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Tikhonov regularization by a reproducing kernel Hilbert space for the Cauchy problem for an elliptic equation

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Abstract

We propose a discretized Tikhonov regularization for a Cauchy problem for an elliptic equation by a reproducing kernel Hilbert space. We prove the convergence of discretized regularized solutions to an exact solution. Our numerical results demonstrate that our method can stably reconstruct solutions to the Cauchy problems even in severe cases of geometric configurations.

1 Introduction

In this paper, we consider a classical ill-posed problem, the Cauchy problem for an elliptic equation: Given h, g_1 and g_2, find u inside of Ω or $\partial_A u|_{\partial\Omega\setminus\Gamma}$ where

$$\begin{cases} Au = h, & x \in \Omega, \\ u|_\Gamma = g_1, & \\ \partial_A u|_\Gamma = g_2, & \end{cases} \tag{1}$$

In (1), the domain $\Omega \subset \mathbb{R}^n$ is a bounded domain whose boundary $\partial\Omega$ is of C^2 class, Γ is an arbitrarily fixed open subset of $\partial\Omega$, and

$$Au(x) = \sum_{i,j=1}^{n} \partial_i(a_{ij}(x)\partial_j u(x)) + c(x)u, \quad x \in \Omega,$$

$\nu = \nu(x)$ is the unit outward normal vector to $\partial\Omega$ at x,

$$\partial_A u = \sum_{i,j=1}^{n} a_{ij}(x)(\partial_j u)\nu_i.$$

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Moreover, we assume that $a_{ij} = a_{ji} \in C^1(\overline{\Omega})$, $1 \le i, j \le n$, $c \in L^\infty(\Omega)$ and that there exists a constant $\gamma_0 > 0$ such that

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \ge \gamma_0 \sum_{j=1}^{n} \xi_j^2, \quad x \in \overline{\Omega},\ \xi_1, ..., \xi_n \in \mathbb{R}.$$

This problem appears in many applications for example in the cardiography, the non-destructive testing, etc. Stable and efficient numerical methods are of high importance. However, it is well-known that the Cauchy problem for an elliptic equation is ill-posed without any *a priori* bounds of u (e.g., Tikhonov and Arsenin [53]). However, under a priori bounds of u, we can restore the stability (see section 5) and, for stable numerical reconstructions of solutions, we can use regularization techniques. There are a large number of works devoting to stable numerical methods. We cannot list all works completely and the following is a partial list of papers which contain numerical tests as well as relevant analysis: Berntsson and Eldén [6], Bourgeois [7], Cheng, Hon, Wei and Yamamoto [9], Engl and Leitão [13], Falk and Monk [14], Hào and Lesnic [22], Hon and Wei [25], Klibanov and Santosa [34], Lattès and Lions [37], Lesnic, Elliott and Ingham [39], Qian, Fu and Xiong [47], Reinhardt, Han and Hào [48]. In particular, [34] uses the quasi-reversibility method as regularization and establishes error analysis for regularized solutions.

See Baumeister [5], Engl, Hanke and Neubauer [12], Groetsch [19, 20], Hofmann [23, 24], as monographs concerning regularizing techniques. Moreover, as for the theoretical results of the uniqueness and the conditional stability in determining u on $\overline{\Omega}$, see Isakov [27], Klibanov and Timonov [35], Lavrent'ev, Romanov and Shishat·skiĭ [38]

For stable reconstruction, we use the conventional Tikhonov regularization and have to discretized it for numerical calculations. The novelty of the paper is the use of the reproducing kernel Hilbert space for the discretization and we can list up the advantages:

(i) We can flexibly set up the accuracy of discretization. Since our problem is ill-posed, we have to choose the discretization size carefully for a noise level in data (e.g., [5, p.109-111]). In particular, it is often that fine discretization breaks stable numerics.

(ii) With a more generous *a priori* choice strategy of the regularizing parameters, we can prove the strong convergence of the regularized solutions, although a general discretization scheme can guarantee only the weak convergence.

(iii) Our methodology is widely applicable to various linear inverse problems.

(iv) The structure of the numerical programming is simple. In particular, thanks to the reproducing kernel Hilbert space, the calculations of the Tikhonov regularizing terms are very fast.

The general theory itself of the reproducing kernel Hilbert space originates for example, from [2], and we can refer to [50], [55] as up-to-date monographs.

In particular, methods by the reproducing kernel Hilbert space are very feasible for the computations of functions from empirical data, and the corresponding numerical method is effectively executed. Our numerical results is satisfactory for the Laplace equation as is seen in section 7.

To the authors' best knowledge, there is no previous work treating the discretized Tikhonov regularization by a reproducing kernel Hilbert space. For other interesting approaches to the Tikhonov regularization for inverse problems, one can see Asaduzzaman, Matsuura and Saitoh [3], Saitoh [50, 51], Saitoh, Matsuura and Asaduzzaman [52]. A special case of a reproducing kernel Hilbert space is the radial basis functions. It often gives accurate and fast numerical solutions for well-posed boundary value problems. See Franke and Schaback [16], Kansa [31], Kansa and Hon [33], Ling and Schaback [41], for example. For applications of the radial basis function to inverse problems, one can see Li [40]. Thanks to the generalized aspect of the choice of V_m, our method can be as flexible as the radial basis functions method.

This paper is composed of 8 sections. In section 2, we give general results for discretized Tikhonov regularization. Our numerical method fully relies on the reproducing kernel, and in section 3, we give a brief introduction of a reproducing kernel Hilbert space. In section 4, for a general inverse problem, we formulate a discretized Tikhonov regularization by a reproducing kernel Hilbert space and prove the convergence of the method. In section 5, we state the conditional stability up to the boundary for the Cauchy problem and the proof is given in appendix. The boundary estimation result for the variable coefficient case (1) is not found in the existing works. In section 6, we show our algorithm for the Cauchy problem (1) for the elliptic equation, and section 7 gives numerical results. Our conditional stability up to the boundary in section 5 interprets numerical results in section 7 in comparison with the existing interior conditional stability. Section 8 gives concluding remarks.

2 Discretized Tikhonov regularization

Many inverse problems can be reduced to a linear ill-posed operator equation

$$Kf = g,$$

by suitably choosing Hilbert spaces V and W and a linear compact operator $K\colon V \to W$. Henceforth $(\cdot,\cdot)_V$ means the inner product in V, and by $\|\cdot\|_V$ we denote the norm in V if we need to specify the space V. Henceforth we do not assume the injectivity of K.

We aim at the reconstruction f_0 satisfying

$$Kf_0 = g_0$$

by means of noisy data g_δ satisfying

$$\|g_0 - g_\delta\|_W \le \delta,$$

where $\delta > 0$ is a noise level. We assume that the value of δ is known *a priori*.

In order to stably reconstruct f_0 from some noisy data g_δ, we consider the Tikhonov regularization [53] and the following discretization. Let V_m be a finite dimensional linear subspace. Let $\{f_j^m\}_{1\le j\le m}$ be a linearly independent set of V_m. We denote P_m to be the orthogonal projection of V onto V_m. We consider the Tikhonov regularization on the finite dimensional space V_m:

$$\min_{f\in V_m} \|Kf - g_\delta\|_W^2 + \alpha\|f\|_V^2, \tag{2}$$

where $\alpha > 0$ is called the regularization parameter. The formulation (2) corresponds to a Ritz approach in [19] where $V_m \subset V_{m+1}$ is assumed. However such monotonicity of V_m may be inconvenient because the monotonicity may make the discretization too fine in the Tikhonov regularization where one usually need to control the accuracy of discretization suitably for the noise level (see e.g., pp.109-111 in [5]) and too fine discretization may not yield the convergence of the regularized solutions. Therefore we should develop the Ritz approach for the Tikhonov regularization without the monotonicity of V_m, which will be done in section 2-4.

Henceforth when we will not mention, the stated results are standard and we can refer for example to Baumeister [5], Bukhgeim [8], Engl, Hanke and Neubauer [12], Groetsch [19], Hofmann [24], Isakov [27], and Tikhonov and Arsenin [53].

We know that there exists a unique minimizer $f_{\alpha,m,\delta}$ of (2) for any $\alpha > 0$, $\delta > 0$ and $m \in \mathbb{N}$. Moreover, the minimizer is given by

$$f_{\alpha,m,\delta} = (K_m^* K_m + \alpha I)^{-1} K_m^* g_\delta,$$

where $K_m = KP_m$. We denote the minimizer when $\delta = 0$ by $f_{\alpha,m}$. With some *a priori* choices of α and m for given $\delta > 0$, we can prove the convergence of the Tikhonov regularized solutions.

The dicsretization of the Tikhonov regularization is similar to the Ritz approach (see [19, Chapter 4]) , but our point is the use of the reproducing kernel Hilbert space. On the other hand, our idea for the discretization can be regarded as a linear version of the reduced basis method, e.g., for an optimization problem subject to a nonlinear constraint (e.g., Ito and Ravindran [28]). Thus we naturally expect that we can extend our method to a stable numerical method for solving nonlinear inverse problems such as the determination of coefficients in partial differential equations by boundary measurements and we will exploit in a forthcoming paper. As for other references concerning the reduced basis method, see Barrett and Reddien [4], Ito and Ravindran [29, 30], and Porsching and Lin Lee [46].

In order to state the convergence results, we need notions for solutions of the equation $Kf = g$.

Definition 1. *Let $K \colon V \to W$ be a bounded linear operator.*

1. *$f \in V$ is called a* least-squares solution *of $Kf = g$ if $\|Kf - g\| = \inf\{\|Kf - g\|; f \in V\}$.*
2. *$f \in V$ is called a* minimum norm solution *of $Kf = g$ if*
   - *(a) f is a least-squares solution of $Kf = g$, and*
   - *(b) $\|f\| = \inf\{\|h\|; h$ is a least-square solution of $Kh = g\}$.*

The minimum norm least-squares solution is uniquely determined if g belongs to a dense subspace $\mathcal{D}(K^\dagger) := R(K) + R(K)^\perp$. We denote such a unique minimum least-squares solution by $K^\dagger g$. If K is injective, we have $f = K^\dagger g$ ([19, 12]).

We can now prove the convergence of the minimizer (2) to the solution $K^\dagger g_0$. Let

$$\gamma_m = \|K(I - P_m)\|.$$

We begin with the following lemma.

Lemma 2. *Suppose that $\lim_{m\to\infty} \|(I - P_m)f\| = 0$ for all $f \in V$. Let $m(\delta) \in \mathbb{N}$ and $\alpha(\delta)$ satisfy $\lim_{\delta\to 0} m(\delta) = \infty$ and $\lim_{\delta\to 0} \alpha(\delta) = 0$. Let $F_\delta := f_{\alpha(\delta),m(\delta)} - f_{\alpha(\delta),m(\delta),\delta}$.*

If $\gamma_m = O(\sqrt{\alpha})$ and $\delta = O(\sqrt{\alpha})$, then $\lim_{\delta\to 0} (F_\delta, z)_V = 0$ for all $z \in R(K^*K)$.

Proof. The proof can be found in [19] where the monotonicity of the finite dimensional subspaces $\{V_m\}$ is assumed.

By our assumptions, there exists a constant $C > 0$ such that $\dfrac{\delta}{\sqrt{\alpha(\delta)}} < C$ for all $\delta > 0$ which is sufficiently small. Let $z \in R(K^*K)$. Then there exists $y \in V$ such that $z = K^*Ky$. Since $\lim_{\delta\to 0} m(\delta) = \infty$, we have $\lim_{m\to\infty} \|(I - P_m)f\| = 0$ for all $f \in V$. Hence, for any $\varepsilon > 0$, there exits $\delta_0 \in \mathbb{N}$ such that

$$\|(I - P_{m(\delta)})y\| < \varepsilon, \quad \sqrt{\alpha(\delta)}\delta < \varepsilon, \quad \delta < \varepsilon,$$

for all $0 < \delta \leq \delta_0$.

On the other hand, we have

$$|(F_\delta, z)| \leq |(F_\delta, K^*K(y - P_{m(\delta)}y))| + |(F_\delta, K^*KP_{m(\delta)}y)| = I_1 + I_2.$$

By [19, Lemma 4.2.7], for all $m \in \mathbb{N}$, $\alpha > 0$ and $\delta > 0$, we have

$$\|f_{\alpha,m} - f_{\alpha,m,\delta}\| \leq \frac{\delta}{\sqrt{\alpha}}.$$

Thus, for all $\delta < \delta_0$, we have

$$I_1 \leq \frac{\delta}{\sqrt{\alpha(\delta)}}\|K^*K\|\varepsilon \leq C\|K^*K\|\varepsilon.$$

On the other hand, by [19, forumla (10) on p. 78], we have

$$\alpha(f_{\alpha,m} - f_{\alpha,m,\delta}, P_m y) + (K(f_{\alpha,m} - f_{\alpha,m,\delta}), KP_m y) = (g_\delta - g, KP_m y),$$

for all $m \in \mathbb{N}$, $\alpha > 0$ and $\delta > 0$, and for all $y \in V$. Hence, for all $\delta < \delta_0$, we have

$$I_2 \leq \alpha|(F_\delta, P_{m(\delta)}y)| + |(g_\delta - g, KP_{m(\delta)}y)| \leq (\|y\| + \|K\|\|y\|)\varepsilon.$$

Thus we complete the proof. □

Unlike [19], we do not assume that $V_m \subset V_{m+1}$, $m \in \mathbb{N}$, so that we do not have $\lim_{m\to\infty} \gamma_m = 0$ (cf. Lemma 4.2.1 in [19]). However the proof of the Theorem 4.2.4 in [19] valid and gives: Suppose that $\lim_{m\to\infty} \gamma_m = 0$ and $\lim_{m\to\infty} \|(I - P_m)f\| = 0$ for all $f \in V$. Let $\lim_{m\to\infty} \alpha_m = 0$. If $\gamma_m = O(\sqrt{\alpha_m})$, then $\lim_{m\to\infty} f_{\alpha_m,m} = K^\dagger g_0$ in V.

In the case where date g_δ is contaminated with noise with level δ and we do not choose a monotone family $\{V_m\}_{m\in\mathbb{N}}$, we can prove the weak convergence:

Proposition 3. *Suppose that $\lim_{m\to\infty} \gamma_m = 0$ and $\lim_{m\to\infty} \|(I - P_m)f\| = 0$ for all $f \in V$. Let $\lim_{\delta\to 0} m(\delta) = \infty$ and $\lim_{\delta\to 0} \alpha(\delta) = 0$. If $\gamma_m = O(\sqrt{\alpha})$, $\delta = O(\sqrt{\alpha})$, then $\lim_{\delta\to 0} f_{\alpha(\delta),m(\delta),\delta} = K^\dagger g_0$ weakly in V.*

Remark 4. *See Theorem 4.2.13 in [19] when the monotonicity of $\{V_m\}_{m\in\mathbb{N}}$ is assumed. If we replace $\delta = O(\sqrt{\alpha})$ by $\delta = o(\sqrt{\alpha})$, then we can prove the strong convergence (Theorem 4.2.8 in [19]).*

Proof. In the following, we do not indicate the dependence on δ in the notations.

Let $f_\alpha = (K^*K+\alpha)^{-1}K^*g$ and let $f_{\alpha,m} = (K_m^*K_m+\alpha)^{-1}K_m^*g$, where $K_m = KP_m$. By [19, Lemma 4.2.7], we have

$$\|f_{\alpha,m,\delta} - f_{\alpha,m}\| \le \frac{\delta}{\sqrt{\alpha}}.$$

Moreover, by [19, Lemma 4.2.3]

$$\|f_{\alpha,m}\| \le \|f_\alpha\| + \left(1+\frac{\gamma_m}{\alpha}\right)^{\frac{1}{2}} \|(I-P_m)f_\alpha\|.$$

By $\gamma_m = O(\sqrt{\alpha})$, we have

$$\|f_{\alpha,m}\| \le C_0\|f_\alpha\|.$$

Here and henceforth, C_j denote constants which are independent of α, m, and δ. Let $\{v_j, u_j, \mu_j\}_{j\in\mathbb{N}}$ be the singular system: $Kv_j = \mu_j u_j$ and $K^*u_j = \mu_j v_j$. Then,

$$f_\alpha = \sum_{j=1}^{\infty} \frac{\mu_j}{\mu_j^2+\alpha}(g,u_j)v_j,$$

so that

$$\begin{aligned}
\|f_\alpha\|^2 &= \sum_{j=1}^{\infty} \frac{\mu_j^2}{(\mu_j^2+\alpha)^2}(g,u_j)^2 = \sum_{j=1}^{\infty} \frac{\mu_j^4}{(\mu_j^2+\alpha)^2}\frac{(g,u_j)^2}{\mu_j^2} \\
&\le \sum_{j=1}^{\infty} \frac{(g,u_j)^2}{\mu_j^2} = \sum_{j=1}^{\infty} \frac{(Kf_0,u_j)^2}{\mu_j^2} = \sum_{j=1}^{\infty} (f_0,v_j)^2 = \|f_0\|^2.
\end{aligned}$$

Since $\delta = O(\sqrt{\alpha})$, we have

$$\|f_{\alpha,m}\| \le C_0\|f_0\|, \tag{3}$$

and

$$\|f_{\alpha,m,\delta}\| \le \|f_{\alpha,m}\| + \frac{\delta}{\sqrt{\alpha}} \le C_0\|f_0\| + \frac{\delta}{\sqrt{\alpha}} \le C_1. \tag{4}$$

Since $f_{\alpha,m,\delta}$ is the minimizer, by (3) we obtain

$$\begin{aligned}
\|Kf_{\alpha,m,\delta} - g_\delta\|^2 + \alpha\|f_{\alpha,m,\delta}\|^2 &\le \|KP_mf_\alpha - g_\delta\|^2 + \alpha\|P_mf_\alpha\|^2 \\
&\le \|KP_mf_\alpha - g_\delta\|^2 + \alpha C_0^2\|f_0\|^2.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\|KP_mf_\alpha - g_\delta\| &\leq \|KP_mf_\alpha - Kf_\alpha\| + \|Kf_\alpha - Kf_0\| + \|Kf_0 - g_\delta\| \\
&\leq \|K(1-P_m)\|\|f_\alpha\| + \|K\|\|f_\alpha - f_0\| + \delta \\
&\leq C\gamma_m\|f_0\| + \|K\|\|f_\alpha - f_0\| + \delta.
\end{aligned}$$

Hence,

$$\|Kf_{\alpha,m,\delta} - g_\delta\| \leq C\gamma_m\|f_0\| + \|K\|\|f_\alpha - f_0\| + \delta + \sqrt{\alpha}C_0\|f_0\|,$$

and we have

$$\begin{aligned}
\|Kf_{\alpha,m,\delta} - Kf_0\| &= \|Kf_{\alpha,m,\delta} - g_\delta + g_\delta - Kf_0\| \leq \|Kf_{\alpha,m,\delta} - g_\delta\| + \|g_\delta - g_0\| \\
&\leq C\gamma_m\|f_0\| + \|K\|\|f_\alpha - f_0\| + 2\delta + \sqrt{\alpha}C_0\|f_0\|.
\end{aligned}$$

By [19, Theorem 2.1.1], we see that $\lim_{\alpha\to 0}\|f_\alpha - f_0\| = 0$. Hence,

$$\lim_{\delta\to 0} Kf_{\alpha,m,\delta} = Kf_0.$$

Choose a subsequence $f'_{\alpha,m,\delta}$ arbitrarily. By (4), we can extract a subsequence $f''_{\alpha,m,\delta}$ such that

$$\lim_{\delta\to 0} f''_{\alpha,m,\delta} = f''_0 \quad \text{weakly in } V.$$

We have $\lim_{\delta\to 0} Kf''_{\alpha,m,\delta} = Kf''_0$ in W because K is compact. Therefore, $Kf''_0 = Kf_0$.

Next, we prove that $f''_0 = K^\dagger g$. Firstly, we observe that

$$f''_0 - K^\dagger g = f''_0 - f''_{\alpha,m,\delta} + f''_{\alpha,m,\delta} - f''_{\alpha,m} + f''_{\alpha,m} - K^\dagger g.$$

From [19, Theorem 4.2.4], we know that $\lim_{\alpha\to 0}\|f''_{\alpha,m} - K^\dagger g\| = 0$. For the proof of $f''_0 = K^\dagger g$, it is sufficient to show that

$$\lim_{\delta\to 0}(f''_{\alpha,m,\delta} - f''_{\alpha,m}, z) = 0, \tag{5}$$

for all $z \in V$.

When $z \in R(K^*K)^\perp = \mathrm{Ker}(K) \subset V$, we have $\lim_{\delta\to 0}(f''_{\alpha,m,\delta} - f''_{\alpha,m}, z) = 0$ by [19, Lemma 4.2.12] in which it is assumed that $V_m \subset V_{m'}$ if $m < m'$, but the proof is valid without it.

On the other hand, when $z \in R(K^*K) \subset V$, by Lemma 2, we see that

$$\lim_{\delta\to 0}(f''_{\alpha,m,\delta} - f''_{\alpha,m}, z) = 0.$$

Since the subspace $R(K^*K) + R(K^*K)^\perp$ is dense in V, by the Banach-Steinhaus theorem, (5) is valid for all $z \in V$.

Thus, an arbitrary subsequence $f'_{\alpha,m,\delta}$ of $f_{\alpha,m,\delta}$ contains a subsequence that weakly converges to the unique limit $K^\dagger g$. Consequently, the original sequence $f_{\alpha,m,\delta}$ itself converges weakly to $K^\dagger g$. □

3 Reproducing kernel Hilbert spaces

In this section, we introduce a reproducing kernel Hilbert space. One can refer to [2, 43, 49, 55] for detailed treatises.

Let E be an arbitrary non-empty subset of $\mathbb{R}^d$. We call a symmetric function $\Phi\colon E \times E \to \mathbb{R}$ a *kernel*. A kernel Φ is said to be *positive definite* (respectively, *positive semi-definite)*, if for all $N \in \mathbb{N}$ and all sets of pairwise distinct points $X = \{x_1, \dots, x_N\} \subset E$, the matrix $[\Phi(x_i, x_j)]_{i,j}$ is positive definite (respectively, positive semi-definite).

Definition 5. Let $\mathcal{H}$ be a real Hilbert space with the inner product $(\cdot,\cdot)_{\mathcal{H}}$ whose elements are some real-valued functions defined in E. A function $\Phi\colon E \times E \to \mathbb{R}$ is called a *reproducing kernel* for $\mathcal{H}$ if

1. $\Phi(\cdot, x) \in \mathcal{H}$ for all $x \in E$,

2. $f(x) = (f, \Phi(\cdot, x))_{\mathcal{H}}$. for all $f \in \mathcal{H}$ and all $x \in E$.

We define the norm by $\|f\|_{\mathcal{H}} = (f, f)_{\mathcal{H}}^{\frac{1}{2}}$.

A Hilbert space of functions which admits a reproducing kernel is called a *reproducing kernel Hilbert space* (in short, *RKHS*). The reproducing kernel of a RKHS is uniquely determined. Conversely, if a symmetric positive definite kernel Φ is given, then one can construct a unique RKHS in which the given kernel acts as the reproducing kernel (see [55] for details). The construction of the RKHS from a given positive definite kernel Φ is achieved as follows. We define a pre-Hilbert space $F_\Phi(E)$ by

$$F_\Phi(E) := \operatorname{span}\left\{\Phi(\cdot, x); x \in E\right\},$$

with inner product $\langle \cdot, \cdot \rangle_\Phi$

$$\langle f, g \rangle_\Phi := \sum_{j=1}^{N} \sum_{k=1}^{M} \alpha_j \beta_k \Phi(x_j, y_k),$$

for functions

$$f = \sum_{j=1}^{N} \alpha_j \Phi(\cdot, x_j), \quad g = \sum_{k=1}^{M} \beta_k \Phi(\cdot, y_k) \in F_\Phi(E).$$

By taking a completion with respect to the norm of $F_\Phi(E)$, we can define the RKHS $\mathcal{H}$ with the inner product $(\cdot,\cdot)_{\mathcal{H}} = \langle \cdot, \cdot \rangle_\Phi$. See [55] for details of the construction.

For later convenience, we collect fundamental properties of RKHS:

Proposition 6 ([55])**. (i)** *For all $x, y \in E$, we have $\Phi(x, y) = (\Phi(\cdot, x), \Phi(\cdot, y))_{\mathcal{H}}$.*

(ii) *For all $f \in \mathcal{H}$ in the form of $f = \sum_{k=1}^{N} \alpha_k \Phi(\cdot, x_k)$ with $x_k \in E$, we have*

$$\|f\|_{\mathcal{H}}^2 = \sum_{k=1}^{N} \sum_{j=1}^{N} \alpha_k \alpha_j \Phi(x_k, x_j).$$

(iii) *$F_\Phi(E)$ is dense in $\mathcal{H}$.*

(iv) *If a sequence $\{f_n\}_{n=1}^{\infty}$ of $\mathcal{H}$ converges to f weakly in $\mathcal{H}$, then $\{f_n(x)\}_{n=1}^{\infty}$ converges to $f(x)$ for all $x \in E$.*

(v) *Suppose that Φ is bounded on $E \times E$. If a sequence $\{f_n\}_{n=1}^{\infty}$ of $\mathcal{H}$ converges to f in $\mathcal{H}$, then $\{f_n\}_{n=1}^{\infty}$ uniformly converges to f on E.*

(vi) *Suppose $\int_E \Phi(x,x) d\mu < \infty$. Then a RKHS has a continuous linear embedding into $L^2(E, \mu)$ where μ is a measure on E.*

For a finite set of points $X := \{x_1, \ldots, x_N\}$ and $f \in \mathcal{H}$, we define $s_{f,X}(x)$ by

$$s_{f,X}(x) := \sum_{k=1}^{N} \alpha_k \Phi(x, x_k),$$

where the coefficients $\{\alpha_k\}_{k=1}^{N}$ are determined by the conditions

$$s_{f,X}(x_k) = f(x_k), \quad 1 \leq k \leq N.$$

Since the matrix $[\Phi(x_i, x_j)]_{i,j}$ is positive definite, $\{\alpha_k\}_{k=1}^{N}$ are uniquely determined.

We define a subspace by

$$\mathcal{V}_X := \operatorname{span}\{\Phi(\cdot, x); x \in X\} \subset \mathcal{H},$$

and an operator $P_X : \mathcal{H} \to \mathcal{V}_X \subset \mathcal{H}$

$$P_X(f)(x) = s_{f,X}(x).$$

Then we have

Proposition 7 ([55])**.** *P_X is an orthogonal projection of $\mathcal{H}$ onto the closed subspace $\mathcal{V}_X$.*

Define the fill distance h_X of X by

$$h_{X,E} = \sup_{x \in E} \min_{x_j \in X} |x - x_j|.$$

We choose some finite sets of points X_m, $m \in \mathbb{N}$ of E such that $h_{X_m,E} > h_{X'_m,E}$ for all $m < m' \in \mathbb{N}$ and $\lim_{m\to\infty} h_{X_m,E} = 0$. We set

$$V_m := \mathcal{V}_{X_m} \quad \text{and} \quad P_m := P_{V_m}.$$

In general, we cannot guarantee that the union $\bigcup_{m=1}^{\infty} V_m$ is dense in $\mathcal{H}$ nor $\lim_{m\to\infty} \|f - P_m(f)\|_{\mathcal{H}} = 0$. However, with a moderate assumption on the kernel Φ, we can prove these properties, which are crucial in our regularization method.

Lemma 8. *If the reproducing kernel Φ is uniformly continuous on $E \times E$, then we have $\lim_{m\to\infty} \|f - P_m(f)\|_{\mathcal{H}} = 0$ for all $f \in \mathcal{H}$.*

Proof. For any $f \in \mathcal{H}$ and any $\varepsilon > 0$, by the density of $F_\Phi(E)$, there exist a finite set of points $X = \{x_1, \dots, x_N\}$ of E and $f_{\varepsilon,X} \in \mathcal{V}_X$ such that $\|f - f_{\varepsilon,X}\|_{\mathcal{H}} < \varepsilon$. Here $f_{\varepsilon,X} \in \mathcal{V}_X$ is of the form

$$f_{\varepsilon,X}(x) = \sum_{k=1}^{N} \alpha_k \Phi(x, x_k)$$

with some $\alpha_k \in \mathbb{R}$.

Since Φ is uniformly continuous on $\Omega \times \Omega$, there exists $\delta > 0$ such that if $|x - y| < \delta$, then

$$\sup_{z\in E} |\Phi(z,x) - \Phi(z,y)| < (\max_{1\le j,k\le N} |\alpha_j \alpha_k| + 1)^{-1} (N\sqrt{2})^{-2} \varepsilon^2.$$

Let $B(x_0, d) = \{x \in \mathbb{R}^d; |x - x_0| < d\}$. Since $\lim_{m\to\infty} h_{X_m,E} = 0$, there exists $m_0 \in \mathbb{N}$ such that $X_m \cap B(x_k, \delta) \neq \emptyset$ for all $m \ge m_0$, $m \in \mathbb{N}$ and for all $x_k \in X$. Thus, for each $x_k \in X$ and for all $m \ge m_0$, we can pick at least one point $y = y(x_k) \in X_m \cap B(x_k, \delta)$.

For $m \ge m_0$, define $g_m \in V_m$ by

$$g_m(x) = \sum_{k=1}^{N} \alpha_k \Phi(x, y(x_k)).$$

Then, by Proposition 6 (i) and (ii), we have

$$\begin{aligned}
\|f_{\varepsilon,X} - g_m\|_{\mathcal{H}}^2 &= \left\| \sum_{k=1}^{N} \alpha_k \Phi(\cdot, x_k) - \sum_{k=1}^{N} \alpha_k \Phi(\cdot, y(x_k)) \right\|_{\mathcal{H}}^2 \\
= \quad & \sum_{j,k=1}^{N} \alpha_j \alpha_k \Phi(x_j, x_k) + \sum_{j,k=1}^{N} \alpha_j \alpha_k \Phi(y(x_j), y(x_k)) - 2 \sum_{j,k=1}^{N} \alpha_j \alpha_k \Phi(x_j, y(x_k)) \\
\le \quad & \left| \sum_{j,k=1}^{N} \alpha_j \alpha_k \Phi(x_j, x_k) - \sum_{j,k=1}^{N} \alpha_j \alpha_k \Phi(x_j, y(x_k)) \right| \\
& + \left| \sum_{j,k=1}^{N} \alpha_j \alpha_k \Phi(x_j, y(x_k)) - \sum_{j,k=1}^{N} \alpha_j \alpha_k \Phi(y(x_j), y(x_k)) \right|
\end{aligned}$$

$$\le \quad N^2 \max_{1\le j,k\le N} |\alpha_j \alpha_k| \sup_{x\in E} |\Phi(x, x_k) - \Phi(x, y(x_k))|$$
$$+N^2 \max_{1\le j,k\le N} |\alpha_j \alpha_k| \sup_{x\in E} |\Phi(x_j, x) - \Phi(y(x_j), x)| < \varepsilon^2.$$

Hence, we have

$$\|f - g_m\|_{\mathcal{H}} \le \|f - f_{\varepsilon,X}\|_{\mathcal{H}} + \|f_{\varepsilon,X} - g_m\|_{\mathcal{H}} < 2\varepsilon,$$

for all $m \ge m_0$. Since $\|f - P_m(f)\|_{\mathcal{H}} \le \|f - g\|_{\mathcal{H}}$ for all $g \in V_m$, we have that

$$\|f - P_m(f)\|_{\mathcal{H}} \le \|f - g_m\|_{\mathcal{H}} < 2\varepsilon$$

for all $m \ge m_0$. □

If we choose Φ and E suitably, then the reproducing kernel Hilbert space $\mathcal{H}$ realizes usual Sobolev spaces (e.g., [50, 55]). For example, $\mathcal{H} = H^1(\mathbb{R})$ if $\Phi(x, y) = \frac{1}{2}\exp(-|x - y|)$.

4 Discretized Tikhonov regularization by reproducing kernel Hilbert spaces

In this section, we apply the general results in section 2 to the case when V is a RKHS.

Let E be a subset of $\mathbb{R}^d$. Let $(E, \mathcal{F}, \mu)$ be a measure space on E. Let $\Phi\colon E\times E \to \mathbb{R}$ be a reproducing kernel. We assume that Φ is uniformly continuous on $E \times E$. We define a RKHS $\mathcal{H}$ on E generated by the kernel Φ. Let $K\colon \mathcal{H} \to W$ be a linear compact operator, where W is a Hilbert space. We consider the problem of finding the solution $f_0 \in \mathcal{H}$ in $Kf_0 = g_0$ by means of noisy data g_δ satisfying $\|g - g_\delta\|_W \le \delta$.

We choose finite sets of points X_m, $m \in \mathbb{N}$ of E such that $\lim_{m\to\infty} h_{X_m,E} = 0$. We set a finite dimensional subspace $V_m := \mathcal{V}_{X_m}$ and the projection $P_m := P_{V_m}$. By Lemma 8, we have $\lim_{m\to\infty} \|(I - P_m)f\| = 0$ for all $f \in \mathcal{H}$. Set $\gamma_m = \|K(I - P_m)\|$. Henceforth we assume that $\lim_{m\to\infty} \gamma_m = 0$, which is satisfied by many reproducing kernels [55].

Let $f_{\alpha,m,\delta}$ be a unique solution of (2) when $V = \mathcal{H}$ and let $f_{\alpha,m}$ be a unique solution of (2) when the data $g_\delta = g_0$. Thanks to the reproducing kernel Hilbert space, we can strengthen the convergence of the discretized Tikhonov regularized solutions:

Theorem 9. *Under the above settings, we have:*

1. *Let* $\lim_{m\to\infty} \alpha_m = 0$. *Suppose* $\sup_{x\in E} \Phi(x, x) < \infty$.

   If $\gamma_m = O(\sqrt{\alpha_m})$, *then* $\lim_{m\to\infty} \|f_{\alpha_m,m} - K^\dagger g_0\|_{L^\infty(E,\mu)} = 0$.

2. *Let* $\lim_{\delta\to 0} m(\delta) = \infty$ *and* $\lim_{\delta\to 0} \alpha(\delta) = 0$. *Suppose* $\int_E \Phi(x, x) d\mu(x) < \infty$.

   If $\gamma_m = O(\sqrt{\alpha})$, $\delta = O(\sqrt{\alpha})$, *then* $\lim_{\delta\to 0} \|f_{\alpha(\delta),m(\delta),\delta} - K^\dagger g_0\|_{L^2(E,\mu)} = 0$.

Proof. Part (i) follows Theorem 4.2.4 in [19] and Proposition 6 (v). We will prove part (ii).

From Definition 5 and Proposition 3, we have

$$\lim_{\delta\to 0}|f_{\alpha,m,\delta}(x)-K^{\dagger}g_0(x)| = \lim_{\delta\to 0}|(f_{\alpha,m,\delta}-K^{\dagger}g_0,\Phi(\cdot,x))_{\mathcal{H}}| = 0,$$

for all $x \in E$.

On the other hand, since a weak convergence sequence is norm bounded, we have

$$\begin{aligned}|f_{\alpha,m,\delta}(x)-K^{\dagger}g_0(x)|^2 &= |(f_{\alpha,m,\delta}-K^{\dagger}g_0,\Phi(\cdot,x))_{\mathcal{H}}|^2\\ &\le \|f_{\alpha,m,\delta}-K^{\dagger}g_0\|_{\mathcal{H}}^2\Phi(x,x) \le C\Phi(x,x)\end{aligned}$$

for all $x \in E$ with a positive constant C which is independent of $\{f_{\alpha,m,\delta}\}$. By assumption, $\Phi(x,x)$ is integrable. Hence, we have $\lim_{\delta\to 0}\|f_{\alpha,m,\delta}-K^{\dagger}g_0\|_{L^2(E)}=0$ by the Lebesgue theorem. □

Since $f_{\alpha,m,\delta}\in V_m$, it can be expressed by

$$f_{\alpha,m,\delta}=\sum_{k=1}^{m}\lambda_k\Phi(\cdot,x_k),$$

with some $\{\lambda_k\}_{k=1}^m$. By the property of a RKHS, the minimization problem (2) is equivalent to $\min_{\lambda\in\mathbb{R}^m}J(\lambda)$, where

$$J(\lambda)=\frac{1}{2}\|\sum_{k=1}^{m}\lambda_k K(\Phi(\cdot,x_k))-g_\delta\|_W^2+\alpha\sum_{k=1}^{m}\sum_{j=1}^{m}\Phi(x_k,x_j).$$

We can reduce the minimization for $J(\lambda)$ to $\dfrac{\partial J}{\partial\lambda_k}(\lambda)=0$, $k=1,\dots,m$. This leads to the system

$$(A+\alpha B)\lambda=G_\delta, \tag{6}$$

where A, B and G_δ are defined, respectively, by

$$A_{i,j}=(K(\Phi(\cdot,x_i)),K(\Phi(\cdot,x_j)))_W,\quad B_{i,j}=\Phi(x_i,x_j),\quad i,j=1,\dots,m,$$

and G_δ is defined by

$$G_{\delta,i}=(K(\Phi(\cdot,x_i)),g_\delta)_W,\quad i=1,\dots,m.$$

As is stated in section 1, the structure of our numerical programme is simple. That is, (i) the discretization of the function in V by the RKHS. (ii) a computation for $K(\Phi(\cdot,x_k))$ which corresponds to well-posed problems. (iii) the Tikhonov regularization of the resulting finite dimensional system (6).

5 Conditional stability

In this section, we show the conditional stability estimates for the Cauchy problem (1). The following is proved and see [27, Theorem 3.3.1].

Theorem 10. (interior conditional stability) *Let Ω_0 be a domain such that $\overline{\Omega_0} \subset \Omega \cup \Gamma$. Then, there exist constants $C > 0$ and $\kappa \in (0,1)$ such that*

$$\|u\|_{H^2(\Omega_0)} \le C(D + \|u\|_{H^1(\Omega)}^{1-\kappa} D^\kappa) \tag{7}$$

where

$$D = \|h\|_{L^2(\Omega)} + \|g_1\|_{H^1(\Gamma)} + \|g_2\|_{L^2(\Gamma)}.$$

Estimate (7) guarantees that $\|u\|_{H^2(\Omega_0)}$ is small if data F is small, provided that $\|u\|_{H^1(\Omega)}$ is *a priori* bounded, and (7) is called a conditional stability estimate.

In Isakov [27], only the first order term $\|u\|_{H^1(\Omega_0)}$ is estimated. We can follow the same argument in [27] with the help of Lemma 17 stated below to estimate also $\|u\|_{H^2(\Omega_0)}$. The theorem is an interior stability estimate that is valid only in Ω_0 as long as $\partial\Omega_0$ does not touch $\partial\Omega \setminus \Gamma$ and so does not give any estimation of u on $\partial\Omega \setminus \Gamma$.

As for interior conditional stability for the Cauchy problem for an elliptic equation, there are many works and see, for example, Fursikov [17], Han and Reinhardt [21], Kubo [36], Lavrent'ev, Romanov and Shishat·skiĭ [38], Payne [44, 45].

In practice, such as determination of corrosion damages (e.g., Fasino and Inglese [15], Cheng, Choulli and Yang [56]), we have to estimate boundary values of u on $\partial\Omega \setminus \Gamma$ which is the inaccessible subboundary. There are only a few papers on such boundary estimation and see Cheng, Choulli and Yang [56], Cheng and Yamamoto [10], and Eller and Yamamoto [11]. The paper [11] treats the stationary anisotropic Maxwell's equations, and the paper [56] discusses only the case where the principal part of A is Δ, and in particular, [10] depends essentially on the analyticity of u.

Theorem 11 (boundary conditional stability)**.** *Let $\eta > \frac{n+2}{2}$. For $0 < \kappa_0 < 1$, there exists a constant $C > 0$ such that*

$$\|u\|_{L^\infty(\partial\Omega\setminus\Gamma)} \le C\|u\|_{H^\eta(\Omega)} \left(\log \frac{1}{\|g_1\|_{L^2(\Gamma)} + \|g_2\|_{L^2(\Gamma)} + \|h\|_{L^2(\Omega)}} + \log \frac{1}{\|u\|_{H^\eta(\Omega)}} \right)^{-\kappa_0}.$$

The theorem says that if the norm $\|g_1\|_{L^2(\Gamma)} + \|g_2\|_{L^2(\Gamma)} + \|h\|_{L^2(\Omega)}$ of data tends to zero, then $\|u\|_{L^\infty(\partial\Omega\setminus\Gamma)}$ approaches 0 provided that we know an *a priori* bound for $\|u\|_{H^\eta(\Omega)}$. The rate of convergence of $\|u\|_{L^\infty(\partial\Omega\setminus\Gamma)}$ is logarithmic. The boundary estimate in Theorem 11 is much weaker than the interior estimate in Theorem 10. The proof is based on a Carleman estimate and is given in appendix.

6 Reconstruction method

We assume that the problem (1) admits a unique solution $u_0 \in H^{\frac{3}{2}}(\Omega)$ for g_1 and g_2. In this section, we show a reconstruction method by means of the discretized Tikhonov regularization proposed in section 4. We assume that $\Omega \subset \mathbb{R}^2$ for simplicity. We also assume that there exists a C^∞ map

$$\Pi\colon [0,1] \to \partial\Omega\backslash\Gamma$$

such that Π is injective and $\Pi([0,1]) = \partial\Omega\backslash\Gamma$. Set $\Sigma := \partial\Omega\backslash\Gamma$. Let

$$\Phi(x,y)\colon [0,1]\times[0,1] \to \mathbb{R}$$

be a positive definite kernel on $[0,1]$. Let $\mathcal{H}$ be the RKHS on $[0,1]$ generated by the kernel Φ. We denote $\varphi(\Pi^{-1}(x))$ by $\Pi_*\varphi(x)$ for $\varphi \in \mathcal{H}$ and $x \in \Sigma$. For $m \in \mathbb{N}$, we define a set of points $X_m \subset [0,1]$. We define the finite subspace V_m by $V_m := \mathcal{V}_{X_m}$ and P_m by $P_m := P_{V_m}$, respectively.

We pose the following two assumptions on the positive definite kernel that is satisfied by many type of positive definite kernels [55].

Assumption 12. *We assume that the kernel Φ is uniformly continuous on* $[0,1]\times[0,1]$.

Assumption 13. *Suppose there exists a function $p\colon \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $\lim_{r\to 0} p(r) = 0$ such that the estimate holds*

$$\|f - P_m f\|_{L^\infty(0,1)} \leq p(h_{X_m})\|f\|_{\mathcal{H}}.$$

for all $f \in \mathcal{H}$. Here $h_{X_m} := \sup_{x\in[0,1]} \min_{x_k\in X_m} |x - x_k|$.

Firstly, we construct an approximation to $\partial_A u_0|_\Sigma$ of the solution of (1). After obtaining the approximation, we solve a boundary value problem which is well-posed and obtain an approximation to the solution of (1). Thus it suffices to approximate $\partial_A u_0|_\Sigma$.

We define a Hilbert space on Σ by

$$\mathcal{H}_\Sigma := \{\Pi_*\varphi\colon \Sigma \to \mathbb{R} \mid \varphi \in \mathcal{H}\},$$

equipped with an inner product

$$(\Pi_*\varphi_1, \Pi_*\varphi_2)_{\mathcal{H}_\Sigma} := (\varphi_1, \varphi_2)_{\mathcal{H}},$$

where $\varphi_i \in \mathcal{H}$. It is easy to check that $\mathcal{H}_\Sigma$ is a RKHS generated by the kernel $\Psi(x,y) := \Phi(\Pi^{-1}(x), \Pi^{-1}(y))$.

Let Γ_0 be a relatively open subset of Γ. Let u_0 denote the unique solution of (1). We assume that $\partial_A u_0(\Pi(t)) \in \mathcal{H}$. Suppose that the noisy data g_1^δ and g_2^δ satisfy

$$\|g_1 - g_1^\delta\|_{L^2(\Gamma)} \leq \delta, \quad \text{and} \quad \|g_2 - g_2^\delta\|_{L^2(\Gamma)} \leq \delta.$$

We first consider the direct problem

$$\begin{cases} Au = h, & x \in \Omega, \\ \partial_A u|_\Sigma = \theta_1, \\ u|_{\Gamma_0} = \theta_2, \\ \partial_A u|_{\Gamma\backslash\Gamma_0} = \theta_3, \end{cases} \tag{8}$$

for $\theta_1 \in L^2(\Sigma)$, $\theta_2 \in L^2(\Gamma_0)$ and $\theta_1 \in L^2(\Gamma\backslash\Gamma_0)$. We denote the solution of (8) by $u(\theta_1, \theta_2, \theta_3, h)$.

Let L and g^δ be defined, respectively, by

$$L\varphi := u(\varphi, 0, 0, 0)|_{\Gamma\backslash\Gamma_0}, \quad g_\delta = g_1^\delta - u(0, g_1^\delta, g_2^\delta, h)|_{\Gamma\backslash\Gamma_0}.$$

Note that the map $\varphi \in L^2(\Sigma) \to u(\varphi, 0, 0, 0)|_{\Gamma\backslash\Gamma_0} \in L^2(\Gamma\backslash\Gamma_0)$ is compact and injective. In fact, the injectivity follows from the unique continuation (e.g., Isakov [27]). The compactness is seen as follows; the map $\varphi \longrightarrow u(\varphi, 0, 0, 0)$ is continuous from $L^2(\Sigma)$ to $H^1(\Omega)$ by a variational formulation or the Lax-Milgram theorem (e.g., [42]). Since the embedding $H^{\frac{1}{2}}(\Gamma \setminus \Gamma_0) \longrightarrow L^2(\Gamma \setminus \Gamma_0)$ is compact, we see from the trace theorem that the map is compact. Moreover, the RKHS $\mathcal{H}_\Sigma$ is continuously embedded into $L^2(\Sigma)$. Therefore, L is a linear and injective compact operator from $\mathcal{H}_\Sigma$ to $L^2(\Gamma\backslash\Gamma_0)$. Let K be defined by $K\varphi := L(\Pi_*\varphi)$. It is clear that K is a linear and injective compact operator from $\mathcal{H}$ to $L^2(\Gamma\backslash\Gamma_0)$. Also, we have $g_\delta \in L^2(\Gamma\backslash\Gamma_0)$. We set

$$g_0 = g_1 - u(0, g_1, g_2, h)|_{\Gamma\backslash\Gamma_0}.$$

Lemma 14. *Let $\varphi \in \mathcal{H}$. Then $K(\varphi) = g_0$ and $\Pi_*\varphi = \partial_A u_0|_\Sigma$ are equivalent.*

Proof. Let $v := u(\Pi_*\varphi, g_1, g_2, h) - u_0$. Suppose $K(\varphi) = g_0$.

Then, we have

$$\begin{aligned} v|_{\Gamma\backslash\Gamma_0} &= u(\Pi_*\varphi, 0, 0, 0)|_{\Gamma\backslash\Gamma_0} + u(0, g_1, g_2, h)|_{\Gamma\backslash\Gamma_0} - u_0|_{\Gamma\backslash\Gamma_0} \\ &= K(\varphi) + u(0, g_1, g_2, h)|_{\Gamma\backslash\Gamma_0} - g_1 = K(\varphi) - g_0 = 0. \end{aligned}$$

On the other hand,

$$\partial_A v|_{\Gamma\backslash\Gamma_0} = g_2 - g_2 = 0.$$

Therefore, by the unique continuation theorem (e.g., [27]), we have $v = 0$ in Ω and $\Pi_*\varphi = \partial_A u_0|_\Sigma$.

Conversely, suppose $\Pi_*\varphi = \partial_A u_0|_\Sigma$. Then, we have

$$\begin{aligned} K(\varphi) &= u(\Pi_*\varphi, g_1, g_2, h)|_{\Gamma\setminus\Gamma_0} - u(0, g_1, g_2, h)|_{\Gamma\setminus\Gamma_0} \\ &= u(\partial_A u_0, g_1, g_2, h)|_{\Gamma\setminus\Gamma_0} - u(0, g_1, g_2, h)|_{\Gamma\setminus\Gamma_0} \\ &= g_1 - u(0, g_1, g_2, h)|_{\Gamma\setminus\Gamma_0} \\ &= g_0. \end{aligned}$$

Thus, the proof is completed. □

From Lemma 14, the problem of finding $\partial_A u_0|_\Sigma$ from g_1^δ and g_2^δ is equivalent to the problem of finding the solution $\varphi \in \mathcal{H}$ in $K\varphi = g_0$ from g_δ. We solve the problem by the method introduced in section 4; that is, we expand the data g_0^δ in terms of $\{K(\Phi(\cdot, x_k)); x_k \in X_m\}$ on $L^2(\Gamma\setminus\Gamma_0)$. In order to circumvent the instability of the inverse problem, the Tikhonov regularization is applied

$$\min_{\varphi\in V_m} \|K(\varphi) - g_\delta\|^2_{L^2(\Gamma\setminus\Gamma_0)} + \alpha\|\varphi\|^2_{\mathcal{H}_\Sigma},$$

where $\alpha > 0$ is a regularization parameter. We know that there exists a unique minimizer which we denote by $\varphi_{\alpha,m,\delta}$. By $\varphi_{\alpha,m}$, we denote the minimizer when $g_\delta = g_0$.

We can apply Theorem 9 in section 4, we show the convergence of $\varphi_{\alpha,m,\delta}$.

Theorem 15. *Under the above settings, we have:*

(i) Let $\lim_{m\to\infty} \alpha_m = 0$. *If* $p(h_m) = O(\sqrt{\alpha_m})$. *Then, we have*

$$\lim_{m\to\infty} \|\Pi_*\varphi_{\alpha,m} - \partial_A u_0\|_{L^2(\Sigma)} = 0.$$

(ii) Let $\lim_{\delta\to 0} m(\delta) = \infty$ *and* $\lim_{\delta\to 0} \alpha(\delta) = 0$. *If* $p(h_m) = O(\sqrt{\alpha})$ *and* $\delta = O(\sqrt{\alpha})$. *Then, we have*

$$\lim_{\delta\to 0} \|\Pi_*\varphi_{\alpha,m,\delta} - \partial_A u_0\|_{L^2(\Sigma)} = 0.$$

Proof. Part (i) follows directly from Theorem 9 (i). We only prove (ii) and we do not indicate the dependence on δ in the notations bellow. We first show that there exists a constant $C > 0$ such that $\gamma_m \le Cp(h_{X_m})$ for all $m \in \mathbb{N}$.

Let $\varphi \in \mathcal{H}$ with $\|\varphi\|_{\mathcal{H}} \leq 1$. We set $g = (I - P_m)\varphi$. By Assumption 13, we have

$$\begin{aligned}
&\|K(I-P_m)\varphi\|^2_{L^2(\Gamma\setminus\Gamma_0)} = \|u(\Pi_*(I-P_m)\varphi, 0, 0, 0)\|^2_{L^2(\Gamma\setminus\Gamma_0)} \\
&\leq C\|\Pi_*(I-P_m)\varphi\|^2_{L^2(\Sigma)} \\
&= C\int_\Sigma \left|g(\Pi^{-1}(x))\right|^2 dS \\
&\leq C\|g\|^2_{L^\infty(I)} \leq Cp^2(h_{X_m}).
\end{aligned}$$

Therefore, we have $\lim\limits_{m\to\infty} \gamma_m = 0$. Hence, from Theorem 9, we have

$$\lim_{\delta\to 0} \|\varphi_{\alpha,m,\delta} - K^\dagger g_0\|_{L^2(I)} = 0.$$

Since K is injective, then by Lemma 14, we have

$$K^\dagger g_0 = K^{-1}g_0 = \partial_A u_0(\Pi(\cdot)).$$

Consequently, we have $\lim\limits_{\delta\to 0} \|\Pi_*\varphi_{\alpha,m,\delta} - \partial_A u_0\|_{L^2(\Sigma)} = 0.$ □

We solve the boundary value problem

$$\begin{cases}
Au = h, & x \in \Omega, \\
\partial_A u|_\Sigma = \Pi_*\varphi_{\alpha,m,\delta}, & \\
u|_{\Gamma_0} = g_1^\delta, & \\
\partial_A u|_{\Gamma\setminus\Gamma_0} = g_2^\delta, &
\end{cases} \tag{9}$$

We denote a unique solution of (9) by $u_{\alpha,m,\delta}$. By $u_{\alpha,m}$, we denote the solution obtained by using $\varphi_{\alpha,m}$ and the noise-free data g_1 and g_2 in (9).

The function $u_0 - u_{\alpha,m,\delta}$ satisfies (8) with $\theta_1 = \partial_A u_0 - \Pi_*\varphi_{\alpha,m,\delta}$, $\theta_2 = g_1 - g_1^\delta$ and $\theta_3 = g_2 - g_2^\delta$. Hence, by Theorem 15, we have

$$\lim_{\delta\to 0} \|u_0 - u_{\alpha,m,\delta}\|_{L^2(\Omega)} = 0.$$

Corollary 16. *Under the above settings, we have:*

(i) Let $\lim\limits_{m\to\infty} \alpha_m = 0$. *If* $p(h_m) = O(\sqrt{\alpha_m})$. *Then, we have*

$$\lim_{m\to\infty} \|u_{\alpha_m,m} - u_0\|_{L^2(\Omega)} = 0.$$

(ii) Let $\lim\limits_{\delta\to 0} m(\delta) = \infty$ *and* $\lim\limits_{\delta\to 0} \alpha(\delta) = 0$. *If* $p(h_m) = O(\sqrt{\alpha})$ *and* $\delta = O(\sqrt{\alpha})$. *Then, we have*

$$\lim_{\delta\to\infty} \|u_{\alpha,m,\delta} - u_0\|_{L^2(\Omega)} = 0.$$

For given data g_0^δ, g_1^δ and a finite set of points X_m of $[0,1]$, the minimizer $\varphi_{\alpha,m,\delta} \in V_m$ can be written in the form:

$$\varphi_{\alpha,m,\delta} = \sum_{k=1}^{m} \lambda_k \Phi(\cdot, x_k).$$

The coefficients $\{\lambda_k\}_{k=1}^m$ are obtained by solving the linear system

$$\frac{\partial J(\lambda)}{\partial \lambda_k} = 0, \qquad k = 1, \dots, m,$$

where

$$J(\lambda) := \|K(\sum_{k=1}^{m} \lambda_k \Phi(\cdot, x_k)) - g_\delta\|^2_{L^2(\Gamma\backslash\Gamma_0)} + \alpha \|\sum_{k=1}^{m} \lambda_k \Phi(\cdot, x_k)\|^2_{\mathcal{H}}.$$

It is easy to check that the resultant system is

$$(A + \alpha B)\lambda = G_\delta. \tag{10}$$

In (10),

$$\begin{aligned} [A]_{i,j} &= \int_{\Gamma\backslash\Gamma_0} K(\Phi(\cdot, x_i))K(\Phi(\cdot, x_j))dS, \quad [B]_{i,j} = \Phi(x_i, x_j), \\ [G_\delta]_i &= \int_{\Gamma\backslash\Gamma_0} K(\Phi(\cdot, x_i))g_\delta dS. \end{aligned}$$

In our numerical computations, the integrals in A and G_δ will be approximated by some quadrature rule with P nodes $\{z_p\} \subset \Gamma\backslash\Gamma_0$ and corresponding weights $\{\omega_p\}$, i.e,

$$\begin{cases} [A]_{i,j} \approx \sum_{p=1}^{P} \omega_p K(\Phi(\cdot, x_i))(z_p) K(\Phi(\cdot, x_j))(z_p), \\ [G_\delta]_i \approx \sum_{p=1}^{P} \omega_p K(\Phi(\cdot, x_i))(z_p) g_\delta(z_p). \end{cases} \tag{11}$$

Thus, system (10) is changed to the following system

$$(V^*V + \alpha B)\lambda = V^* G_\delta, \tag{12}$$

where $[V]_{p,j} = \sqrt{\omega_p} K(\Phi(\cdot, x_j))(z_p)$, for $1 \le p \le P$, $1 \le j \le m$. We note that

$$K(\Phi(\cdot, x_i)) = L(\Pi_* \Phi(\cdot, x_i)), \quad 1 \le i \le m$$

is the trace on $\Gamma\backslash\Gamma_0$ of the solution u_i of the following direct problem

$$\begin{cases} Au_i = 0 & \text{in } \Omega, \\ \partial_A u_i|_\Sigma = \Phi(\Pi^{-1}(\cdot), x_i), & \\ u_i|_{\Gamma_0} = 0, & \\ \partial_A u_i|_{\Gamma\backslash\Gamma_0} = 0. & \end{cases} \tag{13}$$

The direct problem can be solved numerically by using a conventional method such as a finite element method, a finite difference method, a boundary element method, the method of fundamental solution or the Kansa's method [32], etc.

We conclude this section with a brief explanation on the method for the case when $\partial\Omega\backslash\Gamma$ is composed by piecewise smooth parts, i.e,

$$\Sigma = \bigcup_{j=1}^{J} \Sigma_j,$$

with injective C^∞ maps $\Pi^j\colon [0,1] \to \Sigma_j$ such that $\Pi^j([0,1]) = \Sigma_j$ for $j = 1,\dots,J$. For simplicity, we may assume $J = 2$ without loss of generality. Firstly, we choose a positive definite kernel Φ and define the corresponding RKHS $\mathcal{H}$ on $[0,1]$. We define the reproducing kernel Hilbert spaces $\mathcal{H}_{\Sigma_j}$ and the operators $\Pi^j_*\colon \mathcal{H} \to \mathcal{H}_{\Sigma_j}$ for $j = 1,2$ in the same way. For finite sets of points $X_m \subset [0,1]$ such that $\lim_{m\to\infty} h_{X_m,\Omega} = 0$, we define the finite subspaces V_m and the projection P_m on $\mathcal{H}$ in the same way.

For $\varphi = (\varphi_1, \varphi_2) \in \mathcal{H} \times \mathcal{H}$, consider the direct problem

$$\begin{cases} Au = h \quad \text{in } \Omega, \\ \partial_A u|_{\Sigma_j} = \Pi^j_* \varphi_j, \quad j = 1,2, \\ u|_{\Gamma_0} = g_1^\delta, \\ \partial_A u|_{\Gamma\backslash\Gamma_0} = g_2^\delta, \end{cases} \tag{14}$$

and denote the solution by $u(\varphi, g_1^\delta, g_2^\delta, h)$. Define $K\colon \mathcal{H}\times\mathcal{H} \to L^2(\Gamma\backslash\Gamma_0)$ and g_δ, respectively, by

$$K(\varphi_1,\varphi_2) = u(\Pi^1_*\varphi_1, \Pi^2_*\varphi_2, 0, 0, 0),$$

$$g_\delta = u(0, 0, g_1^\delta, g_2^\delta, h),$$

for $(\varphi_1,\varphi_2) \in \mathcal{H}\times\mathcal{H}$. For any $(m_1, m_2) \in \mathbb{N}\times\mathbb{N}$, we construct

$$\{K(\Pi^1_*\Phi(\cdot, x^1), 0), K(0, \Pi^2_*\Phi(\cdot, x^2)); x^1 \in X_{m_1}\ x^2 \in X_{m_2}\}.$$

Then, we seek the unique minimizer of the Tikhonov functional,

$$\varphi_{\alpha,m,\delta} := \arg \min_{\varphi \in V_{m_1}\times V_{m_2}} \|K(\varphi) - g_\delta\|^2_{L^2(\Gamma\backslash\Gamma_0)} + \alpha \sum_{j=1}^{2} \|\varphi_j\|^2_{\mathcal{H}}.$$

The minimizer $\varphi_{\alpha,m,\delta}$ in the space $V_{m_1}\times V_{m_2}$ is of the form

$$\varphi_{\alpha,m,\delta} = (\sum_{i=1}^{m_1} \lambda_{i,1} \Pi^1_* \Phi(\cdot, x_i^1), \sum_{i=1}^{m_2} \lambda_{i,2} \Pi^2_* \Phi(\cdot, x_i^2)).$$

The coefficients $\{\lambda_{i,j}\}$, $1 \le i \le m_j$, for $j = 1, 2$, of the minimizer is obtained by solving the system $\nabla J(\lambda) = 0$ where J is given by

$$J(\lambda) := \|K(\sum_{i=1}^{m_1} \lambda_{i,1}\Phi(\cdot, x_i^1), \sum_{i=1}^{m_2} \lambda_{i,2}\Phi(\cdot, x_i^2)) - g_\delta\|^2_{L^2(\Gamma\backslash\Gamma_0)} + \alpha \sum_{j=1}^{2} \| \sum_{i=1}^{m_j} \lambda_{i,j}\Phi(\cdot, x_i^j)\|^2_{\mathcal{H}}.$$

7 Numerical experiments

In this section, we verify the numerical efficiency of the proposed method for the Cauchy problem (1). We reconstruct an approximate solution to (1) for any given m in X_m. We only focus on the case when $A = \triangle$ and $h = 0$, i.e, the Laplace equation. Firstly, we give an approximation to $\partial_A u_0|_\Sigma$. Then, by using such approximation, we solve equation (9) or (14) to obtain an approximate solution to (1). The regularization parameter α is chosen by the L-curve method (e.g., [12]). We summarize the numerical procedure in Algorithm as follows.

Algorithm 1 Pseudo-code for generating an approximation to u_0.

Choose the reproducing kernel $\Phi(t, s)$.
Choose $m \in \mathbb{N}$ and set $X_m := \{\frac{1}{m}, \frac{2}{m}, \dots, 1\}$.
Define the matrix B by $[B]_{i,j} = \Phi(x_i, x_j)$. Set the nodes $Z_P := \{z_1, \dots, z_P\} \in \Gamma\backslash\Gamma_0$ and the weights $\{\omega_1, \dots, \omega_p\}$ for $P \in \mathbb{N}$ in (11).
for $k = 1, \dots, m$ **do**
  Solve equation (13) to construct $K(\Phi(\cdot, x_k))$, $x_k \in X_m$.
  Define the matrix by $[V]_{p,k} := \sqrt{\omega_p} K(\Phi(\cdot, x_k))(z_p)$ for $p = 1, \dots, m$.
end for
For any given data $\{g_1^\delta, g_2^\delta\}$ on Γ, compute $g_\delta(z_p) := g_1^\delta(z_p) - u(0, g_1^\delta, g_2^\delta, h)(z_p)$, for $p = 1, \dots, P$ to obtain G_δ in (11).
Choose a regularization parameter α by the L-curve method.
Solve system (12) to obtain $\{\lambda_k\}_{k=1}^m$.
Set $\varphi_{\alpha,m,\delta} = \sum_{k=1}^m \lambda_k \Phi(\cdot, x_k)$ and solve equation (9) to obtain an approximation solution $u_{\alpha,m,\delta}$ to (1).

We consider a two-dimensional case where $\Omega = [-1, 1] \times [0, 1]$ and two cases of Γ: (i) $\partial\Omega\backslash\Gamma = [-1, 1] \times \{1\}$ and (ii) $\Gamma = [-1, 1] \times \{0\}$.

We fix the boundary $\Gamma_0 = [-0.1, 0.1] \times \{0\}$ in all the cases. In all numerical experiments, the nodes for the integral approximations in (11) are taken to be $Z_P = \{z_1, z_2, \dots, z_P\} \subset \Gamma\backslash\Gamma_0$ such that $|z_{k+1} - z_k| = 0.02$ for all $1 \le k \le P$. The weights $\{\omega_p\}$ are always chosen to be uniform: $\omega_p = 1$.

We choose the following functions as test examples:

Example 1 $u_0(x,y) = x^3 - 3xy^2 + e^{2y}\sin 2x - e^y \cos x$.

Example 2 $u_0(x,y) = \cos \pi x \cosh \pi y$.

In Klibanov and Santosa [34], the quasi-reversibility method is applied for reconstructing the same function in Example 2 in the case of $\Gamma = [-1,1] \times \{0\}$.

We use two positive definite kernels among Φ_1 and Φ_2:

Kernel 1 $\Phi_1(t,s) := \exp(-10|t-s|^2)$.

Kernel 2 $\Phi_2(t,s) := \varphi(|t-s|)$, where $\varphi(r) := (1-r)_+^3(3r+1)$ and $t_+ = \max\{t, 0\}$.

Each kernel satisfies the Assumption 13 with $p(r) = C_1 \exp(-\frac{C_2}{r})$ for the Kernel 1 and $p(r) = C_3 r^3$ for the Kernel 2, respectively, where C_1, C_2 and C_3 are positive constants [55, Section 11.4].

For the case (i) $\Gamma = [-1,1] \times \{0\}$, the boundary $\Sigma = \partial\Omega \backslash \Gamma$ is composed by three segments: $\Sigma_1 := \{(s,1); s \in [-1,1]\}$, $\Sigma_2 := \{(-1,s); s \in [0,1]\}$ and $\Sigma_3 := \{(1,s); s \in [0,1]\}$. We define maps $\Pi_i \colon [0,1] \to \Sigma_i$, $i = 1,2,3$ by $\Pi_1(t) = (-1,t)$, $\Pi_2(t) = (-1+2t, 1)$ and $\Pi_3(t) = (1,t)$ for $t \in [0,1]$.

We take two finite sets of points X_{10} and X_{20} in $[0,1]$. The fill distances of both $\Pi_1(X_{10})$ and $\Pi_3(X_{10})$ are equal to that of $\Pi_2(X_{20})$.

The noisy data $\{g_1^\delta, g_2^\delta\}$ are obtained by adding random numbers to the exact data $\{g_1, g_2\} = \{u_0|_\Gamma, \partial_A u_0|_\Gamma\}$ by

$$g_i^\delta(\xi) = g_i(\xi) + \frac{\delta}{100} \max_{z\in\Gamma} |g_i(z)| \mathrm{rand}(\xi), \quad i = 1,2,$$

for $\xi \in \Gamma$, where $\mathrm{rand}(\xi)$ is a random number between $[-1,1]$ and $\delta\% \in \{0,1,5,10\}$ is the noise level.

For all given noisy data $\{g_1^\delta, g_2^\delta\}$ with various noisy levels, we apply Algorithm to obtain an approximate solution to u_0 in each example. We denote by u_{Φ_i} the approximate solution obtained with using the kernel Φ_i, $i = 1,2$ in Algorithm. For the numerical error estimations, we compute the relative error of u_{Φ_i} over the whole domain Ω:

$$E_r(u_{\Phi_i}) := \frac{\|u_0 - u_{\Phi_i}\|_{L^2(\Omega)}}{\|u_0\|_{L^2(\Omega)}},$$

for $i = 1,2$. Table 1 shows the relative errors for Example 1 and Example 2. In Figure 1, we show the solution u_0 in Example 2 for the comparison to approximate solution u_{Φ_2}.

	Example1		Example2	
Noise	$E_r(u_{\Phi_1})$	$E_r(u_{\Phi_2})$	$E_r(u_{\Phi_1})$	$E_r(u_{\Phi_2})$
0%	0.0428	0.0338	0.0919	0.0667
1%	0.0507	0.0606	0.1099	0.0781
5%	0.2449	0.2340	0.3055	0.3186
10%	0.2797	0.2682	0.3410	0.3149

Table 1: The relative errors u_{Φ_i} $i = 1, 2$ on the whole domain Ω when the Cauchy data are given on the boundary $\Gamma = [-1, 1] \times \{0\}$.

The solutions u_{Φ_2} obtained by using different noisy data with noise level $\delta = 0, 1, 5, 10$ are given in Figure 2 - Figure 5, respectively.

In order to study the error profiles of our numerical solution to u_{Φ_2}, in Figure 6 and Figure 7, we draw the absolute error

$$E_a(x, y) := |u_0(x, y) - u_{\Phi_2}(x, y)|, \quad (x, y) \in \Omega.$$

In this experiment, the noise level is set to be $\delta = 10$ and both Example 1 and Example 2 are tested. We observe that the errors becomes larger near the boundary Σ in the both examples. This corresponds to the conditional stability estimate up to the boundary as we stated in Theorem 11 where the rate of the convergence to the exact solution is only logarithmic. By the interior conditional stability in Theorem 10, we may expect that the accuracy of the numerical solution will be improved in a small part of the subset $\omega \subset \Omega$ whose boundary $\partial\omega$ does not touch Σ. In [34], the reconstruction was done in a subdomain ω for the same Cauchy problem for the Laplace equation. For comparisons, we choose the same subdomain ω:

$$\omega := \{(x, y); y + 0.6 \left(\frac{x}{0.6}\right)^2 - 0.6 \leq 0,\ y \geq 0\}$$

and consider the relative error in ω

$$e_r(u_{\Phi_i}) := \frac{\|u_0 - u_{\Phi_i}\|_{L^2(\omega)}}{\|u_0\|_{L^2(\omega)}}, \quad i = 1, 2.$$

In Table 2, we can see that all the accuracies have improved.

Finally, we compute the numerical approximate solution to u_0 when the Cauchy data is given on the boundary $\Sigma = \{(x, 1); x \in [-1, 1]\}$. Table 3 and Table 4 show the relative errors in each domain respectively.

	Example1		Example2	
Noise	$e_r(u_{\Phi_1})$	$e_r(u_{\Phi_2})$	$e_r(u_{\Phi_1})$	$e_r(u_{\Phi_2})$
0%	0.0044	0.0040	0.0023	0.0019
1%	0.0041	0.0074	0.0072	0.0052
5%	0.0717	0.0677	0.0638	0.0786
10%	0.0879	0.0830	0.0768	0.0763

Table 2: The relative errors u_{Φ_i}, $i = 1, 2$, in the interior part ω where the Cauchy data is given on the boundary $\Gamma = [-1, 1] \times \{0\}$.

	Example1		Example2	
Noise	$E_r(u_{\Phi_1})$	$E_r(u_{\Phi_2})$	$E_r(u_{\Phi_1})$	$E_r(u_{\Phi_2})$
0%	0.0069	0.0043	0.0037	0.0044
1%	0.0153	0.0106	0.0166	0.0046
5%	0.0375	0.0218	0.0361	0.0198
10%	0.0414	0.0425	0.0539	0.0292

Table 3: The relative errors u_{Φ_i}, $i = 1, 2$, on the whole domain Ω where the Cauchy data is given on the boundary Γ such that $\partial\Omega \backslash \Gamma = [-1, 1] \times \{1\}$.

	Example1		Example2	
Noise	$e_r(u_{\Phi_1})$	$e_r(u_{\Phi_2})$	$e_r(u_{\Phi_1})$	$e_r(u_{\Phi_2})$
0%	0.0012	0.0010	0.0034	0.0037
1%	0.0078	0.0054	0.0078	0.0046
5%	0.0176	0.0098	0.0276	0.0115
10%	0.0207	0.0200	0.0406	0.0138

Table 4: The relative errors u_{Φ_i}, $i = 1, 2$, in the interior part ω where the Cauchy data is given on the boundary Γ such that $\partial\Omega \backslash \Gamma = [-1, 1] \times \{1\}$.

8 Conclusion

In this paper, we propose a reconstruction numerical method for solving the Cauchy problem for elliptic equation. The method is the discretized Tikhonov regularization by the reproducing kernel Hilbert space. The convergence of the method is proven. Numerical examples demonstrate that the method is robust against data noises and reasonably accurate as a solver for the ill-posed problems. The method is non-iterative. Furthermore, the method is applicable to other inverse problem that makes the method practical to handle real-life problems. We also give the conditional stability estimate at the boundary. The argument in our proof can be extended to establish such stability estimates for other inverse problems; for example, the non-characteristic Cauchy problem for parabolic equations which we will study in a forthcoming paper.

Appendix Proof of Theorem 11

Proof. First we show a key Carleman estimate:

Lemma 17. *Let $\psi \in C^2(\overline{\Omega})$ such that $|\nabla\psi| \neq 0$ on $\overline{\Omega}$. Then, for sufficiently large $\lambda > 0$, there exist constants $s_0 > 0$ and $C > 0$ such that*

$$\int_\Omega \left(\frac{1}{s} \sum_{i,j=1}^{n} |\partial_i \partial_j u|^2 + s|\nabla u|^2 + s^3 u^2 \right) \exp(2se^{\lambda\psi}) dx$$
$$\leq C \int_\Omega |Au|^2 \exp(2se^{\lambda\psi}) dx$$

for any $s \geq s_0$ and any $u \in H_0^2(\Omega)$.

As for the proof, see Hörmander [26], Isakov [27] and Klibanov and Timonov [35]. In [27], the second order terms on the left hand side are not estimated, but we can easily include $\partial_i\partial_j u$ by means of an *a priori* estimate for the Dirichlet problem for $A(ue^{s\psi}) = \widetilde{h}$ (e.g., Gilbarg and Trudinger [18]).

We set

$$\begin{cases} M = \|u\|_{H^\eta(\Omega)}, \qquad \eta > \frac{n+2}{2}, \\ D = \|g_1\|_{H^1(\Gamma)} + \|g_2\|_{L^2(\Gamma)} + \|h\|_{L^2(\Omega)}. \end{cases}$$

Without loss of generality, we can assume that $M \geq 1$. Again, $C_j > 0$ and $C > 0$ denote some generic constants. Both constants are independent of choices of $x_0 \in \partial\Omega \setminus \Gamma$ and the parameter $s > 0$ in the Carleman estimate (Lemma 17).

For $y \in \mathbb{R}^n$ and $r > 0$, we set

$$B_r(y) = \{x \in \mathbb{R}^n;\ |x - y| < r\}.$$

Since $\partial\Omega \setminus \Gamma$ satisfies the uniform interior sphere condition, we can choose a ball $B_r(y_0)$ such that $B_r(y_0) \setminus \{x_0\} \subset \Omega$ for any $x_0 \in \partial\Omega \setminus \Gamma$. We can choose $r > 0$ uniformly for $x_0 \in \partial\Omega \setminus \Gamma$. We set

$$\Omega_1 = \bigcup_{x_0 \in \partial\Omega\setminus\Gamma} B_r(y_0) \setminus B_{r/2}(x_0),$$

and we see that $\overline{\Omega_1} \subset \Omega$.

We apply Theorem 10 in Ω_1, so that

$$\|u\|_{H^2(\Omega)} \le C(D + M^{1-\kappa}D^\kappa) \le CM^{1-\kappa}D^\kappa \equiv CD_1. \tag{15}$$

We can assume that $D_1 \le 1$ because we can assume that $D > 0$ is sufficiently small by the homogeneity of (1) in u with respect to the scalar multiplication. Let us fix

$$\max\left\{2, \frac{n+2}{2}\right\} < \eta_1 < \eta.$$

By the interpolation inequality (e.g., Adams [1]), we have

$$\|u\|_{H^{\eta_1}(\Omega_1)} \le \|u\|_{H^2(\Omega_1)}^{\frac{\eta-\eta_1}{\eta-2}} \|u\|_{H^\eta(\Omega_1)}^{\frac{\eta_1-2}{\eta-2}}. \tag{16}$$

Therefore, (15) implies

$$\|u\|_{H^{\eta_1}(\Omega_1)} \le CD_1^{\frac{\eta-\eta_1}{\eta-2}} M^{\frac{\eta_1-2}{\eta-2}}. \tag{17}$$

The Sobolev embedding theorem (e.g., Adams [1]) yields

$$\|u\|_{C^1(\overline{\Omega_1})} \le C\|u\|_{H^{\eta_1}(\Omega_1)}.$$

Hence,

$$\|u\|_{C^1(\overline{\Omega_1})} \le CD_1^{\frac{\eta-\eta_1}{\eta-2}} M^{\frac{\eta_1-2}{\eta-2}}.$$

Let us fix $x_0 \in \partial\Omega \setminus \Gamma$ arbitrarily. By some suitable rotation and translation, we may assume that $B_r(0) \setminus \{x_0\} \subset \Omega$ and $x_0 = (r, 0, ..., 0)$. We set

$$E = B_r(0) \cap \left\{x = (x_1, ..., x_n) \in \mathbb{R}^n;\ x_1 > \frac{r}{4}\right\}.$$

We will apply Lemma 17 in E. First, by the Sobolev extension theorem (e.g., Adams [1]), we can find $u^* \in H^2(E)$ such that

$$u^* = u, \quad \partial_A u^* = \partial_A u \qquad \text{on } \Gamma_0 \equiv \partial E \cap \left\{x_1 = \tfrac{r}{4}\right\},$$

and

$$\|u^*\|_{H^2(E)} \le C(\|u\|_{H^{3/2}(\Gamma_0)} + \|\partial_A u\|_{H^{1/2}(\Gamma_0)}) \le C\|u\|_{H^2(\Omega_1)}.$$

By the definition of Ω_1, we have $\Gamma_0 \subset \Omega_1$ and (17) implies

$$\|u^*\|_{H^2(E)} \le CD_1^{\frac{\eta-\eta_1}{\eta-2}} M^{\frac{\eta_1-2}{\eta-2}}. \tag{18}$$

We set

$$\psi(x) = r^2 - |x|^2, \qquad \varphi(x) = e^{\lambda\psi(x)},$$

and

$$E(\delta) = \{x \in E;\, \psi(x) \ge \delta\},$$

with $\delta > 0$. We note that $E(0) = E$ and $|\nabla\psi| \ne 0$ on $\overline{E}$ because $0 \notin \overline{E}$.

Now, we take $\chi = \chi_\delta \in C^\infty(\mathbb{R}^n)$ such that $0 \le \chi \le 1$ and

$$\chi(x) = \begin{cases} 1, & x \in E(2\delta), \\ 0, & x \in E \setminus E(\delta), \end{cases} \tag{19}$$

and

$$\|\chi\|_{C^2(\mathbb{R}^n)} \le \frac{C}{\delta^2}. \tag{20}$$

In fact, we can choose a function $\widetilde{\chi} \in C^\infty(\mathbb{R})$ such that

$$\widetilde{\chi}(t) = \begin{cases} 1, & t \ge 1, \\ 0, & t \le 0. \end{cases}$$

Set $\chi(x) = \widetilde{\chi}\left(\frac{\psi(x)-\delta}{\delta}\right)$. We can readily verify that (19) and (20) are satisfied.

We put $v = \chi(u - u^*)$. Then, $v \in H_0^2(E)$ and

$$\begin{aligned} Av &= \chi(h - Au^*) + \sum_{i,j=1}^n b_{ij}(x)(\partial_j\chi)\partial_i(u - u^*) \\ &+ \sum_{i,j=1}^n (\widetilde{b}_{ij}(x)(\partial_i\partial_j\chi + \widetilde{c}_j(x)\partial_j\chi)(u - u^*) \\ &\equiv \chi(h - Au^*) + Q(x). \end{aligned} \tag{21}$$

If $\chi(x) =$ constant, then $Q(x) = 0$, that is,

$$Q(x) \ne 0 \quad \text{only if } x \in E \setminus E(2\delta). \tag{22}$$

Moreover, by (20), we have

$$|Q(x)| \le \frac{C}{\delta^2}(|\nabla(u - u^*)(x)| + |(u - u^*)(x)|), \quad x \in E \setminus E(2\delta). \tag{23}$$

We apply Lemma 17 to v with (21), so that (22) and (23) yield

$$\begin{aligned}
&\int_E \left(\frac{1}{s} \sum_{i,j=1}^n |\partial_i \partial_j v|^2 + s|\nabla v|^2 + s^3 v^2 \right) e^{2s\varphi} dx \\
&\le C \int_E |\chi(h - Au^*)|^2 e^{2s\varphi} dx + C \int_E |Q|^2 e^{2s\varphi} dx \\
&\le C \int_E (|h|^2 + |Au^*|^2) e^{2s\varphi} dx + C e^{2se^{2\lambda\delta}} \frac{1}{\delta^4} (\|u\|^2_{H^1(E)} + \|u^*\|^2_{H^1(E)})
\end{aligned}$$

for all large $s > 0$. By (19), we obtain

$$\begin{aligned}
&\int_{E(3\delta)} \left(\frac{1}{s} \sum_{i,j=1}^n |\partial_i \partial_j (u - u^*)|^2 + s|\nabla (u - u^*)|^2 + s^3 |u - u^*|^2 \right) e^{2s\varphi} dx \\
&\le C e^{Cs} (\|h\|^2_{L^2(\Omega)} + \|u^*\|^2_{H^2(E)}) + C e^{2se^{2\lambda\delta}} \frac{1}{\delta^4} (M^2 + \|u^*\|^2_{H^1(E)}).
\end{aligned}$$

Hence, it follows from (18) and (19) that

$$\|u\|^2_{H^2(E(3\delta))} \le C e^{Cs} D_2^2 + \frac{Cs}{\delta^4} e^{-2s\mu_1} M^2 + \frac{Cs}{\delta^4} e^{-2s\mu_1} D_2^2, \qquad s \ge s_0,$$

where we set

$$\mu_1 = e^{3\lambda\delta} - e^{2\lambda\delta} > 0, \quad D_2 = \|h\|_{L^2(\Omega)} + D_1^{\frac{\eta - \eta_1}{\eta - 2}} M^{\frac{\eta_1 - 2}{\eta - 2}}.$$

Similarly to D_1 we can assume that D_2 is small. Since

$$s e^{-s\mu_1} \le \frac{1}{\mu_1} = \frac{1}{e^{2\lambda\delta}(e^{\lambda\delta} - 1)} \le \frac{1}{\lambda\delta}, \qquad s \ge 0,$$

we have

$$\|u\|^2_{H^2(E(3\delta))} \le C e^{Cs} D_2^2 + \frac{C}{\delta^5} e^{-s\mu_1} M^2 + \frac{C}{\delta^5} e^{-s\mu_1} D_2^2, \qquad s \ge s_0. \tag{24}$$

Replacing C by Ce^{Cs_0}, we have (24) for any $s \ge 0$. We choose $s \ge 0$ such that $e^{Cs} D_2^2 = e^{-s\mu_1} M^2$, that is, $s = \frac{2}{C + \mu_1} \log \frac{M}{D_2}$, so that by $D_2 \le 1$ and $M \ge 1$,

$$\begin{aligned}
\|u\|_{H^2(E(3\delta))} &\le \frac{C}{\delta^{\frac{5}{2}}} M^{\frac{C}{\mu_1 + C}} D_2^{\frac{\mu_1}{C + \mu_1}} + \frac{C}{\delta^{\frac{5}{2}}} M^{\frac{C}{\mu_1 + C}} D_2^{\frac{\mu_1}{C + \mu_1}} D_2 \\
&\le \frac{C}{\delta^{\frac{5}{2}}} M D_2^{\frac{\mu_1}{C + \mu_1}}.
\end{aligned}$$

We assume that $\delta \le 1$. Moreover, since

$$\frac{\mu_1}{C + \mu_1} = \frac{e^{2\lambda\delta}(e^{\lambda\delta} - 1)}{C + e^{2\lambda\delta}(e^{\lambda\delta} - 1)} \ge \frac{e^{\lambda\delta} - 1}{C + e^{3\lambda}} \ge \frac{\lambda\delta}{C_1}$$

and $D_2 \le 1$, we have $D_2^{\frac{\mu_1}{C + \mu_1}} \le D_2^{C_2 \delta}$. Hence, changing 3δ to δ, we obtain

$$\|u\|_{H^2(E(\delta))} \le \frac{CM}{\delta^{\frac{5}{2}}} D_2^{C_2 \delta}, \qquad 0 < \delta \le 1. \tag{25}$$

We have

$$u(x_0) = u\left(\frac{r}{4},0\right) + \int_0^1 \frac{\partial}{\partial t}\left(u\left(\frac{r}{4}+\frac{3rt}{4},0\right)\right)dt$$
$$= u\left(\frac{r}{4},0\right) + \frac{3r}{4}\int_0^1 \partial_1 u\left(\frac{r}{4}+\frac{3rt}{4},0\right)dt. \tag{26}$$

In (26), we have $\left(\frac{r}{4}+\frac{3rt}{4},0\right) \in E(d(t))$ where $d(t) = \left(1-\left(\frac{1}{4}+\frac{3t}{4}\right)^2\right)r^2$ and

$$\begin{aligned}\left|\partial_1 u\left(\frac{r}{4}+\frac{3rt}{4},0\right)\right| &\le C\|u\|_{C^1(\overline{E(d(t))})} \le C\|u\|_{H^{\eta_1}(E(d(t)))}\\ &\le C\|u\|_{H^2(E(d(t)))}^{\frac{\eta-\eta_1}{\eta-2}}\|u\|_{H^\eta(E(d(t)))}^{\frac{\eta_1-2}{\eta-2}}\end{aligned}$$

by the Sobolev embedding and the interpolation inequality (16). Therefore, by (25), we obtain

$$\begin{aligned}\left|\partial_1 u\left(\frac{r}{4}+\frac{3rt}{4},0\right)\right| &\le CM^{\frac{\eta_1-2}{\eta-2}}\left(\frac{CM}{d(t)^{\frac{5}{2}}}D_2^{C_2 d(t)}\right)^{\frac{\eta-\eta_1}{\eta-2}}\\ &\le \frac{CM}{d(t)^{\frac{5(\eta-\eta_1)}{2(\eta-2)}}}D_2^{\frac{C_2 d(t)(\eta-\eta_1)}{\eta-2}}.\end{aligned}$$

Now, we choose η_1 such that $\max\left\{2,\frac{n+2}{2}\right\} < \eta_1 < \eta$ and

$$\frac{5(\eta-\eta_1)}{2(\eta-2)} \equiv \theta < 1. \tag{27}$$

Noting that

$$d(t) = \frac{3(1-t)}{4}\frac{5+3t}{4}r^2 \ge \frac{15}{16}(1-t)r^2, \quad 0 < t \le 1,$$

and $D_1 \le 1$, we have

$$\left|\partial_1 u\left(\frac{r}{4}+\frac{3rt}{4},0\right)\right| \le \frac{CM}{(1-t)^\theta}D_2^{C_3(1-t)}, \quad 0 < t \le 1.$$

Hence, (26) yields

$$\begin{aligned}|u(x_0)| &\le \left|u\left(\frac{r}{4},0\right)\right| + CrM\int_0^1 \frac{1}{(1-t)^\theta}D_2^{C_3(1-t)}dt\\ &\le \left|u\left(\frac{r}{4},0\right)\right| + CrM\int_0^1 \xi^{-\theta}\exp\left(-C_3\left(\log\frac{1}{D_2}\right)\xi\right)d\xi\\ &\le \left|u\left(\frac{r}{4},0\right)\right| + CrM\int_0^\infty \xi^{-\theta}\exp\left(-C_3\left(\log\frac{1}{D_2}\right)\xi\right)d\xi\\ &\le \left|u\left(\frac{r}{4},0\right)\right| + CrM\Gamma(1-\theta)\left(\frac{1}{C_3\log\frac{1}{D_2}}\right)^{1-\theta}.\end{aligned} \tag{28}$$

On the other hand, by (25), the Sobolev embedding, the interpolation inequality and $d(0) = \frac{15}{16}r^2$, we have

$$\begin{aligned} \left|u\left(\frac{r}{4},0\right)\right| &\leq C\|u\|_{H^{\frac{n+1}{2}}(E(d(0)))} \leq C\|u\|^{\frac{2\eta-n-1}{2\eta}}_{L^2(E(d(0)))}\|u\|^{\frac{n+1}{2\eta}}_{H^\eta(E(d(0)))} \\ &\leq C\left(\frac{M}{r^5}D_2^{C_2r^2}\right)^{\frac{2\eta-n-1}{2\eta}} M^{\frac{n+1}{2\eta}}. \end{aligned}$$

Hence, (28) yields

$$\begin{aligned} |u(x_0)| &\leq CMD_2^{C_2r^2\frac{2\eta-n-1}{2\eta}} + CrM\Gamma(1-\theta)\left(\frac{1}{C_3\log\frac{1}{D_2}}\right)^{1-\theta} \\ &\leq CrM\Gamma(1-\theta)\left(\frac{1}{C_3\log\frac{1}{D_2}}\right)^{1-\theta}. \end{aligned}$$

In the last inequality, we used the fact that the first term is bounded by the second term for fixed η and r. For any $\kappa_0 \in (0,1)$, we can choose $\eta_1 > 0$ satisfying (27). Then, there exists a constant $C(\kappa_0) > 0$ such that

$$|u(x_0)| \leq C(\kappa_0)M\left(\frac{1}{\log\frac{1}{D_2}}\right)^{\kappa_0}.$$

Here,

$$\begin{aligned} D_2 &= \|h\|_{L^2(\Omega)} + D_1^{\frac{\eta-\eta_1}{\eta-2}}M^{\frac{\eta_1-2}{\eta-2}} \\ &= \|h\|_{L^2(\Omega)} + (M^{1-\kappa}D^\kappa)^{\frac{\eta-\eta_1}{\eta-2}}M^{\frac{\eta_1-2}{\eta-2}} \leq D^{\kappa_1} + MD^{\kappa_1} \leq 2MD^{\kappa_1} \end{aligned}$$

with $\kappa_1 \in (0,1)$, because $D \leq 1$ and $M \geq 1$. Hence,

$$\begin{aligned} \frac{1}{\log\frac{1}{D_2}} &\leq \frac{1}{\kappa_1\log\frac{1}{D} + \log\frac{1}{2M}} \\ &\leq \frac{C}{\log\frac{1}{D} + \log\frac{1}{M}}. \end{aligned}$$

Finally, by the interpolation inequality and the trace theorem, we have

$$\begin{aligned} &\|g_1\|_{H^{3/2}(\Gamma)} + \|g_2\|_{H^{1/2}(\Gamma)} \\ &\leq C\|g_1\|^{\frac{2\eta-4}{2\eta-1}}_{L^2(\Gamma)}\|g_1\|^{\frac{3}{2\eta-1}}_{H^{\eta-\frac{1}{2}}(\Gamma)} + \|g_2\|^{\frac{2\eta-4}{2\eta-3}}_{L^2(\Gamma)}\|g_2\|^{\frac{1}{2\eta-3}}_{H^{\eta-\frac{3}{2}}(\Gamma)} \\ &\leq C\|g_1\|^{\frac{2\eta-4}{2\eta-1}}_{L^2(\Gamma)}M^{\frac{3}{2\eta-1}} + \|g_2\|^{\frac{2\eta-4}{2\eta-3}}_{L^2(\Gamma)}M^{\frac{1}{2\eta-3}}. \end{aligned}$$

Thus, the proof of Theorem 11 is completed. $\square$

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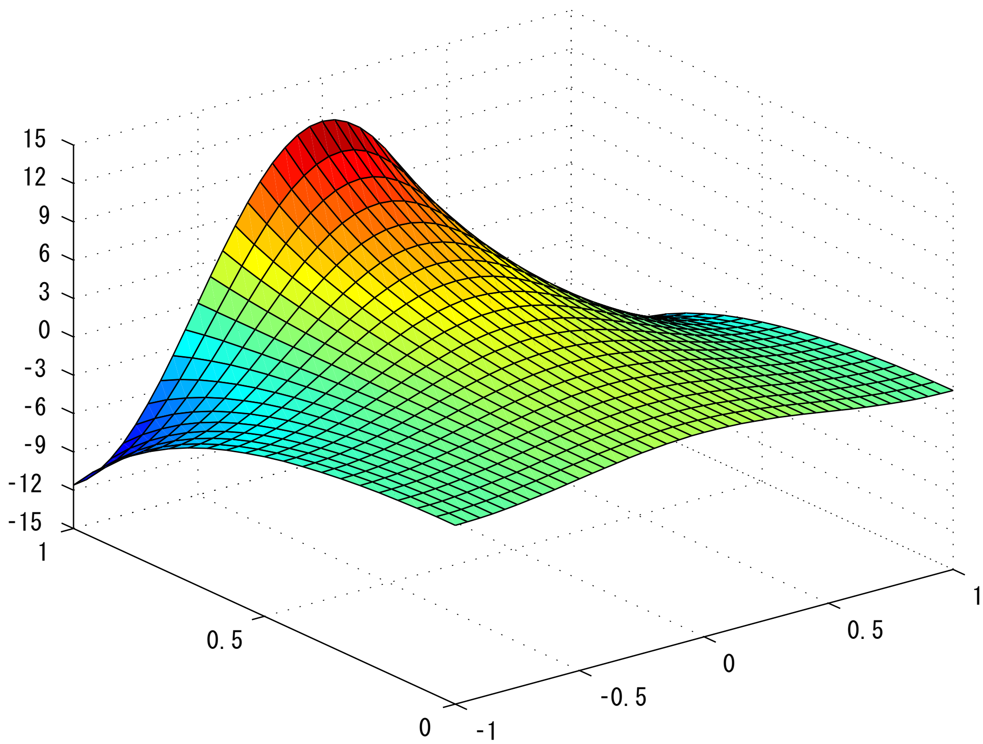

Figure 1: Surface plot for the function $u_0(x, y) = \cos \pi x \cosh \pi y$ in Example 2.

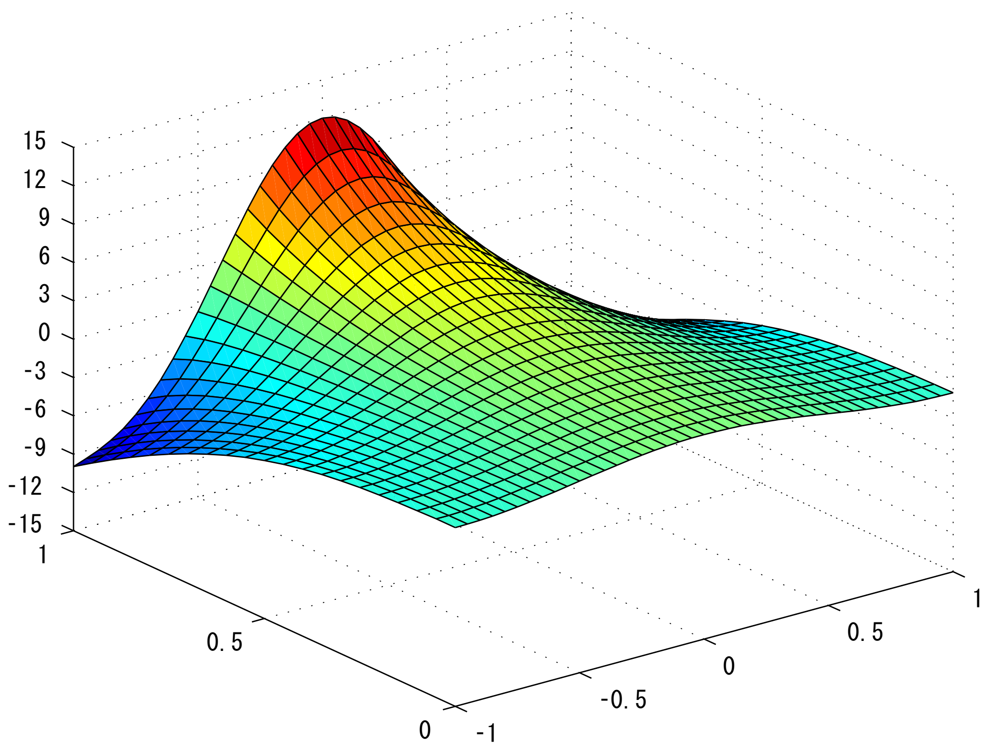

Figure 2: Numerical approximate solution u_{Φ_2} to the solution of Example 2 using noisy data when $\delta = 0$

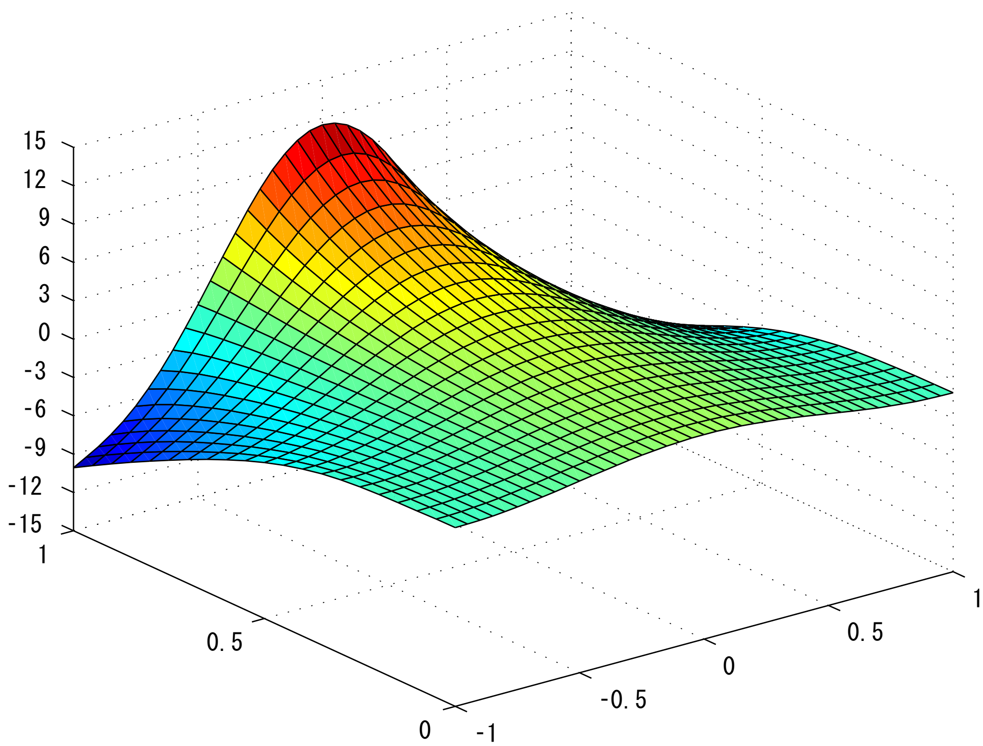

Figure 3: Numerical approximate solution u_{Φ_2} to the solution of Example 2 using noisy data when when $\delta = 1$

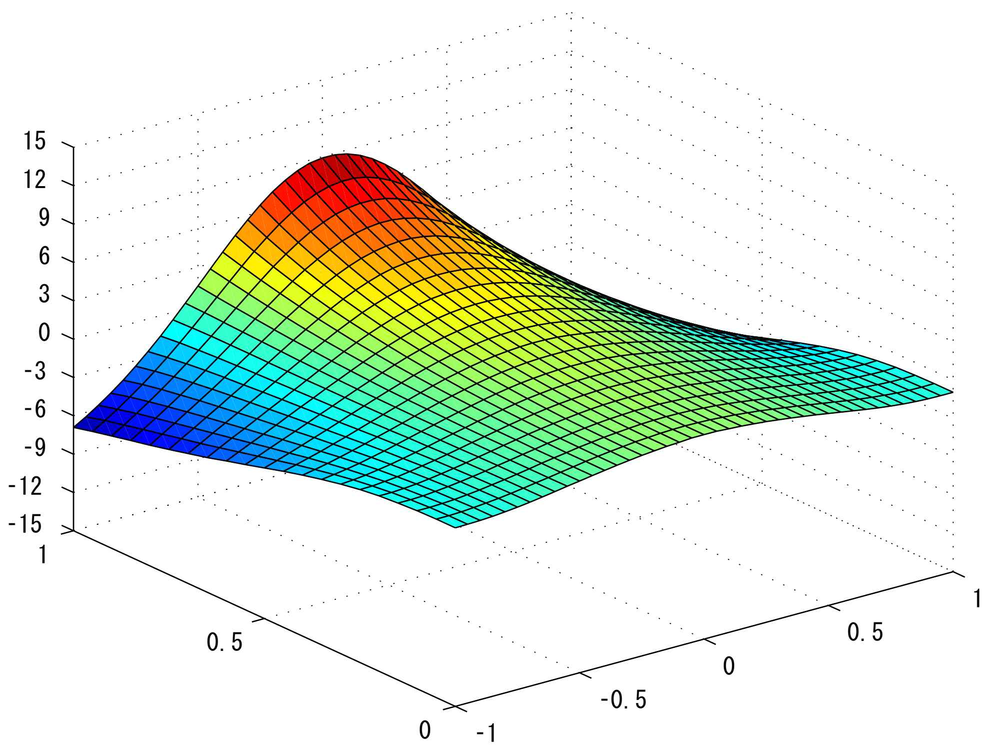

Figure 4: Numerical approximate solution u_{Φ_2} to the solution of Example 2 using noisy data when $\delta = 5$

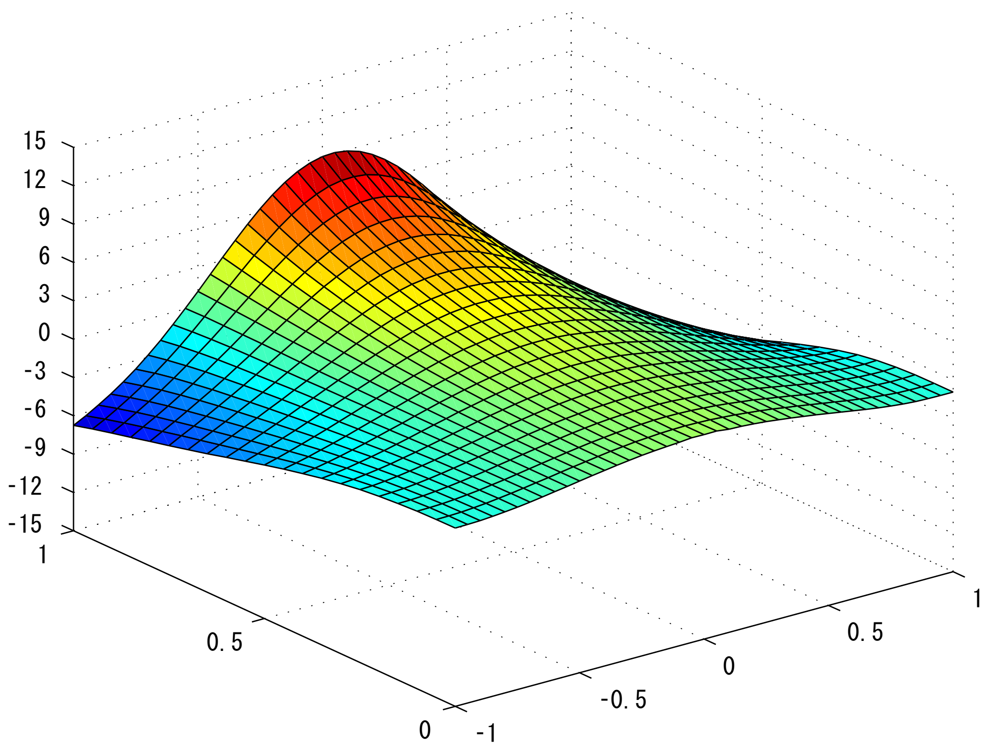

Figure 5: Numerical approximate solution u_{Φ_2} to the solution of Example 2 using noisy data when $\delta = 10$

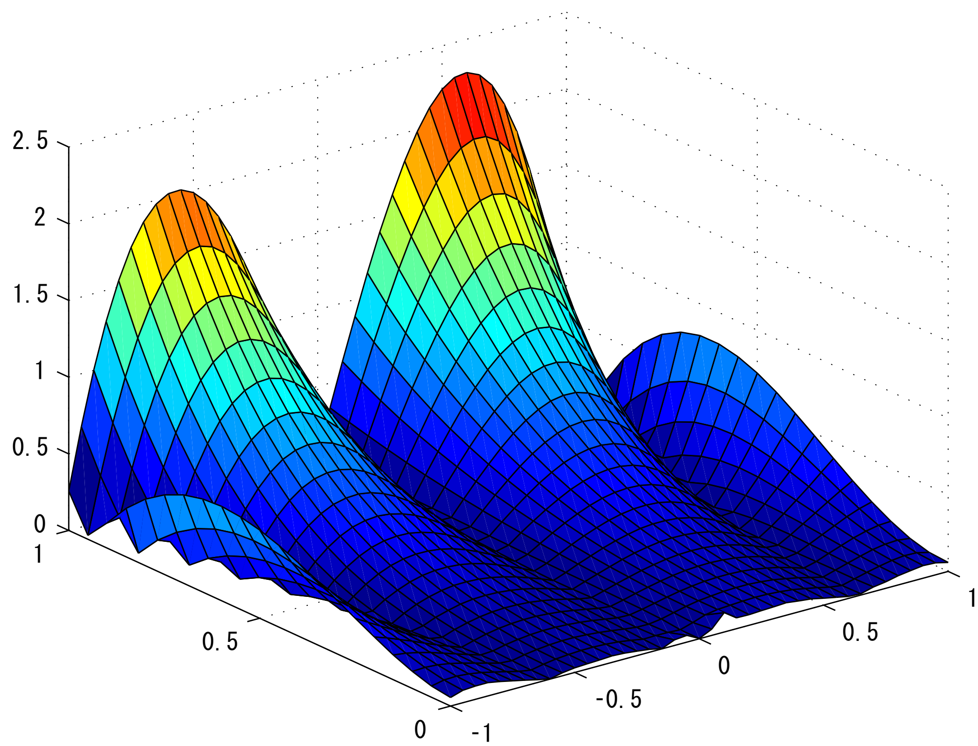

Figure 6: Absolute error $|u_0(x,y) - u_{\Phi_2}(x,y)|$. Here, u_0 is the test function of Example 1 and u_{Φ_2} is an approximation constructed by using the kernel Φ_2 from the Cauchy data on $\Gamma = [-1,1] \times \{0\}$ for noise with level $\delta = 10$.

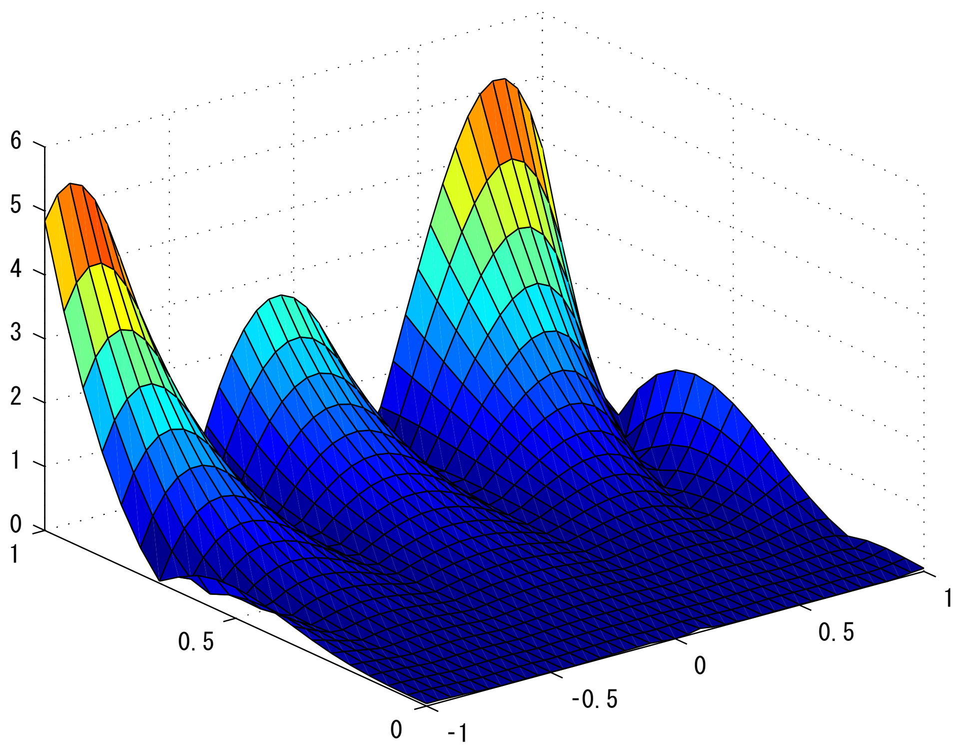

Figure 7: Absolute error $|u_0(x,y) - u_{\Phi_2}(x,y)|$. Here, u_0 is the test function of Example 2 and u_{Φ_2} is an approximation constructed by using the kernel Φ_2 from the Cauchy data on $\Gamma = [-1,1] \times \{0\}$ for noise with level $\delta = 10$.

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