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Existence and nonexistence of global solutions in time for a reaction-diffusion system with inhomogeneous terms

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Existence and Nonexistence of Global Solutions in Time for a Reaction-Diffusion System with Inhomogeneous Terms

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Abstract

We consider the initial value problem for the reaction-diffusion system with inhomogeneous terms. In this paper we show the existence and nonexistence of global solution in time. Especially, for the nonexistence we extend the conditions of the nonlinear terms and the initial data to the weaker conditions. We prove that for the nonlinear term and the initial data whose support is included in some unbounded domain (for instance, the corn), there do not exist the global solutions in time.

Keyword and Phrases: reaction-diffusion, global existence, inhomogeneous term.
AMS subject classifications: 35K45, 35K57.

1 Introduction

We consider nonnegative solutions of the initial value problem for the reaction-diffusion system

$$\begin{cases} u_t = \Delta u + K_1(x,t)v^{p_1}, & x \in \mathbf{R}^n,\ t > 0, \\ v_t = \Delta v + K_2(x,t)u^{p_2}, & x \in \mathbf{R}^n,\ t > 0, \\ u(x,0) = u_0(x) \geq 0, & x \in \mathbf{R}^n, \\ v(x,0) = v_0(x) \geq 0, & x \in \mathbf{R}^n, \end{cases} \tag{1}$$

where $p_1, p_2 \geq 1$ with $p_1 p_2 > 1$. The inhomogeneous terms $K_i(x,t)$ $(i = 1,2)$ are continuous functions satisfying

$$K_i(x,t) \leq C_U \langle x \rangle^{\sigma_i}(t+1)^{q_i} \quad \text{for any } x \in \mathbf{R}^n,\ t \geq 0, \tag{2}$$

$$K_i(x,t) \geq C_L |x|^{\sigma_i} t^{q_i} \quad \text{for any } x \in \bigcup_{m=1}^{\infty} \tilde{B}_{r,m},\ t \geq 0, \tag{3}$$

where $C_U \geq C_L > 0$, $\sigma_i \geq 0$, $q_i \geq 0$ $(i = 1,2)$, $\langle x \rangle = (|x|^2 + 1)^{1/2}$ and

$$\tilde{B}_{r,m} = B_{r|x_m|}(x_m) \tag{4}$$

denotes the ball with radius $r|x_m|$ centered at x_m for some constant $r > 0$ and a sequence $\{x_m\}_{m=1}^{\infty}$ satisfying $0 < |x_m| < |x_{m+1}|$ for any m and $\lim_{m\to\infty} |x_m| = \infty$. The initial data u_0 and v_0 are assumed to satisfy

$$\limsup_{|x|\to\infty} |x|^{\delta_1} u_0(x) < \infty, \quad \limsup_{|x|\to\infty} |x|^{\delta_2} v_0(x) < \infty,$$

with

$$\delta_i = \frac{\sigma_j p_i + \sigma_i}{p_i p_j - 1} \quad ((i,j) = (1,2),(2,1)). \tag{5}$$

Example. *Put $l \subset \mathbf{R}^n$ be a half line such that $|x| \geq 1$ for any $x \in l$. Define $D_1 = \cup_{x\in l} B_{s|x|}(x)$ and $D_2 = \cup_{x\in l} B_{2s|x|}(x)$ with some $s \in (0, 1/4)$. Let K_i $(i = 1,2)$ be nonnegative continuous functions defined by $K_i(x,t) = t^{q_i}|x|^{\sigma_i}$ $(x \in D_1)$, $= 0$ $(x \in \mathbf{R}^n \backslash D_2)$, $\in [0, t^{q_i}|x|^{\sigma_i}]$ $(x \in D_2 \backslash D_1)$. Then K_i satisfy* (2) *and* (3)*. (The supports of K_i are included in some corn in $\mathbf{R}^n$.)*

For given initial values (u_0, v_0), let $T^* = T^*(u_0, v_0)$ be a maximal existence time of the solution of (1). If $T^* = \infty$, the solutions are global in time. On the other hand, if $T^* < \infty$, the solutions are not global in time, and they satisfy

$$\limsup_{t\to T^*} \|u(\cdot,t)\|_\infty = \infty \quad \text{or} \quad \limsup_{t\to T^*} \|v(\cdot,t)\|_\infty = \infty, \tag{6}$$

where $\|\cdot\|_\infty$ denotes the L^∞-norm with respect to space variable. The phenomenon is called that the solutions blow up in finite time, too. We define constants

$$\alpha_i = \frac{(2+\sigma_i+2q_i)+(2+\sigma_j+2q_j)p_i}{p_ip_j-1} \quad ((i,j)=(1,2),(2,1)). \tag{7}$$

Denote by BC the space of all bounded continuous functions in $\mathbf{R}^n$. For $a \geq 0$ we define the following sets;

$$I^a = \left\{ \xi \in BC; \xi(x) \geq 0, \limsup_{|x|\to\infty} |x|^a \xi(x) < \infty \right\},$$
$$\tilde{I}_a = \left\{ \xi \in BC; \xi(x) \geq 0, \limsup_{m\to\infty} \inf_{x\in\tilde{B}_{r,m}} |x|^a \xi(x) > 0 \right\}$$

with $\tilde{B}_{r,m}$ defined in (4). Let L^∞_a be the Banach space of L^∞-functions in $\mathbf{R}^n$ with the norm

$$\|\xi\|_{\infty,a} \equiv \sup_{x\in\mathbf{R}^n} \langle x\rangle^a |\xi(x)|.$$

It is easily seen that $I^a \subset L^\infty_a$. We use the notation operator $S(t)$ of the heat equation defined by

$$S(t)\xi(x) = \int_{\mathbf{R}^n} (4\pi t)^{-\frac{n}{2}} \exp\left(-\frac{|x-y|^2}{4t}\right) \xi(y)dy. \tag{8}$$

As noted in Theorem 3 of section 2 below, if $(u_0, v_0) \in I^{\delta_1} \times I^{\delta_2}$ and $K_i(x,t)$ $(i=1,2)$ satisfy (2), the problem (1) has a unique, nonnegative solution $(u(\cdot,t), v(\cdot,t)) \in L^\infty_{\delta_1} \times L^\infty_{\delta_2}$ at least locally in time.

Now, the results of this paper are summarized in the two theorems. The first theorem asserts nonexistence of the global solution.

Theorem 1. *Assume that $(u_0, v_0) \in I^{\delta_1} \times I^{\delta_2}$, $(u_0, v_0) \not\equiv 0$ and $K_i(x,t)$ $(i=1,2)$ satisfy (2) and (3). Suppose that one of the following three conditions holds;*

(i) $\max\{\alpha_1, \alpha_2\} \geq n$.

(ii) $u_0 \in \tilde{I}_{a_1}$ *with* $a_1 < \alpha_1$ *or* $v_0 \in \tilde{I}_{a_2}$ *with* $a_2 < \alpha_2$.

(iii) $u_0(x)$ *or* $v_0(x) \geq Me^{-\nu_0|x|^2}$ *for some* $\nu_0 > 0$ *and* $M > 0$ *large enough.*

Then, every solution (u, v) of (1) is not global in time.

The method using the sequence of balls in (3) was used in [16, 4] and other papers.

Remark. *Fro proving only non-existence of the global solution in time, it is not needed that* $(u_0, v_0) \in I^{\delta_1} \times I^{\delta_2}$ *and* $K_i(x,t)$ $(i = 1, 2)$ *satisfy* (2).

The second theorem assert the existence of the global solution in time.

Theorem 2. *Assume that* $\max\{\alpha_1, \alpha_2\} < n$ *and* $K_i(x,t)$ $(i = 1, 2)$ *satisfy* (2). *Suppose that*

$$(u_0, v_0) \in I^{a_1} \times I^{a_2} \text{ with } a_1 > \alpha_1,\ a_2 > \alpha_2, \tag{9}$$

and that $\|u_0\|_{\infty,a_1}$ *and* $\|v_0\|_{\infty,a_2}$ *are small enough. Then, every solution* (u, v) *of (1) is global in time. Moreover, we have*

$$u(x,t) \leq CS(t)\langle x\rangle^{-\tilde{a}_1} \quad \text{and} \quad v(x,t) \leq CS(t)\langle x\rangle^{-\tilde{a}_2} \tag{10}$$

in $\mathbf{R}^n \times (0, \infty)$, *where* C *is a positive constant and* $\tilde{a}_1 \leq a_1$ *and* $\tilde{a}_2 \leq a_2$ *are chosen to satisfy*

$$p_i \min\{\tilde{a}_j, n\} - \tilde{a}_i > 2 + \sigma_i + 2q_i \quad ((i,j) = (1,2), (2,1)). \tag{11}$$

Remark. *In Theorem 2, take* $\check{a}_i > a_i$ $(i = 1, 2)$. *If one needs only global existence of solution of* (1) *in time, then by comparison the condition of* (u_0, v_0) *may be replaced by the condition that* $(u_0, v_0) \in I^{\check{a}_1} \times I^{\check{a}_2}$ *with* $\check{a}_i > a_i > \alpha_i$ $(i = 1, 2)$, *and* $\|u_0\|_\infty$ *and* $\|v_0\|_\infty$ *are sufficiently small. In fact since* $a_i > \alpha_i$ $(i = 1, 2)$ *are arbitrary, we need only* $\check{a}_i > \alpha_i$ $(i = 1, 2)$.

Theorems 1 (ii) and 2 assert that there exist both non-global solutions and non-trivial global solutions of (1) when $\max\{\alpha_1, \alpha_2\} < n$. Precisely, assuming the polynomial decay of initial values (u_0, v_0);

$$u_0 \sim \langle x\rangle^{-a_1} \quad \text{and} \quad v_0 \sim \langle x\rangle^{-a_2} \quad \text{as } |x| \to \infty, \tag{12}$$

we obtain the "second critical exponent" $a_1 = \alpha_1$, $a_2 = \alpha_2$ on the decay rate of initial values by which the global existence case and nonexistence case are divided.

We briefly recall a history of the study on global existence and global nonexistence of solutions to the system (1). First, the global existence and nonexistence of solutions in the case $u_0 = v_0$, $p_i = p$ and $K_i = 1$ $(i = 1, 2)$,

$$\begin{cases} u_t = \Delta u + u^p, & x \in \mathbf{R}^n,\ t > 0, \\ u(x,0) = u_0(x) \geq 0, & x \in \mathbf{R}^n, \end{cases} \tag{13}$$

was studied by Fujita[3]. Fujita proved that when $p < 1 + 2/n$ the solution of (13) is not global in time for any $u_0 \not\equiv 0$. On the other hand, he also proved that when $p > 1 + 2/n$ the solution of (13) is global in time if $\|u_0\|_\infty$ is small enough and u_0 has an exponential decay. The number $p = 1 + 2/n$ is called a critical exponent for (13). Fujita's results were also extended by some researcher. Hayakawa[7], Kobayashi-Sirano-Tanaka[9] and Weissler[20] proved that when $p = 1 + 2/n$, the solution of (1) blows up in finite time for any $u_0 \not\equiv 0$.

For the case $p > 1+2/n$, Lee-Ni[10] studied that if $\|u_0\|_\infty$ is large enough or $\liminf_{|x|\to\infty} |x|^a u_0(x) > 0$ with $a < 2/(p-1)$, the solution of (13) is not global in time, and if $\|u_0\|_\infty$ is small enough and $u_0 \in I^a$ with $a > 2/(p-1)$, the solution of (13) is global in time. The number $a = 2/(p-1)$ is called the "second critical exponent".

Fujita's results were extended to the case $K_i(x,t) = K(x,t)$,

$$\begin{cases} u_t = \Delta u + K(x,t)u^p, & x \in \mathbf{R}^n,\ t > 0, \\ u(x,0) = u_0(x) \geq 0, & x \in \mathbf{R}^n. \end{cases}$$

In the case $K(x,t) = |x|^\sigma$ with $\sigma \geq 0$, Bandle-Levine[1] had that when $p < 1+(2+\sigma)/n$ the solution is not global in time for any $u_0 \not\equiv 0$. Hamada[6] had the same result for the critical case $p = 1 + (2 + \sigma)/n$ (see also [13]). Thereafter, Qi[14] extended the result to the case $K(x,t) = t^q|x|^\sigma u^p$ with $q \geq 0$, $\sigma \geq 0$. He proved that the number $p = 1 + (2 + \sigma + 2q)/n$ is a critical exponent in this case.

Moreover, Fujita's result were also extended by Escobedo-Herrero[2] and Mochizuki[11] to the system (1) with $K_1(x,t) = K_2(x,t) = 1$, by Uda[17] to the system (1) with $K_i(x,t) = t^{q_i}$ $(i = 1,2)$, and by Mochizuki-Huang[12] to the system (1) with $K_i(x,t) = |x|^{\sigma_i}$ with $\sigma_i \in [0, n(p_i - 1))$ $(i = 1,2)$. Additionally, Guedda-Kirane[5] and Kirane-Qafsaoui[8] studied in this field. They studied the case $K_i(x,t) \sim t^{q_i}|x|^{\sigma_i}$ as $t \to \infty$ and $|x| \to \infty$. But they needed the condition $\max\{2q_i, \sigma_i\} \leq n(p_i - 1)$ $(i = 1,2)$.

Although the Fujita type critical exponent to the system (1) was established by Escobedo-Herrero[2] and Uda[17], their proofs were rather complicated. Mochizuki[11] and Mochizuki-Huang[12] simplified their proof and also determined the "second critical exponent" on the decay rate of the initial data. Our results are natural extensions of Mochizuki-Huang[12], and are proved by applying the arguments of Mochizuki-Huang[12], Pinsky[13] and Umeda[18, 19].

The rest of the paper is organized as follows. In section 2, we note some preliminary results including the local existence for (1). The result of global nonexistence (Theorem 1) is given in section 3. In section 4, we show the result on global existence (Theorem 2).

2 Preliminaries

In order to show the local solvability of the Cauchy problem (1), we consider the associated integral system

$$u(x,t) = S(t)u_0(x) + \int_0^t S(t-s)K_1(x,s)v(x,s)^{p_1}ds, \tag{14}$$

$$v(x,t) = S(t)v_0(x) + \int_0^t S(t-s)K_2(x,s)u(x,s)^{p_2}ds, \tag{15}$$

where $S(t)$ is defined in (8). Define

$$\Psi(u,v) = (S(t)u_0(x) + \Phi_1(v), S(t)v_0(x) + \Phi_2(u)), \tag{16}$$

where

$$\Phi_1(v) = \int_0^t S(t-s)K_1(x,s)v(x,s)^{p_1}ds,$$

$$\Phi_2(u) = \int_0^t S(t-s)K_2(x,s)u(x,s)^{p_2}ds.$$

For $T > 0$, set

$$E_T = \{(u,v) : [0,T] \to L^\infty_{\delta_1} \times L^\infty_{\delta_2}; \|(u,v)\|_{E_T} < \infty\}$$

with the norm

$$\|(u,v)\|_{E_T} = \sup_{t\in[0,T]} \{\|u(t)\|_{\infty,\delta_1} + \|v(t)\|_{\infty,\delta_2}\}.$$

It is easily seen that E_T is a Banach space.

Lemma 2.1. *Let $\delta \geq 0$. Then,*

$$\|S(t)\langle\cdot\rangle^{-\delta}\|_{\infty,\delta} \leq C$$

for $x \in \mathbf{R}^n$ and $t \in [0,T)$ with every $T < \infty$ and some $C = C(n,\delta,T) > 0$.

Proof. From (8), we have

$$\begin{aligned}
S(t)\langle x\rangle^{-\delta} &= (4\pi t)^{-\frac{n}{2}} \int_{\mathbf{R}^n} \exp\left(-\frac{|x-y|^2}{4t}\right) \langle y\rangle^{-\delta}dy \\
&= (4\pi t)^{-\frac{n}{2}} \int_{\mathbf{R}^n} \exp\left(-\frac{|y|^2}{4t}\right) \langle x-y\rangle^{-\delta}dy \\
&= (4\pi t)^{-\frac{n}{2}} \left(\int_{|x-y|<|x|/2} + \int_{|x-y|>|x|/2}\right) \exp\left(-\frac{|y|^2}{4t}\right) \langle x-y\rangle^{-\delta}dy.
\end{aligned}$$

First, we estimate the first term, and we have

$$
\begin{aligned}
&(4\pi t)^{-\frac{n}{2}} \int_{|x-y|<|x|/2} \exp\left(-\frac{|y|^2}{4t}\right) \langle x-y\rangle^{-\delta} dy \\
&\quad \le (4\pi t)^{-\frac{n}{2}} \int_{|x-y|<|x|/2} \exp\left(-\frac{|y|^2}{4t}\right) dy \le C_1 \left|\frac{x}{\sqrt{t}}\right|^n \exp\left(-\frac{1}{16}\left|\frac{x}{\sqrt{t}}\right|^2\right).
\end{aligned}
$$

Next, the second term is estimated, and we obtain

$$
\begin{aligned}
&(4\pi t)^{-\frac{n}{2}} \int_{|x-y|>|x|/2} \exp\left(-\frac{|y|^2}{4t}\right) \langle x-y\rangle^{-\delta} dy \\
&\quad \le C_2 \langle x\rangle^{-\delta} (4\pi t)^{-\frac{n}{2}} \int_{\mathbf{R}^n} \exp\left(-\frac{|y|^2}{4t}\right) dy \le C_2 \langle x\rangle^{-\delta}.
\end{aligned}
$$

Thus, we have

$$
S(t)\langle x\rangle^{-\delta} \le C_1 \left|\frac{x}{\sqrt{t}}\right|^n \exp\left(-\frac{1}{16}\left|\frac{x}{\sqrt{t}}\right|^2\right) + C_2\langle x\rangle^{-\delta}.
$$

Since $a^n \exp(-a^2/16) \le C\langle a\rangle^{-\delta}$ for $a \ge 0$ with some $C = C(n,\delta) > 0$, we have

$$
\begin{aligned}
S(t)\langle x\rangle^{-\delta} &\le C_1\langle x/\sqrt{t}\rangle^{-\delta} + C_2\langle x\rangle^{-\delta} \le C_1\langle x/\sqrt{T}\rangle^{-\delta} + C_2\langle x\rangle^{-\delta} \\
&\le C_3 T^{\delta/2}\langle x\rangle^{-\delta} + C_2\langle x\rangle^{-\delta} \le C_4\langle x\rangle^{-\delta}
\end{aligned}
$$

with the constants $C_j = C_j(n,\delta,T)$ $(j = 3,4)$. Multiplying both sides above expression by $\langle x\rangle^{\delta}$,

$$
\langle x\rangle^{\delta} S(t)\langle x\rangle^{-\delta} \le C_4.
$$

Hence, we have

$$
\|S(t)\langle\cdot\rangle^{-\delta}\|_{\infty,\delta} \le C_4.
$$

□

Lemma 2.2. (i) *Let* $(u_0, v_0) \in I^{\delta_1} \times I^{\delta_2}$. *Then* $(S(\cdot)u_0, S(\cdot)v_0) \in E_T$ *for some* $T > 0$, *and we have*

$$
\|(S(\cdot)u_0, S(\cdot)v_0)\|_{E_T} \le C\{\|u_0\|_{\infty,\delta_1} + \|v_0\|_{\infty,\delta_2}\}
$$

with some $C = C(n,\delta_1,\delta_2,T) > 0$.
(ii) *Let* $(u,v) \in E_T$. *Then* $(\Phi_1(v), \Phi_2(u)) \in E_T$ *for some* $T > 0$, *and we have*

$$
\|(\Phi_1(v), \Phi_2(u))\|_{E_T} \le C(\tilde{T}_1(T) + \tilde{T}_2(T))\{\|(0,v)\|_{E_T}^{p_1} + \|(u,0)\|_{E_T}^{p_2}\}
$$

with some $C = C(n,\delta_1,\delta_2,T) > 0$, *where* $\tilde{T}_i(s) = \{(s+1)^{q_i+1} - 1\}/(q_i+1)$ $(i = 1,2)$.

Proof. (i) is obvious from Lemma 2.1 with $\delta = \delta_i$ $(i = 1, 2)$.

(ii) Note that

$$\int_0^t S(t-s)K_1(x,s)v(x,s)^{p_1}ds \\ \leq \int_0^t S(t-s)C_U(s+1)^{q_1}\langle x\rangle^{\sigma_1-\delta_2 p_1}ds \sup_{s\in[0,t]} \|v(\cdot,s)\|_{\infty,\delta_2}^{p_1}.$$

By a simple calculation, $-\sigma_1 + \delta_2 p_1 = \delta_1$. Then it follows from Lemma 2.1 that

$$\|S(t-s)\langle\cdot\rangle^{\sigma_1-\delta_2 p_1}\|_{\infty,\delta_1} \leq C$$

with some constant $C = C(n, \delta, T)$. Thus we have

$$\left\|\int_0^t S(t-s)C_U(s+1)^{q_1}\langle\cdot\rangle^{\sigma_1}v(\cdot,s)^{p_1}ds\right\|_{\infty,\delta_1} \leq C\tilde{T}_1(t)\sup_{s\in[0,t]}\|v(s)\|_{\infty,\delta_2}^{p_1}.$$

Similarly, we have

$$\left\|\int_0^t S(t-s)C_U(s+1)^{q_2}\langle\cdot\rangle^{\sigma_2}u(\cdot,s)^{p_2}ds\right\|_{\infty,\delta_2} \leq C\tilde{T}_2(t)\sup_{s\in[0,t]}\|u(s)\|_{\infty,\delta_1}^{p_2}.$$

These inequalities conclude the assertion (ii). □

Now we can prove the following:

Theorem 3. *(The local existence of the solution) Assume that* $(u_0, v_0) \in I^{\delta_1} \times I^{\delta_2}$ *and* K_i $(i = 1, 2)$ *satisfy* (2). *Then there exists a unique solution* $(u, v) \in P_T \equiv \{(u, v) \in E_T; u \geq 0, v \geq 0\}$ *which solves (1) in* $\mathbf{R}^n \times (0, T)$ *for some* $T > 0$.

Proof. Let $B_R = \{(u, v) \in E_T; \|(u, v)\|_{E_T} \leq R\}$, and define $\tilde{T}_i$ same as in Lemma 2.2 (ii). For $(u_1, v_1), (u_2, v_2) \in B_R \cap P_T$ with $R \geq 1$ sufficient large,

$$\|\Psi(u_1, v_1) - \Psi(u_2, v_2)\|_{E_T} = \|(\Phi_1(v_1) - \Phi_1(v_2), \Phi_2(u_1) - \Phi_2(u_2))\|_{E_T}. \tag{17}$$

We consider

$$|\Phi_1(v_1) - \Phi_1(v_2)|\langle x\rangle^{\delta_1} \\ \leq \int_0^t S(t-s)C_U(s+1)^{q_1}\langle x\rangle^{\sigma_1}|v_1(x,s)^{p_1} - v_2(x,s)^{p_1}|ds\langle x\rangle^{\delta_1}.$$

From proof of Lemma 2.2 (ii),

$$\begin{aligned} |\Phi_1(v_1) - \Phi_1(v_2)|\langle x\rangle^{\delta_1} &\leq C_1\tilde{T}_1(T)\sup_{s\in[0,t]}\|v_1^{p_1}(\cdot,s) - v_2^{p_1}(\cdot,s)\|_{\infty,\delta_2} \\ &\leq C_1\tilde{T}_1(T)\sup_{s\in[0,t]}\|R^{p_1-1}p_1(v_1(\cdot,s) - v_2(\cdot,s))\|_{\infty,\delta_2} \end{aligned} \tag{18}$$

with $C_1 = C_1(n, \delta_1, T)$. By same argument, we have

$$|\Phi_2(u_1) - \Phi_2(u_2)|\langle x\rangle^{\delta_2} \le C_2\tilde{T}_2(T) \sup_{s\in[0,t]} \|R^{p_2-1}p_2(u_1(\cdot,s) - u_2(\cdot,s))\|_{\infty,\delta_1} \tag{19}$$

with some $C_2 = C_2(n, \delta_2, T)$. Substitute (18) and (19) into (17). Since we can put T is small enough for R and $\max\{p_1, p_2\} \le p_1p_2$ by $p_1p_2 > 1$, we obtain

$$\begin{aligned}
&\|\Psi(u_1, v_1) - \Psi(u_2, v_2)\|_{E_T}\\
&\le \max\{C_1, C_2\}(\tilde{T}_1(T) + \tilde{T}_2(T))R^{p_1p_2-1}p_1p_2\|(u_1, v_1) - (u_2, v_2)\|_{E_T}\\
&\le \rho\|(u_1, v_1) - (u_2, v_2)\|_{E_T}
\end{aligned}$$

for some $\rho < 1$. Then Ψ is a strict contraction of $B_R \cap P_T$ into itself, whence there exists a unique fixed point $(u, v) \in B_R \cap P_T$ which solves (1). $\square$

3 Proof of Theorem 1

In this section we treat the nonexistence of global solutions in time of (1). Here, we take the same strategy as in [12] and [13].

First, we should consider only the case $r \in (0, 1/2)$ by comparison. Let $\lambda_m > 0$ denote the principal eigenvalue of $-\Delta$ with Dirichlet problem in $\tilde{B}_{r,m}$ defined in (4), and let $\phi_m(x) > 0$ denote the corresponding positive eigenfunction, normalized by $\int_{\tilde{B}_{r,m}} \phi_m(x)dx = 1$. Define

$$F_m(t) = \int_{\tilde{B}_{r,m}} u(x,t)\phi_m(x)dx, \quad G_m(t) = \int_{\tilde{B}_{r,m}} v(x,t)\phi_m(x)dx.$$

We will show that for an appropriate choice of r, $(F_m(t), G_m(t))$ is not global in time, thereby contradicting the assumption that (u, v) is a global solution. Since $\tilde{B}_{r,m}$ is a n-dimensional ball of radius $r|x_m|$, it follows that λ_m satisfies

$$\lambda_m \le \frac{c_1}{|x_m|^2}, \tag{20}$$

where $c_1 > 0$ depends only on the dimension n and r. Let $\nu(x)$ denote the outward unit normal to $\tilde{B}_{r,m}$ at $x \in \partial\tilde{B}_{r,m}$. Integrating by parts, using (20) and the fact that $\phi_m = 0$ and $\partial\phi_m/\partial n \le 0$ on $\partial\tilde{B}_{r,m}$ with a unit normal vector n, and applying Green's formula and Jensen's inequality, we obtain

$$\begin{aligned}
F_m'(t) &= \int_{\tilde{B}_{r,m}} u_t(x,t)\phi_m(x)dx \ge \int_{\tilde{B}_{r,m}} (\Delta u(x,t) + C_L t^{q_1}|x|^{\sigma_1}v^{p_1}(x,t))\phi_m(x)dx\\
&\ge -\lambda_m F_m(t) + C_L t^{q_1}((1-r)|x_m|)^{\sigma_1} G_m(t)^{p_1}\\
&\ge -c_1|x_m|^{-2}F_m(t) + c_2 t^{q_1}|x_m|^{\sigma_1} G_m(t)^{p_1}
\end{aligned}$$

and

$$\begin{aligned} G_m'(t) &\ge -\lambda_m G_m(t) + C_L t^{q_2}((1-r)|x_m|)^{\sigma_2} F_m(t)^{p_2} \\ &\ge -c_1|x_m|^{-2} G_m(t) + c_2 t^{q_2}|x_m|^{\sigma_2} F_m(t)^{p_2} \end{aligned}$$

with $c_2 = C_L(1-r)^{\max\{\sigma_1,\sigma_2\}}$. Thus, we obtain

$$\begin{cases} F_m'(t) \ge -c_1|x_m|^{-2} F_m(t) + c_2 t^{q_1}|x_m|^{\sigma_1} G_m(t)^{p_1}, \\ G_m'(t) \ge -c_1|x_m|^{-2} G_m(t) + c_2 t^{q_2}|x_m|^{\sigma_2} F_m(t)^{p_2}. \end{cases} \tag{21}$$

Let us consider the system of ordinary differential equations

$$\begin{cases} f_m'(t) = -c_1|x_m|^{-2} f_m(t) + c_2 t^{q_1}|x_m|^{\sigma_1} g_m(t)^{p_1}, \\ g_m'(t) = -c_1|x_m|^{-2} g_m(t) + c_2 t^{q_2}|x_m|^{\sigma_2} f_m(t)^{p_2}. \end{cases} \tag{22}$$

By the scaling

$$f(t) = |x_m|^{\alpha_1}\tilde{c}_1 f_m(|x_m|^2 t/c_1), \quad g(t) = |x_m|^{\alpha_2}\tilde{c}_2 g_m(|x_m|^2 t/c_1)$$

with $\tilde{c}_i = c_2^{(p_i+1)/(p_ip_j-1)}/c_1^{(p_i+1+q_jp_i+q_i)/(p_ip_j-1)}$ $((i,j) = (1,2),(2,1))$, we obtain

$$f'(t) = -f(t) + t^{q_1} g(t)^{p_1}, \quad g'(t) = -g(t) + t^{q_2} f(t)^{p_2}. \tag{23}$$

We choose a positive number t_0. Since $t^{q_1} \ge t_0^{q_1}$, $t^{q_2} \ge t_0^{q_2}$ for $t \ge t_0$, we have

$$f'(t) \ge -f(t) + t_0^{q_1} g(t)^{p_1}, \quad g'(t) \ge -g(t) + t_0^{q_2} f(t)^{p_2}, \tag{24}$$

for $t \ge t_0$. Here, let us consider the system of ordinary differential equations

$$a'(t) = -a(t) + t_0^{q_1} b(t)^{p_1}, \quad b'(t) = -b(t) + t_0^{q_2} a(t)^{p_2}, \tag{25}$$

for $t \ge t_0$.

As an application of the standard theory of ordinary differential equations, we have following two lemmas;

Lemma 3.1. *([15, Lemma 4]) Let*

$$Q \equiv \{(a,b) \in \mathbf{R}_+^2; (t_0^{-q_1} a)^{1/p_1} < b < t_0^{q_2} a^{p_2}\}.$$

If $(a(t), b(t))$ solves (25) for $t > t_0$ and $(a(t_0), b(t_0)) \in Q$, then $(a(t), b(t)) \in Q$ for $t > t_0$.

Lemma 3.2. *([15, Lemma 5]) If $(a(t), b(t))$ solves (25) for $t > t_0$ and $(a(t_0), b(t_0)) \in Q$, then $(a(t), b(t))$ is not global in time for $t \ge t_0$.*

Note that there is only one equilibrium point

$$P = (t_0^{-(q_2p_1+q_1)/(p_1p_2-1)}, t_0^{-(q_1p_2+q_2)/(p_1p_2-1)})$$

of system (25) in $\mathbf{R}_+^2$. Then P is a saddle point. Hence one of two separatrices is the stable manifold, whereas another one is the unstable manifold. The unstable manifold starts from 0 and runs to ∞. The stable manifold intersects the a-axis at $A_0 > 0$ and the b-axis at $B_0 > 0$. Consequently if $a(t_0) > A_0$ or $b(t_0) > B_0$, then $(a(t), b(t))$ with $t \geq t_0$ will enter Q in finite time. By Lemmas 3.1 and 3.2, if $a(t_0) > A_0$ or $b(t_0) > B_0$, $(a(t), b(t))$ is not global in time for $t \geq t_0$. Consequently when $(f(t), g(t))$ satisfies (23) for $t \geq t_0$, if $f(t_0) > A_0$ or $g(t_0) > B_0$, then $(f(t), g(t))$ is not global in time. We put

$$A = \tilde{c}_1 A_0, \quad B = \tilde{c}_2 B_0.$$

Hence if $(f_m(t), g_m(t))$ satisfies (22) for $t \geq t_0$ and

$$f_m\left(Y_m^2\right) > A|x_m|^{-\alpha_1} \quad \text{or} \quad g_m\left(Y_m^2\right) > B|x_m|^{-\alpha_2}$$

with

$$Y_m = \sqrt{|x_m|^2 t_0 / c_1}, \tag{26}$$

then $(f_m(t), g_m(t))$ is not global in time.

As a result of these arguments and a comparison principle, we have the following proposition:

Proposition 1. *Let $(F_m(t), G_m(t))$ satisfy differential inequalities (21) for $t \geq t_0$ and $m \in \mathbf{N}$. If for some m*

$$F_m\left(Y_m^2\right) > A|x_m|^{-\alpha_1} \quad \textit{or} \quad G_m\left(Y_m^2\right) > B|x_m|^{-\alpha_2}$$

with some $A, B > 0$ and Y_m defined in (26), then $(F_m(t), G_m(t))$ is not global in time.

Lemma 3.3. *Let u_0 and v_0 are BC and $(u_0, v_0) \not\equiv 0$, and let (u, v) be a solution of (1). Then for any $\tau > 0$ and $x \in \mathbf{R}^n$ and constants $\nu \geq 1$ and $C = C(n, \tau, u_0, v_0, K_1, K_2, p_1, p_2, \nu) > 0$ such that*

$$u(x, \tau) \geq Ce^{-\nu|x|^2} \quad \textit{and} \quad v(x, \tau) \geq Ce^{-\nu|x|^2}.$$

Proof. We may let $u_0(x) \not\equiv 0$ without loss of generality. Then we have

$$\begin{aligned} u(x,t) \geq S(t)u_0(x) &\geq \frac{1}{(4\pi t)^{n/2}} \int_{\mathbf{R}^n} e^{-|x-y|^2/4t} u_0(y)dy \\ &\geq \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/2t} \int_{\mathbf{R}^n} e^{-|y|^2/2t} u_0(y)dy. \end{aligned}$$

Then we have

$$u(x,\tau_1) \geq C_1 e^{-\nu_1 |x|^2} \tag{27}$$

for every $\tau_1 > 0$ with $C_1 = C_1(\tau_1, n, u_0) > 0$ and $\nu_1 \geq 1$.

Next we have for $t \geq 0$

$$\begin{aligned}
v(x,t) &\geq \int_0^t S(t-s) K_2(x,s) u^{p_2}(x,s) ds \\
&\geq \int_0^t \frac{1}{(4\pi(t-s))^{n/2}} \int_{\mathbf{R}^n} e^{-|x-y|^2/4(t-s)} K_2(y,s) u^{p_2}(y,s) dy ds \\
&\geq \int_0^t \frac{1}{(4\pi(t-s))^{n/2}} e^{-|x|^2/2(t-s)} \int_{\mathbf{R}^n} e^{-|y|^2/2(t-s)} K_2(y,s) u^{p_2}(y,s) dy ds.
\end{aligned}$$

Then by (27) we obtain for $\tau_2 > 2\tau_1$

$$\begin{aligned}
v(x,\tau_2) &\geq \int_{\tau_2/2}^{\tau_2} \frac{1}{(4\pi(\tau_2-s))^{n/2}} e^{-|x|^2/2(\tau_2-s)} \int_{\mathbf{R}^n} e^{-|y|^2/2(\tau_2-s)} K_2(y,s) u^{p_2}(y,s) dy ds \\
&\geq C_2 \int_{\tau_2/2}^{\tau_2} \frac{1}{(4\pi(\tau_2-s))^{n/2}} e^{-|x|^2/2(\tau_2-s)} ds \geq C_2 \frac{\tau_2}{2(2\pi\tau_2)^{n/2}} e^{-|x|^2/\tau_2}
\end{aligned}$$

with

$$C_2 = C_2(\tau_2, n, u_0, K_2) = \inf_{t \in (\tau_2/2, \tau_2)} \int_{\mathbf{R}^n} e^{-|y|^2/2(\tau_2-s)} K_2(y,s) u^{p_2}(y,s) dy.$$

Then we have

$$v(x,\tau_2) \geq C_3 e^{-\nu_2 |x|^2}$$

with $\nu_2 = \max\{1, 1/\tau_2\}$ and $C_3 = C_3(\tau_2, n, C_2)$. Put $C = \min\{C_1, C_3\}$ and $\nu = \max\{\nu_1, \nu_2\}$ and $\tau = \tau_2$. Then we have

$$u(x,\tau) \geq C e^{-\nu|x|^2} \quad \text{and} \quad v(x,\tau) \geq C e^{-\nu|x|^2}.$$

□

Lemma 3.4. *For $\sigma \geq 0$, $\nu \geq 1$, $x \in \mathbf{R}^n$ and $t \geq \tau$ with some $\tau > 0$, we have*

$$S(t)\chi_B(x)|x|^\sigma e^{-\nu|x|^2} \geq C(2t)^{\sigma/2}(2\nu t + 1)^{-(n+\sigma)/2} e^{-|x|^2/2t}$$

with some $C = C(\tau, \sigma, r, \nu, a, b) > 0$ and $B = B_a(b)$ with $a > 0$ and $b \in \mathbf{R}^n$, where χ_B is characteristic function of B such that $\chi_B(x) = 1$ ($x \in B$), 0 ($x \in \mathbf{R}^n \backslash B$), and $B_a(b)$ denotes the opened ball of radius a centered at b.

Proof. Since (8), We have

$$
\begin{aligned}
S(t)\chi_B(x)|x|^\sigma e^{-\nu|x|^2} &\geq (4\pi t)^{-n/2}e^{-|x|^2/2t}\int_B |y|^\sigma e^{-(1+2\nu t)|y|^2/2t}dy \\
&\geq \frac{(4\pi t)^{-n/2}e^{-|x|^2/2t}}{\tilde{\nu}^{n+\sigma}(t)}\int_{B_{\tilde{\nu}(t)a}(\tilde{\nu}(t)b)} |z|^\sigma e^{-|z|^2}dz,
\end{aligned}
$$

where $\tilde{\nu}(t) = \sqrt{(1+2\nu t)/2t}$. Since $1 \leq \sqrt{\nu} \leq \tilde{\nu}(t) \leq \tilde{\nu}(\tau)$ for $t \geq \tau$, we have for $t \geq \tau$

$$
\begin{aligned}
S(t)\chi_B(x)|x|^\sigma e^{-\nu|x|^2} &\geq \frac{(4\pi t)^{-n/2}e^{-|x|^2/2t}}{\tilde{\nu}^{\sigma+n}(t)}\int_{B_a(\tilde{\nu}(\tau)b)} |z|^\sigma e^{-|z|^2}dz \\
&\geq C_\tau t^{\sigma/2}(1+2\nu t)^{-(n+\sigma)/2}e^{-|x|^2/2t}
\end{aligned}
$$

with

$$
C_\tau = 2^{(n+\sigma)/2}(4\pi)^{-n/2}\int_{B_a(\tilde{\nu}(\tau)b)} |z|^\sigma e^{-|z|^2}dz.
$$

□

By Lemma 3.3, we can assume

$$
u_0(x) \geq Ce^{-\mu|x|^2}
$$

for some $C > 0$ and $\mu > 0$. Then by a semigroup property of $S(t)$, we have

$$
u(x,t) \geq S(t)u_0(x) \geq C(4\mu t+1)^{-n/2}e^{-|x|^2/(4t+1/\mu)}. \tag{28}
$$

Lemma 3.5. *Let v be second element of the solution of* (1)*. Then for $t \geq \tau$,*

$$
v(x,t) \geq Ct^{1+\sigma_2/2+q_2}(t+1)^{-np_2/2}e^{-|x|^2/t}
$$

with some $\tau > 0$ and $C = C(\tau, u_0, v_0, K_1, K_2, p_1, p_2)$.

Proof. It follows from (15) that

$$
\begin{aligned}
v(x,t) &\geq \int_0^t S(t-s)K_2(x,s)u(x,s)^{p_2}ds \\
&\geq \int_0^t S(t-s)\chi_{\tilde{B}_{r,1}}(x)C_L|x|^{\sigma_2}t^{q_2}u(x,s)^{p_2}ds.
\end{aligned}
$$

By (28), we have

$$
v(x,t) \geq C\int_0^t (4s+1/\mu)^{-np_2/2}s^{q_2}S(t-s)\chi_{\tilde{B}_{r,1}}(x)|x|^{\sigma_2}e^{-p_2|x|^2/(4s+1/\mu)}ds.
$$

By Lemma 3.4 with $\nu = p_2/(4s+1/\mu)$, we then have

$$\begin{aligned} v(x,t) &\geq C\int_0^t (4s+1/\mu)^{-np_2/2} s^{q_2} (t-s)^{\sigma_2/2} \\ &\qquad \times \left\{ \frac{2p_2(t-s)}{4s+1/\mu} + 1 \right\}^{-(n+\sigma_2)/2} e^{-|x|^2/2(t-s)} ds \\ &\geq C(2t+1/\mu)^{-np_2/2} (t/4)^{q_2} (t/2)^{\sigma_2/2} e^{-|x|^2/t} \int_{t/4}^{t/2} ds. \end{aligned}$$

Thus, the inequality of the lemma holds. $\square$

Lemma 3.6. *Let u be first element of the solution of* (1) *and $\alpha_1 \geq n$. Then for $t \geq a$*

$$u(x,t) \geq \begin{cases} Ct^{-n/2} e^{-|x|^2/t} \log(t/2a), & \text{if } \alpha_1 = n, \\ Ct^{-n/2} e^{-|x|^2/t} (t^{\tilde{p}} - (2a)^{\tilde{p}}), & \text{if } \alpha_1 > n \end{cases}$$

with $C = C(a, u_0, v_0, K_1, K_2, p_1, p_2, n)$, where $a > 0$ is a small constant and $\tilde{p} = (p_1 p_2 - 1)(\alpha_1 - n)/2$.

Proof. It follows from Lemmas 3.4 and 3.5 that

$$\begin{aligned} u(x,t) &\geq \int_a^t S(t-s) K_1(x,s) v(x,s)^{p_1} ds \\ &\geq C\int_a^t s^{(1+\sigma_2/2+q_2)p_1+q_1} (s+1)^{-np_1p_2/2} (t-s)^{\sigma_2/2} \\ &\qquad \times \left\{ \frac{2p_1(t-s)}{s} + 1 \right\}^{-(n+\sigma_1)/2} e^{-|x|^2/2(t-s)} ds \\ &\geq C(t/2)^{\sigma_1/2} t^{-(n+\sigma_1)/2} e^{-|x|^2/t} \int_a^{t/2} s^{\{-n(p_1p_2-1)+(2+\sigma_2+2q_2)p_1+\sigma_1+2q_1\}/2} ds \end{aligned}$$

for small $a > 0$. Since

$$\{-n(p_1p_2-1) + (2+\sigma_2+2q_2)p_1 + \sigma_1 + 2q_1\}/2 = (p_1p_2-1)(\alpha_1-n)/2 - 1,$$

this proves the inequality of the lemma. $\square$

Proof of Theorem 1.

Define Y_m same as in (26). First we consider the case (i). We may assume $\alpha_1 \geq \alpha_2$. From the definition, we have $\alpha_1 \geq n$. By Lemma 3.6, since

$x \in \tilde{B}_{r,m}$, we have

$$F_m\left(Y_m^2\right) \geq CY_m^{-n} H_m \int_{\tilde{B}_{r,m}} \exp\left(-\frac{|x|^2}{Y_m^2}\right)\phi_m(x)dx$$
$$\geq C|x_m|^{-n}\left(\frac{t_0}{c_1}\right)^{-n/2} H_m \exp\left(-\frac{(1+r)^2 c_1}{t_0}\right),$$

where

$$H_m = \begin{cases} \log(Y_m/2a), & \text{if } \alpha_1 = n, \\ (Y_m^{\tilde{p}} - (2a)^{\tilde{p}}), & \text{if } \alpha_1 > n \end{cases}$$

with $C = C(a, u_0, v_0, K_1, K_2, p_1, p_2, n)$ and $\tilde{p}$ defined in Lemma 3.6. Since $\alpha_1 \geq n$, it follows that

$$|x_m|^{\alpha_1} F_m\left(Y_m^2\right) \geq C|x_m|^{\alpha_1 - n}\left(\frac{t_0}{c_1}\right)^{-n/2} H_m \exp\left(-\frac{(1+r)^2 c_1}{t_0}\right) > A$$

for m large enough. Thus, $(F_m(t), G_m(t))$ is not global in time by Proposition 1.

Next, we consider the case (ii). From (i), we should consider only the case

$$\max\{\alpha_1, \alpha_2\} < n. \tag{29}$$

Since $u(x,t) \geq S(t)u_0(x)$, it follows that

$$F_m\left(Y_m^2\right) \geq (4\pi Y_m^2)^{-\frac{n}{2}} \int_{\tilde{B}_{r,m}} \left\{\int_{\mathbf{R}^n} \exp\left(-\frac{|x-y|^2}{4Y_m^2}\right) u_0(y)dy\right\}\phi_m(x)dx.$$

Since $x \in \tilde{B}_{r,m}$, we have

$$\exp\left(-\frac{|x-y|^2}{4Y_m^2}\right) \geq \exp\left(-\frac{(|x|^2+|y|^2)}{2Y_m^2}\right) \geq \exp\left(-\frac{((1+r)^2|x_m|^2+|y|^2)}{2Y_m^2}\right)$$
$$= \exp\left(-\frac{(1+r)^2 c_1}{t_0}\right)\exp\left(-\frac{|y|^2}{2Y_m^2}\right).$$

Since $\int_{\tilde{B}_{r,m}} \phi_m(x)dx = 1$, we have

$$F_m\left(Y_m^2\right) \geq (4\pi Y_m^2)^{-n/2} \exp\left(-\frac{(1+r)^2 c_1}{t_0}\right) \int_{\mathbf{R}^n} \exp\left(-\frac{|y|^2}{2Y_m^2}\right) u_0(y)dy$$
$$= \left(\frac{1}{2\pi}\right)^{n/2} \exp\left(-\frac{(1+r)^2 c_1}{t_0}\right) \int_{\mathbf{R}^n} \exp(-|z|^2) u_0\left(|x_m|\sqrt{\frac{2t_0}{c_1}}z\right) dz.$$

Since $u_0 \in \tilde{I}_{a_1}$ with $a_1 < \alpha_1 < n$ by (29), we have

$$|x_m|^{\alpha_1} F_m\left(Y_m^2\right) \geq \left(\frac{1}{2\pi}\right)^{n/2} |x_m|^{\alpha_1 - a_1} \left(\frac{2t_0}{c_1}\right)^{-a_1/2} \times \exp\left(-\frac{(1+r)^2 c_1}{t_0}\right) \int_{\mathbf{R}^n} \exp(-|z|^2)|z|^{-a_1} dz > A$$

for sufficiently large m. If $v_0 \in \tilde{I}_{a_2}$ with $a_2 < \alpha_2 < n$, we similarly have

$$|x_m|^{\alpha_2} G_m(Y_m^2) > B$$

for m large enough. Thus, $(F_m(t), G_m(t))$ is not global in time by Proposition 1.

Finally, we consider the case (iii). We should show the case $u_0 \geq Me^{-\nu_0|x|^2}$. Since

$$F_m\left(Y_m^2\right) \geq (4\pi Y_m^2)^{-\frac{n}{2}} \exp\left(-\frac{(1+r)^2 c_1}{t_0}\right) \int_{\mathbf{R}^n} \exp\left(-\frac{c_1}{2Y_m^2}\right) u_0(y) dy,$$

we have

$$\begin{aligned}
&F_m\left(Y_m^2\right) \\
&\quad \geq M(4\pi Y_m^2)^{-\frac{n}{2}} \exp\left(-\frac{(1+r)^2 c_1}{t_0}\right) \int_{\mathbf{R}^n} \exp\left\{-\left(\frac{1}{2Y_m^2} + \nu_0\right)|y|^2\right\} dy \\
&\quad = M(4\pi Y_m^2)^{-\frac{n}{2}} \left(\frac{1}{2Y_m^2} + \nu_0\right)^{-\frac{n}{2}} \exp\left(-\frac{(1+r)^2 c_1}{t_0}\right) \int_{\mathbf{R}^n} \exp\left(-|z|^2\right) dz \\
&\quad = M(2 + 4\nu_0 Y_m^2)^{-\frac{n}{2}} \exp\left(-\frac{(1+r)^2 c_1}{t_0}\right).
\end{aligned}$$

The other case, we have

$$G_m(Y_m^2) \geq M(2 + 4\nu_0 Y_m^2)^{-\frac{n}{2}} \exp\left(-\frac{(1+r)^2 c_1}{t_0}\right).$$

Thus, if we choose $m = 1$ and $M > \max\{A, B\}(2 + 4\nu_0 Y_1^2)^{n/2} \exp((1 + r)^2 c_1/t_0)$, the condition of Proposition 1 is satisfied. Thus, $(F_1(t), G_1(t))$ is not global in time. $\square$

4 Proof of Theorem 2

In this section we require $\max\{\alpha_1, \alpha_2\} < n$, and treat the existence of global solutions in time of (1). Here, we take the same strategy as in [12] and [19].

First note that condition (9) can be replaced by $(u_0, v_0) \in I^{\tilde{a}_1} \times I^{\tilde{a}_2}$ since we have $I^{a_1} \times I^{a_2} \subset I^{\tilde{a}_1} \times I^{\tilde{a}_2}$. Then, to establish Theorem 2, we have only to consider the special case $\tilde{a}_1 = a_1$ and $\tilde{a}_2 = a_2$. As is easily seen, in this case condition (11) is equivalent to

$$p_i \min\{a_j, n\} - a_i > 2 + \sigma_i + 2q_i \quad ((i,j) = (1,2), (2,1)). \tag{30}$$

We set for $\gamma > 0$

$$\eta_\gamma(x,t) = S(t)\langle x\rangle^{-\gamma}.$$

Lemma 4.1. *([12, Lemma 4.2]) We have in* $\mathbf{R}^n \times (0,\infty)$,

$$(t+1)^q \langle x\rangle^\sigma \eta_a(x,t)^p \\ \leq \begin{cases} C(1+t)^{(\sigma+2q+b-\min\{n,a\}p)/2}\eta_b(x,t), & \text{if } a \neq n, \\ C(1+t)^{(\sigma+2q+b-np)/2}[\log(2+t)]^p \eta_b(x,t), & \text{if } a = n. \end{cases} \tag{31}$$

We define the Banach space X as

$$X = \{v; \|v/\eta_{a_2}\|_\infty < \infty\},$$

where

$$\|w\|_\infty = \sup_{(x,t)\in\mathbf{R}^n\times(0,\infty)} |w(x,t)|.$$

From (14) and (15), we have

$$v(x,t) = V(u_0, v_0, v), \tag{32}$$

where

$$\begin{aligned} V(u_0,v_0,v) =& S(t)v_0(x) + \int_0^t S(t-s)K_2(x,s) \\ & \times \left(S(s)u_0(x) + \int_0^s S(s-r)K_1(x,r)v(x,r)^{p_1}dr\right)^{p_2} ds. \end{aligned}$$

If V is a strict contraction, then its fixed point yields a solution of (1). Moreover, using that $(a+b)^p \leq 2^{p-1}(a^p + b^p)$ for $a > 0$, $b > 0$ and $p \geq 1$, we obtain

$$V(u_0, v_0, v) \leq T(u_0, v_0) + \Gamma(v),$$

where

$$T(u_0,v_0) = S(t)v_0(x) + 2^{p_2-1}\int_0^t S(t-s)K_2(x,s)(S(s)u_0(x))^{p_2} ds,$$

$$\Gamma(v) = 2^{p_2-1}\int_0^t S(t-s)K_2(x,s)\left(\int_0^s S(s-r)K_1(x,r)v(x,r)^{p_1}dr\right)^{p_2} ds.$$

Lemma 4.2. (i) *Let* (u_0, v_0) *satisfy* (9)*. Then* $T(u_0, v_0) \in X$ *and*

$$\|T(u_0, v_0)/\eta_{a_2}\|_\infty \le C_a\{\|v_0\|_{\infty,a_2} + \|u_0\|^{p_2}_{\infty,a_1}\}$$

with some $C_a > 0$.

(ii) *Let* v *be a second element of the solution of* (1)*. Then* Γ *maps* X *into itself and*

$$\|\Gamma(v)/\eta_{a_2}\|_\infty \le C_b\|v/\eta_{a_2}\|^{p_1p_2}_\infty$$

with some $C_b > 0$.

Proof. (i) First, it is easily seen that $S(t)v_0(x) \le \|v_0\|_{\infty,a_2}\eta_{a_2}(x,t)$. Next, from (30) and (31) in Lemma 4.1, we obtain

$$\begin{aligned}
&\int_0^t S(t-s)K_2(x,s)(S(s)u_0(x))^{p_2}ds \\
&\quad \le \|u_0\|^{p_2}_{\infty,a_1}\int_0^t S(t-s)C_U(s+1)^{q_2}\langle x\rangle^{\sigma_2}\eta_{a_1}(x,s)^{p_2}ds \le C\|u_0\|^{p_2}_{\infty,a_1}\eta_{a_2}(x,t).
\end{aligned}$$

Thus, we have

$$|T(u_0, v_0)| \le C\eta_{a_2}(x,t)\{\|v_0\|_{\infty,a_2} + \|u_0\|^{p_2}_{\infty,a_1}\}.$$

This implies assertion (i).

(ii) Similarly as above, it follows from (30) and (31) that

$$\begin{aligned}
\Gamma(v) \le& C\|v/\eta_{a_2}\|^{p_1p_2}_\infty \int_0^t S(t-s)C_U(s+1)^{q_2}\langle x\rangle^{\sigma_2} \\
&\times\left(\int_0^s S(s-r)C_U(r+1)^{q_1}\langle x\rangle^{\sigma_1}\eta_{a_2}(x,r)^{p_1}dr\right)^{p_2}ds \\
\le& C\|v/\eta_{a_2}\|^{p_1p_2}_\infty \int_0^t S(t-s)C_U(s+1)^{q_2}\langle x\rangle^{\sigma_2}\eta_{a_1}(x,s)^{p_2}ds \\
\le& C\|v/\eta_{a_2}\|^{p_1p_2}_\infty\eta_{a_2}(x,t).
\end{aligned}$$

Thus, assertion (ii) is concluded. ☐

Proof of Theorem 2

Let

$$C_a\left(\|v_0\|_{\infty,a_2} + \|u_0\|^{p_2}_{\infty,a_1}\right) \le m,$$

$\|u/\eta_{a_1}\|_\infty \le m$, $\|v/\eta_{a_2}\|_\infty \le m$, $B_m = \{v \in X; \|v/\eta_{a_2}\|_\infty \le 2m\}$ and $P = \{v \in X; v \ge 0\}$, where the constant C_a is appeared in Lemma 4.2 (i). Then we have $\|T(u_0, v_0)/\eta_{a_2}\|_\infty \le m$. We shall show that $V(u_0, v_0, v)$ is a strict contraction of $B_m \cap P$ into itself provided m is small enough.

We shall show that V maps $B_m \cap P$ into $B_m \cap P$. If m is small enough, then

$$\begin{aligned}\|V(u_0,v_0,v)/\eta_{a_2}\|_\infty &\le \|T(u_0,v_0)/\eta_{a_2}\|_\infty + \|\Gamma(v)/\eta_{a_2}\|_\infty \\ &\le m + C_b(2m)^{p_1p_2} \le 2m\end{aligned}$$

by Lemma 4.2 (ii). This proves that V maps $B_m \cap P$ into $B_m \cap P$.

Now, we show that $V(u_0,v_0,v)$ is a strict contraction on $B_m \cap P$. Note that

$$\begin{aligned}|V(u_0,v_0,v_1) - V(u_0,v_0,v_2)| &\le \int_0^t S(t-s)K_2(x,s) \\ \times \Bigg| \left(S(s)u_0(x) + \int_0^s S(s-r)K_1(x,r)v_1(x,r)^{p_1}dr\right)^{p_2} \\ - \left(S(s)u_0(x) + \int_0^s S(s-r)K_1(x,r)v_2(x,r)^{p_1}dr\right)^{p_2}\Bigg| ds.\end{aligned}$$

Since $|a^p - b^p| \le p(a+b)^{p-1}|a-b|$ for $a \ge 0$, $b \ge 0$ and $p \ge 1$, we can estimate as follows,

$$\begin{aligned}|V(u_0,v_0,v_1) - V(u_0,v_0,v_2)| &\le p_2 \int_0^t S(t-s)K_2(x,s) \\ \times \left(2S(s)u_0(x) + \int_0^s S(s-r)K_1(x,r)(v_1(x,r)^{p_1} + v_2(x,r)^{p_1})dr\right)^{p_2-1} \\ \times \left|\int_0^s S(s-r)K_1(x,r)(v_1(x,r)^{p_1} - v_2(x,r)^{p_1})dr\right| ds.\end{aligned}$$

We put

$$\begin{aligned}A(x,s) &= \left(2S(s)u_0(x) + \int_0^s S(s-r)K_1(x,r)(v_1(x,r)^{p_1} + v_2(x,r)^{p_1})dr\right)^{p_2-1}, \\ B(x,s) &= \left|\int_0^s S(s-r)K_1(x,r)(v_1(x,r)^{p_1} - v_2(x,r)^{p_1})dr\right|.\end{aligned}$$

Thus, we may express

$$|V(u_0,v_0,v_1) - V(u_0,v_0,v_2)| \le p_2 \int_0^t S(t-s)K_2(x,s)A(x,s)B(x,s)ds.$$

Since $(a+b)^p \le 2^{\max\{p-1,0\}}(a^p+b^p)$ for $a \ge 0$, $b \ge 0$ and $p \ge 0$, with $v = \max\{v_1, v_2\}$, we obtain

$$
\begin{aligned}
A(x,s) \le & 2^{\max\{p_2-2,0\}}\Big\{(2S(s)u_0(x))^{p_2-1} \\
& + \left(\int_0^s S(s-r)C_U(r+1)^{q_1}\langle x\rangle^{\sigma_1} 2v(x,r)^{p_1}dr\right)^{p_2-1}\Big\}
\end{aligned}
$$

and

$$
\begin{aligned}
B(x,s) &\le \int_0^s S(s-r)C_U(r+1)^{q_1}\langle x\rangle^{\sigma_1}|v_1(x,r)^{p_1} - v_2(x,r)^{p_1}|dr \\
&\le \int_0^s S(s-r)C_U(r+1)^{q_1}\langle x\rangle^{\sigma_1} \\
&\quad \times p_1(v_1(x,r)+v_2(x,r))^{p_1-1}|v_1(x,r)-v_2(x,r)|dr.
\end{aligned}
$$

From (30) and (31) in Lemma 4.1, we have

$$
\begin{aligned}
A(x,s) &\le 2^{\max\{p_2-2,0\}}\{(2\|u_0\|_{\infty,a_1}\eta_{a_1}(x,s))^{p_2-1} \\
&\quad + \left(2\|v\|_{\infty,a_2}^{p_1}\int_0^s S(s-r)C_U(r+1)^{q_1}\langle x\rangle^{\sigma_1}\eta_{a_2}^{p_1}(x,r)dr\right)^{p_2-1} \\
&\le 2^{\max\{p_2-2,0\}}\left\{(2m)^{p_2-1}\eta_{a_1}^{p_2-1}(x,s) + (C_1 m^{p_1})^{p_2-1}\,\eta_{a_1}^{p_2-1}(x,s)\right\}
\end{aligned}
$$

and

$$
\begin{aligned}
B(x,s) &\le \int_0^s S(s-r)2C_U(r+1)^{q_1}\langle x\rangle^{\sigma_1}p_1 v(x,r)^{p_1-1}|v_1(x,r)-v_2(x,r)|dr \\
&\le \int_0^s S(s-r)2C_U(r+1)^{q_1}\langle x\rangle^{\sigma_1}\eta_{a_2}^{p_1}(x,r)(v(x,r)/\eta_{a_2}(x,r))^{p_1-1} \\
&\quad \times p_1(|v_1(x,r)-v_2(x,r)|/\eta_{a_2}(x,r))dr.
\end{aligned}
$$

We can take m satisfying $(2m)^{p_2-1} + (C_1 m^{p_1})^{p_2-1} \le 2^{p_2}m^{(p_2-1)/2}$. Then we have

$$
\begin{aligned}
&|V(u_0,v_0,v_1) - V(u_0,v_0,v_2)| \\
\le & C_2\int_0^t S(t-s)(s+1)^{q_2}\langle x\rangle^{\sigma_2}\left(2^{p_2}m^{(p_2-1)/2}\eta_{a_1}^{p_2-1}(x,s)\right)\eta_{a_1}(x,s) \\
& \times \|v/\eta_{a_2}\|_\infty^{p_1-1}\|v_1/\eta_{a_2} - v_2/\eta_{a_2}\|_\infty ds \\
\le & C_3 m^{p_1+p_2/2-3/2}\int_0^t S(t-s)(s+1)^{q_2}\langle x\rangle^{\sigma_2}\eta_{a_1}^{p_2}(x,s)ds\|v_1/\eta_{a_2} - v_2/\eta_{a_2}\|_\infty \\
\le & C_4 m^{p_1+p_2/2-3/2}\eta_{a_2}(x,t)\|v_1/\eta_{a_2} - v_2/\eta_{a_2}\|_\infty.
\end{aligned}
$$

Since $p_1, p_2 \geq 1$ and $p_1 p_2 > 1$, we obtain for some $\rho < 1$

$$\begin{aligned}
&\|V(u_0, v_0, v_1)/\eta_{a_2} - V(u_0, v_0, v_2)/\eta_{a_2}\|_\infty \\
&\leq C_4 m^{p_1 + p_2/2 - 3/2} \|v_1/\eta_{a_2} - v_2/\eta_{a_2}\|_\infty \leq \rho \|v_1/\eta_{a_2} - v_2/\eta_{a_2}\|_\infty
\end{aligned}$$

with m small enough. Then V is a strict contraction of $B_m \cap P$ into itself. Hence, there exists a unique fixed point $v \in X$ which solves (32). We substitute v into (14). Then (u, v) solves (14) and (15). Moreover, since $v \in B_m$, we find

$$v(x,t) \leq C_5 S(t) \langle x \rangle^{-a_2}.$$

Substituting this into (14), we have

$$\begin{aligned}
u(x,t) \leq & \|u_0\|_{\infty, a_1} \eta_{a_1}(x,t) + C_6 \int_0^t S(t-s) C_U (s+1)^{q_1} \langle x \rangle^{\sigma_1} \eta_{a_2}^{p_1}(x,s) ds \\
\leq & m \eta_{a_1}(x,t) + C_7 \eta_{a_1}(x,t) \leq C_8 \eta_{a_1}(x,t).
\end{aligned}$$

Then $u \in B_m$; that is,

$$u(x,t) \leq C_8 S(t) \langle x \rangle^{-a_1}.$$

Put $C = \max\{C_5, C_8\}$. Then the proof of Theorem 2 is completed. $\square$

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References

[1] C. Bandle and H. A. Levine, *On the existence and nonexistence of global solution of reaction-diffusion equation in sectorial domains,* Trans. Amar. Math. Sec. **316** (1989), 595–622.

[2] M. Escobedo and M. A. Herrero, *Boundness and blow up for a semilinear reaction-diffusion system,* J. Diff. Eqns. **89** (1991), 176–202.

[3] H. Fujita, *On the blowing up of solutions of the Cauchy problem for* $u_t = \Delta u + u^{1+\alpha}$, J. Fac. Sci. Univ. Tokyo Sect. A Math. **16** (1966), 109–124.

[4] Y. Giga and N. Umeda, *Blow-up directions at space infinity for solutions of semilinear heat equations*, Bol. Soc. Parana. Mat. **23** (2005), 9–28.

[5] M. Guedda and M. Kirane, *Criticality for some evolution equations*, Differential Equations **37** (2001), 540–550.

[6] T. Hamada, *Nonexistence of global solutions of parabolic equations in conical domains*, Tsukuba J. Math. **19** (1995), 15-25.

[7] K. Hayakawa, *On nonexistence of global solution of some semilinear parabolic equations*, Proc. Japan. Acad. **49** (1973), 503–505.

[8] M. Kirane and M. Qafsaoui, *Global nonexistence for the Cauchy problem of some nonlinear reaction-diffusion systems*, J. Math. Anal. Appli. **268** (2002), 217-243.

[9] K. Kobayashi, T Sirao and H. Tanaka, *On glowing up problem for semilinear heat equations*, J. Math. Soc. Japan **29** (1977), 407-424.

[10] T.-Y. Lee and W.-M. Ni, *Global existence, large time behavior and life span on solutions of semilinear Cauchy problem*, Trans. Amer. Math. Soc. **333** (1992), 365-378.

[11] K. Mochizuki, *Blow-up, life-span and large time behavior of solutions of a weakly coupled system of reaction-diffusion equations*, Adv. Math. Appl. Sci. **48**, World Scientific 1998, 175-198.

[12] K. Mochizuki and Q. Huang, *Existence and behavior of solutions for a weakly coupled system of reaction-diffusion equations*, Methods and Applications of Analysis **5** (2) (1998), 109-124.

[13] R. G. Pinsky, *Existence and nonexistence of global solutions for $u_t = \Delta u + a(x)u^p$ in $\mathbf{R}^n$*, J. Differential Equations **133** (1997), 152-177.

[14] Y.-W. Qi, *The critical exponents of parabolic equations and blow-up in $\mathbf{R}^n$*, Proc. Roy. Soc. Edinburgh Sect. **128A** (1998), 123-136.

[15] Y.-W. Qi and H. A. Levine, *The critical exponent of degenerate parabolic systems,* Z.Angew Math. Phys. **44** (1993), 249-265.

[16] M. Shimojyo, *On blow-up phenomenon at space infinity and its locality for semilinear heat equations (In Japanese)*, Master's Thesis, The University of Tokyo (2005).

[17] Y. Uda, *The critical exponent for a weakly coupled system of the generalized Fujita type reaction-diffusion equations*, Z.Angew Math. Phys. **46** (1995), 366-383.

[18] N. Umeda, *Blow-up and large time behavior of solutions of a weakly coupled system of reaction-diffusion equations*, Tsukuba J. Math. **27** (2003) 31-46.

[19] N. Umeda, *Existence and nonexistence of global solutions of a weakly coupled system of reaction-diffusion equations*, Comm. Appl. Anal. **10** (2006) 57-78.

[20] F. B. Weissler, *Existence and nonexistence of global solutions for semilinear heat equation*, Israel J. Math. **38** (1981) 29–40.

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