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SOLUTION FORMULA FOR AN INVERSE PROBLEM WITH UNDERDETERMINING DATA

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§1. Introduction and Key Lemma.

In an inverse problem, we are required to determine coefficients in a partial differential equation in order that the solution to the differential equation realizes prescribed data. As the mathematical topics for an inverse problem, we mention the uniqueness and the stability, and additionally the existence of a solution to the inverse problem is important. Usually the solution to an inverse problem is given not by formulae which are involved by algebraic operations and calculi, but is found through limit processes such as iterations (for example, as a solution to an operator equation of the second kind). In this paper, we will show a formula for solutions to an inverse problem which is attached with underdetermining data.

Our formulation for the inverse problem is underdeterming and so cannot guarantee the uniqueness for solutions to the inverse problem. Hence our formula gives "one" solution to the inverse problem under consideration, and does not describe all possible solutions but includes sufficiently many solutions in the sense that it admits a family of coefficients parameterized by free functions in the spatial variable.

Among various inverse problems, for our approach, we will mainly discuss an inverse problem with data at final time. For example, in Bouchouev and Isakov [3], Isakov [5], such an inverse problem for the Black - Scholes equation is considered: Determine a, b, c by $w(x, T)$, $x \in I$, in

$$\alpha \frac{\partial w}{\partial t} = a(x)\frac{\partial^2 w}{\partial x^2} + b(x)\frac{\partial w}{\partial x} + c(x)w.$$

Here $T > 0$ is fixed, I is an interval and $\alpha = const > 0$. As for inverse problems with data at final time, we can further refer to Choulli and Yamamoto [4], Prilepko, Orlovsky and Vasin [6] and the references therein.

In the present paper, we construct a family of solutions of the inverse problems, which is based on representation of solutions and coefficients.

Let $D \in \mathbb{R}^n$ be a domain and let us consider an evolution equation of the following type:

$$\sum_{k=1}^{m} \alpha_k(v(x)) \frac{\partial^k w}{\partial t^k} = \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 w}{\partial x_i \partial x_j} + \sum_{j=1}^{n} b_j(x) \frac{\partial w}{\partial x_j} + c(x)w, \tag{1.1}$$

where $x = (x_1, ..., x_n) \in D \subset \mathbb{R}^n$, $0 \leq t \leq T$, $\alpha_k : \mathbb{R}^n \longrightarrow \mathbb{R}$ are smooth functions for $1 \leq k \leq m$, $v(x) = (v_1(x), \ldots v_n(x))$ is a differentiable vector-valued function such that

$$\left| \frac{\partial(v_1, ..., v_n)}{\partial(x_1, ..., x_n)} \right| \neq 0, x \in \overline{D}$$

and $a_{ij} = a_{ji}$, $1 \leq i, j \leq n$.

First we formulate a general approach for obtaining a representation formula of solution $w(x,t)$ and coefficients $a_{ij}(x)$, $b_j(x)$, $c(x)$.

Note that in the case where

$$\alpha_1 = \alpha = const > 0, \quad \alpha_k = 0, \quad k = 2, 3, \ldots, m,$$

and

$$\sum_{i,j=1}^{n} a_{ij} \eta_j \eta_j > 0, \quad \eta \in \mathbb{R}^n, \eta \neq 0,$$

equation (1.1) becomes parabolic:

$$\alpha \frac{\partial w}{\partial t} = \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 w}{\partial x_i \partial x_j} + \sum_{j=1}^{n} b_j(x) \frac{\partial w}{\partial x_j} + c(x)w.$$

In the case of

$$\alpha_2 = \alpha = const, \quad \alpha_k = 0, k = 1, 3, \ldots, m$$

equation (1.1) becomes hyperbolic when $\alpha > 0$, and elliptic when $\alpha < 0$. Moreover if $\alpha_1 = -\sqrt{-1}$ and $\alpha_2 = \cdots = \alpha_m = 0$, then (1.1) is the Schrödinger equation.

Lemma 1. Let $D_1 \subset \mathbb{R}^n$ be a domain and $\beta_{k\ell}$, β_k, $\beta \in C^1(D_1)$, $1 \leq k, \ell \leq n$ such that $\beta_{k\ell} = \beta_{\ell k}$, be given, and $u \in C^2(D)$ be a given function such that $u(x) \neq 0$ for any $x \in \overline{D}$. Let

$$F(y,t), \quad y \in D_1 \subset \mathbb{R}^n, 0 \leq t \leq T$$

satisfy the equation

$$\sum_{k=1}^{m} \alpha_k(y) \frac{\partial^k F}{\partial t^k} = \sum_{k,\ell=1}^{n} \beta_{k\ell}(y) \frac{\partial^2 F}{\partial y_\ell \partial y_k} + \sum_{k=1}^{n} \beta_k(y) \frac{\partial F}{\partial y_k} + \beta(y) F. \tag{1.2}$$

Then the function

$$w(x,t) = u(x) F(v(x), t)$$

and the coefficients $a_{ij}(x)$, $b_j(x)$, $c(x)$ consecutively defined by linear systems of algebraic equations (1.3), (1.4), (1.5), satisfy equation (1.1):

$$\sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial v_k}{\partial x_i} \frac{\partial v_\ell}{\partial x_j} = \beta_{k\ell}(v(x)) \tag{1.3}$$

$$\sum_{j=1}^{n} b_j(x) \frac{\partial v_k}{\partial x_j} = \frac{1}{u} \left[u\beta_k(v) - \sum_{i,j=1}^{n} a_{ij} \left(\frac{\partial v_k}{\partial x_i} \frac{\partial u}{\partial x_j} + \frac{\partial v_k}{\partial x_j} \frac{\partial u}{\partial x_i} + u \frac{\partial^2 v_k}{\partial x_i \partial x_j} \right) \right], \tag{1.4}$$

$$c(x) = \frac{1}{u} \left[u\beta(v) - \left(\sum_{i,j=1}^{n} a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{j=1}^{n} b_j(x) \frac{\partial u}{\partial x_j} \right) \right]. \tag{1.5}$$

The proof of the lemma is done by direct substitution of the solution $w(x,t) = u(x)F(v(x),t)$ into equation (1.1) in terms of (1.2) - (1.5).

We note that if we choose $v(x) = x$, $u(x) = 1$, $\beta_{k\ell} = a_{k\ell}$, $\beta_k = a_k$ and $\beta = c$ for $1 \le k, \ell \le n$, then (1.3) - (1.5) are true.

After choosing $v = v(x)$, we set $\mathcal{A} = \{u, \{\beta_{k\ell}, \beta_k\}_{1 \le k,\ell \le n}, \beta\}$. Then, by $a_{ij}(\mathcal{A})$, $b_j(\mathcal{A})$, $c(\mathcal{A})$, $1 \le i,j \le n$, we denote a_{ij}, b_j and c defined by (1.3) - (1.5). We note that $\mathcal{A}$ is composed of $\frac{n^2+3n+4}{2}$ functions in x. Then, by Lemma 1 we can represent $\frac{n^2+3n+2}{2}$ functions a_{ij}, b_j, c in x by $\frac{n^2+3n+4}{2}$ functions in x. In other words, our lemma gives representation formulae of coefficients a_{ij}, b_j, c which contains $1 \left(= \frac{n^2+3n+4}{2} - \frac{n^2+3n+2}{2}\right)$ free function in x. Thus by our representation formula, we can give a pair of solution (a_{ij}, b_j, c) which realizes one extra data $w(x,T) = w_1(x)$, $x \in D$. We notice that our formula (1.3) - (1.5) are not involved with limit processes.

In particular, if coefficients α_k, β_{kl}, β_k, β are constant in (1.2), then in some cases it is possible to represent $F(y,t)$ which is a solution to an initial value problem. Moreover, if we a priori know $F(y,t)$, then problem of search for coefficients and solution obviously turns to the determination of only functions $u(x)$, $v(x) = (v_1(x), \ldots, v_n(x))$.

We will consider a one-dimensional variant of Lemma 1 more precisely. Let $w(x,t)$ satisfy the following equation:

$$\sum_{k=1}^{m} \alpha_k(v(x)) \frac{\partial^k w}{\partial t^k} = a(x) \frac{\partial^2 w}{\partial x^2} + b(x) \frac{\partial w}{\partial x} + c(x) w, \tag{1.6}$$

$$x_0 \le x \le x_1, \quad 0 < t < T,$$

where $\alpha_k(y)$, $y \in \mathbb{R}^1$, $v(x)$, $a(x)$, $b(x)$, $c(x)$ are some differentiable functions and α_k may be constant.

Lemma 2. Let

$$u(x), v(x), \alpha_k(y), \beta_1(y), \beta_2(y), \beta_3(y), k = 1, 2, \dots, m$$

be some twice differentiable functions and let

$$x_0 \le x \le x_1, \quad y \in \mathbb{R}^1, \quad u(x) \ne 0, \quad \frac{dv}{dx} \equiv= v'(x) \ne 0.$$

We assume that $F(y,t)$, $y \in \mathbb{R}^1$, $0 < t < T$, satisfies the equation

$$\sum_{k=1}^{m} \alpha_k(y) \frac{\partial^k F}{\partial t^k} = \beta_1(y) \frac{\partial^2 F}{\partial y^2} + \beta_2(y) \frac{\partial F}{\partial y} + \beta_3(y) F, \quad y \in \mathbb{R},\, t > 0. \tag{1.7}$$

Then the functions $w(x,t)$, $a(x)$, $b(x)$, $c(x)$ defined by the following formulae

$$w(x,t) = u(x) F(v(x), t), \quad a(x) = \frac{\beta_1(v(x))}{v'^2},$$

$$b(x) = \frac{v'^2 u \beta_2(v) - \beta_1(v)(2u'v' + uv'')}{uv'^3}$$

$$c(x) = \frac{u^2 v'^3 \beta_3(v) - uu''v'\beta_1(v) - u'v'^2 u\beta_2(v) + 2u'^2 v'\beta_1 + u'uv''\beta_1}{u^2 v'^3}$$

satisfy the equation

$$\sum_{k=1}^{m} \alpha_k(u(x)) \frac{\partial^k w}{\partial t^k} = a(x) \frac{\partial^2 w}{\partial x^2} + b(x) \frac{\partial w}{\partial x} + c(x) w.$$

As for other approaches to inverse problems by means of formulae, we refer to Anikonov [1], [2].

§2. One-dimensional parabolic inverse problem with data at final time.

As an example of using this way we consider an inverse problem for a one-dimensional parabolic equation or a modifed Black-Scholes equation (e.g., [3]): Find functions

$$w(x,t), a(x), b(x), c(x), 0 \le t \le T, x \in \mathbb{R},$$

such that

$$\alpha \frac{\partial w}{\partial t} = a(x) \frac{\partial^2 w}{\partial x^2} + b(x) \frac{\partial w}{\partial x} + c(x) w, \quad x \in \mathbb{R},\, t > 0 \tag{2.1}$$

and

$$w\Big|_{t=0} = w_0(x), w\Big|_{t=T} = w_1(x), \quad x \in \mathbb{R}. \tag{2.2}$$

More precisely, represent the four functions $w(x,t)$, $a(x)$, $b(x)$, $c(x)$ by $w_0(x)$, $w_1(x)$ and one real-valued auxiliary function. Here we assume that

$$w_0(x) > 0, \qquad x \in \mathbb{R}. \tag{2.3}$$

Then we note that $w_1(x) > 0$ for $x \in \mathbb{R}$ by the maximum principle.

Theorem. We choose $\delta \in \mathbb{R}$ and a smooth function $f_0 > 0$ for $x \in \mathbb{R}$ such that

$$\Phi(x) = \frac{f_0(x)}{\int_{-\infty}^{\infty} \frac{1}{2\sqrt{\pi T}} \exp\left(-\frac{y^2}{4T}\right) f_0(x-y)dy} \tag{2.4}$$

is an injective function in $x \geq \delta$. Furthermore we assume that

$$\left[0, \sup_{x\in\mathbb{R}} \frac{w_0(x)}{w_1(x)}\right] \subset \Phi([\delta, \infty)). \tag{2.5}$$

We set

$$v(x) = \Phi^{-1}\left(\frac{w_0(x)}{w_1(x)}\right), \quad u(x) = \frac{w_1(x)}{w_1(v(x))}, \quad x \in \mathbb{R}. \tag{2.6}$$

Then

$$a(x) = \frac{1}{(v'(x))^2},$$

$$b(x) = -\frac{2u'(x)v'(x) + u(x)v''(x)}{u(x)(v'(x))^3},$$

$$c(x) = \frac{-u(x)v'(x)u''(x) + 2(u'(x))^2v'(x) + u(x)u'(x)v''(x)}{(u(x))^2(v'(x))^3},$$

$$w(x,t) = \frac{u(x)}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} \exp\left(-\frac{(x-y)^2}{4t}\right) w_0(v(y))dy, \quad x \in \mathbb{R},\ t > 0,$$

satisfies (2.1) and (2.2).

Example. Setting

$$f_0(x) = e^{-x^2}, \qquad x > 0,$$

we see that for any $\delta > 0$, the function Φ is injective in $x \in \mathbb{R}$. In fact,

$$\int_{-\infty}^{\infty} \frac{1}{2\sqrt{\pi T}} \exp\left(-\frac{y^2}{4T}\right) e^{-(x-y)^2} dy = \frac{1}{\sqrt{4T+1}} \exp\left(-\frac{x^2}{4T+1}\right),$$

and so

$$\Phi(x) = \sqrt{4T+1}\exp\left(-\frac{4Tx^2}{4T+1}\right).$$

Thus under the assumption that

$$\sup_{x\in\mathbb{R}} \frac{w_0(x)}{w_1(x)} < \sqrt{4T+1},$$

we can rewrite (2.6) as

$$v(x) = \left\{\frac{4T+1}{4T}\log\left(\sqrt{4T+1}\frac{w_1(x)}{w_0(x)}\right)\right\}^{\frac{1}{2}}$$

and

$$u(x) = \frac{w_1(x)}{w_1(v(x))},$$

so that the conclusion of the theorem holds.

Proof of Theorem. In Lemma 2, we set $\alpha_1 = 1$, $\alpha_2 = \cdots = \alpha_m = 0$, $\beta_1 = 1$, $\beta_2 = \beta_3 = 0$. Then

$$F(y,t) = \frac{1}{2\sqrt{\pi t}}\int_{-\infty}^{\infty} e^{-\frac{y^2}{4t}} f_0(x-y)dy$$

satisfies (1.7). We have

$$w_0(x) = u(x)f_0(v(x)), \quad w_1(x) = u(x)F(v(x),T).$$

Eliminating $u(x)$ in these equations, we obtain

$$\frac{w_0(x)}{w_1(x)} = \frac{F(v(x),0)}{F(v(x),T)} = \Phi(v(x)).$$

Therefore we have

$$v(x) = \Phi^{-1}\left(\frac{w_0(x)}{w_1(x)}\right)$$

and

$$u(x) = \frac{w_0(x)}{F\left(\Phi^{-1}\left(\frac{w_0(x)}{w_1(x)}\right),0\right)} = \frac{w_1(x)}{F\left(\Phi^{-1}\left(\frac{w_0(x)}{w_1(x)}\right),0\right)}.$$

Hence, by Lemma 2, the proof of the theorem is complete.

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