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Uniqueness and stability in determining the heat radiative coefficient, the initial temperature and a boundary coefficient in a parabolic equation

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Abstract

We consider an inverse parabolic problem. We prove that the heat radiative coefficient, the initial temperature and a boundary coefficient can be simultaneously determined from the final overdetermination, provided that the heat radiative coefficient is a priori known in a small subdomain. Moreover we establish a stability estimate for this inverse problem.

Key words: inverse parabolic problem, heat radiative coefficient, initial temperature, boundary coefficient.

AMS subject classifications: 35R30.

1 Introduction

Let us consider a mixed boundary value problem for a parabolic equation

$$\begin{cases} \Delta u(x,t) + p(x)u(x,t) - \partial_t u(x,t) = f(t,x) & x \in \Omega,\ t > 0 \\ u(x,0) = a(x) & x \in \Omega \\ \partial_\nu u(x,t) + q(x)u(x,t) = 0 & x \in \Gamma,\ t > 0, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain and Γ is its boundary. Here and henceforth ∂_ν will denote the derivative with respect to the outward normal to Γ.

We are interested in the following inverse problem : let $T > 0$ be given. Determine $p(x)$, $a(x)$, $x \in \Omega$ and $q(x)$, $x \in \Gamma$ from observation of final data $u(x,T)$, $x \in \Omega$.

This is an inverse problem with final overdetermination and we can refer to M. Choulli [Ch1], [Ch2], M. Choulli and M. Yamamoto [CY1], V. Isakov [Is], A. B. Kostin and A. I. Prilepko [KP], A. I. Prilepko and V. V. Soloviev [PS], W. Yu [Yu]. However, in these works the authors consider determination of coefficients or right-hand sides in partial differential equations, not of boundary conditions. In this paper, we prove that the final overdetermination can simultaneously determine both the radiative coefficient p, the initial temperature a and the boundary coefficient q under the extra assumption that p is known in some subdomain U[1]. We also prove a stability estimate for our inverse problem.

2 Uniqueness

Henceforth we assume that Ω is of class $C^{2,\alpha}$, for some α, $0<\alpha<1$. Let $p_0 \in C^\alpha(\overline{\Omega})$ and a subdomain $U \subset \Omega$ be fixed, we set

$$\mathcal{P} = \{C^\alpha(\overline{\Omega});\ p = p_0 \text{ in } U\}.$$

We regard $\mathcal{P}$ as the admissible set of unknown coefficients. The definition of $\mathcal{P}$ means that coefficients are assumed to be known in the subdomain U.

We make the following assumptions on the data a and f :
(A) $a \in C^{2,\alpha}(\overline{\Omega})$, $a \geq 0$, $a \not\equiv 0$ and its support is compact in Ω.
(B) $f \in C^{\alpha,\alpha/2}(\overline{\Omega}\times[0,L])$, for some $L > T$, $f \leq 0$, and $t \in (0,L) \to f(\cdot,t) \in L^2(\Omega)$ has an analytic extension in a sector S_θ, $\theta \in (0,\pi/2]$, of the form

$$S_\theta = \{z \in \mathbb{C};\ 0 < |z| < L,\ |\text{arg} z| < \theta\}.$$

We note (see Proposition B.1 in appendix B) that the assumption (B) is equivalent to the following one
(B') $f \in C^{\alpha,\alpha/2}(\overline{\Omega}\times[0,L]) \cap C^\infty((0,L);L^2(\Omega))$ and for each $T \in (0,L)$ there exist two non negative constants $C = C(T)$ and $r = r(T)$ such that

$$\|f^{(k)}(t)\|_{L^2(\Omega)} \leq \frac{Cr^k k!}{t^n}\ t \in (0,T),\ k \geq 0.$$

If $p \in \mathcal{P}$, $q \in \mathcal{Q} = C^{1+\alpha}(\Gamma)$, a and f satisfy (A) and (B), then the initial-boundary value problem (1.1) has a unique solution $u(a,p,q) \in C^{2+\alpha,1+\alpha/2}(D)$, where $D = \overline{\Omega}\times[0,L]$ (e.g. [LSU]). Moreover, modifying slightly the proof of Proposition 3.2 in [Ch3], we find

$$u(a,p,q)(x,t) > 0 \quad (x,t) \in \overline{\Omega}\times(0,L]. \tag{2.1}$$

On the other hand, we know (e.g. [Ou], [Paz] or appendix A) that the operator $A = \Delta + p$, with domain

$$D(A) = \{u \in L^2(\Omega);\ \Delta u \in L^2(\Omega),\ \partial_\nu u + q(x)u = 0 \text{ on } \Gamma\},$$

$q \in \mathcal{Q}$, generates an analytic semigroup e^{tA} on $L^2(\Omega)$. Since $u(a,p,q)$ is given by

$$u(a,p,q)(t) = e^{tA}a - \int_0^t e^{(t-s)A}f(\cdot,s)ds,\ t \in (0,L),$$

we proceed as in the proof of Lemma 1 in [CY2] for deducing

$$t \in (0,L) \to u(a,p,q)(\cdot,t) \in L^2(\Omega)$$

is analytic.

Theorem 2.1 *Let $p_i \in \mathcal{P}$, $q_i \in \mathcal{Q}$ and let a_i, f satisfy (A) and (B), $i = 1,2$. We assume*

$$u_1(x,T) = u_2(x,T),\ x \in \Omega, \tag{2.2}$$

where $u_i = u_i(a_i,p_i,q_i)$, $i = 1,2$. Then

$$a_1 = a_2,\ p_1 = p_2 \text{ and } q_1 = q_2.$$

[1]Note that in general the final overdetermination does not determine the radiative coefficient, see the counterexample in [Is] for the linearized problem.

Proof. Let $u = u_1 - u_2$, $a(x) = a_1(x) - a_2(x)$, $p = p_2 - p_1$ and $q = q_2 - q_1$. Then a straightforward computation shows that u is the solution of the initial-boundary value problem

$$\left\{\begin{array}{ll} \Delta u(x,t) + p_1(x)u(x,t) - \partial_t u(x,t) = p(x)u_2(x,t) & x \in \Omega,\ 0 < t < L \\ u(x,0) = a(x) & x \in \Omega \\ \partial_\nu u(x,t) + q_1(x)u(x,t) = q(x)u_2(x,t) & x \in \Gamma,\ 0 < t < L. \end{array}\right. \tag{2.3}$$

Since $p = 0$ in U, we deduce from the first equation in (2.3)

$$\partial_t u(x,t) = \Delta u(x,t) + p_1(x)u(x,t),\ x \in U,\ 0 < t < L. \tag{2.4}$$

This and (2.2) imply $\partial_t u(x,T) = 0$, $x \in U$. Therefore (2.4) gives

$$\partial_t^2 u(x,T) = \Delta \partial_t u(x,T) + p_1(x)\partial_t u(x,T) = 0,\ x \in U.$$

Repeating this, we obtain

$$\partial_t^m u(x,T) = 0,\ x \in U,\ m \in \mathbb{N}. \tag{2.5}$$

As $t \in (0,L) \to u(\cdot,t)$ is analytic, (2.5) implies that $u \equiv 0$ in $U \times (0,L)$.

We fix $\epsilon > 0$ such that $\epsilon < T$ and $T + \epsilon < L$. We know by (2.1) that

$$u_2(x,t) \geq \delta > 0,\ (x,t) \in \overline{\Omega} \times [T-\epsilon, T+\epsilon],$$

for some constant δ. We can then introduce $v(x,t) = u(x,t)/u_2(x,t)$, $(x,t) \in \overline{\Omega} \times [T-\epsilon, T+\epsilon]$. We easily prove that v is the solution of the following initial-boundary value problem

$$\left\{\begin{array}{ll} \Delta v(x,t) + B(x,t)\cdot\nabla v(x,t) & \\ \qquad + c(x,t)v(x,t) - \partial_t u(x,t) = p(x) & x \in \Omega,\ T-\epsilon < t < T+\epsilon \\ v(x,T-\epsilon) = b(x) & x \in \Omega \\ \partial_\nu v(x,t) + q(x)v(x,t) = q(x) & x \in \Gamma,\ T-\epsilon < t < T+\epsilon, \end{array}\right.$$

where $B(x,t) = -2\nabla u_2(x,t)/u_2(x,t)$, $c(x,t) = f(x,t)/u_2(x,t) - p$ and $b(x) = u(x,T-\epsilon)/u_2(x,T-\epsilon)$.

Since $w = \partial_t v$ is the solution of the following initial-boundary value problem

$$\left\{\begin{array}{ll} \Delta w(x,t) + B(x,t)\cdot\nabla w(x,t) & \\ \qquad + c(x,t)w(x,t) - \partial_t w(x,t) = & \\ \qquad\qquad -\partial_t B(x,t)v(x,t) - \partial_t c(x,t)v(x,t) & x \in \Omega,\ T-\epsilon < t < T+\epsilon \\ w(x,T-\epsilon) = d(x) & x \in \Omega \\ \partial_\nu w(x,t) + q(x)w(x,t) = 0 & x \in \Gamma,\ T-\epsilon < t < T+\epsilon, \end{array}\right.$$

where $d(x) = \Delta b(x) + B(x,T-\epsilon)\cdot\nabla v(x,T-\epsilon) + c(x,T-\epsilon)b(x) - p(x)$, we can proceed as in the proof of Theorem 3.3 in [IY]. We find the following estimate

$$\begin{aligned} \|p\|_{L^2(\Omega)} \leq C(\|v(\cdot,T)\|_{H^2(\Omega)} + \quad & \|v\|_{L^2(U\times(T-\epsilon,T+\epsilon))} \\ & + \|\partial_t v\|_{L^2(U\times(T-\epsilon,T+\epsilon))}). \end{aligned} \tag{2.6}$$

Here C is a positive constant.

We have seen below that $u \equiv 0$ in $(\Omega \times \{T\}) \cup (U \times (0,L))$. Therefore

$$v \equiv 0 \text{ in } (\Omega \times \{T\}) \cup (U \times (0,L)).$$

This and (2.6) imply $p \equiv 0$. Consequently, u satisfies

$$\left\{\begin{array}{ll} \Delta u(x,t) + p_1(x)u(x,t) - \partial_t u(x,t) = 0 & x \in \Omega,\ 0 < t < L \\ u(x,t) = 0 & x \in U,\ 0 < t < L. \end{array}\right.$$

Then $u \equiv 0$ in $\Omega \times (0, L)$ according to the classical unique continuation property for parabolic equations. In particular, we have

$$u = \partial_\nu u = 0 \text{ on } \Gamma \times (0, L),$$

and then $qu_2 \equiv 0$ in $\Gamma \times (0, L)$. Hence $q \equiv 0$ by $(2.1C_0^\infty)$.

Finally, since u is continuous, we get $u(x, 0) = a(x) = 0$. The proof is then complete. □

3 Stability

We use the same notations and assumptions as in the previous section. In addition we assume that there exist two real constants α and β such that

$$p_1 \geq \alpha, \quad q_1 \geq \beta.$$

For some ϵ to be specified later, we deduce from the analyticity of $t \in (0, L) \to u(\cdot, t) \in L^2(\Omega)$

$$u(x, t) = \sum_{m \in \mathbb{N}} \frac{\partial_t^m u(x, T)}{m!}(t - T)^m, \ t \in (T - \epsilon, T + \epsilon).$$

Therefore

$$u(x, t)^2 = \sum_{m \in \mathbb{N}} (\sum_{0 \leq k \leq m} \frac{\partial_t^{m-k} u(x, T)}{(m-k)!} \frac{\partial_t^k u(x, T)}{k!})(t - T)^m, \ t \in (T - \epsilon, T + \epsilon),$$

which implies

$$\int_U u(x, t)^2 dx \leq \sum_{m \in \mathbb{N}} (\sum_{0 \leq k \leq m} \frac{\|\partial_t^{m-k} u(\cdot, T)\|_{L^2(U)}}{(m-k)!} \frac{\|\partial_t^k u(\cdot, T)\|_{L^2(U)}}{k!})(t - T)^m,$$

$t \in (T - \epsilon, T + \epsilon)$. That is

$$\int_U u(x, t)^2 dx \leq (\sum_{m \in \mathbb{N}} \frac{\|\partial_t^m u(\cdot, T)\|_{L^2(U)}}{m!}(t - T)^m)^2 \leq (\sum_{m \in \mathbb{N}} \frac{\|\partial_t^m u(\cdot, T)\|_{L^2(U)}}{m!} \epsilon^m)^2, \tag{3.1}$$

$t \in (T - \epsilon, T + \epsilon)$. From Proposition A.2 in appendix A we know that there exist two positive constants M and ρ such that

$$\|\partial_t^m u(\cdot, T)\|_{L^2(U)} \leq M \rho^m m!, \ m \in \mathbb{N},$$

where $M = M(T, \theta, \alpha, \beta)$ and $\rho = \rho(T, \theta, \alpha, \beta)$ are two positive constants. Here θ is the same as in the assumption (B). Hence the series in (3.1) converges if ϵ is chosen such that $\rho\epsilon < 1$. In the sequel ϵ is assumed to satisfy this condition and it is fixed.

Now we easily derive from (2.4)

$$\partial_t^m u(x, T) = (\Delta + p_1)^m u(x, T), \ x \in U, \ m \in \mathbb{N}. \tag{3.2}$$

We introduce the following new norm for $u(\cdot, T)_{|U}$

$$N(u(\cdot, T)_{|U})(= N_U(u(\cdot, T)_{|U})) = \sum_{m \in \mathbb{N}} \frac{\|(\Delta + p_1)^m u(x, T)\|_{L^2(U)}}{m!} \epsilon^m.$$ [2]

[2] Indeed one can check that the linear space consisting in the functions $h \in L^2(U)$ such that $(\Delta + p_1)^m h \in L^2(U)$, $m \in \mathbb{N}$ and

$$N(h) = \sum_{m \geq 0} \frac{\|(\Delta + p_1)^m h\|_{L^2(U)}}{m!} \epsilon^m < \infty$$

is a Banach space for the norm N.

It follows from (3.1) and (3.2)

$$\|u\|_{L^2(U\times(T-\epsilon,T+\epsilon))} \leq \sqrt{2\epsilon}N(u(\cdot,T)_{|U}). \tag{3.3}$$

Similarly, we have

$$\|\partial_t u\|_{L^2(U\times(T-\epsilon,T+\epsilon))} \leq \sqrt{2\epsilon}N((\Delta+p_1)u(\cdot,T)_{|U}). \tag{3.4}$$

Then (2.6), (3.3) and (3.4) imply

$$\|p\|_{L^2(\Omega)} \leq C\Big[\|u(\cdot,T)\|_{H^2(\Omega)} + N(u(\cdot,T)_{|U}) + N((\Delta+p_1)u(\cdot,T)_{|U})\Big].$$

That is

$$\|p\|_{L^2(\Omega)} \leq C\tilde{N}(u(\cdot,T)_{|U}), \tag{3.5}$$

where $\tilde{N}$ is the following norm

$$\tilde{N}(h) = \|h\|_{H^2(\Omega)} + N(h|_U) + N((\Delta+p_1)h|_U).$$

Next, the estimate

$$u_2(x,T) \geq \gamma = \min_{y\in\overline{\Omega}} u_2(y,T)\ x\in\overline{\Omega}$$

and the identity

$$q(x) = (\partial_\nu(x,T) + q_1(x)u(x,T))/u_2(x,T)$$

lead

$$\|q\|_{C(\Gamma)} \leq E(\|\partial_\nu u(\cdot,T)\|_{C(\Gamma)} + \|u(\cdot,T)\|_{C(\Gamma)}) \leq E'\|u(\cdot,T)\|_{C^1(\overline{\Omega})}, \tag{3.6}$$

where the constants E and E' depend only on γ and M, $M \geq \|q_1\|_{C(\Gamma)}$.

We now consider $y = \partial_t u$. Clearly $y = y_1 + y_2$, where y_1 and y_2 are the respective solutions of initial-boundary value problems

$$\begin{cases} \Delta y_1(x,t) + p_1(x)y_1(x,t) - \partial_t y_1(x,t) = 0 & x\in\Omega,\ 0<t<L \\ y_1(x,0) = \Delta a(x) + p_1(x)a(x) - p(x)a_2(x) & x\in\Omega \\ \partial_\nu y_1(x,t) + q_1(x)y_1(x,t) = 0 & x\in\Gamma,\ 0<t<L, \end{cases}$$

and

$$\begin{cases} \Delta y_2(x,t) + p_1(x)y_2(x,t) - \partial_t y_2(x,t) = p(x)\partial_t u_2(x,t) & x\in\Omega,\ 0<t<L \\ y_2(x,0) = 0 & x\in\Omega \\ \partial_\nu y_2(x,t) + q_1(x)y_2(x,t) = q(x)\partial_t u_2(x,t) & x\in\Gamma,\ 0<t<L. \end{cases} \tag{3.7}$$

We apply the method of the logarithmic convexity (cf [Pay]) to y_1. We find

$$\|y_1(\cdot,t)\|_{L^2(\Omega)} \leq \|\Delta a + p_1 a - pa_2\|_{L^2(\Omega)}^{1-t/T}\|y_1(\cdot,T)\|_{L^2(\Omega)}^{t/T}.$$

If p_1 and a_2 satisfy the following a priori bound

$$\|p_1\|_{C(\overline{\Omega})} + \|a_2\|_{C(\overline{\Omega})} \leq M',$$

M' is some positive constant, then we can find a positive constant δ such that

$$\|\Delta a\|_{L^2(\Omega)} + \|a\|_{L^2(\Omega)} + \|p\|_{L^2(\Omega)} \leq \delta \tag{3.8}$$

implies

$$\|\Delta a + p_1 a - pa_2\|_{L^2(\Omega)} \leq 1.$$

Therefore

$$\|y_1(\cdot,t)\|_{L^2(\Omega)} \le \|y_1(\cdot,T)\|_{L^2(\Omega)}^{t/T} \tag{3.9}$$

when (3.8) is satisfied.

On the other hand, we notice that once again a minor modification of Proposition 3.2 in [Ch3] gives

$$\|y_2(\cdot,t)\|_{L^2(\Omega)} \le \sqrt{L|\Omega|}\|y_2\|_{C(\overline{\Omega}\times[0,L])} \le F(\|p\|_{C(\overline{\Omega})} + \|q\|_{C(\Gamma)}).$$

Here F is some positive constant depending only on Λ,

$$\Lambda \ge \|p_1\|_{C(\overline{\Omega})} + \|q_1\|_{C(\Gamma)}.$$

Given $r > n$, we assume

$$\|p\|_{W^{1,r}(\Omega)} \le G.$$

Then a well known interpolation inequality (e.g. [Ad]) gives

$$\|p\|_{C(\overline{\Omega})} \le G^{1-\mu}\|p\|_{L^2(\Omega)}^{\mu},$$

where $\mu = 2(r-n)/(rn+2(r-n))$. Consequently

$$\|y_2(\cdot,t)\|_{L^2(\Omega)} \le H(\|p\|_{L^2(\Omega)}^{\mu} + \|q\|_{C(\Gamma)}), \tag{3.10}$$

H is a positive constant.

From the identity $y_1 = y - y_2 = \partial_t u - y_2$ we deduce

$$\|y_1(\cdot,T)\|_{L^2(\Omega)} \le \|\Delta u(\cdot,T) + p_1 u(\cdot,T) + p_2 u_2(\cdot,T)\|_{L^2(\Omega)} + \|y_2(\cdot,T)\|_{L^2(\Omega)}$$

This, in combination with (3.10), implies

$$\|y_1(\cdot,T)\|_{L^2(\Omega)} \le H'(\|u(\cdot,T)\|_{H^2(\Omega)} + \|p\|_{L^2(\Omega)}^{\mu} + \|q\|_{C(\Gamma)}),$$

if $\|p\|_{L^2(\Omega)} \le \lambda$, where λ and H' are some positive constants, H' depending on λ. In view of (3.9), we derive from the last estimate

$$\|y_1(\cdot,t)\|_{L^2(\Omega)} \le H''(\|u(\cdot,T)\|_{H^2(\Omega)} + \|p\|_{L^2(\Omega)}^{\mu} + \|q\|_{C(\Gamma)})^{t/T}. \tag{3.11}$$

Here H'' is a positive constant.

Let $\overline{N}(h) = \|h\|_{H^2(\Omega)} + \tilde{N}(h)^{\mu} + \|h\|_{C^1(\overline{\Omega})}$. Then (3.5), (3.6) and (3.11) lead

$$\|y_1(\cdot,t)\|_{L^2(\Omega)} \le H''\overline{N}(u(\cdot,T)_{|U})^{t/T}. \tag{3.12}$$

Now since

$$a(x) = u(x,0) = u(x,T) + \int_T^0 y(x,t)dt,$$

(3.5), (3.6), (3.10) and (3.12), imply

$$\begin{aligned}\|a\|_{L^2(\Omega)} &\le K(\|u(\cdot,T)\|_{L^2(\Omega)} + \textstyle\int_0^T \overline{N}(u(\cdot,T)_{|U})^{t/T}dt + \|p\|_{L^2(\Omega)}^{\alpha} + \|q\|_{C(\Gamma)}) \\ &\le K(\overline{N}(u(\cdot,T)_{|U}) + \textstyle\int_0^T \overline{N}(u(\cdot,T)_{|U})^{t/T}dt,\end{aligned}$$

for some positive constant K. Therefore

$$\|a\|_{L^2(\Omega)} \le \omega(\overline{N}(u(\cdot,T)_{|U})),$$

where we set $\omega(\tau) = KT(\tau-1)/\ln\tau + \tau$, $\tau > 0$.

We then proved the following stability estimate.

Theorem 3.1 *Let $r > n$ and $M > 0$ be given. There exist two constants $\delta > 0$ and $C > 0$ with the property that for all, $i = 1, 2$, $p_i \in \mathcal{P}$, $q_i \in \mathcal{Q}$, a_i, f satisfy (A), (B) and*

$$\|q_i\|_{C(\Gamma)} + \|p_i\|_{C(\overline{\Omega})} + \|p_i\|_{W^{1,r}(\Omega)} + \|a_i\|_{C(\overline{\Omega})} \leq M,$$

if

$$\|a_1 - a_2\|_{L^2(\Omega)} + \|\Delta(a_1 - a_2)\|_{L^2(\Omega)} + \|p_1 - p_2\|_{C(\overline{\Omega})} \leq \delta$$

then

$$\|p_1 - p_2\|_{L^2(\Omega)} \leq C\tilde{N}((u_1 - u_2)(\cdot, T)_{|U})$$

and there exists a constant $\tilde{C}$, depending on a_2, p_2 and q_2 such that

$$\|q_1 - q_2\|_{C(\Gamma)} \leq \tilde{C}\|(u_1 - u_2)(\cdot, T)\|_{C^1(\overline{\Omega})}$$
$$\|a_1 - a_2\|_{L^2(\Omega)} \leq \omega(\overline{N}((u_1 - u_2)(\cdot, T)_{|U}),$$

where $\omega(\tau) = \tilde{C}T(\tau - 1)/\ln\tau + \tau$, $\theta > 0$ and $u_i = u_i(a_i, p_i, q_i)$, $i = 1, 2$.

4 An estimate for the norm N

Let us see why the norm N is not very convenient. For simplicity, we assume that $p_1 = 0$ and we set

$$E = \{h \in L^2(U);\ \Delta^m h \in L^2(U) \text{ for all } m \in \mathbb{N}\}.$$

E is a semi-normed vector space for the familly of semi-norms :

$$p_m(h) = \sum_{k=0}^{m} \|\Delta^k h\|_{L^2(U)}.$$

We define also the subspace E_0 of E as follows

$$E_0 = \{h \in E;\ \sum_{m\geq 0} \frac{\|\Delta^m u\|_{L^2(U)}\epsilon^m}{m!} < \infty\},$$

where ϵ is a given positive real number. E_0 is then a normed vector space for the norm N.

In a classical way one can prove that the topology defined by N is the same as that induced on E_0 by the family of semi-norms (p_m). Moreover, if U is of class C^∞, then $E = C^\infty(\overline{U})$ topologically according to the L^2-elliptic regularity (see for instance [GT]) and the topology defined by the family of semi-norms (p_m) is the same as that given by the family of semi-norms (q_m), where q_m is given by

$$q_m(h) = \max_{|\alpha|\leq m,\ x\in\overline{U}} |D^\alpha h(x)|.$$

In the present section we establish a logarithmic type estimate for the norm $N(h)$ in terms of $\|h\|_{L^2(U)}$.

We first consider the simple case $p_1 = 0$. We start with an interpolation inequality. Let $h \in C_0^\infty(U)$ such that $\|h\|_{L^2(U)} \leq 1$. By the Green theorem we have

$$\int_U h\Delta h dx = -\int_U |Dh|^2 dx.$$

Then the Cauchy-Schwarz inequality yields

$$\|D_i h\|_{L^2(U)} \leq \|Dh\|_{L^2(U)} \leq \|\Delta h\|_{L^2(U)}^{1/2} \|h\|_{L^2(U)}^{1/2}. \tag{4.1}$$

Here and in the sequel D is the gradient, $D_i = \partial/\partial x_i$, $D_{ii} = D_i^2$ and $D_{ij} = D_i D_j$.
Hence

$$\|D_{ii}h\|_{L^2(U)} \leq \|\Delta(D_i h)|_{L^2(U)}^{1/2}\|D_i h\|_{L^2(U)}^{1/2} = \|D_i(\Delta h)\|_{L^2(U)}^{1/2}\|D_i h\|_{L^2(U)}^{1/2}$$

We apply twice (4.1). We find

$$\|D_{ii}h\|_{L^2(U)} \leq \|\Delta^2 h\|_{L^2(U)}^{1/4}\|\Delta h\|_{L^2(U)}^{1/2}\|h\|_{L^2(U)}^{1/4}.$$

Therefore

$$\|\Delta h\|_{L^2(U)} \leq n\|\Delta^2 h\|_{L^2(U)}^{1/4}\|\Delta h\|_{L^2(U)}^{1/2}\|h\|_{L^2(U)}^{1/4},$$

and then

$$\|\Delta h\|_{L^2(U)} \leq n^2\|\Delta^2 h\|_{L^2(U)}^{1/2}\|h\|_{L^2(U)}^{1/2}.$$

As a consequence of the last estimate we have

$$\|\Delta^m h\|_{L^2(U)} \leq n^2\|\Delta^{m+1} h\|_{L^2(U)}^{1/2}\|\Delta^{m-1} h\|_{L^2(U)}^{1/2},\ m \geq 1,$$

Using an induction in m, we easily prove

$$\|\Delta^p h\|_{L^2(U)} \leq n^{2p(m-p)}\|\Delta^m h\|_{L^2(U)}^{p/m}\|h\|_{L^2(U)}^{1-p/m},\ 0 \leq p \leq m.$$

In this estimate if we take $m+1$ in place of m and $p = m$ then

$$\|\Delta^m h\|_{L^2(U)} \leq n^{2m}\|\Delta^{m+1} h\|_{L^2(U)}^{1-1/(m+1)}\|h\|_{L^2(U)}^{1/(m+1)},\ m \geq 0.$$

We assume that h satisfies the following estimate

$$\|\Delta^m h\|_{L^2(U)} \leq M\rho^m m!,\ m \geq 0, \tag{4.2}$$

for some positive constants M and ρ. Then

$$\frac{\|\Delta^m h\|_{L^2(U)}\epsilon^m}{m!} \leq n^{2m}\frac{\epsilon^m}{m!}\|h\|_{L^2(U)}^{1/(m+1)}\left[M\rho^{m+1}(m+1)!\right]^{1-1/(m+1)},\ m \geq 0.$$

Changing ρ by $\max(\rho, 1)$ if necessary, we may assume that $\rho \geq 1$. In this case we have

$$\begin{aligned}\frac{\|\Delta^m h\|_{L^2(U)}\epsilon^m}{m!} &\leq n^{2m}\frac{\epsilon^m}{m!}\|h\|_{L^2(U)}^{1/(m+1)}\left[M\rho^{m+1}(m+1)!\right] \\ &\leq \rho M(m+1)(\epsilon n^2\rho)^m\|h\|_{L^2(U)}^{1/(m+1)},\ m \geq 0.\end{aligned}$$

That is

$$\frac{\|\Delta^m h\|_{L^2(U)}\epsilon^m}{m!} \leq C(m+1)\zeta^m\|h\|_{L^2(U)}^{1/(m+1)},\ m \geq 0,$$

where we set $\zeta = \epsilon n^2\rho$ and $C = \rho M$.
Now we choose ϵ sufficiently small in such way that $\zeta < 1$. Let $x > 0$ be any real number and $N = [x]$. Then

$$\begin{aligned}\sum_{m\geq 0}\frac{\|\Delta^m h\|_{L^2(U)}\epsilon^m}{m!} &\leq \sum_{m=0}^{N} C(m+1)\zeta^m\|h\|_{L^2(U)}^{1/(m+1)} \\ &\qquad + \sum_{m\geq N+1} C(m+1)\zeta^m\|h\|_{L^2(U)}^{1/(m+1)} \\ &\leq \Big(\sum_{m=0}^{N} C(m+1)\zeta^m\Big)\|h\|_{L^2(U)}^{1/(N+1)}\end{aligned}$$

$$
\begin{aligned}
& +\zeta^{N+1}\sum_{m\geq N+1} C(m+1)\zeta^{m-(N+1)}\|h\|_{L^2(U)}^{1/(m+1)} \\
\leq\ & \Big(\sum_{m\geq 0} C(m+1)\zeta^m\Big)\|h\|_{L^2(U)}^{1/(N+1)} \\
& +\zeta^{N+1}\sum_{m\geq 0} C(m+N+2)\zeta^m .
\end{aligned}
$$

Let

$$\alpha=\sum_{m\geq 0} C(m+1)\zeta^m, \quad \beta=\sum_{m\geq 0} C\zeta^m .$$

Then the last estimate gives

$$
\begin{aligned}
\sum_{m\geq 0}\frac{\|\Delta^m h\|_{L^2(U)}\epsilon^m}{m!} &\leq \alpha\|h\|_{L^2(U)}^{1/(N+1)}+\big(\alpha+(N+1)\beta\big)\zeta^{N+1} \\
&\leq \big(\alpha+(N+1)\beta\big)\big(\|h\|_{L^2(U)}^{1/(N+1)}+\zeta^{N+1}\big) \\
&\leq \big(\alpha+(x+1)\beta\big)\big(\|h\|_{L^2(U)}^{1/(x+1)}+\zeta^{x}\big) \\
&\leq \big(\alpha+(x+1)\beta\big)\big(\|h\|_{L^2(U)}^{1/(x+1)}+\frac{1}{\zeta}\zeta^{x+1}\big),
\end{aligned}
$$

i.e.

$$\sum_{m\geq 0}\frac{\|\Delta^m h\|_{L^2(U)}\epsilon^m}{m!}\leq \big(1+\frac{1}{\zeta}\big)\big(\alpha+(x+1)\beta\big)\big(\|h\|_{L^2(U)}^{1/(x+1)}+\zeta^{x+1}\big), \tag{4.3}$$

for all $x>0$. If $\|h\|_{L^2(U)}\neq 0$ we can take x such that $\|h\|_{L^2(U)}^{1/(x+1)}=\zeta^{x+1}$ or equivalently

$$x+1=\Big(\frac{\ln\|h\|_{L^2(U)}}{\ln\zeta}\Big)^{1/2}.$$

This particular choice of x in (4.3) implies

$$N_U(h)=N(h)=\sum_{m\geq 0}\frac{\|\Delta^m h\|_{L^2(U)}\epsilon^m}{m!}\leq \kappa_0(\|h\|_{L^2(U)}).$$

Here

$$\kappa_0(\tau)=2\big(1+\frac{1}{\zeta}\big)\big(\alpha+\beta\Big(\frac{\ln\tau}{\ln\zeta}\Big)^{1/2}\big)\zeta^{\left(\frac{\ln\tau}{\ln\zeta}\right)^{1/2}}.$$

We see that one can find two positive constants c_0 and c_1 such that

$$\kappa_0(\tau)\leq\kappa(\tau)=c_0\zeta^{c_1(\ln\frac{1}{\tau})^{1/2}},$$

provided that τ is sufficiently small.

Therefore

$$N_U(h)=N(h)\leq\kappa(\|h\|_{L^2(U)}). \tag{4.4}$$

Note that one can prove that κ is non decreasing in a neighborhood of the origin.

Now let $h\in C^\infty(\overline{U})$ satisfying (4.2) and $\|h\|_{L^2(U)}\leq 1$, V an open subset of U with $\overline{V}\subset U$ and $\varphi\in C_0^\infty(U)$, $0\leq\varphi\leq 1$ and $\varphi=1$ in $\overline{V}$. Clearly we have $N_V(h)\leq N_U(\varphi h)$ and, since κ is non decreasing in a neighborhood of the origin, we have

$$\kappa(\|\varphi h\|_{L^2(U)})\leq\kappa(\|h\|_{L^2(U)}).$$

But

$$N_U(\varphi h)\leq\kappa(\|\varphi h\|_{L^2(U)})$$

by (4.4). Therefore

$$N_V(h) \le \kappa(\|h\|_{L^2(U)}).$$

We sum up this in the following proposition.

Proposition 4.1 *Let $p_1 = 0$ and assume that ϵ is sufficiently small. Then there exist positive constants $c_0 = c_0(\rho, M, U, n, \epsilon)$, $c_1 = c_1(\rho, M, U, n, \epsilon)$ and $\theta = \theta(\epsilon, \rho)$ with the property that if $h \in C^\infty(\overline{U})$, $\|h\|_{L^2(\Omega)} \le \delta$ and h satisfies (4.2) then*

$$N(h) \le \kappa(\|h\|_{L^2(\Omega)}),$$

where $\kappa(\tau) = c_0 \zeta^{c_1(\ln \frac{1}{\tau})^{1/2}}$

As a consequence of this proposition we have

Corollary 4.1 *Let β, K be two positive constants and let θ be as in the assumption (B). Assume that U is of class C^∞. Let ϵ be sufficiently small. Then there exist positive constants $\sigma = \sigma(\epsilon, \beta, T, K, U)$, $c_0 = c_0(\epsilon, \beta, T, K, U)$ and $c_1 = c_1(\epsilon, \beta, T, K, U)$ with the property that for all $i = 1, 2$, $q_i \in \mathcal{Q}$, a_i, f satisfy (A), (B)*

$$\|f\|_{C(\overline{S_\theta}; L^2(\Omega))},\ \|a_i\|_{L^\infty(\Omega)},\ \|q_i\|_{L^\infty(\Omega)} \le K$$
$$\|q_1 - q_2\|_{L^\infty(\Gamma)},\ \|p_2\|_{L^\infty(\Omega)},\ \|a_1 - a_2\|_{L^\infty(\Omega)} \le \sigma$$

then

$$N((u_1 - u_2)(\cdot, T)|_U) \le \kappa(\|(u_1 - u_2)(\cdot, T)|_U\|_{L^2(\Omega)}),$$

where $\kappa(\tau) = c_0 \zeta^{c_1(\ln \frac{1}{\tau})^{1/2}}$, $u_1 = u_1(a_1, 0, q_1)$ and $u_2 = u_2(a_2, p_2, q_2)$.

Proof. Let $u = u_1 - u_2$ and $h = u(\cdot, T)|_U$. Note that as we have seen in the beginning of this section that $h \in C^\infty(\overline{U})$. On the other hand, according to Proposition A.2 in the appendix, we have

$$\|\Delta^m h\|_{L^2(U)} \le M \rho^m m!,$$

for some positive constants $M = M(T, \theta, \beta, K)$ and $\rho = \rho(T, \theta, \beta, K)$.

From Proposition 3.2 in [Ch3], there exists a positive constant $c = c(T, \Omega, \beta)$ such that

$$\|u_2\|_{L^\infty(\Omega \times (0,T))} \le c(\|f\|_{L^\infty(\Omega \times (0,T))} + \|a_2\|_{L^\infty(\Omega)}) \le 2cK,$$

and, since u solves the initial boundary value problem (2.3), the same proposition implies

$$\|u\|_{L^\infty(\Omega \times (0,T))} \le c'(\|q_1 - q_2\|_{L^\infty(\Gamma)} + \|p_2\|_{L^\infty(\Omega)} + \|a_1 - a_2\|_{L^\infty(\Omega)}),$$

where $c' = c(T, \Omega, \beta, K)$ is a positive constant. Therefore $\|h\| \le \delta$, δ as in the previous proposition, provided that

$$\|q_1 - q_2\|_{L^\infty(\Gamma)} + \|p_2\|_{L^\infty(\Omega)} + \|a_1 - a_2\|_{L^\infty(\Omega)} \le \sigma,$$

for some $\sigma = \sigma(\epsilon, \beta, T, K, U)$ sufficiently small. The conclusion follows by applying Proposition 4.1. □

We turn now our attention to the general case. We need the following lemma. We set $L = \Delta + p_1$ and we assume that $p_1 \in W^{2,\infty}(\Omega)$,

$$\|p_1\|_{W^{2,\infty}(\Omega)} \le \Phi,$$

for some positive constant Φ.

Lemma 4.1 *There exists a constant C depending only on Ω and Φ such that*

$$\|h\|_{H^4(U)} \leq C(\|h\|_{L^2(U)} + \|Lh\|_{L^2(U)} + \|L^2h\|_{L^2(U)}),$$

for all $h \in C_0^\infty(U)$.

Proof. Let $h \in C_0^\infty(U)$. From the classical H^2 estimate for the Laplace operator we have

$$\|h\|_{H^2(U)} \leq C_0\|\Delta h\|_{L^2(U)},$$

for some constant C_0 depending only on Ω. Therefore

$$\|h\|_{H^2(U)} \leq C_0(\|Lh\|_{L^2(U)} + \|p_1\|_{L^\infty(U)}\|h\|_{L^2(U)}).$$

That is

$$\|h\|_{H^2(U)} \leq C_1(\|Lh\|_{L^2(U)} + \|h\|_{L^2(U)}), \tag{4.5}$$

where $C_1 = C_0\max(1, \Phi)$.

On the other hand, we have

$$\begin{aligned} D_iLh &= LD_ih + [D_i, L]h, \\ D_{ij}Lh &= LD_{ij}h + [D_{ij}, L]h. \end{aligned}$$

Here $[\cdot,\cdot]$ is the usual commutator.

We can easily check that there exists a positive constant C_2, depending only on Φ, such that

$$\|[D_i, L]h\|_{L^2(U)} + \|[D_{ij}, L]h\|_{L^2(U)} \leq C_2\|h\|_{H^1(U)}$$

Therefore

$$\|LD_ih\|_{L^2(U)} + \|LD_{ij}h\|_{L^2(U)} \leq \|D_iLh\|_{L^2(U)} + \|D_{ij}Lh\|_{L^2(U)} + C_2\|h\|_{H^1(U)}.$$

This, (4.5) and the following inequality

$$\|h\|_{H^4(U)} \leq \|h\|_{H^2(U)} + \sum_i \|D_ih\|_{H^2(U)} + \sum_{i,j}\|D_{ij}h\|_{H^2(U)}.$$

lead to the desired estimate. $\square$

Lemma 4.2 *Let $h \in C_0^\infty(U)$ satisfying*

$$\|L^mh\|_{L^2(U)} \leq a_m = M\rho^m m!,\ m \in \mathbb{N}, \tag{4.6}$$

for some positive constants M and $\rho \geq 1$. Then

$$\|L^mu\|_{L^2(U)} \leq 3^{\frac{1}{2}+\cdots\frac{1}{2^m}} C^{1+\frac{1}{2}+\cdots+\frac{1}{2^{m-1}}} a_{m+1}^{\frac{1}{2}+\cdots+\frac{1}{2^m}} \|h\|_{L^2(U)}^{\frac{1}{2^m}},\ m \in \mathbb{N}.$$

In particular,

$$\|L^mu\|_{L^2(U)} \leq 3C^2a_{m+1}\|h\|_{L^2(U)}^{\frac{1}{2^m}}.$$

Proof. We note that $a_{m-1} \leq a_m$ for $m \in \mathbb{N}$. For $m = 1$, by the usual interpolation inequality

$$\|w\|_{H^2(U)} \leq C_0\|w\|_{H^4(U)}^{\frac{1}{2}}\|w\|_{L^2(U)}^{\frac{1}{2}},\ w \in H^4(U),$$

and Lemma 4.1, we have

$$\begin{aligned}
\|Lh\|_{L^2(U)} &\leq C_1\|h\|_{H^2(U)} \\
&\leq C_2\|h\|^{\frac{1}{2}}_{H^4(U)}\|h\|^{\frac{1}{2}}_{L^2(U)} \\
&\leq C(\|h\|_{L^2(U)} + \|Lh\|_{L^2(U)} + \|L^2h\|_{L^2(U)})^{\frac{1}{2}}\|h\|^{\frac{1}{2}}_{L^2(U)}.
\end{aligned}$$

Therefore

$$\|Lu\|_{L^2(U)} \leq C(3a_2)^{\frac{1}{2}}\|h\|^{\frac{1}{2}}_{L^2(U)}. \tag{4.7}$$

Thus the case $m = 1$ is proved. Let the case of $m = k$ be proved. Then (4.7) yields

$$\begin{aligned}
\|L^{k+1}h\|_{L^2(U)} &= \|L(L^kh)\|_{L^2(U)} \\
&\leq C(\|L^kh\|_{L^2(U)} + \|L^{k+1}h\|_{L^2(U)} + \|L^{k+2}h\|_{L^2(U)})^{1/2}\|L^kh\|^{\frac{1}{2}}_{L^2(U)} \\
&\leq C3^{1/2}a^{1/2}_{k+2}(3^{1/2+\cdots 1/2^k}C^{1+1/2+\cdots+1/2^{k-1}})^{1/2}a^{1/2\left(1/2+\cdots+1/2^k\right)}_{k+1}\|h\|^{1/2^{k+1}}_{L^2(U)} \\
&\leq 3^{1/2+\cdots 1/2^{k+1}}C^{1+1/2+\cdots+1/2^k}a^{1/2+\cdots+1/2^{k+1}}_{k+2}\|h\|^{1/2^{k+1}}_{L^2(U)}.
\end{aligned}$$

Thus the proof for $m = k+1$ is finished and the proof of the lemma is complete.
□

Proposition 4.2 *Assume that ϵ is sufficienty small. For any $\mu \in (0,1)$, there exists a positive constant $C = C(\mu,\rho,M,U,n,\epsilon)$ such that if $h \in C^\infty(\overline{U})$ satisfies* (4.6) *and $\|h\|_{L^2(U)} \leq 1$ then*

$$N(h) \leq \chi(\|h\|_{L^2(U)}).$$

where

$$\chi(\tau) = \frac{C}{(\ln\frac{1}{\tau})^\mu}.$$

Proof. In this proof C_i is a positive constant depending on data. Arguing as before, we see that the proof can be reduced to functions from $C^\infty_0(U)$ satisfying (4.6). Let then $h \in C^\infty_0(U)$ satisfies (4.6). For simplicity, we set $d = \|h\|_{L^2(U)}$. From Lemma 4.2 we derive

$$N(h) \leq 3C^2M\rho\sum_{m=1}^\infty(m+1)(\rho\epsilon)^m d^{1/2^m}.$$

We further choose $\epsilon > 0$ such that $\gamma^2 \equiv \rho\epsilon < (1/2)^2$. Since $\sup_{m\in\mathbb{N}}(m+1)\gamma^m < \infty$, we have

$$\begin{aligned}
N(h) &\leq C_1\sum_{m=1}^\infty\gamma^m d^{1/2^m} = C_1\left(\sum_{m=1}^N + \sum_{m=N+1}^\infty\right)\gamma^m d^{\frac{1}{2^m}} \\
&\leq C_2\left(d^{1/2^N} + \gamma^N\right), \quad N \in \mathbb{N}.
\end{aligned}$$

We choose $N = [x]$, where

$$x = \ln_2\left(\left(\ln_2\frac{1}{d}\right)^\mu\right)$$

(recall that $\ln_2 t = \ln t/\ln 2$). Therefore

$$N(h) \leq C_3\left(d^{1/2^x} + \gamma^x\right).$$

We have $2^x = \left(\ln_2 \frac{1}{d}\right)^{\mu}$. Setting $y = \ln_2 \frac{1}{d}$, we have

$$d^{\frac{1}{2^x}} = (2^{-y})^{y^{-\mu}} = 2^{-y^{1-\mu}} \le \frac{C_4}{y}$$

as y goes to ∞ by $1-\mu > 0$. Moreover

$$\gamma^x = \gamma^{\ln_2\left(\left(\ln_2 \frac{1}{d}\right)^{\mu}\right)} \le \frac{C_5}{2^{\ln_2\left(\left(\ln_2 \frac{1}{d}\right)^{\mu}\right)}}$$

because for fixed $\gamma \in \left(0, \frac{1}{2}\right)$, there exists $C_6 > 0$ such that

$$\gamma^y \le \frac{C_6}{2^y}$$

for any $y > 0$. Therefore

$$\gamma^x \le \frac{C_6}{\left(\ln_2 \frac{1}{d}\right)^{\mu}}.$$

Hence

$$N(h) \le \frac{C_7}{\left(\ln_2 \frac{1}{d}\right)^{\mu}} + \frac{C_7}{\ln_2 \frac{1}{d}} \le \frac{2C_7}{\left(\ln_2 \frac{1}{d}\right)^{\mu}}$$

by $0 < \mu < 1$. Thus the proof is completed. □

Similarly as before, we deduce from the last proposition the following corollary.

Corollary 4.2 *Let β, K be two positive constants, $\mu \in (0,1)$ and let θ be as in the assumption (B). Assume that U is of class C^∞. Let ϵ be sufficiently small. Then there exist positive constants $\sigma = \sigma(\epsilon, \beta, T, K, U)$ and $C = C(\epsilon, \beta, T, K, U, \mu)$ with the property that for all $i = 1, 2$, $p_i \in \mathcal{P}$, $q_i \in \mathcal{Q}$, a_i, f satisfy (A), (B), $p_1 \in C^\infty(\overline{U})$ and*

$$\|f\|_{C(\overline{S_\theta};L^2(\Omega))},\ \|a_i\|_{L^\infty(\Omega)},\ \|q_i\|_{L^\infty(\Omega)} \le K$$
$$\|q_1 - q_2\|_{L^\infty(\Gamma)},\ \|p_1 - p_2\|_{L^\infty(\Omega)},\ \|a_1 - a_2\|_{L^\infty(\Omega)} \le \sigma$$

then

$$N((u_1 - u_2)(\cdot, T)|_U) \le \chi(\|(u_1 - u_2)(\cdot, T)|_U\|_{L^2(\Omega)}),$$

where

$$\chi(\tau) = \frac{C}{\left(\ln \frac{1}{\tau}\right)^{\mu}}.$$

and $u_i = u_i(a_i, p_i, q_i)$, $i = 1, 2$.

Remark 4.1 *We note that the estimate in Corollary 4.1 (the case $p_1 = 0$) is better than that given in Corollary 4.2. In fact, one can see that $\kappa(\tau) \le \chi(\tau)$ if τ is small enough.*

Appendix

A The semigroup generated by the Laplacian with Robin BC

If $p \in L^\infty(\Omega)$ and $q \in L^\infty(\Gamma)$, let us consider the following (bounded) bilinear form

$$a_{p,q}(u,v) = \int_\Omega Du \cdot Dv dx + \int_\Omega puv dx + \int_\Gamma quv,\ u, v \in H^1(\Omega).$$

We note that $a_{p,q}$ is the bilinear form associated to the operator $-\Delta+p$ with Robin boundary condition $\partial_\nu u + qu = 0$. We denote this operator by $A_{p,q}$ and we recall that the spectrum of $A_{p,q}$ consists in a countable sequence of eigenvalues

$$-\infty < \lambda^1_{p,q} \leq \lambda^2_{p,q} \leq \dots \lambda^k_{p,q} \to +\infty.$$

We have the following comparaison principle

Proposition A.1 *Let $p_i \in L^\infty(\Omega)$ and $q_i \in L^\infty(\Gamma)$, $i = 1, 2$. Then*

$$p_1 \leq p_2 \text{ and } q_1 \leq q_2 \tag{A.1}$$

implies

$$\lambda^k_{p_1,q_1} \leq \lambda^k_{p_2,q_2},\ k \geq 1.$$

Proof. Under the assumption $(A.1)$ we have

$$a_{p_1,q_1}(u,u) \leq a_{p_2,q_2}(u,u), \text{ for each } u \in H^1(\Omega).$$

The conclusion follows by applying Proposition 30 in [DL] p. 126. □

As a consequence of this proposition we have the following corollary.

Corollary A.1 *Let α and β be two constants. Then there exists a real constant μ depending only on α and β such that for each $p \in L^\infty(\Omega)$ and $q \in L^\infty(\Gamma)$ satisfying $p \geq \alpha$ and $q \geq \beta$,*

$$\mu \leq \lambda^k_{p,q}, \text{ for all } k \geq 1.$$

Proof. In view of the last proposition, we can take $\mu = \lambda^1_{\alpha,\beta}$. □

When $p \in L^\infty(\Omega)$ and $q \in L^\infty(\Gamma)$, the operator $-A_{p,q}$ generates an analytic semigroup $(e^{-zA_{p,q}})$ in the half plane

$$\Pi = \{z \in \mathbb{C};\ \Re z > 0\}.$$

This semigroup is explicitly given by

$$e^{-zA_{p,q}}f = \sum_{k\geq 1} e^{-z\lambda_{p,q}}(\varphi^k_{p,q}, f)_{L^2(\Omega)}\varphi^k_{p,q},$$

where $(\varphi^k_{p,q})$ is an orthonormal basis of $L^2(\Omega)$ consisting in eigenfunctions, with $\varphi^k_{p,q}$ associated to $\lambda^k_{p,q}$, and $(\cdot,\cdot)_{L^2(\Omega)}$ is the usual scalar product on $L^2(\Omega)$.

Then an elementary calculations show

$$\|\frac{d^m}{dz^m}e^{-zA_{p,q}}\|_{B(L^2(\Omega))} \leq \begin{cases} \frac{m!}{(\Re z)^m}, \text{ if } \mu \geq 0 \\ e^{2|\mu|\Re z}\frac{m!}{(\Re z)^m}, \text{ if } \mu = -|\mu| < 0. \end{cases} \tag{A.2}$$

Here $\|\cdot\|_{B(L^2(\Omega))}$ is the operator norm.

Let $\mathcal{A}(S_\theta; L^2(\Omega))$, $\theta \in (0, \pi/2]$, denote the set of the analytic functions from S_θ into $L^2(\Omega)$, where

$$S_\theta = \{z \in \mathbb{C};\ 0 < |z| < L,\ |\arg z| < \theta\}.$$

Proposition A.2 *Let K be a given constant. Let $f \in \mathcal{A}(S_\theta; L^2(\Omega)) \cap C(\overline{S_\theta}; L^2(\Omega))$ and $a \in L^2(\Omega)$ such that*

$$\|f\|_{C(\overline{S_\theta}; L^2(\Omega))},\ \|a\|_{L^2(\Omega)} \leq K.$$

Then u given by

$$u(z) = e^{-zA_{p,q}}a + \int_0^z e^{-wA_{p,q}} f(z-w)dw,\ z \in S_\theta$$

is in $\mathcal{A}(S_\theta; L^2(\Omega)) \cap C(\overline{S_\theta}; L^2(\Omega))$. Moreover for any $0 < T < L$, there exist two constants $C = C(T, \theta, \mu, K)$ and $\rho = \rho(T, \theta, \mu, K)$ such that

$$\|\frac{d^m}{dz^m} u(T)\| \leq C\rho^m m!. \tag{A.3}$$

Proof. We write $u = u_0 + u_1$, where

$$u_0(z) = e^{-zA_{p,q}}a \text{ and } u_1(z) = \int_0^z e^{-wA_{p,q}} f(z-w)dw,\ z \in S_\theta$$

Clearly u_0 is in $\mathcal{A}(S_\theta; L^2(\Omega)) \cap C(\overline{S_\theta}; L^2(\Omega))$. For ϵ small enough we set

$$u_1^\epsilon(z) = \int_\epsilon^{z-\epsilon} e^{wA} f(z-w)dw.$$

We can see that $u_1^\epsilon \in \mathcal{A}(S_\theta; L^2(\Omega)) \cap C(\overline{S_\theta}; L^2(\Omega))$ and, using the properties of e^{zA} and the assumptions of f, we deduce that u_1^ϵ converges uniformly to u_1. Therefore $u_1 \in \mathcal{A}(S_\theta; L^2(\Omega)) \cap C(\overline{S_\theta}; L^2(\Omega))$.

The proof of $(A.3)$ follows again from the properties of $e^{-zA_{p,q}}$, the assumptions on a, f and a Cauchy formula :

$$\frac{d^m}{dz^m} u(T) = \frac{m!}{2i\pi} \int_\gamma \frac{u(z)}{(z-T)^{m+1}} dz,$$

where γ is a circle around T contained in S_θ and oriented in the counter-clockwise direction. □

B Characterization of analyticity in a sector

Let X be a Banach space. Then we have the following proposition

Proposition B.1 *The following are equivalent :*

(i) $f \in C^\infty((0, L); X) \cap C([0, L]; X)$ and for each $T \in (0, L)$ there exist two non negative constants $C = C(T)$ and $r = r(T)$ such that

$$\|f^{(k)}(t)\| \leq \frac{Cr^k k!}{t^n}\ t \in (0, T),\ k \geq 0. \tag{B.1}$$

(ii) For each $T \in (0, L)$ there exists a sector

$$S(\theta, T) = \{z \in \mathbb{C};\ |z| \in (0, T) \text{ and } |arg(z)| < \theta\}, \tag{B.2}$$

for some $\theta \in (0, \pi/2]$, such that f has an extension, still denoted by f, in $\mathcal{A}(S(\theta, T); X) \cap C(\overline{S(\theta, T)}; X)$.

Proof. Let us first introduce the set

$$\Omega(r,T)=\cup_{t\in(0,T)}D(t,t/r).$$

Let $S(\theta,T)$ be a sector of the form $(B.2)$ and $\theta=\arctan(1/r)$. If $z\in S(\theta,T)$, $t=\Re z$, $s=\Im z$ and $\varphi=\arg(z)$ then

$$\frac{|s|}{t}=\frac{|\sin\varphi|}{\cos\varphi}=|\tan\varphi|<\tan\theta=1/r.$$

Therefore $|z-t|=|s|<t/r$. That is $z\in B(t,t/r)\subset\Omega(r,T)$. In the other words we proved that $S(\theta,T)\subset\Omega(r,T)$.

We assume that (i) is satisfied. Let $t\in(0,L)$. Then by $(B.1)$ we deduce that the series $\sum(z-t)^m\frac{f^{(m)}(t)}{m!}$ converges in $D(t,t/r)$. Consequently f has an analytic extension in $\Omega(r,T)\supset S(\theta,T)$, still denoted by f. Reducing θ if necessary we can choose $f\in\mathcal{A}(S(\theta,T);X)\cap C(\overline{S(\theta,T)};X)$.

Conversely, we suppose that (ii) is satisfied. Let $T\in(0,L)$ and $T'\in(T,L)$. We can easily see that if t is small enough then the distance from t to $\partial S(\theta,T')$ is attained at a point of the part of the boundary given by $\{z=re^{i\theta};\ 0\le r\le T'\}$. That is we can find $t_0\in(0,T)$ such that $\mathrm{dist}(t,\partial S(\theta,T'))=t/\tan\theta,\ 0\le t\le t_0$. For $0<t\le t_0$, we have by applying a Cauchy formula

$$f^{(m)}(t)=\frac{m!}{2i\pi}\int_\gamma\frac{f(z)}{(z-t)^{m+1}}dz,$$

where γ is any circle of center t and radius r, oriented in the counter-clockwise direction, with $0<r<t/\tan\theta$. Therefore

$$\|f^{(m)}(t)\|\le Mm!r^{-m},\ 0<t\le t_0, \tag{B.3}$$

where $M=\max_{z\in\overline{S(\theta,T)}}\|f(z)\|$. Taking in $(B.3)$ the limit when r goes to $t/\tan\theta$, we find

$$\|f^{(m)}(t)\|\le M(\tan\theta)^m m!t^{-m},\ 0<t\le t_0. \tag{B.4}$$

On the other, based again on a Cauchy formula, we can derive that for each $t_0\in[t_0,T]$, there exists a small segment $J(t)\subset\mathbb{R}$ around t and two positive constants $C(t)$ and $\rho(t)$ such that

$$\|f^{(m)}(s)\|\le C(t)\rho(t)^m m!,\ s\in J(t) \tag{B.5}$$

Since $[t_0,T]$ est compact, there exist $t_1,\dots t_p$ in $[t_0,T]$ such that $[t_0,T]\subset\cup_{i=1}^pJ(t_i)$. Let $C=\max(C(t_1),\dots C(t_p))$ and $\rho=\max(\rho(t_1),\dots\rho(t_p))$. Then it follows from $(B.5)$

$$\|f^{(m)}(t)\|\le C\rho^m m!,\ t\in[t_0,T],$$

and then

$$\|f^{(m)}(t)\|\le C(T\rho)^m m!t^{-m}\ t\in[t_0,T]. \tag{B.6}$$

The desired estimate follows from a combination of $(B.4)$ and $(B.6)$. $\square$

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