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General asymptotics of Wiener functionals and application to mathematical finance

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General Asymptotics of Wiener Functionals and Application to Mathematical Finance

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Abstract

In the present paper, we give an asymptotic expansion of probability density for a component of general diffusion models. Our approach is based on infinite dimensional analysis on the Malliavin calculus and Kusuoka-Stroock's asymptotic expansion theory for general Wiener functionals [10]. The initial term of the expansion is given by the 'energy of path' and we calculate the energy by solving Hamilton equation. We apply our approach to the problems of mathematical finance. In particular, we obtain general asymptotic expansion formulae of implied volatilities for general diffusion models, e.g. CEV model, displaced diffusion and SABR model.

1 Introduction

There are many applications of asymptotic expansion theory to mathematical finance. Most popular application is the singular perturbation approach. For example, Hagan-Woodward [4] gave an asymptotic expansion formula of implied volatilities for local volatility models and Hagan-Kumar-Lesniewski-Woodward [5] gave a formula for a stochastic volatility model named SABR model. Their formula is well-known for practitioners. Berestycki-Busca-Florent [2] applied non-linear PDE analysis to this problem. Henry-Labordère [6] applied a heat kernel expansion method and gave an asymptotic expansion formula for mean-reverting SABR model.

In this paper, we take another approach based on Malliavin calculus. The theory of asymptotic expansions of probability densities based on Malliavin calculus was originated by Bismut [3] and was developed by Watanabe [14] and Kusuoka-Stroock [10], [11]. Many applications of this theory to finance were given by Yoshida [15], Takahashi-Kunitomo [8] and Jaeckel-Kawai [7]. In [12], we gave an asymptotic expansion of implied volatilities of call options for dynamic SABR model by using this theory.

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In this paper, we apply the methods of Kusuoka-Stoock [11] to mathematical finance. The key theorem is given in [9]. We consider the asymptotic expansion of implied volatilities of call options. Finally we give some explicit formulae for general diffusion models including SABR model.

Let us explain our results. Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\{W^1(t), \cdots, W^d(t)\ ;\ t \in [0,T]\}$ be a d-dimensional Brownian motion. Let $V_0, \cdots, V_d \in C_b^\infty([0,T] \times \mathbb{R}^N; \mathbb{R}^N)$. Here $C_b^\infty([0,T] \times \mathbb{R}^N; \mathbb{R}^N)$ denotes the space of $\mathbb{R}^N$-valued smooth functions defined in $[0,T] \times \mathbb{R}^N$ whose derivatives of any order are bounded.

Now let $X_\varepsilon(t), t \in [0,T], \varepsilon \in (0,1]$, be the solution to the stochastic differential equation

$$\begin{aligned} (1.1) \qquad & dX_\varepsilon^i(t) = \sum_{k=1}^d \varepsilon V_k^i(t, X_\varepsilon(t)) dW^k(t) + V_0^i(t, X_\varepsilon(t)) dt, \quad 1 \le i \le N, \\ & X_\varepsilon(0) = x_0 = (x_0^1, \ldots, x_0^N), \quad x_0 \in \mathbb{R}^N. \end{aligned}$$

We assume

$$\text{(A1)} \qquad V_0^1 \equiv 0,$$

and the ellipticity of $V_1, \cdots, V_d$, at x_0, i.e. there exists a constant $\delta > 0$ such that

$$\text{(A2)} \qquad \sum_{k=1}^d V_k(0, x_0) \otimes V_k(0, x_0) \ge \delta I,$$

where I denotes the identity matrix. Then there exists a unique solution to this equation. Moreover we assume that $X_\varepsilon(t)$ is continuous in t with probability one.

We investigate the distribution of $X_\varepsilon^1(T)$. From the ellipticity condition(A2), the law of $X_\varepsilon^1(T)$, denoted by ν_ε, is absolutely continuous and has a smooth density $p_\varepsilon(y)$. Let H be the Cameron-Martin space. We consider the associated ordinary differential equation

$$\begin{aligned} (1.2) \qquad & \frac{d}{dt} y^i(t; h) = \sum_{k=1}^d V_k^i(t, y(t; h)) \dot{h}^k(t) + V_0^i(t, y(t; h)), \quad t \in [0,T], \quad h \in H, \\ & y(0; h) = x_0, \quad x_0 \in \mathbb{R}^n. \end{aligned}$$

We define the energy function $e : \mathbb{R} \to \mathbb{R}$ by

$$(1.3) \qquad e(y) = \inf \Big\{ \frac{1}{2} \sum_{i=1}^d \int_0^T |\dot{h}^i(s)|^2 ds; h \in H,\ y^1(T; h) = y \Big\}.$$

Since $V_0^1 \equiv 0$, this energy function satisfies $e(x_0^1) = 0$. Let us define a flow $\phi : [0,T] \times \mathbb{R}^N \to \mathbb{R}^N$ by

$$\begin{aligned} (1.4) \qquad & \frac{d}{dt} \phi(t, x) = V_0(t, \phi(t, x)), \quad t \in [0,T], \quad x \in \mathbb{R}^N, \\ & \phi(0, x) = x. \end{aligned}$$

Then the map $\phi(t,\cdot) : \mathbb{R}^N \to \mathbb{R}^N,\ t \in [0,T]$ is a diffeomorphism denoted by ϕ_t. Note that ϕ_t^1 is an identity map. We define

$$(1.5)\qquad \tilde{V}_k^i(t,y) = \sum_{j=1}^N \frac{\partial \phi^i}{\partial x^j}(-t,\phi(t,y))V_k^j(t,\phi(t,y)), \quad 1 \le i \le N, \quad 1 \le k \le d,$$

which is the push-forward of the vector field V by the map ϕ_t. Let us define $(g^{ij})_{1\le i,j\le N}$: $[0,T] \times \mathbb{R}^N \to \mathbb{R}$ by

$$g^{ij}(t,x) = \sum_{k=1}^d \tilde{V}_k^i(t,x)\tilde{V}_k^j(t,x), \quad 1 \le i,j \le N.$$

From (A2), the matrix $(g^{ij})_{1\le i,j\le N}$ is non-negative definite corresponding to Riemaniann metric on $\mathbb{R}^N$. We define the generating operator $L_t,\ t \in [0,T]$ by

(1.6)

$$(L_t f)(x) = \frac{1}{2}\sum_{i,j=1}^N g^{ij}(t,x)\frac{\partial^2 f}{\partial x^i \partial x^j}(x) + \sum_{i=1}^N b^i(t,x)\frac{\partial f}{\partial x^i}(x),\ f \in C_b^\infty(\mathbb{R}^N), \quad x \in \mathbb{R}^N,\ t \in [0,T],$$

where $b \in C_b^\infty([0,T]\times\mathbb{R}^N;\mathbb{R}^N)$ is given by

$$(1.7)\qquad b^i(t,y) = \frac{1}{2}\sum_{k,l=1}^N \sum_{m=1}^d \frac{\partial^2\phi^i}{\partial x^k \partial x^l}(-t,\phi(t,y))V_m^k(t,\phi(t,y))V_m^l(t,\phi(t,y)),\ 1 \le i \le N.$$

Let us define linear operators $V : C_b^\infty([0,T]\times\mathbb{R}^N) \to C_b^\infty([0,T]\times\mathbb{R}^N)$ and $\Gamma : C_b^\infty([0,T]\times\mathbb{R}^N) \otimes C_b^\infty([0,T]\times\mathbb{R}^N) \to C_b^\infty(\mathbb{R}^N)$ by

$$(1.8)\qquad (Vf)(t,x) \equiv \sum_{i=1}^N g^{1i}(t,x)\int_t^T \frac{\partial f}{\partial x^i}(s,x)ds,$$

$$(1.9)\qquad \Gamma(f,g)(x) \equiv \sum_{i,j=1}^N \int_0^T g^{ij}(t,x)\Big(\int_t^T \frac{\partial f}{\partial x^i}(s,x)ds\Big)\Big(\int_t^T \frac{\partial g}{\partial x^j}(s,x)ds\Big)dt.$$

Our main result is the following.

THEOREM 1.1. *There is a constant $r_0 > 0$ satisfying the following.*
(1) *The energy function $e \in C^2([x_0^1 - r_0, x_0^1 + r_0])$ and there is a constant $C_0 > 0$ such that the asymptotic expansion of energy e satisfies*

$$(1.10)\qquad \Big|e(y) - \Big[\frac{1}{2b_1}(y-x_0^1)^2 - \frac{b_2}{3b_1^3}(y-x_0^1)^3 + \Big(-\frac{b_3}{4b_1^4} + \frac{b_2^2}{2b_1^5}\Big)(y-x_0^1)^4\Big]\Big| \le C_0|y-x_0^1|^5,$$
$$y \in [x_0^1 - r_0, x_0^1 + r_0],$$

where

$$(1.11)\qquad b_1 = \int_0^T g^{11}(t,x_0)dt,\ b_2 = \frac{3}{2}\int_0^T (Vg^{11})(t,x_0)dt,$$
$$b_3 = 2\int_0^T (V^2 g^{11})(t,x_0)dt + \frac{1}{2}\Gamma(g^{11},g^{11})(x_0).$$

(2) *There are constants $C_1,\ C_2 > 0$ such that the probability density $p_\varepsilon(y)$ satisfies following.*

$$\Big|(2\pi\varepsilon^2)^{\frac{1}{2}}\exp\Big(\frac{e(y)}{\varepsilon^2}\Big)p_\varepsilon(y) - a_0(y) - \varepsilon^2 a_2(y)\Big| \le \varepsilon^4 C_1, \quad y \in [x_0^1 - r_0, x_0^1 + r_0]. \tag{1.12}$$

Here, a_0 and a_2 are continuous functions such that

$$\Big|a_0(y) - \Big(\frac{\partial^2 e(y)}{\partial y^2}\Big)^{\frac{1}{2}}\exp\Big(\frac{L(y-x_0^1)^2}{2b_1^2}\Big)\Big| \le C_2|y-x_0^1|^3, \quad y \in [x_0^1 - r_0, x_0^1 + r_0], \tag{1.13}$$

and

$$a_2(x_0^1) = \frac{1}{\sqrt{b_1}}\Big(-\frac{L}{2b_1} - \frac{5}{6}\frac{b_2^2}{b_1^3} + \frac{3}{4}\frac{b_3}{b_1^2}\Big), \tag{1.14}$$

where

$$L = \int_{0<u<t<T} L_u(g^{11}(t,\cdot))(x_0)\,du\,dt. \tag{1.15}$$

Next, we apply our results to mathematical finance. We investigate the asymptotic expansion of the value of call options and their implied volatilities. We regard X_ε^1 as the underlying of options. Then the forward value of a call option of strike rate K and maturity T is given by

$$C_\varepsilon(T,K) = E[(X_\varepsilon^1(T) - K)^+], \quad \varepsilon \in (0,1], \quad K > 0.$$

We define smooth functions $\varphi_n \in C_b^\infty([0,\infty)),\ n \ge 0$, by

$$\varphi_n(x) = \int_0^\infty z^n \exp(-xz - \frac{z^2}{2})dz, \quad x \ge 0. \tag{1.16}$$

Some properties of φ_n are given in Appendix A. Since (1.10), we can define the following function $q \in C^2([x_0^1 - r_0, x_0^1 + r_0];\mathbb{R}_+)$ such that

$$e(x) = \frac{1}{2}\Big(\int_{x_0^1}^x \frac{dy}{q(y)}\Big)^2, \quad x \in [x_0^1 - r_0, x_0^1 + r_0]. \tag{1.17}$$

Then the asymptotic expansion of values of call options are given by the following.

Theorem 1.2. *There are constants $K_1 < K_0$ and C_1 such that the value of the call option with strike rate K, maturity T satisfies*

$$\big|\sqrt{2\pi}\exp(\frac{e(K)}{\varepsilon^2})C_\varepsilon(T,K) - \varepsilon a_0(K)q(K)^2\varphi_1(\frac{\sqrt{2e(K)}}{\varepsilon})R_2(\varepsilon,K)\big| \le C_1\varepsilon^4,$$

where

(1.18)

$$R_2(\varepsilon,K) = \varepsilon q(K)\Big(\frac{a_0'(K)}{a_0(K)} + \frac{3}{2}\frac{q'(K)}{q(K)}\Big)\frac{\varphi_2(\sqrt{2e(K)}/\varepsilon)}{\varphi_1(\sqrt{2e(K)}/\varepsilon)} + \varepsilon^2 q(K)^2\Big[\frac{1}{2}\frac{a_0''(K)}{a_0(K)} + 2\frac{a_0'(K)}{a_0(K)}\frac{q'(K)}{q(K)}$$
$$+ \frac{7}{6}\Big(\frac{q'(K)}{q(K)}\Big)^2 + \frac{2}{3}\frac{q''(K)}{q(K)}\Big]\frac{\varphi_3(\sqrt{2e(K)}/\varepsilon)}{\varphi_1(\sqrt{2e(K)}/\varepsilon)} + \varepsilon^2\frac{a_2(K)}{a_0(K)}.$$

Next we calculate the asymptotic expansion of implied volatilities of call options. Let us define $f \in C^\infty(\mathbb{R}_+;\mathbb{R}_+)$ by

$$f(x) = \frac{1}{x}\frac{1}{\sqrt{2\pi}}\exp(-\frac{x^2}{2})\varphi_1(x), \quad x > 0. \tag{1.19}$$

We can easily check that f is strictly decreasing and

$$f(0_+) = \infty, \quad f(\infty) = 0.$$

Therefore the inverse function $f^{-1} : \mathbb{R}_+ \to \mathbb{R}_+$ is well defined. When we consider the following normal model

$$d\tilde{X}(t) = \sigma d\tilde{W}(t), \quad \tilde{X}(0) = x_0^1,$$

the value of the call option with strike rate K and maturity T is given by

$$C_N(T,K) = \frac{1}{\sqrt{2\pi\sigma^2 T}}\int_{-\infty}^{\infty}(z + x_0^1 - K)^+ \exp(-\frac{z^2}{2\sigma^2 T})dx = (K - x_0^1)\cdot f\Big(\frac{K - x_0^1}{\sigma\sqrt{T}}\Big).$$

Therefore the implied normal volatility can be written as

$$\sigma_N^\varepsilon(T,K) = \frac{K - x_0^1}{f^{-1}(C_\varepsilon(T,K)/(K - x_0^1))\sqrt{T}}, \quad K > x_0^1.$$

The asymptotic expansion of the implied normal volatilities are given by the following.

THEOREM 1.3. *The asymptotic expansion of implied normal volatilities are given by*

$$\Big|\Big(\frac{\varepsilon|K - x_0^1|}{\sqrt{2e(K)T}}\Big)^{-1}\sigma_N(T,K) - \exp(J)\Big| \le C(\varepsilon + |K - x_0^1|)^3, \quad K \in [x_0^1, K_1], \tag{1.20}$$

where

$$\begin{aligned} J = {} & \frac{|K - x_0^1|^2}{b_1^2}\Big(\frac{L}{2} + \frac{1}{6}\frac{b_2^2}{b_1^2} - \frac{1}{4}\frac{b_3}{b_1}\Big)\varphi_1\big(\frac{\sqrt{2e(K)}}{\varepsilon}\big) + \frac{\varepsilon^2}{b_1}\Big(-\frac{L}{2} - \frac{5}{6}\frac{b_2^2}{b_1^2} + \frac{3}{4}\frac{b_3}{b_1}\Big)\varphi_1\big(\frac{\sqrt{2e(K)}}{\varepsilon}\big) \\ & + \frac{\varepsilon}{\sqrt{b_1}}\frac{|K - x_0^1|}{b_1}\Big(L + \frac{2}{3}\frac{b_2^2}{b_1^2} - \frac{3}{4}\frac{b_3}{b_1}\Big)\varphi_2\big(\frac{\sqrt{2e(K)}}{\varepsilon}\big) + \frac{\varepsilon^2}{b_1}\Big(\frac{L}{2} + \frac{b_2^2}{2b_1^2} - \frac{b_3}{2b_1}\Big)\varphi_3\big(\frac{\sqrt{2e(K)}}{\varepsilon}\big). \end{aligned} \tag{1.21}$$

REMARK 1.4. Since we can give the same formula for put options, Theorem 1.3 still holds in the case $K < x_0^1$. The implied volatility for a put option of strike rate K and maturity T is the same as the implied volatility for a call option with the same strike rate and maturity because we have put-call parity. See Appendix B for the details.

We also give a relation with the expansion based on Watanabe [14] and Yoshida [15]. Finally we apply our theorem to some known models e.g. CEV model, displaced diffusion and SABR model.

2 Hamilton equation and the energy of path

In this section, we investigate the correspondence between the Hamilton equation and the energy of path defined by (1.3). It is enough to discuss in the case $T = 1$.

Let H be a separable real Hilbert space defined by

$$H = \Big\{ h \in C_0([0,1];\mathbb{R}^N) : \ h \text{ is absolutely continuous and } \sum_{i=1}^{N} \int_0^1 \big| \frac{d}{dt} h^i(t) \big|^2 dt < \infty \Big\}.$$

The inner product is given by

$$(h,k)_H = \sum_{i=1}^{N} \int_0^1 \dot{h}^i(s) \dot{k}^i(s) ds.$$

This Hilbert space H is called the Cameron-Martin space.

Let $y(t;h),\ t \in [0,1],\ h \in H$, be the solution to the ordinary differential equation

$$\frac{d}{dt} y^i(t;h) = \sum_{k=1}^{d} V_k^i(t, y(t;h)) \frac{d}{dt} h^k(t) + b^i(t, y(t;h)), \quad 1 \le i \le N, \quad t \in [0,1],$$
$$y(0;h) = x_0, \quad x_0 \in \mathbb{R}^n.$$

Here we consider the Hamiltonian with potential term. Let $c : [0,1] \times \mathbb{R}^N \to \mathbb{R}$ be a smooth function. We define the functional $f : H \to \mathbb{R}$ by

$$f(h) = \int_0^1 c(t; y(t;h)) dt.$$

We define $(g^{ij})_{1 \le i,j \le N} : [0,1] \times \mathbb{R}^N \to \mathbb{R}$ by

$$g^{ij}(t,x) = \sum_{k=1}^{d} V_k^i(t,x) V_k^j(t,x).$$

We define Hamilitonian $\mathcal{H} : [0,1] \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ by

$$\mathcal{H}(t,x,p) = \frac{1}{2} \sum_{i,j=1}^{N} g^{ij}(t,x) p_i p_j + \sum_{i=1}^{N} b^i(t,x) p_i + c(t,x). \tag{2.1}$$

Then the correspondence between Hamilton equation and the energy of path is given by the following.

Proposition 2.1. *Let $J_j^i : [0,1] \times H \to \mathbb{R}$ be the solution to the following ordinary differential equation*

$$\frac{d}{dt} J_j^i(t;h) = \sum_{k=1}^{d} \sum_{r=1}^{N} \frac{\partial V_k^i}{\partial x^j}(t, y(t;h)) J_j^r(t;h) \dot{h}^k(t) + \sum_{r=1}^{N} \frac{\partial b^i}{\partial x^r}(t, y(t;h)) J_j^r(t;h),$$
$$J_j^i(0;h) = \delta_{ij}, \quad 1 \le i,j \le N,$$

where δ_{ij} is Kronecker's delta. Let $\bar{J}(t;h) = J^{-1}(t;h)$. We assume there is $h_0 \in H$ and $\lambda \in \mathbb{R}^N$ such that

$$h_0 = \sum_{k=1}^{N} \lambda_k Dy^k(1, h_0) + Df(h_0). \tag{2.2}$$

We define $x, p \in C^\infty([0,T]; \mathbb{R}^N)$ by

$$\begin{aligned} &x(t) = y(t;h_0), \\ &p_i(t) = \sum_{j=1}^{N} \bar{J}_i^j(t;h_0)\Big(\sum_{k=1}^{N} J_j^k(1;h_0)\lambda_k + \int_t^1 \frac{\partial c}{\partial x^k}(s, y(s;h_0))J_j^k(s;h_0)ds\Big). \end{aligned} \tag{2.3}$$

Then (x,p) satisfies the following Hamilton equation, where Hamiltonian $\mathcal{H}$ is given by (2.1).

$$\begin{aligned} &\frac{d}{dt}x_i(t) = \frac{\partial}{\partial p_i}\mathcal{H}(t, x(t), p(t)), \\ &\frac{d}{dt}p_i(t) = -\frac{\partial}{\partial x^i}\mathcal{H}(t, x(t), p(t)), \quad t \in [0,1], \quad 1 \le i \le N, \\ &x(0) = x_0, \quad x_0 \in \mathbb{R}^n. \end{aligned} \tag{2.4}$$

Furthermore, we have $\lambda = p(1)$ and

$$\begin{aligned} &\frac{d}{dt}h_0^k(t) = \sum_{i=1}^{N} p_i(t)V_k^i(t;x(t)), \quad 0 \le t \le 1, \quad 1 \le k \le d, \\ &\|h_0\|^2 = \sum_{i,j=1}^{N} \int_0^1 g^{ij}(t, x(t))p_i(t)p_j(t)dt. \end{aligned} \tag{2.5}$$

Proof. We note that $\bar{J}_j^i : [0,1] \times H \to \mathbb{R}$ satisfies the following ordinary differential equation.

$$\begin{aligned} &\frac{d}{dt}\bar{J}_j^i(t;h) = -\sum_{k=1}^{d}\sum_{r=1}^{N} \frac{\partial}{\partial x^j}V_k^r(t, y(t;h))\bar{J}_r^i(t;h)\dot{h}^k(t) - \sum_{r=1}^{N} \frac{\partial}{\partial x^j}V_0^r(t, y(t;h))\bar{J}_r^i(t;h), \\ &\bar{J}_j^i(0;h) = \delta_{ij}, \quad 1 \le i, j \le N. \end{aligned}$$

From Proposition 6.6 in Shigekawa [13], we have

$$Dy^i(1;h)[k] = \sum_{l=1}^{d}\sum_{r,j=1}^{N} J_r^i(1;h)\int_0^1 \bar{J}_j^r(t;h)V_l^j(t, y(t;h))\dot{k}^l(t)dt, \quad 1 \le i \le N, \tag{2.6}$$

$$\begin{aligned} Df(h)[k] &= \sum_{i=1}^{N}\int_0^1 \frac{\partial c}{\partial x^i}(t, y(t;h))Dy^i(t;h)[k]dt \\ &= \sum_{l=1}^{d}\sum_{i,j,r=1}^{N}\int_0^1 \dot{k}^l(t)\bar{J}_j^r(t;h)V_l^j(t, y(t;h))\Big(\int_t^1 \frac{\partial c}{\partial x^i}(s, y(s;h))J_r^i(s;h)ds\Big)dt. \end{aligned} \tag{2.7}$$

We can check that $(x(t),p(t)),\ 0\le t\le 1$, satisfies (2.4) as follows.

$$\begin{aligned}\frac{d}{dt}x^i(t) &= \sum_{k=1}^{d} V_k^i(t,y(t;h_0))\dot{h}_0^k(t)+b^i(t;y(t;h_0))\\ &= \sum_{j=1}^{N} g^{ij}(t,x(t))p_j(t)+b^i(t;x(t)),\end{aligned}$$

$$\begin{aligned}\frac{d}{dt}p_i(t) =& -\sum_{k=1}^{d}\sum_{j,r=1}^{N}\frac{\partial V_k^j}{\partial x^i}(t,y(t;h))p_j(t)\dot{h}^k(t)-\sum_{r=1}^{N}\frac{\partial b^r}{\partial x^i}(t;y(t;h))p_r(t)\\ &-\sum_{j=1}^{N}\sum_{r=1}^{N}\bar{J}_i^j(t;h_0)\frac{\partial c}{\partial x^r}(t,y(t;h_0))J_j^r(t;h_0)\\ =& -\sum_{j,r=1}^{N}\frac{\partial g^{jr}}{\partial x^i}(t,x(t))p_j(t)p_r(t)-\sum_{j=1}^{N}\frac{\partial b^j}{\partial x^i}(t;x(t))p_j(t)-\frac{\partial c}{\partial x^i}(t;x(t)).\end{aligned}$$

Since the definition of p, we have $\lambda=p(1)$. Since $h_0=\sum_{i=1}^{N}\lambda_i Dy^i(h_0)+Df(h_0)$, we see that

$$(h_0,k)=\sum_{i=1}^{N}\sum_{l=1}^{d}\int_0^1 p_i(t)V_l^i(t,y(t;h_0)\dot{k}^l(t)dt.$$

Therefore we have (2.5). $\square$

REMARK 2.2. We define the functional $E:H\to\mathbb{R}$ by

$$E(h)=\frac{1}{2}\sum_{k=1}^{d}\int_0^1\left|\frac{d}{dt}h^k(t)\right|^2dt-f(h).$$

Then the condition (2.2) is obtained by the Euler-Lagrange equation associated with

$$\inf\{E(h);y(1;h)=x\}.$$

Let us define the following notations.

$$f\underset{k}{\sim}g\overset{\text{def}}{\Longleftrightarrow}\lim_{w\downarrow 0}\frac{f(w)-g(w)}{w^k}=0,\ k\ge 0,\ f,g\in C([0,1]).$$

In the following case, we obtain the asymptotic solutions.

PROPOSITION 2.3. *Suppose that Hamiltonian is given by* (2.1) *with* $b\equiv 0,\ c\equiv 0$ *and*

$$\lambda_i=\begin{cases} w & (i=1),\quad w\in\mathbb{R}\\ 0 & (2\le i\le N).\end{cases}\tag{2.8}$$

Let $x(t;w),\ p(t;w)$ *be the solution to the associated Hamilton equation. Then the asymptotic expansion of* $x^1(1;w)$ *is given as follows.*

$$x^1(1;w)\underset{3}{\sim}x_0+b_1w+b_2w^2+b_3w^3,\tag{2.9}$$

where $b_1,\ b_2,\ b_3$ *are defined by* (1.11).

Proof. The solution can be written as

$$x^i(t;w) = x_0^i + \sum_{j=1}^{N} \int_0^t g^{ij}(s, x(s;w))p_j(s;w)ds, \tag{2.10}$$

$$p_i(t;w) = p_i(1;w) + \frac{1}{2}\sum_{j,r=1}^{N} \int_0^t \frac{\partial g^{jr}}{\partial x^i}(s, x(s;w))p_j(s;w)p_r(s;w)ds. \tag{2.11}$$

We calculate the asymptotic expansion inductively. Since $x(t;0) = x_0,\ p(t;0) = 0$, we have

$$x(t;w) \underset{0}{\sim} x_0,\ p(t;w) \underset{0}{\sim} 0. \tag{2.12}$$

Since the integral term in (2.11) is second order in w and the boundary condition (2.8), we have the first order expansion of p.

$$p_i(t;w) \underset{1}{\sim} p_i(1;w) = \begin{cases} w & (i=1) \\ 0 & (2 \le i \le N). \end{cases} \tag{2.13}$$

We substitute (2.13) for (2.10), we have the first order expansion of x.

$$x^i(t;w) \underset{1}{\sim} x_0^i + \Big(\int_0^t g^{i1}(s, x_0)ds\Big)w. \tag{2.14}$$

We also substitute (2.13) for (2.11), we have second order expansion of p.

$$\begin{aligned} p_i(t;w) &\underset{2}{\sim} p_i(1;w) + \frac{1}{2}\sum_{j,r=1}^{N} \int_t^1 \Big(\frac{\partial g^{jr}}{\partial x^i}(s, x(s;w))ds\Big)p_j(1;w)p_r(1;w) \\ &\underset{2}{\sim} p_i(1;w) + \frac{1}{2}\Big(\int_t^1 \frac{\partial g^{11}}{\partial x^i}(s, x_0)ds\Big)w^2. \end{aligned} \tag{2.15}$$

We substitute (2.14) for (2.10), we have second order expansion of x.

$$\begin{aligned} x^i(t;w) &\underset{2}{\sim} x_0^i + \sum_{j=1}^{N} \int_0^t g^{ij}(s, x(s;w))\Big\{p_j(1) + \frac{1}{2}\Big(\int_s^1 \frac{\partial g^{11}}{\partial x^j}(r, x_0)dr\Big)w^2\Big\}ds \\ &\underset{2}{\sim} x_0^i + \Big(\int_0^t g^{i1}(s, x_0)ds\Big)w + \sum_{j=1}^{N}\Big(\int_0^t \int_0^s \frac{\partial g^{i1}}{\partial x^j}(s, x_0)g^{j1}(u, x_0)duds \\ &\quad + \frac{1}{2}\int_0^t \int_s^1 g^{ij}(s, x_0)\frac{\partial g^{11}}{\partial x^j}(u, x_0)duds\Big)w^2. \end{aligned} \tag{2.16}$$

From second order expansion of p and first order expansion of x, we have third order expansion of p.

(2.17)

$$\begin{aligned} p_i(t;w) &\underset{3}{\sim} p_i(1;w) + \frac{1}{2}\Big(\int_t^1 \frac{\partial g^{11}}{\partial x^i}(s, x_0)ds\Big)w^2 \\ &+ \frac{1}{2}\sum_{j=1}^{N}\Big\{\frac{\partial g^{j1}}{\partial x^i}(s, x_0)\Big(\int_s^1 \frac{\partial g^{11}}{\partial x^j}(u, x_0)du\Big)ds + \int_t^1 \frac{\partial^2 g^{11}}{\partial x^i \partial x^j}(s, x_0)\Big(\int_0^s g^{j1}(u, x_0)du\Big)\Big\}w^3. \end{aligned}$$

Finally we have third order expansion of x.

(2.18)

$$
\begin{aligned}
x^i(t;w) \underset{3}{\sim} & \; x_0^i + \Big(\int_0^t g^{i1}(s,x_0)ds\Big)w \\
& + \sum_{j=1}^N \Big(\int_0^t \int_0^s \frac{\partial g^{i1}}{\partial x^j}(s,x_0) g^{j1}(u,x_0)duds + \frac{1}{2}\int_0^t \int_s^1 g^{ij}(s,x_0)\frac{\partial g^{11}}{\partial x^j}(u,x_0)duds\Big)w^2 \\
& + \sum_{j,k=1}^N \Big[\frac{1}{2}\int_0^t g^{ij}(s,x_0)\Big(\int_s^1 \frac{\partial g^{k1}}{\partial x^j}(u,x_0)\Big(\int_u^1 \frac{\partial g^{11}}{\partial x^k}(r,x_0)dr\Big)du\Big)ds \\
& + \frac{1}{2}\int_0^t g^{ij}(s,x_0)\Big(\int_s^1 \frac{\partial^2 g^{11}}{\partial x^j \partial x^k}(u,x_0)\Big(\int_0^u g^{k1}(r,x_0)dr\Big)du\Big)ds \\
& + \frac{1}{2}\int_0^t \frac{\partial g^{ij}}{\partial x^k}(s,x_0)\Big(\int_s^1 \frac{\partial g^{11}}{\partial x^j}(u,x_0)du\Big)\Big(\int_0^s g^{k1}(r,x_0)dr\Big)ds \\
& + \frac{1}{2}\int_0^t \frac{\partial g^{i1}}{\partial x^j}(s,x_0)\Big(\int_0^s g^{jk}(u,x_0)\Big(\int_u^1 \frac{\partial g^{11}}{\partial x^k}(r,x_0)dr\Big)du\Big)ds \\
& + \int_0^t \frac{\partial g^{i1}}{\partial x^j}(s,x_0)\Big(\int_0^s \frac{\partial g^{j1}}{\partial x^k}(u,x_0)\Big(\int_0^u g^{k1}(r,x_0)dr\Big)du\Big)ds \\
& + \frac{1}{2}\int_0^t \frac{\partial^2 g^{i1}}{\partial x^j \partial x^k}(s,x_0)\Big(\int_0^s g^{j1}(u,x_0)du\Big)\Big(\int_0^s g^{k1}(r,x_0)dr\Big)ds\Big]w^3.
\end{aligned}
$$

From the definition of linear operator V given in (1.8), we have

$$x^1(1;w) \underset{3}{\sim} x_0^1 + b_1 w + b_2 w^2 + b_3 w^3, \tag{2.19}$$

where b_1, b_2, b_3 are defined in (1.11). □

3 Asymptotic expansion of energy term

In this section, we give a proof of Theorem 1.1 (1). We apply Proposition 2.3 in the case energy function given by (1.3).

Lemma 3.1. *Let $y(t;h) : [0,1] \times H \to \mathbb{R}$, be the solution defined by* (1.2). *Let us define*

$$\tilde{y}(t;h) = \phi(-t, y(t;h)), \quad 1 \le i \le N, \quad t \in [0,1],$$

then $\tilde{y}$ satisfies the ordinary differential equation

$$\frac{d}{dt}\tilde{y}^i(t;h) = \sum_{k=1}^d \tilde{V}_k^i(t, \tilde{y}(t;h)\frac{d}{dt}h^k(t), \quad 1 \le i \le N, \quad t \in [0,1], \tag{3.1}$$

where $\tilde{V}$ is defined by (1.5).

Proof. From the definition of ϕ given by (1.4), we have

$$-V_0^i(t, \phi(-t, \phi(t,y)) + \sum_{j=1}^d \nabla_j \phi^i(-t, \phi(t,y)) V_0^j(t, \phi(t,y)) = 0.$$

Therefore we have our lemma. □

Since $V_0^1 \equiv 0$, we have $\tilde{y}^1(t;h) = y^1(t;h)$, and the energy function can be defined as follows.

$$e(x) = \frac{1}{2} \inf\Big\{\sum_{k=1}^{d} \int_0^1 \big|\frac{d}{dt} h_0^k(t)\big|^2 dt : \tilde{y}^1(1;h) = x\Big\}.$$

Therefore it is enough to prove in the driftless case, i.e. $V_0 \equiv 0$.

Proof of Theorem 1.1(1). Let h_0 be defined by

$$h_0(x) \equiv argmin\{e(h); h \in H, y^1(1;h) = x\}. \tag{3.2}$$

We denote $h_0(x)(t) \equiv h_0(t,x)$. Then from non-degeneracy condition, there is an $r > 0$ such that $h_0(x)$ is unique in $x \in (x_0 - r, x_0 + r)$. Using Lagrange multiplier theorem, we have

$$h_0(x) = \lambda(x) DF^1(0, h_0(x)), \tag{3.3}$$

where $\lambda : (x_0 - r, x_0 + r) \to \mathbb{R}$ is a smooth function. Applying Proposition 2.3, we have

$$\big|x^1(1;\lambda(x)) - \Big(x_0^1 + b_1\lambda(x) + b_2\lambda(x)^2 + b_3\lambda(x)^3\Big)\big| = O(|x - x_0|^4).$$

Therefore we have the asymptotic expansion of λ in x.

$$\lambda(x) \underset{3}{\sim} c_1(x - x_0^1) + c_2(x - x_0^1)^2 + c_3(x - x_0^1)^3, \tag{3.4}$$

where

$$c_1 = \frac{1}{b_1},\ c_2 = -\frac{b_2}{b_1^3},\ c_3 = -\frac{b_3}{b_1^4} + 2\frac{b_2^2}{b_1^5}. \tag{3.5}$$

From [9] we have

$$\lambda(x) = \frac{\partial e(x)}{\partial x}. \tag{3.6}$$

Since $e(x_0^1) = 0$, we can calculate the path of energy by

$$e(x) = \int_{x_0^1}^{x} \lambda(y) dy \underset{4}{\sim} \frac{c_1}{2}(x - x_0^1)^2 + \frac{c_2}{3}(x - x_0^1)^3 + \frac{c_3}{4}(x - x_0^1)^4.$$

Therefore we have Theorem 1.1 (1). □

Let us define $\alpha : [0,1] \to \mathbb{R}$ by

$$\alpha(t) = c_1\Big(\int_0^t V_k^1(u;x_0)du\Big). \tag{3.7}$$

Then we have the following.

Corollary 3.2. *Let $h_0 \in H$ be the element defined in* (3.2), *then we have*

$$\|h_0^k(x) - \alpha(\cdot)(x - x_0^1)\|_H = O(|x - x_0^1|^2).$$

Proof. From (2.5) and the proof of Theorem 1.1(1), we have

$$\begin{aligned} h_0^k(t,x) &= \sum_{i=1}^{N} \int_0^t p_i(u;w) V_k^i(u, x(u;w)) dt \\ &\underset{1}{\sim} \Big(\int_0^t V_k^1(u;x_0)du\Big) w \underset{1}{\sim} \Big(\int_0^t V_k^1(u;x_0)du\Big) c_1(x - x_0^1). \end{aligned}$$

□

4 Proof of Theorem 1.1

Let X be the solution to the stochastic differential equation

$$(4.1) \qquad dX_s^i(t,\theta) = \sum_{k=1}^{d} V_k^i(t, X_s(t,\theta))d\theta^k(t) + sb^i(t, X_s(t,\theta))dt, \quad 1 \le i \le N, \quad t \in [0,1],$$
$$X_s(0) = x_0.$$

Let us define Wiener functionals $F^i : (0,1) \times \Theta \times [-r_0, r_0] \to \mathbb{R}$, $1 \le i \le N$, by

$$(4.2) \qquad F^i(s,\theta,y) = X_s^i(1,\theta) - y.$$

To apply the main theorem in [9], it is necessary to check the assumptions (A-1), ..., (A-5) given in [9]. Since $f \equiv 0$, we can check (A-1). Since $h(0) = 0$, we can check (A-2), (A-3) and (A-4) in the neighborhood of origin. Since the ellipticity condition at origin, we can check (A-5) in [9], using the same discussion given in Appendix B in [12]. Then we have the following.

For each $(s,y) \in (0,1] \times [-r_0, r_0]$, the density function $p_s(y)$ satisfies

$$\big|(2\pi s)^{1/2} \exp(\frac{e(y)}{s})p_s(y) - a_0(y)\big| \le K_0 s^{1/2}, \ (s,y) \in (0,1] \times [-r_0, r_0].$$

The function $a_0 \in C([-r_0, r_0])$ is given by

$$(4.3) \qquad a_0(y) = \Big(\frac{\partial^2 e(y)}{\partial y^2}\Big)^{\frac{1}{2}} \det{}_2(I_H - B(y))^{-\frac{1}{2}} \exp\big(\frac{\partial e(y)}{\partial y}\mathcal{A}F^1(0, h_0(y), y)\big).$$

Here $\mathcal{A}$ is called the heat operator defined by

$$\mathcal{A}f(s,\theta) = [\frac{\partial f}{\partial s} + \frac{1}{2} trace_H D^2 f](s,\theta),$$

and

$$(4.4) \qquad B(y) \equiv \frac{\partial e(y)}{\partial y} D^2 F^1(0, h_0(y), y).$$

In this section, we calculate each terms in right hand side of (4.3) explicitly. First we calculate the heat operator.

LEMMA 4.1. *There are constants $C > 0$ and $r > 0$ such that*

$$\Big|\mathcal{A}F^1(0, h_0(y), y) - \frac{(y - x_0^1)}{2b_1}\Big\{\sum_{i=1}^{N} \int_0^1 \int_0^t b^i(u, x_0) \nabla_i g^{11}(t, x_0) du dt$$
$$+ \sum_{k=1}^{d} \sum_{i,j=1}^{N} \int_0^1 V_k^1(t, x_0) \nabla_{i,j}^2 V_k^1(t, x_0) \Big(\int_0^t g^{ij}(u, x_0) du\Big) dt\Big\}\Big| = O(|y - x_0^1|^2), \quad y > x_0^1.$$

Proof. Since the adaptivity of Y, we have

$$\mathcal{A}F^i(s,\theta,y) = \sum_{k=1}^{d} \int_0^1 \mathcal{A}[V_k^i(u, X_s(u,\theta))] d\theta_s^k(u) + \int_0^1 b^i(u, X_s(u,\theta)) du$$
$$+ s\int_0^1 \mathcal{A}[b_s^i(u, X_s(u,\theta))] du, \quad 1 \le i \le N.$$

Therefore we have

$$\mathcal{A}F^1(0,h_0(y),y)=\sum_{j=1}^{N}\sum_{k=1}^{d}\int_0^1 \nabla_j V_k^1(u,X_0(u,h_0(u;y)))\mathcal{A}X_0^j(u,h_0(u;y))\dot{h}_0^k(u;y)du$$
$$+\frac{1}{2}\sum_{k=1}^{d}\sum_{i,j=1}^{N}\int_0^1 \nabla_{i,j}^2 V_k^1(u,X_0(u,h_0(u;y)))\langle DX_0^i(u),DX_0^j(u)\rangle\dot{h}_0^k(u;y)du.$$

Then using Corollary 3.2, we have the following.

$$\big|\mathcal{A}F^1(0,h_0(y),y)-(y-x_0^1)\Big(\sum_{j=1}^{N}\sum_{k=1}^{d}\int_0^1 \nabla_j V_k^1(u,X_0(u;0))\mathcal{A}X_0^j(u;0)\dot{\alpha}^k(u)du$$
$$+\frac{1}{2}\sum_{k=1}^{d}\sum_{i,j=1}^{N}\int_0^1 \nabla_{i,j}^2\tilde{V}_k^1(u,X_0(u;0))\langle DX_0^i(u;0),DX_0^j(u;0)\rangle\dot{\alpha}^k(u)du\Big)\big|=O(|y-x_0^1|^2),$$

where $\mathcal{A}X_0^j(t;0)=\int_0^t b^j(u,X_0(u;0))du$. □

Lemma 4.2. *Hilbert-Schmidt norm of D^2F^1 is given by*

$$\|D^2F^1(0,0,x_0)\|_{HS}^2=2\sum_{m=1}^{d}\sum_{l_1,l_2=1}^{N}\int_0^1\int_0^t g^{l_1l_2}(u,x_0)\nabla_{l_1}V_m^1(t,x_0)\nabla_{l_2}V_m^1(t,x_0)dudt. \tag{4.5}$$

Proof. The Malliavin derivatives of $X_0^i,\ 1\le i\le N$, to the direction $k\in H$ is given by

$$DX_0^i(t;h)[k]=\sum_{l=1}^{N}\sum_{m=1}^{d}\int_0^t \nabla_l V_m^i(u,X_0(u;h))DX_0^l(u;h)[k]\dot{h}^m(u)du+\sum_{m=1}^{d}\int_0^t V_m^i(u,X_0(u;h))\dot{k}^m(u)du.$$

The second Malliavin derivative of F^1 to the direction $k_1,k_2\in H$ is given by

$$D^2F^1(0,0,x_0)[k_1][k_2]=\sum_{l=1}^{N}\sum_{m=1}^{d}\int_0^1 \nabla_l V_m^1(u,x_0)DX_0^l(u;0)[k_1]\dot{k}_2^m(u)du$$
$$+\int_0^1 \nabla_l V_m^1(u,x_0)DX_0^l(u;0)[k_2]\dot{k}_1^m(u)du$$
$$=\sum_{l=1}^{N}\sum_{m_1,m_2=1}^{d}\int_0^1 \nabla_l V_{m_1}^1(t,x_0)\Big(\int_0^t V_{m_2}^l(u,x_0)\dot{k}_2^{m_2}(u)du\Big)\dot{k}_1^{m_1}(t)dt$$
$$=\sum_{l=1}^{N}\sum_{m_1,m_2=1}^{d}\int_0^1\int_0^1(\nabla_l V_{m_1}^1(t,x_0)V_{m_2}^l(u,x_0)1_{t>u}+\nabla_l V_{m_2}^1(u,x_0)V_{m_1}^l(t,x_0)1_{t<u})\dot{k}_1^{m_1}(t)\dot{k}_2^{m_2}(u)dtdu.$$

Therefore we have the Hilbert-Schmidt norm as following.

$$\begin{aligned}
&\|D^2F^1(0,0,x_0)\|_{HS}^2\\
&=\sum_{l=1}^{N}\sum_{m_1,m_2=1}^{d}\int_0^1\int_0^1((\nabla_l V_{m_1}^1(t,x_0)V_{m_2}^l(u,x_0)1_{t>u}+\nabla_l V_{m_2}^1(u,x_0)V_{m_1}^l(t,x_0)1_{t<u})^2dtdu\\
&=2\sum_{l=1}^{N}\sum_{m_1,m_2=1}^{d}\int_0^1\int_0^t(\sum_{l=1}^{N}\nabla_l V_{m_1}^1(t,x_0)V_{m_2}^l(u,x_0))^2dudt\\
&=2\sum_{l_1,l_2=1}^{N}\sum_{m_1,m_2=1}^{d}\int_0^1\int_0^t\nabla_{l_1}V_{m_1}^1(t,x_0)V_{m_2}^{l_1}(u,x_0)\nabla_{l_2}V_{m_1}^1(t,x_0)V_{m_2}^{l_2}(u,x_0)dudt\\
&=2\sum_{l_1,l_2=1}^{N}\sum_{m=1}^{d}\int_0^1\int_0^t g^{l_1l_2}(u,x_0)\nabla_{l_1}V_m^1(t,x_0)\nabla_{l_2}V_m^1(t,x_0)dudt.
\end{aligned}$$

□

Finally we will complete the proof of Theorem 1.1.

Proof of Theorem 1.1(2). Using (4.3), we have

$$\log a_0(y)=-\frac{1}{2}\log\bigl(\det{}_2(I_H-B(y))\bigr)+\frac{\partial e(y)}{\partial y}\mathcal{A}F^1(0,h(y),y)+\frac{1}{2}\log\Bigl(\frac{\partial^2 e(y)}{\partial y^2}\Bigr).$$

In the right hand side, the asymptotic expansion of second term is given by Lemma 4.1, so we will give the asymptotic expansion of first term.

Since B is defined by (4.4) and $\frac{\partial e(y)}{\partial y}\underset{1}{\sim}c_1(y-x_0^1)$, we have

$$\bigl|B(y)-c_1D^2F^1(0,0,x_0)(y-x_0^1)\bigr|=O(|y-x_0^1|^2).$$

Since $B(x_0^1)=0$, if $|y-x_0^1|$ is sufficiently small we have

$$\det{}_2(I-B(y))=\exp(-\sum_{n=2}^{\infty}\frac{1}{n}trace_H(B(y)^n)).$$

Therefore we have

$$\Bigl|\log\bigl(\det{}_2(I_H-B(y))\bigr)+\frac{c_1^2(y-x_0^1)^2}{2}\|D^2F(0,0,x_0)\|_{HS}^2\Bigr|=O(|y-x_0|^3). \tag{4.6}$$

The Hilbert-Schmidt norm of D^2F is given by Lemma 4.2. Therefore we have

$$\begin{aligned}
\log a_0(y)&\underset{2}{\sim}\frac{(y-x_0^1)^2}{4b_1^2}\|D^2F^1(0,0,x_0)\|_{HS}^2+\frac{(y-x_0^1)}{b_1}\mathcal{A}F^1(0,h_0(y),y)+\frac{1}{2}\log\Bigl(\frac{\partial^2 e(y)}{\partial y^2}\Bigr)\\
&=\frac{1}{2}\frac{(y-x_0^1)^2}{b_1^2}\sum_{l_1,l_2=1}^{N}\sum_{m=1}^{d}\int_0^1\int_0^t g^{l_1l_2}(u,x_0)\nabla_{l_1}V^m(t,x_0)\nabla_{l_2}V^m(t,x_0)dudt\\
&+\frac{(y-x_0^1)^2}{2b_1^2}\sum_{j=1}^{N}\int_0^1\int_0^t b^j(u,x_0)\nabla_j g^{11}(t,x_0)dudt\\
&+\frac{1}{2}\frac{(y-x_0^1)^2}{b_1^2}\sum_{l_1,l_2=1}^{N}\sum_{m=1}^{d}\int_0^1\int_0^t V_m^1(t,0)g^{l_1l_2}(u,x_0)\nabla_{l_1,l_2}V_m^1(t,x_0)dudt+\frac{1}{2}\log\Bigl(\frac{\partial^2 e(y)}{\partial y^2}\Bigr).
\end{aligned}$$

From the definition of (1.6), we have

$$\log a_0(y) \underset{2}{\sim} \frac{(y-x_0^1)^2}{2b_1^2} \int_{0<u<t<1} L_u(g^{11}(t,\cdot))dudt + \frac{1}{2}\log\Big(\frac{\partial^2 e(y)}{\partial y^2}\Big).$$

Then we have (1.13).

Finally we calculate $a_2(x_0^1)$. First we give an asymptotic expansion of the density using Hermite polynomials. Let $y = x_0^1 + \varepsilon\frac{z}{\sqrt{c_1}}$. Then the asymptotic expansion in ε up to second order is given as follows.

$$\begin{aligned}
p_\varepsilon(y)dy &= p_\varepsilon(x_0^1 + \frac{\varepsilon z}{\sqrt{c_1}})\frac{\varepsilon dz}{\sqrt{c_1}} \\
&\underset{2}{\sim} (a_0(x_0^1 + \frac{\varepsilon z}{\sqrt{c_1}}) + \varepsilon^2 a_2(x_0^1))\frac{1}{\sqrt{2\pi}}\exp\Big[-(\frac{c_1}{2}(\frac{z}{\sqrt{c_1}})^2 + \frac{\varepsilon c_2}{3}(\frac{z}{\sqrt{c_1}})^3 + \frac{\varepsilon^2 c_3}{4}(\frac{z}{\sqrt{c_1}})^4)\Big]dx \\
&= \Big[1 - \frac{c_2}{3c_1^{3/2}}\varepsilon(z^3-3z) + \varepsilon^2\Big\{\frac{c_2^2}{18c_1^3}z^6 - (\frac{c_2^2}{3c_1^3} + \frac{c_3}{4c_1^2})z^4 \\
&+ (-\frac{c_2^2}{2c_1^3} + \frac{3c_3}{2c_1^2} + \frac{Lc_1}{2})z^2 + \varepsilon^2 a_2(x_0^1)\Big\}\Big]\phi(z)dz \\
&= \Big[1 - \varepsilon\frac{c_2}{3c_1^{3/2}}H_3(z) + \varepsilon^2\frac{c_2^2}{18c_1^3}H_6(z) + \varepsilon^2(\frac{c_2^2}{2c_1^3} - \frac{c_3}{4c_1^2})H_4(z) + \varepsilon^2(\frac{c_1 L}{2})H_2(z) \\
&+ \varepsilon^2(a_2(x_0^1) - \frac{c_2^2}{18c_1^3}H_6(0) - (\frac{c_2^2}{2c_1^3} - \frac{c_3}{4c_1^2})H_4(0) - (\frac{c_1 L}{2})H_2(0))\Big] \\
&\cdot \sqrt{\frac{c_1}{2\pi\varepsilon^2}}\exp(-\frac{z^2}{2})dz,
\end{aligned}$$

where $H_n,\ n \in \mathbb{N}$, are Hermite polynomials e.g.

$$\begin{aligned}
H_2(x) &= x^2 - 1, \\
H_3(x) &= x^3 - 3x, \\
H_4(x) &= x^4 - 6x^2 + 3, \\
H_6(x) &= x^6 - 15y^4 + 45y^2 - 15.
\end{aligned}$$

Since p_ε is probability density, we have

$$1 = \int_{-\infty}^{\infty} p_\varepsilon(y)dy = \int_{-\infty}^{\infty} p_\varepsilon(\varepsilon z)\varepsilon dz.$$

The orthogonality of Hermite polynomials implies

$$\int_{-\infty}^{\infty} H_n(z)\frac{1}{\sqrt{2\pi}}\exp(-\frac{z^2}{2})dz = 0, \quad n \geq 1,$$

then we have

$$a_2(x_0^1) = \frac{c_2^2}{18c_1^3}H_6(0) - (\frac{c_2^2}{2c_1^3} - \frac{c_3}{4c_1^2})H_4(0) - (\frac{c_1 L}{2})H_2(0).$$

This completes the proof of Theorem 1.1. □

The asymptotic expansion of the probability density in ε using Hermite polynomials is given as follows.

Corollary 4.3. *For each $z \in \mathbb{R}$, let $y = x_0^1 + \varepsilon \frac{z}{\sqrt{c_1}}$, $\varepsilon \in (0,1]$. For any $r \geq 0$, there is a constant $C > 0$ such that*

$$\Big|\sqrt{\frac{2\pi\varepsilon^2}{c_1}} \exp(\frac{z^2}{2}) p_\varepsilon(y) - \Big[1 - \varepsilon \frac{c_2}{3c_1^{3/2}} H_3(z) + \varepsilon^2 \frac{c_2^2}{18c_1^3} H_6(z) + \varepsilon^2 \big(\frac{c_2^2}{2c_1^3} - \frac{c_3}{4c_1^2}\big) H_4(z) + \varepsilon^2 \big(\frac{c_1 L}{2}\big) H_2(z)\Big]\Big| \leq \varepsilon^3 C, \quad \varepsilon \in (0,1], \quad z \in [-r,r].$$

5 Asymptotic expansion of call options

In this section, we prove Theorem 1.2. First we prove the following theorem.

Theorem 5.1. *We assume $X_\varepsilon^1(T)$ has a density $p_\varepsilon(y)$, $y \in \mathbb{R}$ and let*

$$a_\varepsilon(y) = (2\pi\varepsilon^2)^{1/2} \exp\big(\frac{e(y)}{\varepsilon^2}\big) p_\varepsilon(y), \quad y \in \mathbb{R}.$$

We assume that there are constants $N \in \mathbb{N}$, $C_0 > 0$ and $K_0 > 0$ such that

$$\big|a_\varepsilon(y) - \sum_{k=0}^{N} a_{2k}(y)\varepsilon^{2k}\big| \leq C_0 \varepsilon^{2N+2}, \quad y \in [x_0^1, K_0],$$

and assume that the energy function e satisfies $e'(x) > 0$, $x \in (x_0^1, K_0]$. We define $g : \mathbb{R} \to \mathbb{R}$ by

$$e(g(x)) = \frac{x^2}{2}.$$

Since e is strictly increasing, g is well defined. Then there are constants $K_1 < K_0$ and C_1, such that the value of the call option satisfies following.

$$\Big|\sqrt{2\pi} \exp(\frac{e(K)}{\varepsilon^2}) C_\varepsilon(T,K) - \varepsilon \varphi_1\big(\frac{g^{-1}(K)}{\varepsilon}\big) a_0(K) q(K)^2 R_N(\varepsilon, K)\Big| \leq C_1 \varepsilon^{N+1}, \quad \varepsilon \in (0,1], \ K \in [x_0^1, K_1].$$

where

$$R_N(\varepsilon, K) = \sum_{\substack{n,m \geq 0, n+m \geq 1 \\ 2n+m+1 \leq N}} \frac{c_{n,m}(g^{-1}(K))}{c_{0,0}(g^{-1}(K))} \frac{\varphi_{m+1}(g^{-1}(K)/\varepsilon)}{\varphi_1(g^{-1}(K)/\varepsilon)} \varepsilon^{2n+m}. \tag{5.1}$$

Here $c_{n,m} \in C(\mathbb{R})$ is given by

$$c_{n,m}(x) = \sum_{k=0}^{m} \frac{1}{(k+1)!(m-k)!} \big(\frac{d}{dx}\big)^{k+1} g(x) \cdot \big(\frac{d}{dx}\big)^{m-k} A_n(x), \tag{5.2}$$

where

$$A_k(x) = a_{2k}(g(x)) g'(x), \quad n \in \mathbb{N}, \quad x \in [x_0^1, K_1]. \tag{5.3}$$

We have the following.

LEMMA 5.2.

$$A_0(x_0^1)=1.$$

Proof. Since

$$1=\int_{-\infty}^{\infty}p_\varepsilon(y)dy=\frac{1}{(2\pi\varepsilon^2)^{1/2}}\int_{-\infty}^{\infty}a_\varepsilon(y)\exp(-\frac{e(y)}{\varepsilon^2})dy,$$

we have

$$1=\frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty}a_\varepsilon(g(\varepsilon y))\exp(-\frac{y^2}{2})g'(\varepsilon y)dy.$$

Since the right hand side is bounded, taking the limit of $\varepsilon\downarrow 0$, we have $a_0(g(0))g'(0)=1$. □

Proof of Theorem 5.1. We can divide the value of a call option into two parts.

$$C_\varepsilon(T,K)=\tilde{C}_\varepsilon(T,K)+R_\varepsilon(K_0),$$

where

$$\tilde{C}_\varepsilon(T,K)=\int_K^{K_0}(y-K)p_\varepsilon(y)dy=\int_K^{K_0}(y-K)\big(\frac{1}{2\pi\varepsilon^2}\big)^{\frac{1}{2}}\exp(-\frac{e(y)}{\varepsilon^2})a_\varepsilon(y)dy,$$

and

$$R_\varepsilon(K_0)=E[X_\varepsilon^1(T)-K:X_\varepsilon^1(T)>K_0].$$

Since $e(g(x))=\frac{x^2}{2}$, we have

$$\tilde{C}_\varepsilon(T,K)=\int_{g^{-1}(K)}^{g^{-1}(K_0)}(g(x)-K)\big(\frac{1}{2\pi\varepsilon^2}\big)^{\frac{1}{2}}\exp(-\frac{x^2}{2\varepsilon^2})a_\varepsilon(g(x))g'(x)dx.$$

Let $A_\varepsilon(x)=a_\varepsilon(g(x))g'(x)$ and $\tilde{K}_\varepsilon=\frac{1}{\varepsilon}(g^{-1}(K_0)-g^{-1}(K))$. Putting $x=\varepsilon z+g^{-1}(K)$, we have

$$\begin{aligned}&\exp\big(\frac{g^{-1}(K)^2}{2\varepsilon^2}\big)\tilde{C}_\varepsilon(T,K)\\&=\int_0^{\tilde{K}_\varepsilon}\big(g(\varepsilon z+g^{-1}(K))-K\big)\frac{1}{\sqrt{2\pi}}\exp\big(-\frac{z^2}{2}-\frac{zg^{-1}(K)}{\varepsilon}\big)A_\varepsilon(\varepsilon z+g^{-1}(K))dz.\end{aligned}$$

We define

$$\tilde{A}_{\varepsilon,n}(x)=\bar{a}_{\varepsilon,n}(g(x))g'(x)=\sum_{k=0}^{n}A_k(x)\varepsilon^{2k}.$$

We also define

$$\begin{aligned}&\tilde{C}_{\varepsilon,n}(T,K)\\&=\exp\big(-\frac{g^{-1}(K)^2}{2\varepsilon^2}\big)\int_0^{\tilde{K}_\varepsilon}\big(g(\varepsilon z+g^{-1}(K))-K\big)\frac{1}{\sqrt{2\pi}}\exp\big(-\frac{z^2}{2}-\frac{zg^{-1}(K)}{\varepsilon}\big)\tilde{A}_{\varepsilon,n}(\varepsilon z+g^{-1}(K))dz.\end{aligned}$$

Then there exist constants $C_1,\ C_2 > 0$ such that

$$\exp\big(\frac{g^{-1}(K)^2}{2\varepsilon^2}\big)\Big|\tilde{C}_\varepsilon(T,K) - \tilde{C}_{\varepsilon,n}(T,K)\Big| \leq C_1\varepsilon^{2n+2}.$$

Since

$$\Big|(g(\varepsilon z + g^{-1}(K)) - K)\tilde{A}_{\varepsilon,n}(\varepsilon z + g^{-1}(K)) - \sum_{\substack{n,m\geq 0\\ 2n+m+1\leq N}} c_{n,m}(g^{-1}(K))\varepsilon^{2n+m+1}z^{m+1}\Big| \leq C_2\varepsilon^{N+1},$$
$$K \in [x_0^1, K_1],$$

we have

$$\Big|\exp(\frac{e(K)}{\varepsilon^2})\tilde{C}_{\varepsilon,n}(T,K) - \sum_{\substack{n,m\geq 0\\ 2n+m+1\leq N}} c_{n,m}(g^{-1}(K))\varepsilon^{2n+m+1}\frac{1}{\sqrt{2\pi}}\varphi_{m+1}\big(\frac{g^{-1}(K)}{\varepsilon}\big)\Big| \leq R\varepsilon^{N+1},$$
$$K \in [x_0^1, K_1].$$

For any $\delta > 0$, we have

$$\begin{aligned} R_\varepsilon(K_0) &\leq E[X_\varepsilon^1(T); X_\varepsilon^1(T) > K_0] \\ &\leq E[X_\varepsilon^1(T)^{1/\delta}]^\delta P(X_\varepsilon^1(T) > K_0)^{1-\delta}. \end{aligned}$$

Therefore we have

$$\lim_{\varepsilon\downarrow 0}\varepsilon^2 \log R_\varepsilon(K_0) \leq \lim_{\varepsilon\downarrow 0}\varepsilon^2(1-\delta)\log P(X_\varepsilon^1(T) > K_0) = -(1-\delta)e(K_0).$$

Note that $e(K_0) > e(K_1)$, we have

$$\limsup_{\varepsilon\downarrow 0}\varepsilon^2 \log R_\varepsilon(K_0) < -e(K_1).$$

The function q defined by (1.17) can be written as

$$q(K) = g'(g^{-1}(K)) = \big(\frac{d}{dK}g^{-1}(K)\big)^{-1}. \tag{5.4}$$

Then we have our assertion. □

Finally we prove Theorem 1.2.

Proof of Theorem 1.2. From the definition given in (5.1), we have

$$R_2(\varepsilon,K) = \varepsilon\frac{c_{0,1}(g^{-1}(K))}{c_{0,0}(g^{-1}(K))}\frac{\varphi_2(g^{-1}(K)/\varepsilon)}{\varphi_1(g^{-1}(K)/\varepsilon)} + \varepsilon^2\frac{c_{0,2}(g^{-1}(K))}{c_{0,0}(g^{-1}(K))}\frac{\varphi_3(g^{-1}(K)/\varepsilon)}{\varphi_1(g^{-1}(K)/\varepsilon)} + \varepsilon^2\frac{c_{1,0}(g^{-1}(K)/\varepsilon)}{c_{0,0}(g^{-1}(K)/\varepsilon)}.$$

The second and third derivatives of g at $g^{-1}(K)$ are given as follows:

$$\begin{aligned} &\frac{d^2}{dK^2}g(g^{-1}(K)) = q(K)q'(K), \\ &\frac{d^3}{dK^3}g(g^{-1}(K)) = q(K)q'(K)^2 + q(K)^2q''(K). \end{aligned}$$

Using the definition of $c_{n,m}$ given in (5.2), we can calculate $c_{0,0},\ c_{0,1}, c_{1,0}, c_{0,2}$ explicitly as follows.

$$
\begin{aligned}
c_{0,0}(g^{-1}(K)) &= a_0(K)q(K)^2,\\
c_{1,0}(g^{-1}(K)) &= a_2(K)q(K)^2,\\
c_{0,1}(g^{-1}(K)) &= a_0'(K)q(K)^3 + \frac{3}{2}a_0(K)q(K)^2q'(K),\\
c_{0,2}(g^{-1}(K)) &= \frac{1}{2}a_0''(K)q(K)^4 + 2a_0'(K)q(K)^3q'(K) + \frac{7}{6}a_0(K)q(K)^2q'(K)^2\\
&\quad + \frac{2}{3}a_0(K)q(K)^3q''(K).
\end{aligned}
$$

Then we have our theorem. $\square$

6 Asymptotic expansion of implied volatilities

In this section, we will prove Theorem 1.3. First, we define smooth functions $\theta_n,\ n \in \mathbb{N}$, inductively by

$$
\begin{aligned}
\theta_1(x) &= \varphi_1(x),\\
\theta_{n+1}(x) &= -n\theta_n(x) + \theta_n'(x)\theta_1(x)x.
\end{aligned}
\tag{6.1}
$$

We define the function $h : [0,1]\times\mathbb{R}\to\mathbb{R}$ by

$$
h(t,y) \equiv f^{-1}(tf(y)),
\tag{6.2}
$$

where f is defined by (1.19). The properties of h are given in Appendix A. Then we have the following.

Proposition 6.1. *The implied normal volatilities of call options are given as follows.*

$$
\sigma_N^{\varepsilon}(T,K) = \frac{\varepsilon(K-x_0^1)}{g^{-1}(K)\sqrt{T}}\exp\bigl(-\int_1^{1+l(\varepsilon,K)}\frac{1}{t}\varphi_1(h(t,\frac{g^{-1}(K)}{\varepsilon}))dt\bigr), \quad K > x_0^1.
$$

Here

$$
l(\varepsilon,K) = (1+R(\varepsilon,K)(1+r(K)) - 1,
$$

where

$$
R(\varepsilon,K) = \frac{\sqrt{2\pi}\exp(\frac{e(K)}{\varepsilon^2})C_\varepsilon(T,K)}{\varepsilon c_{0,0}(g^{-1}(K))\varphi_1(g^{-1}(K)/\varepsilon)} - 1,
$$

and

$$
r(K) = \frac{g^{-1}(K)c_{0,0}(g^{-1}(K))}{(K-x_0^1)} - 1.
$$

R and r satisfies following respectively:

$$
\bigl|R(\varepsilon,K) - R_N(\varepsilon,K)\bigr| \le C\varepsilon^N,
\tag{6.3}
$$

and

$$
\lim_{K\downarrow x_0^1} r(K) = 0.
$$

Proof. Since Theorem 5.1 and

$$\sup_{x \geq 0} \frac{\varphi_n(x)}{\varphi_1(x)} < \infty$$

we have (6.3). Using l'Hospital's rule, we have

$$\lim_{K \downarrow 0} \frac{g^{-1}(K) c_{0,0}(g^{-1}(K))}{K - x_0^1} = g'(x_0^1) a_0(x_0^1) = 1.$$

By definition of R, we can rewrite the value of call option as

$$C_\varepsilon(T, K) = f(g^{-1}(K)/\varepsilon) g^{-1}(K) c_{0,0}(g^{-1}(K))(1 + R(\varepsilon, K)).$$

On the other hand, the value of call option under the normal model is given by

$$V = (K - x_0^1) f\big(\frac{K - x_0^1}{\sigma\sqrt{T}}\big).$$

Therefore we have

$$f\big(\frac{K - x_0^1}{\sigma\sqrt{T}}\big) = (1 + r(K))(1 + R(\varepsilon, K)) f\big(\frac{g^{-1}(K)}{\varepsilon}\big).$$

Using the definition of h given by (6.2) and Lemma A.4, we have our assertion. $\square$

Next we will give the asymptotic expansion of implied volatilities.

THEOREM 6.2. *For any $N \in \mathbb{N}$, there is a constant $C > 0$ such that the asymptotic expansion of implied volatilities satisfy following.*

$$\Big| \Big(\frac{\varepsilon(K - x_0^1)}{g^{-1}(K)\sqrt{T}} \Big)^{-1} \sigma_N(T, K) - \exp\Big(\sum_{n=0}^{N} \frac{l_N(\varepsilon, K)^{n+1}}{(n+1)!} \theta_{n+1}\big(\frac{g^{-1}(K)}{\varepsilon}\big) \Big) \Big| < C(\varepsilon + |K - x_0^1|)^{N+1},$$
$$K \in [x_0^1, K_1].$$

Here

$$l_N(\varepsilon, K) = (1 + R_N(\varepsilon, K))(1 + r(K)) - 1, \tag{6.4}$$

where

$$r(K) = \frac{g^{-1}(K) c_{0,0}(g^{-1}(K))}{K - x_0^1} - 1. \tag{6.5}$$

Proof. Using Lemma A.4, we have

$$\Big(\frac{\partial}{\partial t} \Big)^n \frac{1}{t} \varphi_1(h(t, y)) \Big|_{t=1} = \theta_n(y), \ n \geq 1.$$

Therefore

$$
\begin{aligned}
&\Big|\int_1^{1+l(\varepsilon,K)} \frac{1}{t}\theta_1(h(t,\frac{g^{-1}(K)}{\varepsilon}))dt - \sum_{n=0}^{N}\int_1^{1+l_N(\varepsilon,K)} \frac{\theta_n(y)}{n!}(t-1)^n dt\Big| \\
\le&\Big|\int_1^{1+l(\varepsilon,K)} \frac{1}{t}\theta_1(h(t,\frac{g^{-1}(K)}{\varepsilon}))dt - \int_1^{1+l_N(\varepsilon,K)} \frac{1}{t}\theta_1(h(t,\frac{g^{-1}(K)}{\varepsilon}))dt\Big| \\
&+\Big|\int_1^{1+l_N(\varepsilon,K)} \frac{1}{t}\theta_1(h(t,\frac{g^{-1}(K)}{\varepsilon}))dt - \sum_{n=0}^{N}\int_1^{1+l_N(\varepsilon,K)} \frac{\theta_n(y)}{n!}(t-1)^n dt\Big| \\
&\le C_1|l(\varepsilon,K)-l_N(\varepsilon,K)| + C_2|l_N(\varepsilon,K)|^N \le C(\varepsilon+|K-x_0^1|)^N.
\end{aligned}
$$

□

Finally we prove Theorem 1.3.

LEMMA 6.3. *The derivatives of $q,\ a_0,\ a_2$ at x_0 are given as follows.*

$$
\begin{aligned}
&q(x_0^1)=\frac{1}{\sqrt{c_1}}, \quad \frac{q'(x_0^1)}{q(x_0^1)}=-\frac{2}{3}\frac{c_2}{c_1}, \quad \frac{q''(x_0^1)}{q(x_0^1)}=\frac{11}{9}\Big(\frac{c_2}{c_1}\Big)^2-\frac{3}{2}\frac{c_3}{c_1}, \\
&\frac{a_0'(x_0^1)}{a_0(x_0^1)}=\frac{c_2}{c_1}, \quad \frac{a_0''(x_0^1)}{a_0(x_0^1)}=c_1^2L-\Big(\frac{c_2}{c_1}\Big)^2+\frac{3c_3}{c_1}, \\
&\frac{a_2(x_0^1)}{a_0(x_0^1)}=\frac{1}{c_1}\Big(-\frac{c_1^2L}{2}+\frac{2}{3}\Big(\frac{c_2}{c_1}\Big)^2-\frac{3}{4}\frac{c_3}{c_1}\Big),
\end{aligned}
$$

where $c_i,\ (i=1,2,3)$ are given by (3.5).

Proof. Since

$$e(g(x))=\frac{1}{2}x^2,$$

and $g'(x)>0$, the derivatives are given by

$$
\begin{aligned}
&x=e'(g(x))g'(x), \\
&1=e''(g(x))g'(x)^2+e'(g(x))g''(x), \\
&0=e'''(g(x))g'(x)^3+3e''(g(x))g'(x)g''(x)+e'(g(x))g'''(x), \\
&0=e^{(4)}(g(x))g'(x)^4+6e'''(g(x))g'(x)^2g''(x)+3e''(g(x))g''(x)^2 \\
&\quad+4e''(g(x))g'(x)g'''(x)+e'(g(x))g^{(4)}(x).
\end{aligned}
$$

Furthermore, since

$$e'(x_0^1)=0,\ e''(x_0^1)=\frac{1}{b_1},\ e'''(x_0^1)=-\frac{2b_2}{b_1^3},\ e^{(4)}(x_0^1)=-\frac{6b_3}{b_1^4}+\frac{12b_2^2}{b_1^5},$$

we have

$$g'(0)=\sqrt{b_1},\ g''(0)=\frac{2}{3}\frac{b_2}{b_1},\ g'''(0)=\frac{\sqrt{b_1}}{6}(9b_1b_3-8b_2^2).$$

□

LEMMA 6.4.

$$\begin{aligned}&\left|R_2(\varepsilon,K)-R_2^0(\varepsilon,K)\right|\le C(\varepsilon+|K-x_0^1|)^3,\\&\left|r(K)-r^0(K)\right|\le C|K-x_0^1|^3,\end{aligned}$$

where

$$\begin{aligned}R_2^0(\varepsilon,K)=&\frac{\varepsilon(K-x_0^1)}{\sqrt{c_1}}\Big[c_1^2L-\frac{5}{6}\Big(\frac{c_2}{c_1}\Big)^2+\frac{3}{4}\frac{c_3}{c_1}\Big]\frac{\varphi_2(g^{-1}(K)/\varepsilon)}{\varphi_1(g^{-1}(K)/\varepsilon)}\\&+\frac{\varepsilon^2}{c_1}\Big[\frac{c_1^2L}{2}-\frac{1}{2}\Big(\frac{c_2}{c_1}\Big)^2+\frac{1}{2}\frac{c_3}{c_1}\Big]\frac{\varphi_3(g^{-1}(K)/\varepsilon)}{\varphi_1(g^{-1}(K)/\varepsilon)}+\frac{\varepsilon^2}{c_1}\Big[-\frac{c_1^2L}{2}+\frac{2}{3}\Big(\frac{c_2}{c_1}\Big)^2-\frac{3}{4}\frac{c_3}{c_1}\Big],\end{aligned}$$

and

$$r^0(K)=\Big[-\frac{1}{3}\Big(\frac{c_2}{c_1}\Big)^2+\frac{1}{4}\frac{c_3}{c_1}+\frac{c_1^2L}{2}\Big](K-x_0^1)^2.$$

Proof. We will calculate each terms of R_2 given by (1.18). From Lemma A.1, the functions φ_2/φ_1 and φ_3/φ_1 are bounded above. Since the first term is $O(\varepsilon)$ and other terms are $O(\varepsilon^2)$, it is enough to calculate the first order of K in the first term and 0th order in the other terms. Using Lemma 6.3, we have

$$\frac{c_{0,1}(x_0^1)}{c_{0,0}(x_0^1)}=0,$$

and the first derivative is given by

$$\frac{d}{dK}\frac{c_{0,1}(g^{-1}(K))}{c_{0,0}(g^{-1}(K))}=q(K)\Big[\frac{a_0''(K)}{a_0(K)}+\Big(\frac{a_0'(K)}{a_0(K)}\Big)^2+\frac{3}{2}\frac{q''(K)}{q(K)}+\frac{a_0'(K)}{a_0(K)}\frac{q'(K)}{q(K)}\Big],$$

and use Lemma 6.3 again, we have

$$\begin{aligned}\frac{c_{0,1}(g^{-1}(K))}{c_{0,0}(g^{-1}(K))}&\underset{1}{\sim}\frac{(K-x_0^1)}{\sqrt{c_1}}\Big[c_1^2L-\frac{5}{6}\Big(\frac{c_2}{c_1}\Big)^2+\frac{3}{4}\frac{c_3}{c_1}\Big],\\\frac{c_{0,2}(g^{-1}(K))}{c_{0,0}(g^{-1}(K))}&\underset{0}{\sim}\frac{1}{c_1}\Big[\frac{c_1^2L}{2}-\frac{1}{2}\Big(\frac{c_2}{c_1}\Big)^2+\frac{1}{2}\frac{c_3}{c_1}\Big],\\\frac{c_{1,0}(g^{-1}(K))}{c_{0,0}(g^{-1}(K))}&\underset{0}{\sim}\frac{1}{c_1}\Big[-\frac{c_1^2L}{2}+\frac{2}{3}\Big(\frac{c_2}{c_1}\Big)^2-\frac{3}{4}\frac{c_3}{c_1}\Big].\end{aligned}$$

We can calculate $r(K)$ in the same way and we have our results. □

Proof of Theorem 1.3. Using (6.4) we have

$$l_2(\varepsilon,K)\underset{2}{\sim}R_2^0(\varepsilon,K)+r^0(K)$$

Since R_2^0 and r^0 are second order in ε, K, we have

$$\sum_{n=0}^{2}\frac{l_2(\varepsilon,K)^{n+1}}{(n+1)!}\theta_{n+1}(\frac{g^{-1}(K)}{\varepsilon})\underset{2}{\sim}(R_2^0(\varepsilon,K)+r^0(K))\varphi_1(\frac{g^{-1}(K)}{\varepsilon}),$$

then we have our result. □

7 Watanabe's Theorem

In this section, we give the correspondence between our result and the expansion given in Watanabe [14], Yoshida [15], Takahashi-Kunitomo [8] and [12].

We define

$$G(y) = \int_y^\infty (x-y)\phi(x)dx.$$

PROPOSITION 7.1. *For each $y \in \mathbb{R}$, let $K_\varepsilon = x_0^1 + \varepsilon \frac{y}{\sqrt{c_1}}$, $\varepsilon \in (0,1]$. For any $r \geq 0$, there is a constant $C > 0$ such that the value of a call option of strike K_ε, maturity T satisfies following.*

$$\Big| E[(X_\varepsilon^1(T) - K_\varepsilon)_+] - \frac{\varepsilon}{\sqrt{c_1}}\Big[G(y) - \varepsilon\frac{c_2}{3c_1^{3/2}}H_1(y)\phi(y) + \varepsilon^2\frac{c_2^2}{18c_1^3}H_4(y)\phi(y)$$
$$+ \varepsilon^2\big(\frac{c_2^2}{2c_1^3} - \frac{c_3}{4c_1^2}\big)H_2(y)\phi(y) + \varepsilon^2\big(\frac{c_1 L}{2}\big)H_0(y)\phi(y)\Big]\Big| \leq \varepsilon^3 C,\ \varepsilon \in (0,1],\ y \in [-r,r].$$

Proof. The asymptotic expansion of the probability density is given by Corollary 4.3. Since

$$\int_y^\infty (x-y)H_n(x)\phi(x)dx = H_{n-2}(y)\phi(y), \quad n \geq 2,$$

we have our assertion. □

This formula coincides with the formula given in [12]. We can calculate the implied normal volatility in the same way as the proof of [12] Theorem 1.1.

PROPOSITION 7.2. *For each $y \in \mathbb{R}$, let $K_\varepsilon = x_0^1 + \varepsilon \frac{y}{\sqrt{c_1}}$, $\varepsilon \in (0,1]$. For any $r \geq 0$, there is a constant $C > 0$ such that the asymptotic expansion of the implied normal volatility satisfies following.*

$$\Big|\frac{\sigma_N(T,K_\varepsilon)}{\varepsilon} - \frac{1}{\sqrt{c_1 T}}\Big[1 - \varepsilon\frac{c_2}{3c_1^{3/2}}y + \varepsilon^2\Big\{\Big(\frac{1}{6}\frac{c_2^2}{c_1^3} - \frac{1}{4}\frac{c_3}{c_1^2}\Big)y^2 + \frac{c_1 L}{2} - \frac{1}{3}\frac{c_2^2}{c_1^3} + \frac{1}{4}\frac{c_3}{c_1^2}\Big\}\Big]\Big|$$
$$\leq \varepsilon^3 C, \quad \varepsilon \in (0,1], \quad y \in [-r,r]. \tag{7.1}$$

REMARK 7.3. The Taylor expansion of (1.20) to the second order is given by

$$\Big|\frac{K_\varepsilon - x_0^1}{\sqrt{2e(K_\varepsilon)T}} - \frac{1}{\sqrt{c_1 T}}\Big[1 - \frac{1}{3}\frac{c_2}{c_1}(K_\varepsilon - x_0^1) + \Big(\frac{1}{6}\big(\frac{c_2}{c_1}\big)^2 - \frac{1}{4}\frac{c_3}{c_1}\Big)(K_\varepsilon - x_0^1)^2\Big]\Big| = O(|K_\varepsilon - x_0^1|^3).$$

Then we have

$$\frac{\sigma_N(T,K_\varepsilon)}{\varepsilon} \sim \frac{K_\varepsilon - x_0^1}{\sqrt{2e(K)T}}\Big[1 + \varepsilon^2\Big\{\frac{c_1 L}{2} - \frac{1}{3}\frac{c_2^2}{c_1^3} + \frac{1}{4}\frac{c_3}{c_1^2}\Big\}\Big]$$
$$= \frac{K_\varepsilon - x_0^1}{\sqrt{2e(K)T}}(1 + R_2(\varepsilon, x_0^1)). \tag{7.2}$$

As we see in the next section, applying this formula for SABR model we obtain SABR formula.

8 Examples

In this section, we apply our results to some known models.

8.1 Local volatility models

We assume the following model. Let $\sigma : \mathbb{R} \to \mathbb{R}_+$ be a smooth function whose derivatives of any order are bounded. Let λ be continuous $\mathbb{R}_+$-valued functions defined on $[0, T]$.

$$
\begin{aligned}
&dX^\varepsilon(t) = \varepsilon\lambda(t)\sigma(X^\varepsilon(t))dW_t,\\
&X^\varepsilon(0) = x_0.
\end{aligned}
$$

In this case we can solve the energy as following.

$$
e(y) = \frac{1}{2\Lambda}\Big(\int_{x_0}^{y} \frac{dx}{\sigma(x)}\Big)^2,
$$

where

$$
\Lambda = \int_0^T \lambda^2(t)dt.
$$

Here minimum energy path h is given by

$$
h(t) = \frac{1}{\Lambda}\Big(\int_{x_0}^{y} \frac{dx}{\sigma(x)}\Big)\int_0^t \lambda(s)ds.
$$

We can easily calculate the coefficients.

$$
\begin{aligned}
&b_1 = \sigma(x_0)^2\Lambda, \quad b_2 = \frac{3}{2}\sigma(x_0)^3\sigma'(x_0)\Lambda^2, \quad b_3 = \Big(\frac{8}{3}\sigma(x_0)^4\sigma'(x_0)^2 + \frac{2}{3}\sigma(x_0)^5\sigma''(x_0)\Big)\Lambda^3,\\
&L = \Big(\frac{1}{2}\sigma(x_0)^2\sigma'(x_0)^2 + \frac{1}{2}\sigma(x_0)^3\sigma''(x_0)\Big)\Lambda^2, \quad g^{-1}(y) = \frac{1}{\sqrt{\Lambda}}\Big(\int_{x_0}^{y} \frac{dx}{\sigma(x)}\Big).
\end{aligned}
$$

Then using Theorem 1.1 and Theorem 1.3 we can calculate the density function and implied normal volatilities. We illustrate some cases.

Example 8.1 (CEV model). This is in the case $\lambda(t) \equiv \alpha$ and

$$
\sigma(x) = x^\beta.
$$

and each terms are given by

$$
\begin{aligned}
&\Lambda = \alpha^2 T, \quad b_1 = x_0^{2\beta}\Lambda, \quad b_2 = \frac{3}{2}\beta x_0^{4\beta-1}\Lambda^2, \quad b_3 = \frac{2}{3}(\beta^2 - \beta + 4)x_0^{6\beta-2}\Lambda^3,\\
&L = (\beta^2 - \frac{\beta}{2})x_0^{4\beta-2}\Lambda^2, \quad e''(y) = \frac{\beta(1+\beta)}{2\alpha^2 T y^{\beta+2}},
\end{aligned}
$$

$$
g^{-1}(y) = \begin{cases} \frac{1}{\sqrt{\Lambda}}\Big(\frac{y^{1-\beta} - x_0^{1-\beta}}{1-\beta}\Big) & (\beta \neq 1)\\ \frac{1}{\sqrt{\Lambda}}\log(\frac{y}{x_0}) & (\beta = 1). \end{cases}
$$

Example 8.2 (Displaced diffusion). This is the case $\lambda(t) \equiv \sigma$ and

$$\sigma(x) = qx + (1-q)x_0.$$

Each terms are given by

$$\Lambda = \sigma^2 T, \quad b_1 = x_0^2\Lambda, \quad b_2 = \frac{3}{2}x_0^3 q\Lambda^2, \quad b_3 = \frac{8}{3}x_0^4 q^2\Lambda^3, \quad L = \frac{1}{2}x_0^2 q^2\Lambda^2,$$
$$g^{-1}(y) = \frac{1}{\sqrt{\Lambda}}\int_{x_0}^{y}\frac{dx}{qx+(1-q)x_0} = \frac{1}{q\sqrt{\Lambda}}\log(\frac{qy+(1-q)x_0}{x_0}),$$
$$e''(y) = \frac{1+g^{-1}(y)q\sqrt{\Lambda}}{\Lambda(qy+(1-q)x_0)^2}.$$

Black-Scholes model is the case $q = 1$. We present a numerical results of the asymptotic expansion formula comparing with analytical solution.

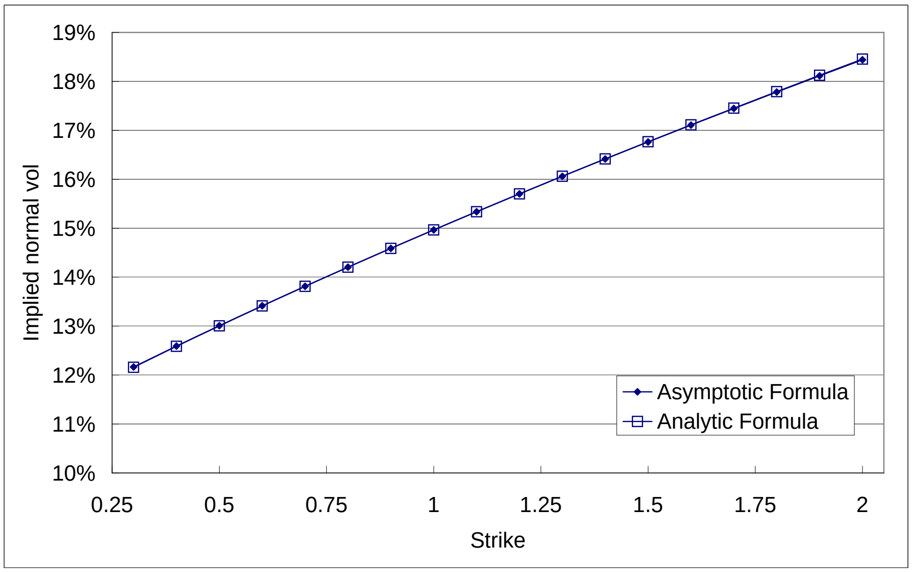

Figure 1: Implied volatility smile of displaced diffusion, asymptotic expansion vs analytic solution with $x_0 = 1.0,\ q = 0.5,\ \sigma = 0.15,\ T = 10$.

8.2 SABR model

We investigate the following model which is called SABR model.

$$dX^\varepsilon(t) = \varepsilon\alpha^\varepsilon(t)\sigma(X^\varepsilon(t))(\rho dW(t) + \sqrt{1-\rho^2}dZ(t)),$$
$$d\alpha^\varepsilon(t) = \varepsilon\nu\alpha^\varepsilon(t)dW(t),$$
$$X^\varepsilon(0) = x_0,\ \alpha^\varepsilon(0) = \alpha.$$

This model was investigated in Hagan-Kumar-Lesniewski-Woodward [4] and [12]. In Theorem 3.1 [12], we gave the energy function as follows.

$$e(y) = \frac{1}{2\nu^2 T}\log(\frac{\sqrt{1-2\rho\zeta+\zeta^2}-\rho+\zeta}{1-\rho})^2 = \frac{\hat{x}(\zeta(y))^2}{2\nu^2 T},$$

where

$$\zeta(y) = -\frac{\nu}{\alpha}\int_{x_0}^{y}\frac{dz}{\sigma(z)}.$$

Then the parameters are given by

$$\begin{aligned}
b_1 &= \alpha^2\sigma(x_0)^2 T, \quad b_2 = \frac{3}{2}\sigma(x_0)^3\alpha^3(\alpha\sigma'(x_0)+\nu\rho)T^2,\\
b_3 &= \Big(\frac{8}{3}\alpha^6\sigma(x_0)^4\sigma'(x_0)^2 + \frac{2}{3}\alpha^6\sigma(X_0)^5\sigma''(x_0) + 6\nu\rho\sigma(x_0)^4\sigma'(x_0)\alpha^5\\
&\quad + 2\nu^2\rho^2\sigma(x_0)^4\alpha^4 + \frac{2}{3}\alpha^4\sigma(x_0)^4\nu^2\Big)T^3,\\
L &= \frac{\alpha^2\sigma(x_0)^2T^2}{2}\Big(\alpha^2(\sigma'(x_0)^2+\sigma(x_0)\sigma''(x_0)) + 4\nu\rho\alpha\sigma'(x_0)+\nu^2\Big),\\
g^{-1}(y) &= \frac{1}{\nu\sqrt{T}}\log(\frac{\sqrt{1-2\rho\zeta(y)+\zeta(y)^2}-\rho+\zeta(y)}{1-\rho}).
\end{aligned}$$

We present a numerical results of the asymptotic expansion formula comparing with Monte Carlo simulation. Here we assume $\sigma(x) = x^\beta$.

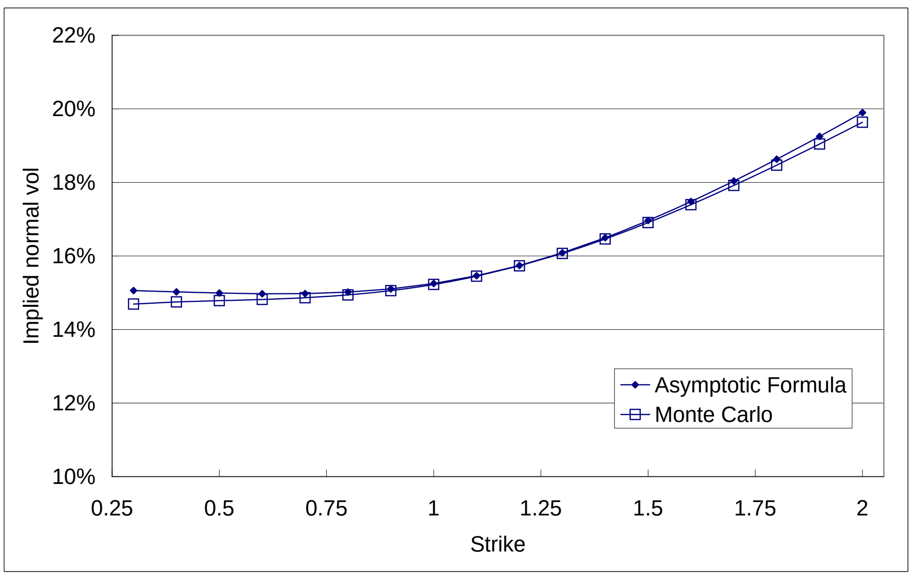

Figure 2: Implied volatility smile of SABR model, asymptotic expansion vs Monte Carlo simulation with $x_0 = 1,\ \alpha = 0.15,\ \beta = 0.5,\ \nu = 0.2,\ \rho = -0.2,\ T = 10$.

Applying the formula (7.2) for SABR model, we obtain

$$\sigma_N(T,K) = \frac{K-x_0}{\sqrt{2e(K)T}}\Big(1+\Big[\frac{2\sigma(x_0)\sigma''(x_0)-\sigma'(X_0)^2}{24}\alpha^2 + \frac{1}{4}\rho\nu\alpha\sigma'(x_0) + \frac{2-3\rho^2}{24}\nu^2\Big]T\Big).$$

This is almost the same as original SABR formula given in [4].

Appendix

A Special functions

In this section, we investigate some properties of functions defined in Section 1. First we consider $\varphi_n,\ n \geq 0$ defined by (1.16).

LEMMA A.1. *The functions φ_n have the following properties.*

(1) $\varphi_n(x) > 0,\ x \geq 0$.

(2) $\lim_{x\to\infty} x^{n+1}\varphi_n(x) = n!$.

(3) $\displaystyle\sup_x x^n \frac{\varphi_n(x)}{\varphi_1(x)} < \infty, \qquad n \geq 1.$

Proof. (1) is easy to check. We prove (2). Putting $y = xz$

$$\varphi_n(x) = \int_0^\infty \exp(-\frac{y^2}{2x^2} - y)(\frac{y}{x})^n \frac{dy}{x} = \frac{1}{x^{n+1}} \int_0^\infty y^n \exp(-y - \frac{y^2}{2x^2})dy$$

Then we know

$$\lim_{x\to\infty} x^{n+1}\varphi_n(x) = \int_0^\infty y^n e^{-y} dy = n!.$$

□

The following is easy to check.

LEMMA A.2. *The functions $\{\varphi_n\}$ satisfy the following recurrence relations.*

$$\begin{aligned}
\varphi_{n+1}(x) &= -x\varphi_n(x) + n\varphi_{n-1}(x), \\
\varphi_n'(x) &= -\varphi_{n-1}(x).
\end{aligned}$$

EXAMPLE A.3. φ_i $(0 \leq i \leq 3)$ are given as follows:

$$\begin{aligned}
\varphi_0(x) &= \exp(\frac{x^2}{2}) \int_x^\infty \exp(-\frac{z^2}{2})dz, \\
\varphi_1(x) &= -x\varphi_0(x) + 1, \\
\varphi_2(x) &= (x^2+1)\varphi_0(x) - x, \\
\varphi_3(x) &= -(x^3+3x)\varphi_0(x) + x^2 + 2.
\end{aligned}$$

Next we consider the function $h \in C^\infty([0,1] \times \mathbb{R}_+)$ defined by (6.2).

LEMMA A.4. *The n-times differentiation of $\log h(t,y)$ with respect to t is given as follows. We define θ in (6.1).*

$$(\frac{\partial}{\partial t})^n \log h(t,y) = \frac{1}{t^n}\theta_n(h(t,y)), \quad t \in [0,1],\ y > 0,$$

where $\theta_n \in C_b[0,\infty],\ n \geq 1$ are given inductively as follows:

$$\begin{aligned}
\theta_1(x) &= \varphi_1(x), \\
\theta_{n+1}(x) &= n\theta_n(x) + \theta_n'(x)\theta_1(x)x.
\end{aligned}$$

Proof. In the case $n = 1$, since $f(h(t, y)) = tf(y)$, we have

$$\frac{\partial h}{\partial t}(t, y) = \frac{f(h(t, y))}{tf'(h(t, y))}.$$

Since

$$f'(x) = -(\frac{1}{x} + x + \frac{\varphi_2(x)}{\varphi_1(x)})f(x) < 0, \ x > 0,$$

we have

$$\theta_1(x) = \frac{f(x)}{xf'(x)} = \Big(1 + x^2 + x\frac{\varphi_2(x)}{\varphi_1(x)}\Big)^{-1} = \varphi_1(x).$$

It is easy to check that $\theta_1 \in C_b([0, \infty])$ and $x\theta_1(x) \in C_b([0, \infty])$. We have

$$\frac{\partial}{\partial t} \log h(t, y) = \frac{1}{t}\theta_1(h(t, y)).$$

Since

$$\frac{\partial}{\partial t}\big(\frac{1}{t^n}\theta_n(h(t, y))\big) = \frac{1}{t^{n+1}}\big(-n\theta_n(h(t, y)) + \theta_n'(h(t, y))\theta_1(h(t, y))h(t, y)\big),$$

it is easy to prove our lemma. □

B The implied volatilities of put options

In this section, we discuss about the implied volatilities for the case $K < x_0^1$. We define the forward value of a put option of strike rate K and maturity T by

$$P_\varepsilon(T, K) = E[(K - X_\varepsilon^1(T))^+]$$

Since we have put-call parity, the implied volatility of the put option is the same as the implied volatility of a call option with strike rate K and maturity T. Since

$$P_\varepsilon(T, K) = E[(-X_\varepsilon^1(T) - (-K))_+] = E[(-(X_\varepsilon^1(T) - x_0^1) - (-(K - x_0^1)))_+]$$

It is enough to discuss in the case $x_0^1 = 0$.

Let $x = (x^1, \ldots, x^n) \in \mathbb{R}^n$. We denote $\bar{x} = (-x^1, x^2, \ldots, x^n)$. We define $\bar{X}_\varepsilon(t) = \bar{X_\varepsilon}(t)$. Then we have

$$d\bar{X}_\varepsilon^i(t) = \sum_{k=1}^d \varepsilon\bar{V}_k^i(t, \tilde{X}_\varepsilon(t))dW^k(t) + \bar{V}_0^i(t, \tilde{X}_\varepsilon(t))dt, \quad 1 \leq i \leq N,$$

where

$$\bar{V}_k^j(t, x) = \begin{cases} -V_k^1(t, \bar{x}) & (1 \leq k \leq d) \\ V_k^j(t, \bar{x}) & (1 \leq k \leq d, \ j \neq 1). \end{cases}$$

Since the associated Riemaniann metric $\bar{g}^{ij}(t, x) = \sum_{k=1}^d \tilde{\bar{V}}_k^i \tilde{\bar{V}}_k^j$ is given by

$$\bar{g}^{11}(t, x) = g^{11}(t, x), \quad \bar{g}^{1i}(t, x) = -g^{1i}(t, x) \ (i \neq 1), \quad \bar{g}^{ij}(t, x) = g^{ij}(t, x) \ (i, j \neq 1),$$

we have

$$\bar{b}_1 = b_1, \quad \bar{b}_2 = -b_2, \quad \bar{b}_3 = b_3, \quad \bar{L} = L..$$

Therefore Theorem 1.1 and Theorem 1.3 still hold for $K < x_0^1$.

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