \in \mathbf{R} \setminus \{0\}$ *and* $b > 0$

$$E[\exp(sY1_{\{|Y| \leqq b\}}) \leqq 1 + |s|^\gamma \exp(|s|b)E[|Y|^\gamma].$$

Proposition 15 *Let* X *be a random variable and* $\gamma \in (0, 1]$. *We assume that*

$$E[|X|^\gamma] < \infty.$$

Then for any $t > 0$

$$n \log E[\exp(\pm\frac{1}{tn^{1/\gamma}} X1_{\{|X| \leqq tn^{1/\gamma}\}})] \leqq \frac{6}{t^\gamma} E[|X|^\gamma].$$

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