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THE DIMENSION OF THE SPACE OF SIEGEL EISENSTEIN SERIES OF WEIGHT ONE

KEIICHI GUNJI

Abstract. In general, it is difficult to determine the dimension of the space of Siegel modular forms of low weights. In particular, the dimension of the space of cusp forms are known in only a few cases. In this paper, we calculate the dimension of the space of Siegel Eisenstein series of weight 1, which is a certain subspace of a complement of the space of cusp forms.

1. Introduction

Let p be an odd prime number. In this paper, we determine the dimension of the space of Siegel Eisenstein series of weight 1 associated with the principal congruence subgroup $\Gamma^g(p)$ of level p, degree g. More precisely, let $M_1(\Gamma^g(p))$ be the space of holomorphic Siegel modular forms of weight 1, degree $g \geq 2$, and let $E_1^0(\Gamma^g(p))$ be a complement of the space of all functions which vanish at all 0-dimensional cusps.

Theorem (Theorem 3.1). *For $g \geq 2$, we have*

$$\dim E_1^0(\Gamma^g(p)) = \begin{cases} \frac{1}{2}(p^g+1) & p \equiv 3 \bmod 4 \\ 0 & p \equiv 1 \bmod 4. \end{cases}$$

In other words, we may take as $E_1^0(\Gamma^g(p))$ the space of theta functions of quadratic forms of level p.

The representation theory of the finite group $Sp(g, \mathbb{F}_p)$ is crucial in our proof. The representation of $Sp(g, \mathbb{F}_p)$ on $E_k^0(\Gamma^g(p))$ is isomorphic to a sub-representation of the induced representation of a certain character of the subgroup $\overline{P}_0$, which is the image of the Siegel parabolic subgroup P_0 of $Sp(g, \mathbb{Z})$ (Lemma 3.2). Moreover, each irreducible component of the induced representation is generated by the elements of $E_k(\Gamma_0^g(p), \psi)$ for some Dirichlet character ψ. Thus Theorem 3.1 is reduced to computing the dimension of $E_1(\Gamma_0^g(p), \psi)$.

Proposition (Proposition 3.4).

$$\dim E_1^0(\Gamma_0^g(p), \psi) = \begin{cases} 1 & \psi^2 \equiv 1, \ \psi(-1) = -1; \\ 0 & \textit{otherwise}. \end{cases}$$

The structure of the boundary of the Satake compactification of $\Gamma_0^g(p)\backslash\mathbb{H}_2$ is very simple: there are $g-1$ one-dimensional cusps and g zero-dimensional cusps. We prove Proposition 3.4 by using this fact and properties of elliptic modular forms of weight 1, which we consider in §2.

We remark that if $g \geq 3$, J.-S. Li already determined the dimension of $M_1(\Gamma^g(p^r))$ ([L]). Moreover, by the theory of singular series, studied by Resnikoff ([R]) and Freitag ([F1], [F2]), we know that $M_1(\Gamma_0^g(p), \psi)$ is generated by theta functions of quadratic forms, if $g \geq 3$. Thus the essential part of our result is the case of $g = 2$. We also remark that Weissauer asserts that the space $S_1(\Gamma_0^2(N), \psi)$ of cusp forms of weight 1, degree 2 is generated by theta series ([W, Theorem 4]).

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Notations: Let $\mathbb{H}_g = \{Z \in M_g(\mathbb{C}) \mid {}^tZ = Z,\ \operatorname{Im} Z > 0\}$ be the Siegel upper half space. We put $\Gamma^g = Sp(g,\mathbb{Z}) := \{\gamma \in SL(2g,\mathbb{Z}) \mid {}^t\gamma J_g \gamma = J_g\}$ for $J_g = \begin{pmatrix} 0 & 1_g \\ -1_g & 0 \end{pmatrix}$. The congruence subgroups $\Gamma^g(N)$ and $\Gamma_0^g(N)$ are given by

$$\Gamma^g(N) := \{\gamma \in \Gamma^g \mid \gamma \equiv 1_{2g} \bmod N\}, \quad \Gamma_0^g(N) := \left\{\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \,\middle|\, C \equiv 0 \bmod N\right\}.$$

For a function f on $\mathbb{H}_g$ and $\gamma \in \Gamma^g$, we put $(f|_k\gamma)(Z) = j(\gamma,z)^{-k} f(\gamma\langle Z\rangle)$, here $j(\gamma,Z) = \det(CZ+D)$ and $\gamma\langle Z\rangle = (AZ+B)(CZ+D)^{-1}$ for $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. For a congruence subgroup $\Gamma' \subset \Gamma^g$ we define the space of Siegel modular forms of weight k as:

$$M_k(\Gamma') = \left\{ f : \text{a holomorphic function on } \mathbb{H}_g \,\middle|\, \begin{array}{l} f|_k\gamma = f \text{ for all } \gamma \in \Gamma', \\ f \text{ is holomorphic at each cusp if } g = 1 \end{array} \right\}.$$

We define the Siegel Φ-operators for $0 \le r \le g-1$ by

$$\Phi^r(f)(z_r) := \lim_{\lambda\to\infty} f\begin{pmatrix} z_r & 0 \\ 0 & i\lambda 1_{g-r} \end{pmatrix}, \quad z_r \in \mathbb{H}_r,$$

and define the space of cusp forms $S_k(\Gamma') := \{f \in M_k(\Gamma') \mid \Phi^{g-1}(f|_k\gamma) = 0 \text{ for all } \gamma \in \Gamma^g\}$.

2. Elliptic modular forms

First, we recall the classical theory of elliptic modular forms. Some of the facts in this section are first proved by Hecke (cf. [H]). We mainly refer to the book of Schoeneberg [Sc, Chapter VII].

Let p be an odd prime number, and $\mathbf{a} = (a_1,a_2) \in \mathbb{Z}^2$. For $z \in \mathbb{H}_1$ and $s \in \mathbb{C}$, we define

$$\phi^1_{\mathbf{a}}(z,s) := \sum_{\substack{m_1,m_2\in\mathbb{Z} \\ (m_1,m_2)\equiv(a_1,a_2) \bmod p}} (m_1z+m_2)^{-1}|m_1z+m_2|^{-s}.$$

If $\operatorname{Re} s > 1$, the infinite sum of the right hand side converges absolutely for every $z \in \mathbb{H}_1$.

It is easy to see that

$$\begin{aligned} &\phi^1_{-\mathbf{a}}(z,s) = -\phi^1_{\mathbf{a}}(z,s), \\ &\phi^1_{\mathbf{a}}(z,s) = \phi^1_{\mathbf{b}}(z,s) \quad \text{if } \mathbf{a} \equiv \mathbf{b} \bmod p, \\ &j(\gamma,z)^{-1}\phi^1_{\mathbf{a}}(\gamma\langle z\rangle,s) = |j(\gamma,z)|^{-s}\phi^1_{\mathbf{a}'}(z,s), \quad \text{with } \mathbf{a}' = \mathbf{a}\gamma,\ \gamma \in SL(2,\mathbb{Z}). \end{aligned} \tag{2.1}$$

Theorem 2.1 (Hecke)**.** *The function $\phi^1_{\mathbf{a}}(z,s)$ is continued meromorphically on the whole s-plane and it is holomorphic at $s=0$. Moreover, $\phi^1_{\mathbf{a}}(z,0)$ is holomorphic in z.*

We put $e^1_{\mathbf{a}}(z) = \phi^1_{\mathbf{a}}(z,0)$. From (2.1), we see that $e^1_{\mathbf{a}}|_1\gamma = e^1_{\mathbf{a}}$ for $\gamma \in \Gamma^1(p)$. In order to show $e^1_{\mathbf{a}} \in M_1(\Gamma^1(p))$, we write down the Fourier expansion of $e^1_{\mathbf{a}}$ explicitly. For this we put

$$\delta\left(\frac{a}{p}\right) = \begin{cases} 1 & \text{if } a \equiv 0 \bmod p, \\ 0 & \text{otherwise,} \end{cases}$$

$$\widetilde{\zeta}(s,\alpha) = \sum_{n>-\alpha} (n+\alpha)^{-s} \quad \alpha \in \mathbb{R}.$$

The right hand side of $\widetilde{\zeta}(s,\alpha)$ converges absolutely for $\operatorname{Re} s > 1$, and is continued to the whole s-plane as a meromorphic function, which has a simple pole only at $s=1$ with residue 1. Notice that $\widetilde{\zeta}(s,\alpha+1) = \widetilde{\zeta}(s,\alpha)$ and, for $0 < \alpha \le 1$, we write

$$\widetilde{\zeta}(s,\alpha) = \zeta(s,\alpha) := \sum_{n=0}^{\infty}(n+\alpha)^{-s}.$$

This is the usual Hurwitz zeta function.

The Fourier expansion of $e^1_{\mathbf{a}}$ is given as follows ([Sc, (27) §2, Chapter VII]:

$$e^1_{\mathbf{a}}(z) = \sum_{\nu=0}^{\infty} \alpha_\nu(p,\mathbf{a}) e^{2\pi i \nu z/p},$$

where

$$\begin{aligned} \alpha_0(p,\mathbf{a}) &= \frac{1}{p}\delta\left(\frac{a_1}{p}\right) \lim_{s\to 1}\left\{\widetilde{\zeta}\left(s,\frac{a_2}{p}\right) - \widetilde{\zeta}\left(s,-\frac{a_2}{p}\right)\right\} \\ &\quad - \frac{\pi i}{p}\left\{\widetilde{\zeta}\left(0,\frac{a_1}{p}\right) - \widetilde{\zeta}\left(0,-\frac{a_1}{p}\right)\right\}, \\ a_\nu(p,\mathbf{a}) &= -\frac{2\pi i}{p} \sum_{\substack{m\mid\nu \\ \frac{\nu}{m}\equiv a_1 \bmod p}} (\operatorname{sgn} m) e^{2\pi i a_2 m/p} \quad \text{for } \nu \geq 1. \end{aligned} \tag{2.2}$$

In particular we have $e^1_{\mathbf{a}} \in M_1(\Gamma^1(p))$.

Let ψ be a Dirichlet character modulo p such that $\psi(-1) = -1$. We put

$$M_1(\Gamma^1_0(p),\psi) = \left\{ f \in M_1(\Gamma^1(p)) \;\middle|\; f|_1\gamma = \psi(d)f, \text{ for } \gamma = \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in \Gamma^1_0(p)\right\}.$$

It is easy to show that

$$M_1(\Gamma^1_1(p)) = \bigoplus_{\psi(-1)=-1} M_1(\Gamma^1_0(p),\psi), \tag{2.3}$$

where $\Gamma^1_1(p) = \left\{ \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in SL(2,\mathbb{Z}) \;\middle|\; a\equiv d \equiv 1 \bmod p,\ c\equiv 0 \bmod p\right\}$.

There are two cusps on $\Gamma^1_0(p)\backslash\mathbb{H}$ corresponding to 1_2 and J_1. For $f \in M_1(\Gamma^1_0(p),\psi)$, the 0-th Fourier coefficients of f and $f|_1 J_1$ are called the *values of f at the cusps* ∞ and 0 respectively.

Now we define

$$f_\psi(z) := \sum_{u=1}^{p-1}\sum_{v=0}^{p-1} \psi(u) e^1_{(u,v)}(z).$$

Clearly $f_\psi \in M_1(\Gamma^1_0(p),\psi)$.

Lemma 2.2. *For $\psi(-1) = -1$, f_ψ takes non-zero values at both cusps ∞ and 0. In particular $f_\psi \neq 0$.*

Proof. First we consider the cusp ∞. Since $\delta(\frac{u}{p}) = 0$ for all (u,v), the 0-th Fourier coefficient $c_0(f_\psi)$ of f_ψ is

$$\begin{aligned} c_0(f_\psi) &= -\frac{\pi i}{p}\sum_{u,v}\psi(u)\left\{\widetilde{\zeta}\left(0,\frac{u}{p}\right) - \widetilde{\zeta}\left(0,-\frac{u}{p}\right)\right\} \\ &= -\pi i \sum_{u=1}^{p-1}\psi(u)\left\{\zeta\left(0,\frac{u}{p}\right) - \zeta\left(0,\frac{p-u}{p}\right)\right\}. \end{aligned}$$

We use the formula

$$\zeta(0,\alpha) = -B_1(\alpha) = \frac{1}{2} - \alpha,$$

where $B_1(x)$ is the first Bernoulli polynomial. Then

$$\begin{aligned} c_0(f_\psi) &= -\pi i \sum_{u=1}^{p-1}\psi(u)\left(1 - \frac{2u}{p}\right) \\ &= \frac{2\pi i}{p}\sum_{u=1}^{p-1}\psi(u)u = \frac{2\pi i}{p}B_{1,\psi}. \end{aligned}$$

The value $B_{1,\psi}$ is called the generalised Bernoulli number and it is well-known that $B_{1,\psi} \neq 0$ for $\psi(-1) = -1$. Thus we have $c_0(f_\psi) \neq 0$.

Next we consider the cusp 0. Notice that

$$f_\psi = 2\sum_{u=1}^{\frac{p-1}{2}}\sum_{v=0}^{p-1}\psi(u)e^1_{(u,v)}$$

by (2.1) and $\psi(-1)=-1$. From (2.1) we have

$$f_\psi|_1 J_1 = 2\sum_{u=1}^{\frac{p-1}{2}}\sum_{v=0}^{p-1}\psi(u)e^1_{(-v,u)}$$

and, by (2.2), the value of f_ψ at 0 is given by

$$\frac{2}{p}\sum_{u=1}^{\frac{p-1}{2}}\lim_{s\to 1}\psi(u)\left\{\widetilde{\zeta}\left(s,\frac{u}{p}\right)-\widetilde{\zeta}\left(s,-\frac{u}{p}\right)\right\}-\frac{2\pi i}{p}\sum_{u=1}^{\frac{p-1}{2}}\sum_{v=0}^{p-1}\psi(u)\left\{\widetilde{\zeta}\left(0,-\frac{v}{p}\right)-\widetilde{\zeta}\left(0,\frac{v}{p}\right)\right\}.$$

The second term is 0, since $\widetilde{\zeta}(0,\alpha)=\widetilde{\zeta}(0,\alpha+1)$. For the first term, since $\psi(-1)=-1$, we have

$$\begin{aligned}&\frac{2}{p}\sum_{u=1}^{\frac{p-1}{2}}\lim_{s\to 1}\left\{\psi(u)\zeta\left(s,\frac{u}{p}\right)+\psi(p-u)\zeta\left(s,\frac{p-u}{p}\right)\right\}\\ =&\frac{2}{p}\lim_{s\to 1}\sum_{u=1}^{p-1}\psi(u)\zeta\left(s,\frac{u}{p}\right)\\ =&2L(1,\psi).\end{aligned}$$

Here $L(s,\psi)$ denotes the Dirichlet L-function and it is well-known that $L(1,\psi)\neq 0$ for any non-trivial Dirichlet character ψ. Hence we complete the proof. $\square$

We fix a decomposition

$$M_1(\Gamma^1(p)) = S_1(\Gamma^1(p))\oplus E_1(\Gamma^1(p)), \tag{2.4}$$

where $E_1(\Gamma^1(p))$ is a complement space of $S_1(\Gamma^1(p))$ and we assume that it is closed under the action of Γ^1. Such a decomposition exists since $\Gamma^1/\Gamma^1(p)=SL(2,\mathbb{F}_p)$ is a finite group. We write $E_1(\Gamma_1^1(p)) = M_1(\Gamma_1^1(p))\cap E_1(\Gamma^1(p))$ and $E_1(\Gamma_0^1(p),\psi)=M_1(\Gamma_0^1(p),\psi)\cap E_1(\Gamma^1(p))$. It is known that

$$\dim E_1(\Gamma_1^1(p)) = \frac{1}{2}\times\{\text{the number of (regular) cusps of } \Gamma_1^1(p)\backslash\mathbb{H}_1\} = \frac{1}{2}(p-1).$$

By (2.3), we have the following theorem.

Theorem 2.3. *Let ψ be a Dirichlet character modulo p such that $\psi(-1)=-1$. Then $E_1(\Gamma_0^1(p),\psi)$ is one-dimensional, and the basis f_ψ of $E_1(\Gamma_0^1(p),\psi)$ takes non-zero values at both cusps ∞ and 0.*

3. Siegel modular forms of degree g

Let p be an odd prime number. In this section we always assume $g\geq 2$. We put $G=Sp(g,\mathbb{F}_p)\simeq \Gamma^g/\Gamma^g(p)$ and consider the action of G on $M_k(\Gamma^g(p))$ as follows: for $\gamma\in G$ and $f\in M_k(\Gamma^g(p))$, G acts on the left on $M_k(\Gamma^g(p))$ via $(\gamma,f)\mapsto f|_k\widetilde{\gamma}^{-1}$, here $\widetilde{\gamma}\in\Gamma^g$ is a lift of γ.

The space $M_k(\Gamma^g(p))$ is decomposed as

$$M_k(\Gamma^g(p)) = S_k(\Gamma^g(p))\oplus E_k^{g-1}(\Gamma^g(p))\oplus\cdots\oplus E_k^0(\Gamma^g(p)),$$

where $E_k^r(\Gamma^g(p))$ is the subspace of a complement space of $S_k(\Gamma^g(p))\oplus\bigoplus_{i=r-1}^{g-1}E^r(\Gamma^g(p))$ consisting of those elements f such that $\Phi^r(f)\in S_k(\Gamma^r(p))$. We assume that all $E_k^r(\Gamma^g(p))$ is closed under the action of Γ^g, or equivalently, under the action of $G=Sp(g,\mathbb{F}_p)$. This decomposition exists because of the complete reducibility of the representations of finite groups.

Our main result is:

Theorem 3.1. *For $g \geq 2$,*

$$\dim E_1^0(\Gamma^g(p)) = \begin{cases} \frac{1}{2}(p^g+1) & p \equiv 3 \bmod 4, \\ 0 & p \equiv 1 \bmod 4. \end{cases}$$

Remark . If $g \geq 3$, we already know the dimension of $M_1(\Gamma^g(p))$ by the theorem of Li ([L]). His result is

$$\dim M_1(\Gamma^g(p)) = \begin{cases} \frac{1}{4}(h(p)+1)(p^g+1), & p \equiv 3 \bmod 4, \\ 0 & p \equiv 1 \bmod 4, \end{cases}$$

where $h(p)$ is the class number of $\mathbb{Q}(\sqrt{-p})$. One can show that $M_1(\Gamma^g(p))$ is generated by theta functions of quadratic forms of level p (see Proposition 3.7). Thus our result follows from the theorem of Li if $g \geq 3$, however the case of $g = 2$ is the essential part.

In order to prove the theorem, we need some preparations. We define for $0 \leq r \leq g-1$,

$$P_r = \left\{ \gamma = \left(\begin{array}{cc|cc} a_1 & 0 & b_1 & b_2 \\ a_3 & a_4 & b_3 & b_4 \\ \hline c_1 & 0 & d_1 & d_2 \\ 0 & 0 & 0 & d_4 \end{array}\right) \begin{array}{l} \updownarrow r \\ \updownarrow g-r \\ \updownarrow r \\ \updownarrow g-r \end{array} \in Sp(g,\mathbb{Z}) \;\middle|\; \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \in Sp(r,\mathbb{Z}),\ d_4 \in GL(g-r,\mathbb{Z}) \right\}.$$

Let $u_r \colon P_r \to \{\pm 1\}$ be the character of P_r defined by $\gamma \mapsto \det(d_4)$. We define homomorphisms $\pi_r \colon P_r \to Sp(r,\mathbb{Z})$ and $\iota_r \colon Sp(r,\mathbb{Z}) \to P_r$ by

$$\pi_r \colon \left(\begin{array}{cc|cc} a_1 & 0 & b_1 & b_2 \\ a_3 & a_4 & b_3 & b_4 \\ \hline c_1 & 0 & d_1 & d_2 \\ 0 & 0 & 0 & d_4 \end{array}\right) \longmapsto \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \qquad \iota_r \colon \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \left(\begin{array}{cc|cc} a & 0 & b & 0 \\ 0 & 1_{g-r} & 0 & 0 \\ \hline c & 0 & d & 0 \\ 0 & 0 & 0 & 1_{g-r} \end{array}\right).$$

We decompose Γ^g into double cosets by $\Gamma^g(p)$ and P_r:

$$\Gamma^g = \coprod_{\mu} \Gamma^g(p) M_\mu^r P_r$$

(assume that $1_{2g} \in \{M_\mu^r\}_\mu$). Each representative element M_μ^r corresponds to an r-dimensional cusp of $\Gamma^g(p)\backslash\mathbb{H}_g$. We fix the above decomposition and we write $\Phi_\mu^r(f) = \Phi^r(f|_k M_\mu^r)$ for $f \in M_k(\Gamma^r(p))$. It is easy to see

$$\Phi^r(f|_k\gamma) = u_r(\gamma)^k \Phi^r(f)|_k \pi_r(\gamma) \quad \text{for } \gamma \in P_r. \tag{3.1}$$

Lemma 3.2. *The representation of G on $E_k^0(\Gamma^g(p))$ is isomorphic to a sub-representation of $\mathrm{Ind}_{\overline{P}_0}^G(u_0^k)$, where $\overline{P}_0$ is the image of P_0 under the natural map $\Gamma^g \to G$.*

Proof. The proof is given in [G, Proposition 5.2], but we recall the proof here. First we assume that there exists $f \in E_k^0(\Gamma^g(p))$ such that $\Phi^0(f) = 1$ and $\Phi_\mu^0(f) = 0$ for $M_\mu^0 \neq 1_{2g}$. We put $f_\mu = f|_k(M_\mu^0)^{-1}$, then $f_\mu \in E_k^0(\Gamma^g(p))$ and

$$\Phi_\mu^0(f_{\mu_0}) = \begin{cases} 1 & \mu = \mu_0, \\ 0 & \mu \neq \mu_0. \end{cases}$$

By definition, $\{f_\mu\}_\mu$ form a basis of $E_k^0(\Gamma^g(p))$.

Now since $\{M_\mu^0\}_\mu$ is a representative system of $\Gamma^g(p)\backslash\Gamma^g/P_0$, $\{(M_\mu^0)^{-1}\}_\mu$ is a representative system of $\Gamma^g(p)P_0\backslash\Gamma^g$. Fix $\gamma \in \Gamma^g$, then for each μ there exists μ' such that

$$(M_\mu^0)^{-1}\gamma = x p_\mu (M_{\mu'}^0)^{-1} \quad x \in \Gamma^g(p),\ p_\mu \in P_0 \tag{1}$$

and, when μ runs through the representative system, μ' also runs through the system. We have $f_\mu|_k\gamma = f|_k(M_\mu^0)^{-1}\gamma = f|_k p_\mu (M_{\mu'}^0)^{-1}$. Since $\{p_\mu M_\mu^0\}_\mu$ is a representative system of $\Gamma^g(p)\backslash\Gamma^g/P_0$,

$$\Phi_\mu(f|_k p_\mu) = \Phi(f|_k(p_\mu M_\mu^0)) = \begin{cases} \Phi(f|_k p_\mu) = u_0(p_0)^k & \text{if } M_\mu^0 = 1_4; \\ 0 & \text{otherwise.} \end{cases}$$

Thus $f|_k p_\mu = u_0(p_\mu)^k f|_k$ and we have

$$f_\mu|_k\gamma = u_0(p_\mu)^k f_{\mu'}. \tag{2}$$

This shows that the representation of G is isomorphic to $\mathrm{Ind}_{\overline{P}_0}^G(u_0^k)$ in this case.

In general, we consider the $\mathbb{C}$-vector space V spanned by the free basis $\{f_\mu\}$ and induce the action of Γ^g by (1) and (2). We define a morphism $\varphi\colon E_k^0(\Gamma^g(p)) \to V$ as follows: for $f \in E_k^0(\Gamma^g(p))$ such that $\Phi_\mu^0(f) = a_\mu$, put $\varphi(f) = \sum_\mu a_\mu f_\mu$. Then φ is injective by the definition of $E_k^0(\Gamma^g(p))$ and, by the construction of φ, φ is a homomorphism of G-modules. Hence the representation of G on $E_k^0(\Gamma^g(p))$ is isomorphic to the sub-representation of $\mathrm{Ind}_{\overline{P}_0}^G(u_0^k)$. □

Let

$$H = \left\{\gamma = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \in G\right\}$$

be a subgroup of G. For a Dirichlet character ψ modulo p, we put $\widetilde{\psi}(\gamma) = \psi(\det D)$ for $\gamma \in H$.

Lemma 3.3. *We have a decomposition of the representation of G:*

$$\mathrm{Ind}_{\overline{P}_0}^G(u_0^k) = \bigoplus_{\psi(-1)=(-1)^k} \mathrm{Ind}_H^G(\widetilde{\psi}).$$

Proof. The condition $\psi(-1) = (-1)^k$ means that $\widetilde{\psi}|_{\overline{P}_0} = u_0^k$. Thus we have a non-zero H-homomorphism $\widetilde{\psi} \to \mathrm{Ind}_{\overline{P}_0}^H(u_0^k)$, by the Frobenius reciprocity law. However $[H : \overline{P}_0] = (p-1)/2$ and the number of Dirichlet characters ψ such that $\psi(-1) = (-1)^k$ is just $(p-1)/2$ for a fixed k. Hence we have

$$\mathrm{Ind}_{\overline{P}_0}^H(u_0^k) = \bigoplus_{\psi(-1)=(-1)^k} \widetilde{\psi}$$

by comparing the dimensions of both sides. Now our lemma follows from the associativity of induced representations.

□

Now we put

$$M_k(\Gamma_0^g(p), \psi) = \{f \in M_k(\Gamma^g(p)) \mid f|_k(\gamma) = \widetilde{\psi}(\gamma) f,\ \gamma \in \Gamma_0^g(p)\},$$

and $E_k^r(\Gamma_0^g(p), \psi) = M_k(\Gamma_0^g(p), \psi) \cap E_k^r(\Gamma^g(p))$. Notice that H is the image of $\Gamma_0^g(p)$ under the canonical map $\Gamma^g \to G$. Thus by the Frobenius reciprocity law we have

$$E_k^0(\Gamma_0^g(p), \psi) \simeq \mathrm{Hom}_H(E_k^0(\Gamma^g(p)), \widetilde{\psi}) \simeq \mathrm{Hom}_G(E_k^0(\Gamma^g(p)), \mathrm{Ind}_H^G(\widetilde{\psi})). \tag{3.2}$$

By Lemma 3.3, our problem is reduced to considering the space $E_1^0(\Gamma_0^g(p), \psi)$.

Proposition 3.4. *Let ψ be a Dirichlet character modulo p such that $\psi(-1) = -1$. Then:*

$$\dim E_1^0(\Gamma_0^g(p), \psi) = \begin{cases} 1 & \psi^2 \equiv 1, \\ 0 & \textit{otherwise}. \end{cases}$$

To prove the proposition, we investigate the structure of the boundary of Satake compactification of $\Gamma_0^g(p)\backslash\mathbb{H}_g$. We use the following notations: $I_r = I_r^g = \mathrm{diag}(\underbrace{1,\dots,1}_{r}, \underbrace{0,\dots,0}_{g-r})$, $E_r = E_r^g = \mathrm{diag}(\underbrace{0,\dots,0}_{r}, \underbrace{1,\dots,1}_{g-r})$, and $\mathcal{M}_r = \mathcal{M}_r^g = \begin{pmatrix} E_r & I_r \\ -I_r & E_r \end{pmatrix}$. Notice that $\mathcal{M}_0^g = 1_{2g}$, $\mathcal{M}_g^g = J_g$ and $\mathcal{M}_r \in \bigcup_{k \geq r} P_k$.

Lemma 3.5. (1) *A representative system of the double coset $\Gamma_0^g(p)\backslash\Gamma^g/P_0$ is given by $\{\mathcal{M}_r\}$, $0 \leq r \leq g$.*

(2) *A representative system of the double coset $\Gamma_0^g(p)\backslash\Gamma^g/P_1$ is given by $\{\mathcal{M}_r\}$, $1 \leq r \leq g$.*

Proof. In both cases, it suffices to consier $H\backslash G/\overline{P}_0$ or $H\backslash G/\overline{P}_1$. For simplicity, we shall write A_γ (resp. C_γ) for the left-upper (resp. the left-lower) $g \times g$-block of $\gamma \in G$.

First we consider (1). For $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G$, $x \in H$ and $p \in \overline{P}_0$, $C_{x\gamma p}$ is of the form UCV with $U, V \in GL(g, \mathbb{F}_p)$, $\det V = \pm 1$. Thus we can take suitable x and p such that $C_{x\gamma p} = -I_r$ for some $0 \le r \le g$. If $r = 0$, then $x\gamma p \in H$. We assume $r \ge 1$. Then $A_{x\gamma p}$ is of the form $A' = \begin{pmatrix} a_1 & 0 \\ a_3 & a_4 \end{pmatrix}$, $a_1 = {}^t a_1 \in M_r(\mathbb{F}_p)$. If we put

$$y = \begin{pmatrix} 1_g & T \\ 0 & 1_g \end{pmatrix} \in H \quad \text{with } T = \begin{pmatrix} a_1 & {}^t a_3 \\ a_3 & 0 \end{pmatrix},$$

then $A_{yx\gamma p} = E_r$. Finally we can take a suitable $q = \begin{pmatrix} 1_g & S \\ 0 & 1_g \end{pmatrix} \in \overline{P}_0$ such that $yx\gamma pq = \mathcal{M}_r$. Hence, we have a decomposition $G = \bigcup_{0 \le r \le g} H\mathcal{M}_r\overline{P}_0$. Since C_γ and $C_{x\gamma p}$ have the same rank for any $\gamma \in G$, $x \in H$ and $p \in \overline{P}_0$, the above decomposition is a disjoint union. This proves (1).

Next we prove (2). For $\gamma \in G$ we put $r = \operatorname{rank} C_\gamma$. If $r = 0$ then $\gamma \in H$; if $r = g$, it is easy to see that $\gamma \in H\mathcal{M}_g\overline{P}_1$. We assume that $1 \le r \le g-1$. By considering $x\gamma$ for suitable $x \in H$, we may assume that C_γ is of the form $\begin{pmatrix} 0 & 0 \\ u & C' \end{pmatrix}$ for $u \in M_{g-1,1}(\mathbb{F}_p)$ and $C' \in M_{g-1}(\mathbb{F}_p)$. Next we consider $y\gamma p$ for $y \in H$ and $p \in \overline{P}_1$ such that

$$D_y = \begin{pmatrix} 1 & 0 \\ 0 & U \end{pmatrix}, \ A_p = \begin{pmatrix} 1 & 0 \\ 0 & V \end{pmatrix}, \quad U, V \in GL(g-1, \mathbb{F}_p), \ \det V = \pm 1,$$

then we may assume that C_g is of the form

$$\begin{array}{cl} \left(\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline c & -1_r & 0 \\ \hline 0 & 0 & 0 \end{array}\right) & \begin{array}{l} \updownarrow 1 \\ \updownarrow r \\ \updownarrow g-r-1 \end{array} \end{array}$$

The left-lower $(g-r-1) \times 1$ block is zero, since $\operatorname{rank} C_\gamma = r$. Moreover we consider γq for $q \in \overline{P}_1$ such that

$$q = \begin{pmatrix} A & 0 \\ 0 & {}^t A^{-1} \end{pmatrix}, \quad A = \begin{array}{cl} \left(\begin{array}{c|c|c} 1 & 0 & 0 \\ \hline -c & -1_r & 0 \\ \hline 0 & 0 & 1_{g-r-1} \end{array}\right) & \begin{array}{l} \updownarrow 1 \\ \updownarrow r \\ \updownarrow g-r-1 \end{array} \end{array}$$

we may assume that

$$(*) \qquad C_\gamma = \begin{array}{cl} \left(\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline 0 & 1_r & 0 \\ \hline 0 & 0 & 0 \end{array}\right) & \begin{array}{l} \updownarrow 1 \\ \updownarrow r \\ \updownarrow g-r-1 \end{array} \end{array}$$

Then the lower-right $g \times g$ block D_γ of γ is of the form

$$\begin{array}{cl} \left(\begin{array}{c|c|c} d_1 & 0 & d_3 \\ \hline d_4 & d_5 & d_6 \\ \hline d_7 & 0 & d_9 \end{array}\right) & \begin{array}{l} \updownarrow 1 \\ \updownarrow r \\ \updownarrow g-r-1 \end{array} \end{array}, \quad d_5 = {}^t d_5.$$

We remark that it does not happen $d_1 = 0$ and $d_7 = 0$; indeed if $d_7 = 0$ then $\gamma \in P_{r+1}(\mathbb{F}_p)$, thus

$$\begin{array}{cl} \left(\begin{array}{cc|cc} \multicolumn{2}{c|}{*} & \multicolumn{2}{c}{*} \\ \hline 0 & 0 & d_1 & 0 \\ 0 & -1_r & d_4 & d_5 \end{array}\right) & \begin{array}{l} \updownarrow r+1 \\ \updownarrow 1 \\ \updownarrow r \end{array} \end{array} \in Sp(r+1, \mathbb{F}_p),$$

this shows $d_1 \neq 0$. Hence we may assume $d_1 \neq 0$ by exchanging low vectors of C_γ and D_γ, again C_γ is of the form $(*)$. We take $p \in \overline{P}_1$ such that

$$p = \begin{pmatrix} 1_g & 0 \\ T & 1_g \end{pmatrix}, \quad T = \begin{pmatrix} d_1^{-1} & 0 \\ 0 & 0 \end{pmatrix},$$

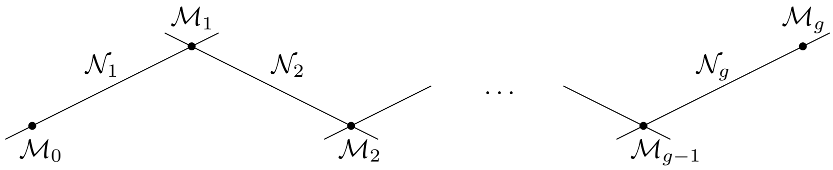
then

$$C_{\gamma p} = \left(\begin{array}{c|c|c} 1 & 0 & 0 \\ \hline * & -1_r & 0 \\ \hline 0 & 0 & 0 \end{array}\right).$$

Take a suitable $q \in \overline{P}_1$, then we have $C_{\gamma pq} = I_{r+1}$. Finally by the same process as (1), we can change γ of the form $\mathcal{M}_{r+1}$ by multiplying suitable $x \in H$, $p \in \overline{P}_1$. This shows that $\Gamma^g = \bigcup_{r=1}^{g} H\mathcal{M}_r\overline{P}_1$.

We show that this is a disjoint union. Assume that $h\mathcal{M}_{r_1} = \mathcal{M}_{r_2}p$ for $h \in H$, $p \in \overline{P}_1$. By comparing left-lower $g \times g$ block, we have $-VI_{r_1} = -I_{r_2}A_p + E_{r_2}C_p = -I_{r_2}A_p$ for $V \in GL(g, \mathbb{F}_p)$, hence $r_1 \leq r_2$; since $h^{-1}M_{r_2} = M_{r_1}p^{-1}$, also we have $r_2 \leq r_1$, thus $r_1 = r_2$. This complete the proof. □

We put $\mathcal{N}_r = \mathcal{M}_r$ for $1 \leq r \leq g$; 0-dimensional cusps are represented by $\mathcal{M}'s$, and 1-dimensional cusps are represented by $\mathcal{N}'s$. Then the structure of the boundary of $\Gamma_0^g(p)\backslash\mathbb{H}_g$ around 0 and 1-dimensional cusps is described as in the figure.



We explain the figure more precisely. The lines $\mathcal{N}_r$ $(1 \leq r \leq g)$ are modular curves $\Gamma^{1,0}(p)\backslash\mathbb{H}_1$, with

$$\Gamma^{g,0}(p) = J_g^{-1}\Gamma_0^g(p)J_g = \left\{\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(g,\mathbb{Z}) \;\middle|\; B \equiv 0 \bmod p\right\}.$$

The points $\mathcal{M}_r$ $(0 \leq r \leq g)$ are 0-dimensional cusps of $\Gamma_0^g(p)\backslash\mathbb{H}_g$. Also $\mathcal{M}_{r-1}$ and $\mathcal{M}_r$ are cusps of $\mathcal{N}_r$ corresponding to J_1 and 1_2 respectively for $1 \leq r \leq g$.

We fix a decomposition $M_k(\Gamma^1(p)) = S_k(\Gamma^1(p)) \oplus E_k(\Gamma^1(p))$ as in (2.4), and we write $E_k(\Gamma^{1,0}(p), \psi) = M_k(\Gamma^{1,0}(p), \psi) \cap E_k(\Gamma^1(p))$. We consider the map:

$$\Psi\colon E_k^0(\Gamma_0^g(p), \psi) \longrightarrow \prod_{r=1}^{g} E_k(\Gamma^{1,0}(p), \psi^{-1}), \quad f \longmapsto \prod_{r=1}^{g} q(\Phi_r^1(f)),$$

where $\Phi_r^1(f) = \Phi^1(f|_k\mathcal{N}_r)$, and $q\colon M_k(\Gamma^{1,0}(p), \psi) \to E_k(\Gamma^{1,0}(p), \psi)$ is the projector. Then Ψ is injective by the definition of E_k^0, and the image of Ψ is contained in ∂M_k where ∂M_k is the subspace consisting of those elements $(h_r)_{1 \leq r \leq g}$ which satisfy the following condition: if we write $\iota_1(1_2)\mathcal{N}_r = x\mathcal{M}_r p_1$ and $\iota_1(J_1)\mathcal{N}_{r+1} = y\mathcal{M}_r p_2$ with $x, y \in \Gamma_0^g(p)$, $p_1, p_2 \in P_0$, then

$$\widetilde{\psi}(x)^{-1}u_0(p_1)^{-1}\Phi^0(h_r) = \widetilde{\psi}(y)^{-1}u_0(p_2)^{-1}\Phi^0(h_{r+1}). \tag{3.3}$$

This value is called *the value of* $(h_r)_r$ *at cusp* $\mathcal{M}_r$. For $\Psi(f) \in \partial M_k$, the value of (3.3) coincides with $\Phi_r^0(f)$.

Lemma 3.6. *Assume that* $\psi^2 \not\equiv 1$. *Then any element of* $M_k(\Gamma_0^g(p), \psi)$ *takes value* 0 *at the* 0-*dimensional cusp* $\mathcal{M}_r^g$ *for* $1 \leq r \leq g-1$.

Proof. We assume that $x\mathcal{M}_r^g p = \mathcal{M}_r^g$ for some $x \in \Gamma_0^g(p)$ and $p \in P_0$. Then $\Phi_r^0(f) = \Phi^0(f|_k\mathcal{M}_r) = \psi(x)u_0(p)^k\Phi_r^0(f)$. Hence for the character $\widetilde{\psi}_r(y) = \widetilde{\psi}(\mathcal{M}_r y(\mathcal{M}_r)^{-1})$ on $(\mathcal{M}_r)^{-1}\Gamma_0^g(p)\mathcal{M}_r$, if $\widetilde{\psi}_r^{-1}u_0^k \not\equiv 1$ on $P_0 \cap (\mathcal{M}_r)^{-1}\Gamma_0^g(p)\mathcal{M}_r$, then $\Phi_r^0(f) = 0$ for any $f \in M_k(\Gamma_0^g(p), \psi)$. We see that

$$(\mathcal{M}_r)^{-1}P_0\mathcal{M}_r = \left\{\gamma = \begin{array}{cl} \left(\begin{array}{cc|cc} a_1 & a_2 & b_1 & b_2 \\ a_3 & a_4 & b_3 & b_4 \\ \hline c_1 & c_2 & d_1 & d_2 \\ c_3 & c_4 & d_3 & d_4 \end{array}\right) & \begin{array}{l} \updownarrow r \\ \updownarrow g-r \\ \updownarrow r \\ \updownarrow g-r \end{array} \end{array} \;\middle|\; \begin{pmatrix} b_1 & a_2 \\ d_3 & c_4 \end{pmatrix} \equiv 0 \in M_g(\mathbb{Z}) \bmod p\right\},$$

and

$$\widetilde{\psi}_r(\gamma) = \psi\left(\det\begin{pmatrix} a_1 & -b_2 \\ -c_3 & d_4 \end{pmatrix}\right).$$

Thus for $\gamma \in P_0 \cap (\mathcal{M}_r)^{-1}\Gamma_0^g(p)\mathcal{M}_r$, $\widetilde{\psi}^{-1}(\gamma)u_0(\gamma)^k = \psi(\det(a_1)\det(d_4))^{-1}u_0(\gamma)^k$. Since $\psi(-1) = (-1)^k$ and $\{\pm 1\} \ni u_0(\gamma) \equiv \det(d_1)\det(d_4) \bmod p$, we have $u_0(\gamma)^k = \psi(\det(d_1)\det(d_4))$. Therefore $\widetilde{\psi}(\gamma)^{-1}u_0(\gamma)^k = \psi(\det(a_1)^{-1}\det(d_1)) = \psi(\det(d_1))^2$, since γ is of the form $\begin{pmatrix} V & T \\ 0 & {}^tV^{-1}\end{pmatrix}$. This proves the lemma. □

Proof of Proposition 3.4. By Theorem 2.3 and (3.3), we see that $\dim \partial M_1 \leq 1$ from the structure of the boundary of $\Gamma_0^g(p)\backslash\mathbb{H}_g$. If $\psi^2 \not\equiv 1$ then $\dim \partial M_1 = 0$ by Lemma 3.6, thus $\dim E_1^0(\Gamma_0^g(p), \psi) = 0$. If $\psi^2 \equiv 1$ then $p \equiv 3 \bmod 4$ and $\psi(x) = (\frac{x}{p})$ with Legendre symbol $(\frac{\cdot}{p})$. Now we define

$$\theta_Q(Z) = \sum_{N \in M_{2,g}(\mathbb{Z})} \exp \pi i \operatorname{Tr}({}^tNQNZ), \quad Q = \begin{pmatrix} 2 & 1 \\ 1 & (p+1)/2 \end{pmatrix}$$

then $\theta_Q(Z) \in M_1(\Gamma_0^g(p), \psi)$ and $\Phi^0(\theta_Q(Z)) = 1$. Hence we have $\dim E_1^0(\Gamma_0^g(p), \psi) = 1$ for $\psi^2 \equiv 1$. □

In order to prove Theorem 3.1, we review the theory of theta functions of quadratic forms.

Proposition 3.7. *Let $Q \in M_m(\mathbb{Z})$ be a symmetric positive definite matrix with even diagonal entries, and let q be a level of Q, that is, the minimum positive integer such that qQ^{-1} is integral with even diagonal entries. We put $T^g(Q) = \{T \in M_{m,g}(\mathbb{Z}) \mid QT \equiv 0 \mod q\}$. We define for $Z \in \mathbb{H}_g$ and $T \in T^g(Q)$,*

$$\theta_Q^T(Z) = \sum_{N \in M_{m,g}(\mathbb{Z})} \exp \pi i \operatorname{Tr}\left({}^t(N + \frac{1}{q}T)Q(N + \frac{1}{q}T)Z\right).$$

Then $\theta_Q^T(Z) \in M_{m/2}(\Gamma^g(q))$. Moreover the following properties hold.

$$\theta_Q^T(\left(\begin{smallmatrix} V & 0 \\ 0 & {}^tV^{-1}\end{smallmatrix}\right)\langle Z\rangle) = \theta_Q^{TV}(Z) \quad \text{for } V \in GL_g(\mathbb{Z}).$$

$$\theta_Q^T(\left(\begin{smallmatrix} 1 & S \\ 0 & 1\end{smallmatrix}\right)\langle Z\rangle) = \exp \pi i \operatorname{Tr}\left(\frac{1}{q^2}{}^tTQTS\right)\theta_Q^T(Z).$$

$$\theta_Q^T(\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0\end{smallmatrix}\right)\langle Z\rangle) = (\det Q)^{-g/2}(\det(-iZ))^{m/2} \sum_{\substack{T' \in T^g(Q) \\ \bmod q}} \exp 2\pi i \operatorname{Tr}\left(\frac{1}{q^2}{}^tTQT'\right)\theta_Q^{T'}(Z).$$

For the proof, see [A, Proposition 1.3.14, Exercise 2.2.3].

For $p \equiv 3 \bmod 4$, we put $Q_0 = \begin{pmatrix} 2 & 1 \\ 1 & (p+1)/2\end{pmatrix}$. Then the level of Q_0 is p, and

$$T^g(Q_0) = \left\{ \begin{pmatrix} a_1 & a_2 & \cdots & a_g \\ -2a_1 & -2a_2 & \cdots & -2a_g \end{pmatrix} \;\middle|\; a_i \in \mathbb{Z} \right\}.$$

Let V be the vector space spanned by $\{\theta_{Q_0}^T\}$ for $T \in T^g(Q_0)$. Obviously V, is closed under the action of Γ^g. Since $\theta_{Q_0}^T = \theta_{Q_0}^{-T}$, we have $\dim V \leq (p^g+1)/2$. We write $T_1, \ldots, T_{(p^g+1)/2}$ for the representative elements of $\{T^g(Q_0) \bmod p\}/\{\pm 1\}$.

We put

$$U = \left\{ \gamma(S) = \begin{pmatrix} 1_g & S \\ 0 & 1_g \end{pmatrix} \in \Gamma^g \;\middle|\; {}^tS = S \right\}.$$

Then V is decomposed into the eigen space by the action of U. Actually,

$$\theta_{Q_0}^T|_k\gamma(S)(Z) = \exp\frac{2\pi i}{p}\operatorname{Tr}\left(\sum_{i,j=1}^g a_ia_js_{ij}\right)\theta_{Q_0}^T(Z), \quad \text{for } T = \begin{pmatrix} a_1 & \cdots & a_g \\ -2a_1 & \cdots & -2a_g\end{pmatrix}, S = (s_{ij}).$$

Thus $\theta_{Q_0}^{T_i}$ and $\theta_{Q_0}^{T_j}$ are contained in relatively distinct eigenspaces for $i \neq j$. In particular $\dim V = (p^g+1)/2$.

Lemma 3.8. *The representation of $G = Sp(g, \mathbb{F}_p)$ on V is irreducible.*

Proof. First notice that for a character ξ of U, the projector of V to each eigenspace V_ξ of U is contained in $\mathbb{C}[U]$. Indeed we write $\xi_1, \ldots, \xi_n$ for all the characters of U. If we take elements u_i of U such that $\xi_1(u_i) \neq \xi_i(u_i)$ for $2 \leq i \leq n$, then $\prod_{i=2}^{n}(u_i - \xi_i(u_i)) \in \mathbb{C}[U]$ maps elements of V_{ξ_i} to 0 for $2 \leq i \leq n$, and acts on V_{ξ_1} by non-zero scalar multiple.

Now take $0 \neq v \in V$. Let W be the subspace of V generated by v as G-module. We shall show that $V = W$. By the above remark, at least one non-zero element, say $v_{\xi_1} \in V_{\xi_1}$ is contained in W. Then by Proposition 3.7, for all ξ such that $V_\xi \neq 0$, the V_ξ components of $v_{\xi_1}|J_g$ are not zero. Thus again by the above remark, we see that all $V_\xi \subset W$, hence $V = W$. $\square$

Proof of Theorem 3.1. By Lemma 3.3, (3.2) and Proposition 3.4, $E_k^0(\Gamma^g(p)) = 0$ if $p \equiv 1 \bmod 4$, and contains only one irreducible representation corresponding to $E_k^0(\Gamma_0^g(p), \psi)$ if $p \equiv 3 \bmod 4$. However by Lemma 3.8, this representation is isomorphic to V, thus $\dim E_k^0(\Gamma^g(p)) = (p^g + 1)/2$. $\square$

For the rest of this paper we remark for some results. First the following lemma follows from the proof of Theorem 3.1.

Lemma 3.9. *Let $Q \in M_2(\mathbb{Z})$ be a symmetric even matrix of $\det Q = p$. Let V_Q be the subspace of $M_1(\Gamma^g(p))$ spanned by θ_Q^T. Then the the representation of G on V_Q is irreducible and the equivalence class is independent on the choice of Q.*

Lemma 3.10. *Let ψ be a Dirichlet character modulo p. If $\psi^2 \not\equiv 1$ then $\mathrm{Ind}_H^G(\widetilde{\psi})$ is a irreducible representation of G.*

To prove the lemma, we use the following Mackey's criterion.

Theorem 3.11 (Mackey)**.** *Let $H \subset G$ be finite groups. For a representation ρ of H, $\mathrm{Ind}_H^G(\rho)$ is irreducible if and only if the following two conditions hold;*

1. *ρ is irreducible.*
2. *For $s \in G \setminus H$ we put $H_s = sHs^{-1} \cap H$, and ρ^s is the representation of H_s defined by $\rho(x) = \rho(s^{-1}xs)$. Then, $\mathrm{Hom}_{H_s}(\rho|_{H_s}, \rho^s) = 0$.*

Proof of Lemma 3.10. In our case, condition (1) is obvious. For the condition (2), it suffices to consider s in $H\backslash G/H$. By Lemma 3.5, we put $s = \mathcal{M}_r$ for $1 \leq r \leq g$. By a direct computation we have

$$H^s = \left\{ \gamma = \begin{array}{c} \left(\begin{array}{cc|cc} {}^t d_1^{-1} & 0 & 0 & b_2 \\ a_3 & {}^t d_4^{-1} & b_3 & b_4 \\ \hline 0 & 0 & d_1 & d_2 \\ 0 & 0 & 0 & d_4 \end{array}\right) \end{array} \begin{array}{l} \updownarrow r \\ \updownarrow g-r \\ \updownarrow r \\ \updownarrow g-r \end{array} \right\},$$

and $\widetilde{\psi}(\gamma) = \psi(\det(d_1)\det(d_4))$, $\widetilde{\psi}^s(\gamma) = \widetilde{\psi}(\det(d_1)^{-1}\det(d_4))$. Thus $\widetilde{\psi} = \widetilde{\psi}^s$ if and only if $\psi^2 \equiv 1$. $\square$

For $k \geq g+2$, we define

$$E_{\mathcal{M}_r}^{k,\psi}(Z) = \sum_{\gamma \in P_0 \cap \mathcal{M}_r^{-1}\Gamma_0^g(p)\mathcal{M}_r \backslash \mathcal{M}_r^{-1}\Gamma_0^g(p)\mathcal{M}_r} \widetilde{\psi}^r(\gamma)^{-1} j(\gamma, Z)^{-k},$$

with $\widetilde{\psi}^r(x) = \widetilde{\psi}(\mathcal{M}_r x \mathcal{M}_r^{-1})$. This summation is well-defined if $\psi^2 \equiv 1$ or $r = 0, g$. In theses cases, the infinite sum converges absolutely and uniformly on $V(d)$ for any $d > 0$, where

$$V(d) = \{Z = X + iY \in \mathbb{H}_g \mid X = (x_{ij}),\ |x_{ij}| < d,\ Y > d^{-1}1_g\}.$$

Then $E_r = E_{\mathcal{M}_r}^{k,\psi}|_k(\mathcal{M}_r)^{-1} \in M_k(\Gamma_0^g(p), \psi)$. Moreover, since

$$\lim_{\lambda \to \infty} |j(\gamma, i\lambda 1_g)|^{-k} = 0 \quad \text{if } \gamma \notin P_0,$$

one sees that E_r takes value 1 at 0-dimensional cusp $\mathcal{M}_r$ and takes value 0 at the other 0-dimensional cusps.

By the above discussion and (3.2) shows the following lemma.

Lemma 3.12. *If $\psi^2 \equiv 1$ then $\mathrm{Ind}_H^G(\widetilde{\psi})$ contains $g+1$ irreducible representations with multiplicity.*

In the case of $g=2$, all the irreducible characters of the group $Sp(2,\mathbb{F}_p)$ were determined by Srinivasan ([Sr]). Using this result, we can decompose the representations explicitly.

Lemma 3.13. (1) *The representation of $Sp(2,\mathbb{F}_p)$ on $E_k^0(\Gamma^2(p))$ is the sub-representation of*

$$\mathrm{Ind}_{P_0}^G(u_0^k)=\begin{cases}\underbrace{\theta_3}_{(p^2+1)/2}\oplus\underbrace{\theta_4}_{(p^2+1)/2}\oplus\underbrace{\Phi_9}_{p(p^2+1)}\oplus\bigoplus_{\substack{1\le l\le (p-3)/2\\ l:odd}}2\underbrace{\chi_8(l)}_{(p+1)(p^2+1)} & \text{if } k \text{ is odd, } p\equiv 3 \bmod 4;\\ \bigoplus_{\substack{1\le l\le (p-3)/2\\ l:odd}}2\chi_8(l) & \text{if } k \text{ is odd, } p\equiv 1 \bmod 4;\\ 1_G\oplus\underbrace{\theta_9}_{p(p+1)^2/2}\oplus\underbrace{\theta_{11}}_{p(p^2+1)/2}\oplus\bigoplus_{\substack{1\le l\le (p-3)/2\\ l:even}}2\chi_8(l) & \text{if } k \text{ is even, } p\equiv 3 \bmod 4;\\ 1_G\oplus\theta_9\oplus\theta_{11}\oplus\theta_3\oplus\theta_4\oplus\Phi_9\oplus\bigoplus_{\substack{1\le l\le (p-3)/2\\ l:even}}2\chi_8(l) & \text{if } k \text{ is even, } p\equiv 1 \bmod 4.\end{cases}$$

(2) *We fix a generator ξ of the cyclic group $\mathbb{F}_p^\times$, and define a Dirichlet character ψ_l modulo p by $\psi_l(\xi^a)=e^{2\pi i a l/(p-1)}$. Then*

$$\mathrm{Ind}_H^G(\widetilde{\psi_l})=\begin{cases}1_G\oplus\theta_9\oplus\theta_{11} & l=0;\\ \theta_3\oplus\theta_4\oplus\Phi_9 & l=(p-1)/2;\\ \chi_8(|l|) & -(p-3)/2\le l\le (p-3)/2,\ l\neq 0.\end{cases}$$

References

[A] A. N. Andrianov, *Quadratic forms and Hecke operators*, Grundl. math. Wiss. 286, Springer-Verlag, 1987.

[F1] E. Freitag, "Holomorphe Differentialformen zu Kongruenzgruppen der Siegelschen Modulgruppe.". Invent. math. **30** (1975), 181-196.

[F2] E. Freitag, "Die Invarianz gewisser von Thetareihen erzeugter Vektorräume unter Heckeoperatoren", Math. Z. **156** (1977), No.2, 141-155.

[G] K. Gunji, "On the graded ring of Siegel modular forms of degree 2, level 3", J. Math. Soc. Japan, **56** (2004), no. 2, 375-403.

[H] E. Hecke, "Theorie der Eisensteinschen Reihen höherer Stufe und ihre Anwendung auf Funktionentheorie und Arithmetik". Abh. Math. Sem. Univ. Hamburg **5** (1927), 199-224; "*Mathematische Werke*" Göttingen Vandenhoeck & Ruprecht 1970, 461-486.

[L] J.-S. Li, "On the dimensions of spaces of Siegel modular forms of weight one", Geometric And Functional Analysis **6** (1996), No.3, 512-555.

[R] H. L. Resnikoff, "Automorphic forms of singular weight are singular forms", Math. Ann. **215** (1975), 173-193.

[Sc] B. Schoeneberg, *Elliptic Modular Functions*, Grundle. math. Wiss. 203, Springer-Verlag, 1974.

[Sr] B. Srinivasan, "The character of the finite symplectic group $Sp(4,q)$", Trans. Amer. Math. Soc. **131** (1968), 488-525.

[W] R. Weissauer, "Modular forms of genus 2 and weight 1", Math. Z. **210** (1992), 91-96.

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