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by

Shigeo Kusuoka and Yasufumi Osajima

UNIVERSITY OF TOKYO

GRADUATE SCHOOL OF MATHEMATICAL SCIENCES

KOMABA, TOKYO, JAPAN

A Remark on the Asymptotic Expansion of density function of Wiener Functionals

Shigeo KUSUOKA *and Yasufumi OSAJIMA †

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Abstract

We consider asymptotic expansion of density function of Wiener functionals as in [4] and give a formula for the first coefficient.

1 Introduction

Let $(\Theta, \|\cdot\|_\Theta)$ be a separable Banach space and $(H, \|\cdot\|_H)$ be a separable Hilbert space such that H is a dense subspace of Θ and the inclusion map is continuous. Let μ_s, $s \in [0,\infty)$, be the (necessarily unique) probability measure on $(\Theta, \mathcal{B}_\Theta)$ with the property that

$$\int_\Theta \exp[\sqrt{-1}\langle u, \theta\rangle]\mu_s(d\theta) = \exp(-\frac{s}{2}\|u\|_H^2),\ u \in \Theta^*.$$

Then (Θ, H, μ_1) is an abstract Wiener space in the sense of L. Gross.

Given a separable Hilbert space E and an $n \in \mathbb{Z}_{\geqq 1}$, let $C^\infty_\nearrow(\mathbb{R}^n; E)$ be the space of smooth E-valued functions f on $\mathbb{R}^n$ with the property that, for each multi-index $\alpha \in \mathbb{Z}^n_{\geqq 0}$, there exist ν_α, $C_\alpha \in (0,\infty)$ such that

$$\left\|\frac{\partial^\alpha f}{\partial x^\alpha}(x)\right\|_E \leqq C_\alpha(1+|x|^2)^{\nu_\alpha/2},\ x \in \mathbb{R}^n.$$

*Graduate School of Mathematical Sciences, The University of Tokyo, Komaba 3-8-1, Meguro-ku, Tokyo 153-8914, Japan, research supported by the 21st century COE project, Graduate School of Mathematical Sciences, The University of Tokyo

†Mitsubishi UFJ Securities Co., Ltd. Research & Development Division, 2-4-1 Marunouchi, Chiyoda-ku, Tokyo 100-6317, Japan, E-mail: osajima-yasufumi@sc.mufg.jp

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Next, define $\mathcal{F}C^\infty_{\nearrow}([0,\infty)\times\Theta;E)$ to be the space of $f:[0,\infty)\to E$ for which there exists an $n\in\mathbb{N}$, an $\tilde{f}\in C^\infty_{\nearrow}(\mathbb{R}^{1+n})$, and a continuous linear map $A:\Theta\to\mathbb{R}^n$ such that

$$f(s,\theta)=\tilde{f}(s,A\theta),\ (s,\theta)\in[0,\infty)\times\Theta.$$

We use $\mathcal{H}(E)$ to denote $H\otimes E$ (or equivalently, the space $H.S.(H;E)$ of Hilbert-Schmidt operators from H into E). We define an operator $D:\mathcal{F}C^\infty_{\nearrow}([0,\infty)\times\Theta;E)\to\mathcal{F}C^\infty_{\nearrow}([0,\infty)\times\Theta;\mathcal{H}(E))$ by

$$Df(s,\theta)(h)=\lim_{\tau\to 0}\frac{f(s,\theta+\tau h)-f(s,\theta)}{\tau},\ (s,\theta)\in[0,\infty)\times\Theta \textit{ and } h\in H.$$

We define $\mathcal{H}^m(E)$ inductively for $m\geqq 2$ so that $\mathcal{H}^m(E)=\mathcal{H}(\mathcal{H}^{m-1}(E))$. Then D^m can be defined inductively so that $D^{m+1}=D\circ D^m$. Noting that, for any $f\in\mathcal{F}C^\infty_{\nearrow}([0,\infty)\times\Theta;E),(s,\theta)\in[0,\infty)\times\Theta$, and complete orthonormal basis $\{h_i\}\subset H$, the Laplacian Δf of f given by

$$\Delta f(s,\theta)=trace_H D^2f(s,\theta)\equiv\sum_i D^2f(s,\theta)(h_i,h_i)\in E$$

is well defined and independent of the choices of basis $\{h_i\}$, we now define the heat operator $\mathcal{A}:\mathcal{F}C^\infty_{\nearrow}([0,\infty)\times\Theta;E)\to\mathcal{F}C^\infty_{\nearrow}([0,\infty)\times\Theta;E)$ by

$$\mathcal{A}f(s,\theta)=\frac{\partial f}{\partial s}(s,\theta)+\frac{1}{2}\Delta f(s,\theta),\ (s,\theta)\in[0,\infty)\times\Theta.$$

We consider a certain class of seminorms on the vector space $\mathcal{F}C^\infty_{\nearrow}([0,\infty)\times\Theta;E)$ and its completion $\mathcal{G}^\infty(\mathcal{A};E)$, and also introduce a notion, complete P-regularity for functions in $\mathcal{G}^\infty(\mathcal{A};E)$ (see Section 2 for the precise definitions).

Now let $f,\ g\in\mathcal{G}^\infty(\mathcal{A};\mathbb{R})$ and $F\in\mathcal{G}^\infty(\mathcal{A};\mathbb{R}^N)$ be completely P-regular functions and Y be a compact subset in $\mathbb{R}^N$.

First we assume the following.

(A1) there is an $\alpha>0$ such that

$$\sup_{s\in(0,1]} s\log\Big(\int_\Theta\exp\Big(\frac{(1+\alpha)f(s,\theta)}{s}\Big)\mu_s(d\theta)\Big)<\infty.$$

We define $e:\mathbb{R}^N\to(-\infty,\infty]$ by

$$e(x)\equiv\inf\{\frac{\|h\|_H^2}{2}-f(0,h):F(0,h)=x\},\qquad x\in\mathbb{R}^N.$$

We also assume the following.

(A2) For each $y\in Y$,

$$M(y)\equiv\{h\in H;F(0,h)=y\}\neq\emptyset$$

and that

$$e(y) = \frac{\|h(y)\|^2}{2} - f(0, h(y))$$

for precisely one $h(y) \in M(y)$.

We assume moreover the following.

(A3) $T(y) \equiv DF(0, h(y))$ has rank N for every $y \in Y$.

Let $\pi(y) = T(y)^*(T(y)T(y)^*)^{-1}T(y)$, $y \in Y$. $\pi(y)$ is an orthogonal projection in H. Let $\pi(y)^\perp = I_H - \pi(y)$. Then $\pi(y)^\perp$ is also an orthogonal projection in H onto $ker\ T(y)$. Let $V(y) : H \times H \to \mathbb{R}$ be a bilinear form given by

$$\begin{aligned} & V(y)(h, h') \\ &= D^2 f(0, h(y))(\pi(y)^\perp h, \pi(y)^\perp h') \\ &+ (h(y) - Df(0, h(y)), T(y)^*(T(y)T(y)^*)^{-1} D^2 F(0, h(y))(\pi(y)^\perp h, \pi(y)^\perp h'))_H. \end{aligned}$$

We assume the following furthermore.

(A4) For all $y \in Y$ and $h \in H \setminus \{0\}$

$$V(y)(h, h) < ||h||_H^2.$$

Finally we define

$$\begin{aligned} A(s, \theta) &= DF(s, \theta) DF(s, \theta)^* \\ &= ((DF_i(s, \theta), DF_j(s, \theta))_H)_{1 \leqq i,j \leqq N} \end{aligned}$$

and assume the following.

(A5) For any $p \in [1, \infty)$

$$\varlimsup_{s \downarrow 0} s \log(\int_\Theta |\det A(s, \theta)|^{-p} \mu_s(d\theta)) \leqq 0.$$

Then Kusuoka-Stroock [4] proved the following.

Theorem 1.1. *For each $s \in (0, 1]$, a signed measure $P_s(\cdot)$ on $\mathbb{R}^N$ given by*

$$P_s(\Gamma) = \int_{F(s,\theta) \in \Gamma} g(s, \theta) \exp\left(\frac{f(s, \theta)}{s}\right) \mu_s(d\theta), \ \Gamma \in \mathcal{B}(\mathbb{R}^N),$$

admits a smooth density $p_s(\cdot)$ with respect to Lesbegue's measure. Moreover, there exist sequence $\{a_n\}_{n=0}^\infty \subseteq C(Y; \mathbb{R})$ and $\{K_n\}_{n=0}^\infty \subseteq (0, \infty)$ with the property that, for every $n \in \mathbb{N}$,

$$\left| (2\pi s)^{N/2} e^{e(y)/s} p_s(y; 0) - \sum_{m=0}^n s^{m/2} a_m(y) \right| \leqq K_n s^{(n+1)/2}, \ (s, y) \in (0, 1] \times Y.$$

Note that the relation of functions ρ in [4] and e in this paper is given by $\rho(y) = -e(y)$, $y \in Y$.

Our main result is the following.

Theorem 1.2. *e is smooth in the neighborhood of Y and*

$$a_0(y) = (\det \nabla^2 e(y))^{1/2} \mathrm{det}_2(I_H - B(y))^{-1/2} \exp\Big(\sum_{l=1}^N \frac{\partial e}{\partial y_l}(y)\mathcal{A}F^l(0,h(y)) + \mathcal{A}f(0,h(y))\Big),$$

where

$$B(y) \equiv \sum_{l=1}^N \frac{\partial e}{\partial y_l}(y) D^2 F^l(0,h(y)) + D^2 f(0,h(y)).$$

Here we identify a continuous symmetric bilinear form $B : H \times H \to \mathbb{R}$ with a bounded symmetric linear operator $\tilde{B} : H \to H$ given by

$$(\tilde{B}h, k)_H = B(h,k), \qquad h,k \in H,$$

and $\det_2$ is a Carleman-Fredfolm determinant (c.f. Dunford-Schwartz [3] pp.1106).

An application of this theorem to finance will be given in Osajima [5].

2 Definitions

In this section and the next section, we summarize the results in [4]. Let $(\Omega_\Theta, ||\cdot||_{\Omega_\Theta})$ be a Banach space given by

$$\Omega_\Theta = \{w \in C([0,\infty);\Theta);\ w(0) = 0,\ \text{and}\ \lim_{t\to\infty} \frac{||w(t)||_\Theta}{t} = 0\},$$

and

$$||w||_{\Omega_\Theta} = \sup_{t\in[0,\infty)} \frac{||w(t)||_\Theta}{1+t}.$$

Let P be a (unique) probability measure on Ω_Θ such that for any $n \geqq 1$, and $0 = t_0 < t_1 < \cdots < t_n$, $w(t_i) - w(t_{i-1})$, $i = 1,\dots,n$ are independent under P and that the probability law of $w(t_i) - w(t_{i-1})$ under P is $\mu_{t_i - t_{i-1}}$, $i = 1,\dots,n$.

Let E be a separable real Hilbert space. For any measurable map $f : [0,\infty)\times\Theta \to E$, $p \in (1,\infty)$ and $R \in (0,\infty)$, let us define $||f||_{p,R;E}$ by

$$||f||_{p,R;E} = \sup_{0\leqq s\leqq R} \sup_{||h||_H \leqq R} (\int_{\Omega_\Theta} ||f(s,w(s)+h)||_E^p P(dw))^{1/p}.$$

Let $\mathcal{G}^1(\mathcal{A};E)$ be a set of measurable maps $f:[0,\infty)\times\Theta\to E$ such that there are measurable maps $Df:[0,\infty)\times\Theta\to\mathcal{H}(E)$, $\mathcal{A}f:[0,\infty)\times\Theta\to E$ and a sequence $\{f_n\}_{n=1}^\infty$ in $\mathcal{F}C^\infty([0,\infty)\times\Theta;E)$ such that

$$||f-f_n||_{p,R;E}\to 0,\quad ||Df-Df_n||_{p,R;\mathcal{H}(E)}\to 0,\quad ||\mathcal{A}f-\mathcal{A}f_n||_{p,R;E}\to 0$$

as $n\to\infty$ for all $p\in(1,\infty)$ and $R\in(0,\infty)$. We define seminorms $||\cdot||^{(1)}_{p,R;E}$, $p\in(1,\infty)$ and $R\in(0,\infty)$, on $\mathcal{G}^1(\mathcal{A};E)$ by

$$||f||^{(1)}_{p,R;E}=\{||f||^p_{p,R;E}+||Df||^p_{p,R;\mathcal{H}(E)}+||\mathcal{A}f||^p_{p,R;E}\}^{1/p}.$$

The closability of the linear operators D and $\mathcal{A}$ is guaranteed by Ito's formula

$$f(s,h+w(s))=f(0,h)+\int_0^s Df(t,h+w(t))dw(t)+\int_0^s \mathcal{A}f(t,h+w(t))dt,\ P{-}a.s.w\in\Omega_\Theta\quad h\in H.$$

We define $\mathcal{G}^n(\mathcal{A};E)$, $n\geqq 2$, inductively in the following. We say that $f\in\mathcal{G}^n(\mathcal{A};E)$, if $f\in\mathcal{G}^1(\mathcal{A};E)$, $Df\in\mathcal{G}^{n-1}(\mathcal{A};\mathcal{H}(E))$ and $\mathcal{A}f\in\mathcal{G}^{n-1}(\mathcal{A};E)$. We define seminorms $||\cdot||^{(n)}_{p,R;E}$, $p\in(1,\infty)$ and $R\in(0,\infty)$, on $\mathcal{G}^n(\mathcal{A};E)$, $n\geqq 2$, inductively by

$$||f||^{(n)}_{p,R;E}=\{||f||^p_{p,R;E}+||Df||^{(n-1)\ p}_{p,R;\mathcal{H}(E)}+||\mathcal{A}f||^{(n-1)\ p}_{p,R;E}\}^{1/p}.$$

Finally we define $\mathcal{G}^\infty(\mathcal{A};E)$ by

$$\mathcal{G}^\infty(\mathcal{A};E)=\bigcap_{n=1}^\infty\mathcal{G}^n(\mathcal{A};E).$$

We regard $\mathcal{G}^\infty(\mathcal{A};E)$ as a topological vector space with seminorms $||\cdot||^{(n)}_{p,R;E}$, $n\geq 1$, $p\in(1,\infty)$ and $R\in(0,\infty)$. Then $D:\mathcal{G}^\infty(\mathcal{A};E)\to\mathcal{G}^\infty(\mathcal{A};\mathcal{H}(E))$ and $\mathcal{A}:\mathcal{G}^\infty(\mathcal{A};E)\to\mathcal{G}^\infty(\mathcal{A};E)$ are continuous linear operators.

Let Y be a compact metric space. We say that a measurable map $f:[0,\infty)\times\Theta\times Y\to E$ is P-regular uniformly on Y into E, if there exists a sequence $\{f_n\}_{n=1}^\infty\subset C([0,\infty)\times\Theta\times Y;E)$ with the property that

$$\lim_{n\to\infty}\sup\{||f(0,h,y)-f_n(0,h,y)||_E;\ (h,y)\in H\times Y\text{ with }||h||_H\leqq L\}=0$$

for any $L>0$, and

$$\lim_{n\to\infty}\overline{\lim_{s\downarrow 0}}\sup_{y\in Y}s\log(P(\{||f(s,w(s),y)-f_n(s,w(s),y)||_E>\delta\}))=-\infty$$

for any $\delta>0$.

We say that a map $f:[0,\infty)\times\Theta\times Y\to E$ is completely P-regular uniformly on Y, if $y\in Y\mapsto f(\cdot,*,y)$ is a continuous mapping into $\mathcal{G}^\infty(\mathcal{A};E)$ and, for each $n,m\in\mathbb{Z}_{\geqq 0}$, $D^n\mathcal{A}^mf:[0,\infty)\times\Theta\times Y\to\mathcal{H}^n(E)$ is P- regular uniformly on Y into $\mathcal{H}^n(E)$.

3 Asymptotic Expansions

Let Y be a compact metric space, and let $f:[0,\infty)\times\Theta\times Y\to\mathbb{R}$, $F:[0,\infty)\times\Theta\times Y\to\mathbb{R}^N$ and $g:[0,\infty)\times\Theta\times Y\to\mathbb{R}$ be completely P-regular uniformly on Y.

We assume that there is an $\alpha>0$ such that

$$\sup_{y\in Y}\sup_{s\in(0,1]} s\log(\int_\Theta \exp(\frac{(1+\alpha)f(s,\theta,y)}{s})\mu_s(d\theta))<\infty.$$

We define $\tilde{e}:\mathbb{R}^N\times Y\to(-\infty,\infty]$ by

$$\tilde{e}(x,y)\equiv\inf\{\frac{\|h\|_H^2}{2}-f(0,h,y):F(0,h,y)=x\},\qquad x\in\mathbb{R}^N,\ y\in Y.$$

Remind again that the function ρ in [4] is expressed by $\rho(y)=-\tilde{e}(0,y)$, $y\in Y$. We assume that for each $y\in Y$

$$\tilde{M}(y)\equiv\{h\in H;F(0,h,y)=0\}\neq\emptyset$$

and

$$\tilde{e}(0,y)=\frac{\|\tilde{h}(y)\|^2}{2}-f(0,h(y),y)$$

for precisely one $h(y)\in M(y)$. We assume moreover that

$$\tilde{T}(y)\equiv DF(0,h(y),y)$$

has rank N for every $y\in Y$. Let $\tilde{\pi}(y)=\tilde{T}(y)^*(\tilde{T}(y)\tilde{T}(y)^*)^{-1}\tilde{T}(y)$, $y\in Y$. $\tilde{\pi}(y)$ is an orthogonal projection in H. Let $\tilde{\pi}(y)^\perp=I_H-\tilde{\pi}(y)$. Then $\tilde{\pi}(y)^\perp$ is an orthogonal projection in H onto $ker\tilde{T}(y)$. Let $\tilde{V}(y):H\times H\to\mathbb{R}$ be a bilinear form given by

$$\begin{gathered}\tilde{V}(y)(h,h')\\ =D^2f(0,h(y),y)(\tilde{\pi}(y)^\perp h,\tilde{\pi}(y)^\perp h')\\ +(h(y)-Df(0,h(y),y),\tilde{T}(y)^*(\tilde{T}(y)\tilde{T}(y)^*)^{-1}D^2F(0,h(y),y)(\tilde{\pi}(y)^\perp h,\tilde{\pi}(y)^\perp h'))_H\end{gathered}$$

We assume furthermore that

$$\tilde{V}(y)(h,h)<||h||_H^2 \text{ for all } y\in Y \text{ and } h\in H\setminus\{0\}\ .$$

Finally we define

$$\begin{aligned}\tilde{A}(s,\theta,y)&=DF(s,\theta,y)DF(s,\theta,y)^*\\&=((DF_i(s,\theta,y),DF_j(s,\theta,y))_H)_{1\leqq i,j\leqq N}\end{aligned}$$

and assume that

$$\varlimsup_{s\downarrow 0} s\log(\sup_{y\in Y}\int_\Theta|\det\tilde{A}(s,\theta,y)|^{-p}\mu_s(d\theta))\leqq 0,\ p\in[1,\infty).$$

The following has been shown in [4].

Theorem 3.1. *For each $s \in (0,1]$ and $y \in Y$, a signed measure $P_s(\cdot, y)$ on $\mathbb{R}^N$ given by*

$$P_s(\Gamma, y) = \int_{F(s,\theta,y)\in\Gamma} g(s,\theta,y)\exp[\frac{f(s,\theta,y)}{s}]\mu_s(d\theta),\ \ \Gamma \in \mathcal{B}(\mathbb{R}^N),$$

admits a smooth density $p_s(\cdot, y)$ with respect to Lebesgue's measure. Moreover, there exist sequences $\{a_n\}_{n=0}^{\infty} \subseteq C(Y;\mathbb{R})$ and $\{K_n\}_{n=0}^{\infty} \subseteq (0,\infty)$ with the property that, for every $n \in \mathbb{N}$,

$$\Big|(2\pi s)^{N/2} e^{e(0,y)/s} p_s(0,y) - \sum_{m=0}^{n} s^{m/2} a_m(y)\Big| \leqq K_n s^{(n+1)/2},\ \ (s,y) \in (0,1] \times Y.$$

We will show the following theorem in the following sections.

Theorem 3.2. *$\tilde{e}(\cdot, y)$ is smooth in the neighborhood of 0 for each $y \in Y$, and*

$$a_0(y) = (\det \nabla_x^2 \tilde{e}(0,y))^{\frac{1}{2}} \det{}_2(I_H - B(y))^{-\frac{1}{2}} \exp\Big(\sum_{l=1}^{N} \frac{\partial \tilde{e}}{\partial x^l}(0,y)\mathcal{A}F^l(0,h(y),y) + \mathcal{A}f(0,h(y),y)\Big),$$

where

$$B(y) \equiv \sum_{l=1}^{N} \frac{\partial \tilde{e}}{\partial x^l}(0,y) D^2 F^l(0,h(y),y) + D^2 f(0,h(y),y).$$

We have Theorem 1.2 as an immediate corollary to Theorem 3.2, applying Theorem 1.2 to the Wiener functional $F(s,\theta,y) = F(s,\theta) - y$.

4 Preparations

We make some preparations to prove Theorem 3.2. The statement in Theorem 3.2 is just an equation for each $y \in Y$. So we may assume that Y consists of one point y_0. For simplicity, we denote $\tilde{e}(\cdot, y_0)$, $\tilde{h}(y_0)$, $\tilde{T}(y_0)$ and $\tilde{\pi}(y_0)$, by $e_0(\cdot)$, h_0, T_0 and π_0 respectively. Also, we denote $f(s,\theta,y_0)$, $F(s,\theta,y_0)$ and $g(s,\theta,y_0)$ by $f(s,\theta)$, $F(s,\theta)$ and $g(s,\theta)$.

We have to follow the argument in p.49-59 in [4]. For any completely P-regular map $G : [0,\infty) \times \Theta \to E$, $\tilde{G} : [0,\infty) \times \Theta \to C_c^{\infty}(\mathbb{R}^N; E)$ is defined in Theorem 4.19 in [4]. Then $\Xi(s,\theta)(\cdot)$ is defined as a modified inverse function of $\tilde{F}(s,\theta)(\cdot)$ in p.57 in [4]. Then $J(s,\theta)$ is given by

$$J(s,\theta) = |\det(\nabla \Xi(s,\theta)(0))|.$$

Finally $\bar{g}$ and $\bar{f}$ are defined in the following.

$$\bar{g}(s,\theta) = J(s,\theta)\tilde{g}(s,\theta)(\Xi(s,\theta)(0)),$$

and

$$\bar{f}(s,\theta) = \tilde{f}(s,\theta)(\Xi(s,\theta)(0)) - \frac{1}{2}|U_0^*\Xi(s,\theta) + \pi_0 h_0|^2.$$

Then it is shown in [4] that

$$\overline{\lim_{s\downarrow 0}}\, s\log|p_s(0,y_0) - \int_\Theta \bar{g}(s,\theta)\exp(\frac{\bar{f}(s,\theta)}{s})\mu_s(d\theta)| < e_0(0).$$

So by (3.16) in [4], we see that

$$a_0(y_0) = \bar{g}(0,h_0)\mathrm{det}_2(I_H - D^2\bar{f}(0,h_0))^{-1/2}\exp(\mathcal{A}\bar{f}(0,h_0)). \tag{4.1}$$

Therefore what we have to do is to compute the right hand side of Equation (4.1).

Since $h_0 \in H$ is a minimizer of $\frac{1}{2}||h||^2 - f(0,h)$ subject to the condition $F(0,h) = 0$, and T_0 has rank N, we can apply Lagrange's method and there is a $\lambda_0 \in \mathbb{R}^N$ such that

$$h_0 = Df(0,h_0) + \sum_{i=1}^N \lambda_0^i DF^i(0,h_0).$$

Let $U_0 = (T_0T_0^*)^{-1/2}T_0$. Then $\pi_0 = U_0^*U_0$. Remind that $\pi_0 : H \to H$ is an orthogonal projection onto the image of $DF(0,h_0)^*$ and that $\pi_0^\perp = I_H - \pi_0$ is an orthogonal projection onto $kerDF(0,h_0)$.

Let $v_0 \in \mathbb{R}^N$ be given by

$$v_0 = (T_0T_0^*)^{-1}T_0Df(0,h_0). \tag{4.2}$$

Then we have

$$(T_0T_0^*)^{-1}T_0\pi_0h_0 = v_0 + (T_0T_0^*)^{-1}T_0(\sum_{i=1}^N \lambda_0^i DF^i(0,h_0)) = v_0 + \lambda_0 \tag{4.3}$$

So we see that

$$\lambda_0 = (T_0T_0^*)^{-1}T_0(\pi_0h_0 - Df(0,h_0)). \tag{4.4}$$

In particular, we have

$$V(y_0)(h,h') = D^2f(0,h_0)(h,h') + \lambda_0 \cdot D^2F(0,h_0)(h,h'), \qquad h,h' \in H. \tag{4.5}$$

Several cut-off functions and modified procedures are used in the definitions of $\tilde{G}$ and Ξ in [4]. To avoid complexity, we use the following notion. For any separable real Hilbert space E and completely P-regular maps, $f_i : [0,\infty) \times \Theta \to E$, $i = 1,2$, we denote $f_1(s,\theta) \simeq f_2(s,\theta)$ if

$$\mathcal{D}^n\mathcal{A}^m f_1(0,\pi_0^\perp h_0) = \mathcal{D}^n\mathcal{A}^m f_2(0,\pi_0^\perp h_0)$$

for all $n, m \in \mathbb{Z}_{\geqq 0}$.

Let $B_r = \{x \in \mathbb{R};\ |x| < r\}$, $r > 0$, and let $W_2^n(B_r; E)$, $n \geqq 1$, denote L^2-Sobolev spaces of E-valued functions defined in B_r (e.g. Adams [1]). Then there is a natural map $j_{n,r}$ corresponding $\varphi \in C^\infty(\mathbb{R}^N; E)$ to $\varphi|_{B_r} \in W_2^n(B_r; E)$.

Then for any completely P-regular map $G : [0, \infty) \times \Theta \to E$, $j_{n,r} \circ \tilde{G} : [0, \infty) \times \Theta \to W_2^n(B_r; E)$ is also completely P-regular. Let us define a map $G'_{n,r} : [0, \infty) \times \Theta \to W_2^n(B_r; E)$ be given by

$$G'_{n,r}(s, \theta)(\xi) = G(s, U_0^*\xi + \pi_0 h_0 + \pi_0^\perp \theta), \qquad \xi \in B_r.$$

Checking the definitions in [4], we have the following.

Proposition 4.1. *Let $n > N + 2$. Then there is an $r > 0$ satisfying the following.*
(1) *For any completely P-regular map $G : [0, \infty) \times \Theta \to E$,*

$$j_{n,r} \circ \tilde{G}(s, \theta) \simeq G'_{n,r}(s, \theta).$$

(2)

$$\tilde{F}(s, \theta) \circ \Xi(s, \theta) \simeq Id_{B_r}.$$

Here $Id_{B_r} \in W_2^n(B_r; \mathbb{R}^N)$ is given by $Id_{B_r}(\xi) = \xi$, $\xi \in B_r$.

Then we have the following.

Proposition 4.2. *For any completely P-regular map $G : [0, \infty) \times \Theta \to E$, we have the following.*
(1)

$$\tilde{G}(0, \pi_0^\perp h_0)(0) = G(0, h_0).$$

(2)

$$D\tilde{G}(0, \pi_0^\perp h_0)(0) = DG(0, h_0)(0)\pi_0^\perp.$$

(3)

$$D^2\tilde{G}(0, \pi_0^\perp h_0)(0)(h_1, h_2) = D^2G(0, h_0)(0)(\pi_0^\perp h_1, \pi_0^\perp h_2), \qquad h_1, h_2 \in H.$$

(4)

$$\nabla_\xi \tilde{G}(0, \pi_0^\perp h_0)(0) = DG(0, h_0)(0)U_0^*.$$

(5)

$$\nabla_\xi^2 \tilde{G}(0, \pi_0^\perp h_0)(0)(\xi_1, \xi_2) = D^2G(0, h_0)(U_0^*\xi_1, U_0^*\xi_2), \qquad \xi_1, \xi_2 \in \mathbb{R}^N.$$

(6)

$$\mathcal{A}\tilde{G}(0, \pi_0^\perp h_0)(0) = \mathcal{A}G(0, h_0) - \frac{1}{2} trace_H D^2G(0, h_0)(0)(\pi_0 \cdot, \pi_0 \cdot).$$

Proof. The assertion (1) is obvious. Since

$$DG'_{n,r}(s,\theta)(\xi) = DG(s, U_0^*\xi + \pi_0 h_0 + \pi_0^\perp\theta)\pi_0^\perp,$$

we see that

$$D(j_{n,r} \circ \tilde{G}(s,\theta)(\xi)) \simeq j_{n,r} \circ \tilde{DG}(s,\theta)(\xi)\pi_0^\perp$$

So we have the assertions (2) and (3).

Since

$$\nabla_\xi G'_{n,r}(s,\theta)(\xi) = DG(s, U_0^*\xi + \pi_0 h_0 + \pi_0^\perp\theta)U_0^*,$$

we see that

$$\nabla_\xi(j_{n,r} \circ \tilde{G})(s,\theta)(\xi) \simeq j_{n,r} \circ \tilde{DG}(s,\theta)(\xi)U_0^*.$$

So we have the assertions (4) and (5).

Finally we have

$$\mathcal{A}G'_{n,r}(s,\theta)(\xi)$$

$$= \mathcal{A}G(s, U_0^*\xi + \pi_0 h_0 + \pi_0^\perp\theta) - \frac{1}{2} trace_H \tilde{DG}(s, U_0^*\xi + \pi_0 h_0 + \pi_0^\perp\theta)(\pi_0\cdot, \pi_0\cdot).$$

So we have the assertion (6). ∎

Proposition 4.3. (1) $\nabla_\xi \Xi(0, \pi_0^\perp h_0)(0) = (T_0 U_0^*)^{-1}$.
(2) $D\Xi(0, \pi_0^\perp h_0)(0) = 0$.
(3) $D^2\Xi(0, \pi_0^\perp h_0)(0)(\pi_0^\perp h_1, \pi_0^\perp h_2) = -(T_0 T_0^*)^{-1/2} D^2 F(0, h_0)(\pi_0^\perp h_1, \pi_0^\perp h_2)$ *for any* $h_1, h_2 \in H$.
(4)$\mathcal{A}\Xi(0, \pi_0^\perp h_0)(0) = -(T_0 T_0^*)^{-1/2} \mathcal{A}F(0, h_0) + \frac{1}{2} trace_H((T_0 T_0^*)^{-1/2} D^2 F(0, h_0)(0)(\pi_0\cdot, \pi_0\cdot))$.

Proof. By Proposition 4.1 (2), we have

$$Identity_{\mathbb{R}^N} \simeq \nabla_\xi(F'_{n,r}(s,\theta)(\Xi(s,\theta)(\xi)) = DF(s, U_0^*\xi + \pi_0 h_0 + \pi_0^\perp\theta)(U_0^* \nabla_\xi \Xi(s,\theta)(\xi)).$$

This implies our assertion (1).

By Proposition 4.1 (2), we also have

$$0 \simeq D\{F'_{n,r}(s,\theta)(\Xi(s,\theta)(\xi))\}$$

$$= DF'_{n,r}(s,\theta)(\Xi(s,\theta)(\xi) + \nabla F'_{n,r}(s,\theta)(\Xi(s,\theta)(\xi))(D\Xi(s,\theta)(\xi)).$$

Therefore we have

$$0 = DF'_{n,r}(0, \pi_0^\perp h_0)(0) + \nabla F'_{n,r}(0, h_0)(0)(D\Xi(0, h_0)(0)).$$

$$= DF(0, h_0)\pi_0^\perp + DF(0, h_0)U_0^* D\Xi(0, \pi_0^\perp h_0)(0) = (T_0 T_0^*)^{1/2} D\Xi(0, \pi_0^\perp h_0)(0).$$

This implies the assertion (2).

By Proposition 4.1 (2), we have

$$0 \simeq D^2\{F'_{n,r}(s,\theta)(\Xi(s,\theta)(\xi))\}$$

$$= D^2F'_{n,r}(s,\theta)(\Xi(s,\theta))(\xi) + 2\nabla DF'_{n,r}(s,\theta)(\Xi(s,\theta)(\xi))(D\Xi(s,\theta)(\xi))$$

$$+\nabla F'_{n,r}(s,\theta)(\Xi(s,\theta)(\xi))(D^2\Xi(s,\theta)(\xi))+\nabla^2 F'_{n,r}(s,\theta)(\Xi(s,\theta)(\xi))(D\Xi(s,\theta)(\xi), D\Xi(s,\theta)(\xi)).$$

So we see that

$$0 = D^2F(0,h_0)(\pi_0^\perp h_1, \pi_0^\perp h_2) + (T_0T_0^*)^{1/2}D^2\Xi(0,\pi_0^\perp h_0)(0)(\pi_0^\perp h_1, \pi_0^\perp h_2).$$

This implies the assertion (3).

By Proposition 4.1 (2), we have

$$0 \simeq \mathcal{A}\{F'_{n,r}(s,\theta)(\Xi(s,\theta)(\xi))\}$$

$$= \mathcal{A}F'_{n,r}(s,\theta)(\Xi(s,\theta)(\xi) + \nabla F'_{n,r}(s,\theta)(\Xi(s,\theta)(\xi))(\mathcal{A}\Xi(s,\theta)(\xi))$$

$$-trace_H(D\nabla F'_{n,r}(s,\theta)(\Xi(s,\theta)(\xi))(D\Xi(s,\theta)(\xi)))-\frac{1}{2}\nabla^2 F'_{n,r}(s,\theta)(\Xi(s,\theta)(\xi))(D\Xi(s,\theta)(\xi), D\Xi(s,\theta)(\xi)).$$

So we have

$$0 = \mathcal{A}\tilde{F}(0,\pi_0^\perp h_0) + \nabla\tilde{F}(0,\pi_0^\perp h_0)(0)(\mathcal{A}\Xi(0,\pi_0^\perp h_0)(0))$$

$$= \mathcal{A}F(0,h_0) - \frac{1}{2}trace_H D^2F(0,h_0)(0)(\pi_0\cdot,\pi_0\cdot) + (T_0T_0^*)^{1/2}\mathcal{A}\Xi(0,\pi_0^\perp h_0)(0).$$

This implies the assertion (4).

This completes the proof. ∎

Proposition 4.4. (1) $D^2\bar{f}(0,\pi_0^\perp h_0)$

$$= D^2f(0,h_0)(\pi_0^\perp\cdot,\pi_0^\perp\cdot) + (D^2F(0,h_0)(\pi_0^\perp\cdot,\pi_0^\perp\cdot),\lambda_0)_{\mathbb{R}^N}.$$

(2) $\mathcal{A}\bar{f}(0,\pi_0^\perp h_0)$

$$= \mathcal{A}f(0,h_0) - \frac{1}{2}trace_H D^2f(0,h_0)(\pi_0\cdot,\pi_0\cdot)) + (\mathcal{A}F(0,h_0)),\lambda_0)_{\mathbb{R}^N}$$

$$-\frac{1}{2}(trace_H(D^2F(0,h_0)(\pi_0\cdot,\pi_0\cdot)),\lambda_0)_{\mathbb{R}^N}.$$

Proof. Let

$$\bar{f}_1(s,\theta) = \tilde{f}(s,\theta)(\Xi(s,\theta)(0)),$$

$$\bar{f}_2(s,\theta) = ||U_0^*\Xi(s,\theta)(0)||_H^2,$$

and

$$\bar{f}_3(s,\theta) = (U_0^*\Xi(s,\theta)(0), \pi_0 h_0)_H.$$

Then we see that

$$\bar{f}(s,\theta) = \bar{f}_1(s,\theta) - \frac{1}{2}\bar{f}_2(s,\theta) - \bar{f}_3(s,\theta) - \frac{1}{2}||\pi_0 h_0||_H^2.$$

Since $\Xi(0,\pi_0^\perp h_0) = 0$ and $D\Xi(0,\pi_0^\perp h_0) = 0$, we have

$$D^2\bar{f}_2(0,\pi_0^\perp h_0) = 0, \text{ and}$$

$$\mathcal{A}\bar{f}_2(0,\pi_0^\perp h_0) = 0.$$

By Proposition 4.3, we see that

$$\begin{aligned} D^2\bar{f}_3(0,\pi_0^\perp h_0) &= -(T_0^*(T_0T_0^*)^{-1}D^2F(0,h_0)(\pi_0^\perp\cdot,\pi_0^\perp\cdot),\pi_0 h_0)_H \\ &= -(D^2F(0,h_0)(\pi_0^\perp\cdot,\pi_0^\perp\cdot),\lambda_0+v_0)_{\mathbb{R}^N}, \end{aligned}$$

and

$$\begin{aligned} &\mathcal{A}\bar{f}_3(0,\pi_0^\perp h_0) \\ &= -(T_0^*(T_0T_0^*)^{-1}\mathcal{A}F(0,h_0),\pi_0 h_0)_H + \frac{1}{2}(trace_H(T_0^*(T_0T_0^*)^{-1}D^2F(0,h_0)(\pi_0\cdot,\pi_0\cdot),\pi_0 h_0)_H \\ &= -(\mathcal{A}F(0,h_0),\lambda_0+v_0)_{\mathbb{R}^N} + \frac{1}{2}(trace_H(D^2F(0,h_0)(\pi_0\cdot,\pi_0\cdot),\lambda_0+v_0)_{\mathbb{R}^N}. \end{aligned}$$

Note that

$$\begin{aligned} &D^2\bar{f}_1(s,\theta) \\ &= D^2\tilde{f}(s,\theta)(\Xi(s,\theta)(0)) + \nabla\tilde{f}(s,\theta)(\Xi(s,\theta)(0))(D^2\Xi(s,\theta)(0)) + 2D\nabla\tilde{f}(s,\theta)(\Xi(s,\theta)(0))(D(\Xi(s,\theta)(0)), \end{aligned}$$

and

$$\begin{aligned} &\mathcal{A}\bar{f}_1(s,\theta) \\ &= \mathcal{A}\tilde{f}(s,\theta)(\Xi(s,\theta)(0)) + \nabla\tilde{f}(s,\theta)(\Xi(s,\theta)(0))(\mathcal{A}\Xi(s,\theta)(0)) \\ &\quad + 2trace_H(D\nabla\tilde{f}(s,\theta)(\Xi(s,\theta)(0))(D(\Xi(s,\theta)(0))) \end{aligned}$$

Also, we see that

$$\begin{aligned} &D^2\bar{f}_1(0,\pi_0^\perp h_0) \\ &= D^2\tilde{f}(0,\pi_0^\perp h_0)(0) + \nabla\tilde{f}(0,\pi_0^\perp h_0)(0)(D^2(\Xi(0,\pi_0^\perp h_0)(0)) \\ &= D^2f(0,h_0)(\pi_0^\perp\cdot,\pi_0^\perp\cdot) - Df(0,h_0)(U_0^*(T_0T_0^*)^{-1/2}D^2F(0,h_0)(\pi_0^\perp\cdot,\pi_0^\perp\cdot)) \\ &= D^2f(0,h_0)(\pi_0^\perp\cdot,\pi_0^\perp\cdot) - (D^2F(0,h_0)(\pi_0^\perp\cdot,\pi_0^\perp\cdot),v_0)_{\mathbb{R}^N}, \end{aligned}$$

and

$$\mathcal{A}\bar{f}_1(0,\pi_0^\perp h_0)$$

$$= \mathcal{A}\tilde{f}(0,\pi_0^\perp h_0)(0) + \nabla\tilde{f}(0,\pi_0^\perp h_0)(0)(\mathcal{A}(\Xi(0,\pi_0^\perp h_0)(0))$$

$$= \mathcal{A}f(0,h_0) - \frac{1}{2}trace_H D^2 f(0,h_0)(\pi_0\cdot,\pi_0\cdot)) - Df(0,h_0)(U_0^*(T_0T_0^*)^{-1/2}\mathcal{A}F(0,h_0))$$

$$+\frac{1}{2}D^2 f(0,h_0)(U_0^*(T_0T_0^*)^{-1/2}trace_H(D^2F(0,h_0)(\pi_0\cdot,\pi_0\cdot)))$$

$$= \mathcal{A}f(0,h_0) - \frac{1}{2}trace_H D^2 f(0,h_0)(\pi_0\cdot,\pi_0\cdot)) - (\mathcal{A}F(0,h_0)), v_0)_{\mathbb{R}^N}$$

$$+\frac{1}{2}(trace_H(D^2F(0,h_0)(\pi_0\cdot,\pi_0\cdot)), v_0)_{\mathbb{R}^N}.$$

Combining these equations, we have our assertions.

This completes the proof. ∎

By Equation (4.1) and Propositions 4.2, 4.3, 4.4, we have the following.

Proposition 4.5. $a_0(y_0)$

$$= g(0,h_0)\det(T_0T_0^*)^{-1/2}\mathrm{det}_2(I_H - \pi_0^\perp B_0\pi_0^\perp)^{-1/2}$$

$$\times\exp(\mathcal{A}f(0,h_0) + \sum_{i=1}^N \lambda_0^i\mathcal{A}F^i(0,h_0) - \frac{1}{2}trace_H(\pi_0 B_0)),$$

where

$$B_0 = D^2 f(0,h_0) + \sum_{i=1}^N \lambda_0^i D^2 F^i(0,h_0).$$

5 Proof of Theorem 3.2

Proposition 5.1. *There is an $r > 0$ and smooth maps $\hat{h} : B_r \to H$ and $\hat{\lambda} : B_r \to \mathbb{R}$ satisfying the following.*

(1)

$$e_0(x) = \frac{1}{2}||\hat{h}(x)||_H^2 - f(0,\hat{h}(x)).$$

(2)

$$\hat{h}(x) - Df(0,\hat{h}(x)) = \sum_{i=1}^N \hat{\lambda}^i(x)DF^i(0,\hat{h}(x)).$$

(3)

$$F(0,\hat{h}(x)) = x \qquad \textit{for each } x \in B_r.$$

Moreover,

$$\hat{h}(0) = h_0 \textit{ and } \hat{\lambda}(0) = \lambda_0.$$

Proof. Let us define a smooth map $\Phi : H \times \mathbb{R}^N \to H \times \mathbb{R}^N$ by

$$\Phi(h,\lambda) = (h - Df(0,h) - \sum_{i=1}^N \lambda^i DF^i(0,h), F(0,h)) \qquad (h,\lambda) \in H \times \mathbb{R}^N.$$

Note that $\Phi(h_0,\lambda_0) = (0,0)$. Also we see that the Frechét derivative $\Phi'(h_0,\lambda_0)$ of Φ at (h_0,λ_0) is

$$\Phi'(h_0,\lambda_0)(k,z)$$

$$= (k - D^2 f(0,h_0)(k,\cdot) - \sum_{i=1}^N \lambda_0^i D^2 F^i(0,h_0)(k,\cdot) - \sum_{i=1}^N z^i DF^i(0,h_0), DF(0,h_0)(k)), \quad (k,z) \in H \times \mathbb{R}.$$

First, we prove that $\Phi'(h_0,\lambda_0) : H \times \mathbb{R}^N \to H \times \mathbb{R}^N$ is nondegenerate. If $\Phi'(h_0,\lambda_0)(k,z) = 0$, $DF(0,h_0)(k) = 0$, and so $\pi_0^\perp k = k$, and we have

$$k - D^2 f(0,h_0)(k,\cdot) - \sum_{i=1}^N \lambda_0^i D^2 F^i(0,h_0)(k,\cdot) - \sum_{i=1}^N z^i DF^i(0,h_0)(k) = 0.$$

Taking the inner product with $k = \pi_0^\perp k$, we see by Equation (4.5) that

$$||k||_H^2 - V(y_0)(k,k) = 0.$$

This implies $k = 0$. Then it is easy to see that $z = 0$. So we see that $\Phi'(h_0,\lambda_0)$ is nondegenerate.

So by the inverse function theorem, we see that there is an $r' > 0$ and smooth maps $\hat{h} : B_{r'} \to H$ and $\hat{\lambda} : B_{r'} \to \mathbb{R}$ such that

$$\Phi(\hat{h}(x), \hat{\lambda}(x)) = (0,x), \text{ and } (\hat{h}(0), \hat{\lambda}(0)) = (h_0,\lambda_0).$$

Let $E : H \to \mathbb{R}$ be given by

$$E(h) = \frac{1}{2}||h||_H^2 - f(0,h), \qquad h \in H.$$

It is sufficient to show that there is an $r \in (0,r')$ such that $e_0(x) = E(\hat{h}(x))$, for any $x \in B(r)$.

Assume that such an r does not exist. Since f and F are completely P-regular, we see that $f(0,\cdot) : H \to \mathbb{R}$, $F(0,\cdot) : H \to \mathbb{R}^N$, $DF(0,\cdot) : H \to \mathcal{H}(\mathbb{R}^N)$ are weakly continuous on bounded sets in H.

It is shown in [4] that there are $c_0, c_1 > 0$ such that

$$\frac{1}{2}||h||_H^2 - f(0,h) \leqq c_0 - c_1||h||_H^2 \text{ for any } h \in H.$$

Since the function $E : H \to \mathbb{R}$ is lower semicontinuous in weak topology, we see that for any $x \in B(r')$ there are $h \in H$ such that $F(0,h) = x$ and $E(h) = e_0(x)$.

So from our assumption, there are $x_n \in \mathbb{R}^N$ and $h_n \in H$, $n = 1, 2, \ldots$, such that $x_n \to 0$, $n \to \infty$, $F(0, h_n) = x_n$, $e_0(x_n) = E(h_n)$, and $h_n \neq \hat{h}(x_n)$. Since $||h_n||_H$, $n = 1, 2, \ldots$, are bounded, we may assume that h_n, $n = 1, 2, \ldots$, converges weakly to a certain $h_\infty \in H$. Noting

$$E(h_\infty) \leqq \overline{\lim_{n\to\infty}} E(h_n) \leqq \overline{\lim_{n\to\infty}} E(\hat{h}(x_n)) = E(\hat{h}(0)) = e_0(0),$$

we see that $h_\infty = h_0$, and $||h_n||_H \to ||h_0||_H$, $n \to \infty$. Therefore we see that $h_n \to h_0$ in H as $n \to \infty$. Then we see that $DF(0, h_n) : H \to \mathbb{R}^N$ is nondegenerate for sufficiently large n. Then we can apply Lagrange's principle and so there are $\lambda_n \in \mathbb{R}^N$ such that $h_n - Df(0, h_n) - \lambda_n \cdot DF(0, h_n) = 0$ for sufficiently large n. Then we see that $\lambda_n \to \lambda_0$, $n \to \infty$. These imply that $\Phi(h_n, \lambda_n) = (0, x_n)$ for sufficiently large n, and $(h_n, \lambda_n) \to (h_0, \lambda_0)$, $n \to \infty$. But the inverse function theorem implies that $h_n = \hat{h}(x_n)$ for sufficiently large n. This is the contradiction. So we have our assertion. This completes the proof. ∎

Proposition 5.2. *$I_H - \pi_0^\perp B_0 : H \to H$ is bijective.*

Proof. By the definition of $\tilde{V}(y_0)$, B_0 and Equation (4.4) we have

$$||h||_H^2 - \tilde{V}(y_0)(h, h) = ((I_H - \pi_0^\perp B_0 \pi_0^\perp)h, h)_H, h \in H.$$

If $(I_H - \pi_0^\perp B_0)h = 0$ for some $h \in H$, then we see that $\pi_0 h = 0$. So we see that $||h||_H^2 - \tilde{V}(y_0)(h, h) = 0$. This implies that $h = 0$ by the assumption on $\tilde{V}$. This proves our assertion.

Proposition 5.3. (1) $(\pi_0 \dfrac{\partial}{\partial x^i}\hat{h}(x), DF^j(0, \hat{h}(x)))_H = \delta_{ij}, \qquad i, j = 1, \ldots, N.$

(2) $\hat{\lambda}^i(x) = \dfrac{\partial e_0}{\partial x^i}(x),\ i = 1, \ldots, N.$

(3)

$$\sum_{j=1}^N \frac{\partial^2 e_0}{\partial x^i \partial x^j}(0) DF^j(0, h_0) = (I_H - B_0)(I_H - \pi_0^\perp B_0)^{-1} \pi_0 \frac{\partial}{\partial x_i}\hat{h}(0),\ i = 1, \ldots, N.$$

Proof. Acting $\partial/\partial x^i$ to Proposition 5.1(3), we have

$$DF^i(0, \hat{h}(x)) \frac{\partial}{\partial x^j}\hat{h}(x) = \delta_{ij}.$$

This implies the assertion (1)

Acting $\partial/\partial x^i$ to Proposition 5.1(1), we have

$$\frac{\partial e_0}{\partial x^i}(x) = (\hat{h}(x) - Df(0, \hat{h}(x)), \frac{\partial}{\partial x^i}\hat{h}(x))_H.$$

Then we have the assertion (2) by Proposition 5.1(2) and the assertion (1).

Acting $\partial/\partial x^i$ to Proposition 5.1(2), we have by the assertion (2)

$$(I_H - D^2 f(0,\hat{h}(x))\frac{\partial}{\partial x^i}(x)\hat{h}(x)$$

$$= \sum_{j=1}^N \hat{\lambda}^j(x) D^2 F^j(0,\hat{h}(x))\frac{\partial}{\partial x^i}\hat{h}(x) + \sum_{j=1}^N \frac{\partial^2 e_0}{\partial x^i \partial x^j}(x) DF^j(0,\hat{h}(x)).$$

This implies that

$$(I_H - B_0)\frac{\partial}{\partial x^i}\hat{h}(0) = \sum_{j=1}^N \frac{\partial^2 e_0}{\partial x^i \partial x^j}(0) DF^j(0,\hat{h}(0)). \tag{5.1}$$

Acting $\pi_0^\perp$, we have

$$\pi_0^\perp (I_H - B_0)\frac{\partial}{\partial x^i}\hat{h}(0) = 0,$$

which implies that

$$(I_H - \pi_0^\perp B_0)\frac{\partial}{\partial x^i}\hat{h}(0) = \pi_0 \frac{\partial}{\partial x^i}\hat{h}(0).$$

Therefore

$$\frac{\partial}{\partial x^i}\hat{h}(0) = (I_H - \pi_0^\perp B_0)^{-1}\pi_0 \frac{\partial}{\partial x^i}\hat{h}(0).$$

Combining this with Equation (5.1), we have the assertion (3). ∎

The following is easy to check.

Proposition 5.4. *Let A be a bounded operator on $\mathbb{R}^N$. Assume that $\{e_i\}_{i=1}^N$ and $\{f_i\}_{i=1}^N$ are basis on $\mathbb{R}^N$ satisfying*

$$(e_i, f_j) = \delta_{ij}, \qquad i,j = 1,\dots,N.$$

Then

$$\det A = \det(((Ae_i, f_j)_{i,j=1,\dots,N}).$$

Proposition 5.5.

$$\det(T_0T_0^*)\mathrm{det}_2(I_H - \pi_0^\perp B_0 \pi_0^\perp) = (\det \nabla^2 e_0(0))^{-1}\mathrm{det}_2(I_H - B_0)\exp(-trace_H(\pi_0 B_0)).$$

Proof. Note that

$$I_H - \pi_0 + \pi_0(I_H - B_0)(I_H - \pi_0^\perp B_0)^{-1}$$

$$= I_H - \pi_0 B_0 (I_H - \pi_0^\perp B_0)^{-1} = (I_H - B_0)(I_H - \pi_0^\perp B_0)^{-1}.$$

Let $S = \pi_0 B_0 (I_H - \pi_0^\perp B_0)^{-1}$. By Propositions 5.4 and 5.3, we have

$$\det(I_H - S) = \det(I_H - \pi_0 + \pi_0(I_H - B_0)(I_H - \pi_0 B_0)^{-1}\pi_0)$$

$$= \det(((I - B_0)(I - \pi_0^\perp B_0)^{-1}\pi_0)\frac{\partial}{\partial x_i}\hat{h}(0), DF^j(0,h_0))_H)_{i,j=1,\dots,N})$$
$$= \det(\nabla^2 e_0(0))\det(T_0 T_0^*).$$

On the other hand, we have

$$\det{}_2(I_H - B_0) = \det{}_2((I_H - S)(I_H - \pi_0^\perp B_0))$$
$$= \det{}_2(I_H - S)\det{}_2(I_H - \pi_0^\perp B_0)\exp(-trace_H(S(\pi_0^\perp B_0)))$$
$$= \det(I_H - S)\det{}_2(I_H - \pi_0^\perp B_0)\exp(tr(S(I_H - \pi_0^\perp B_0)).$$
$$= \det(\nabla^2 e_0(0))\det(T_0 T_0^*)\det{}_2(I_H - \pi_0^\perp B_0 \pi_0^\perp)\exp(trace_H(\pi_0 B_0)).$$

Thus we have our assertion. ∎

Now Theorem 3.2 is a direct consequence of Propositions 4.5 and 5.5.

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ADDRESS:
Graduate School of Mathematical Sciences, The University of Tokyo
3–8–1 Komaba Meguro-ku, Tokyo 153-8914, JAPAN
TEL +81-3-5465-7001 FAX +81-3-5465-7012